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# Induced subgraphs of hypercubes

*Geir Agnarsson* *

**Abstract**

Let $Q_k$ denote the $k$-dimensional hypercube on $2^k$ vertices. A vertex in a subgraph of $Q_k$ is *full* if its degree is $k$. We apply the Kruskal-Katona Theorem to compute the maximum number of full vertices an induced subgraph on $n \leq 2^k$ vertices of $Q_k$ can have, as a function of $k$ and $n$. This is then used to determine $\min(\max(|V(H_1)|, |V(H_2)|))$ where (i) $H_1$ and $H_2$ are induced subgraphs of $Q_k$, and (ii) together they cover all the edges of $Q_k$, that is $E(H_1) \cup E(H_2) = E(Q_k)$.

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## 1 Introduction

The maximum number $f(n)$ of edges of an induced subgraph on $n$ vertices of the hypercube $Q_k$, where $k \geq \lceil \lg n \rceil$, has been studied extensively in [8], [14], [5], [4], and [3] to name a few articles. The function $f(n)$ satisfies, and is determined by, the well-known divide-and-conquer maximin recurrence

$$f(n) = \max_{\substack{n_1+n_2=n \\ n_1,n_2\geq 1}} (\min(n_1, n_2) + f(n_1) + f(n_2)), \quad (1)$$

and can be expressed compactly by the formula $f(n) = \sum_{i=0}^{n-1} s(i)$, where $s(i)$ is the sum of the digits of $i$ when expressed as a binary number. The function $f$ and its number sequence $(f(n))_{n=0}^{\infty} = (0, 1, 2, 4, 5, 7, 9, 12, 13, 15, 17, 20, \dots)$ is given in [2, A000788], where it is presented by a different recursion. The divide-and-conquer maximin recurrence (1) is one of the most studied recurrences, especially since it occurs naturally when analysing wort-case scenarios in sorting algorithms [13]. The maximin recurrence (1) is also one of the few such maximin recurrences that have a solution $f(n)$ that can be expressed explicitly by a formula.

Clearly the hypercube $Q_k$ is a subgraph of the $k$-dimensional rectangular grid graph $\mathbb{Z}^k$. It is interesting to note that for $k \geq \lceil \lg n \rceil$ the maximum number of edges of an induced subgraph on $n$ vertices of $\mathbb{Z}^k$ is the same if we restrict to $Q_k$, namely $f(n)$. However, if we consider $k$ fixed and consider the maximum number $g_k(n)$ of edges of an induced subgraph on $n$ vertices of the grid graph $\mathbb{Z}^k$, then the only cases where a formula for $g_k(n)$ is known is for $k \in \{1, 2\}$: trivially $g_1(n) = n-1$, and $g_2(n) = \lceil 2n - 2\sqrt{n} \rceil$ as proved in [7]. For $k \geq 3$ no formula for $g_k(n)$ is known, but the first few terms of $(g_3(n))_{n=1}^{\infty} = (0, 1, 2, 4, 5, 7, 9, 12, 13, 15, 17, 20, \dots)$ is given heuristically in [1, A007818]. – In short, considering $k$ fixed (and hence not allowing conveniently large dimensions) makes it harder to solve such maximin problems.

*Department of Mathematical Sciences, George Mason University, MS 3F2, 4400 University Drive, Fairfax, VA 22030, geir@math.gmu.edu

The purpose of this article is to consider a related problem of induced subgraphs on $n$ vertices of the hypercube $Q_k$ where we consider $k$ fixed. A vertex of a subgraph of $Q_k$ is called *full* in the subgraph if its degree is $k$. If we let $\phi_k(n)$ be the maximum number of full vertices an induced subgraph on $n$ vertices of $Q_k$ can have, then (i) we show that $\phi_k(n)$ satisfies a divide-and-conquer maximin recurrence (8), and (ii) we derive its solution, namely the formula for $\phi_k(n)$ given in Theorem 3.2. We then apply the formula for $\phi_k(n)$ to (iii) determine the min-max function $\min(\max(|V(H_1)|, |V(H_2)|))$ where both $H_1$ and $H_2$ are induced subgraphs of $Q_k$, and together they cover all the edges of $Q_k$. We show that this min-max function is given by the formula in Theorem 4.1.

The remainder of the paper is organized as follows.

In Section 2 we recall the celebrated Katona-Kruskal Theorem that describes when exactly an integral vector of $\mathbb{Z}^{d+1}$ is an $f$-vector of a $(d-1)$-dimensional simplicial complex. We then derive some helpful tools: Claim 2.5 and Lemmas 2.7, 2.8, and 2.9, that we will use in the following section.

In Section 3 we use what we have derived in Section 2 to derive our main Theorem 3.2 that determines the exact maximum number of full vertices an induced subgraph on $n$ vertices of $Q_k$ can have.

In the final Section 4 we apply Theorem 3.2 from the previous section prove Theorem 4.1, that determines $\min(\max(|V(H_1)|, |V(H_2)|))$, the function of $k \in \mathbb{N}$ where $H_1$ and $H_2$ are induced subgraphs of $Q_k$, and together $H_1$ and $H_2$ cover all the edges of $Q_k$.

**Notation and terminology** The set of integers will be denoted by $\mathbb{Z}$ and the set of natural numbers $\{1, 2, 3, \ldots\}$ by $\mathbb{N}$. For $n \in \mathbb{N}$ let $[n] = \{1, \ldots, n\}$. For a set $X$ denote the set of all subsets of $X$ by $2^X$. Denote the subsets of $X$ of cardinality $i$ by $\binom{X}{i}$, so for $X$ finite we have $\left|\binom{X}{i}\right| = \binom{|X|}{i}$. For $\mathcal{S} \subseteq 2^X$ and $y \notin X$, let $\mathcal{S} \uplus \{y\} = \{S \cup \{y\} : S \in \mathcal{S}\}$.

Unless otherwise stated, all graphs in this article will be finite, simple and undirected. For a graph $G$, its set of vertices will be denoted by $V(G)$ and its set of edges by $E(G)$. Clearly $E(G) \subseteq \binom{V(G)}{2}$ the set of all 2-element subsets of $V(G)$. We will denote an edge with endvertices $u$ and $v$ by $uv$ instead of the actual 2-set $\{u, v\}$. By an *induced subgraph* $H$ of $G$ we mean a subgraph $H$ such that $V(H) \subseteq V(G)$ in the usual set theoretic sense, and such that if $u, v \in V(H)$ and $uv \in E(G)$, then $uv \in E(H)$. If $U \subseteq V(G)$ then the subgraph of $G$ induced by $V$ will be denoted by $G[U]$.

For $k \in \mathbb{N}$ the *hypercube* $Q_k$ in our context is a simple graph with the $2^k$ vertices $\{0, 1\}^k$, and where two vertices $\tilde{x}, \tilde{y} \in \{0, 1\}^k$ are adjacent iff the *Manhattan distance* $d(\tilde{x}, \tilde{y}) = \sum_{i=1}^{k} |x_i - y_i| = 1$. So, two vertices are connected iff they only differ in one coordinate, in which they differ by $\pm 1$. The vertices of the hypercube $Q_k$ are more commonly viewed as binary strings of length $k$ instead of actually points in the $k$-dimensional Euclidean space. In that case the Manhattan distance is called the called the *Hamming distance*. We will not make a specific distinction between these two slightly different presentations of the hypercube $Q_k$. In many situations it will be convenient to partition the hypercube $Q_k$ into two copies of $Q_{k-1}$ where corresponding vertices in each copy are connected by and edge. If $b \in \{0, 1\}$ and $B_b = \{\tilde{x} \in \{0, 1\}^k : x_k = b\}$ is the set of binary strings of length $k$ with $k$-th bit equal to $b$, then clearly each of $Q^0_{k-1} := Q_k[B_0]$ and $Q^1_{k-1} := Q_k[B_1]$ are induced subgraphs isomorphic to $Q_{k-1}$, and (i) $V(Q_k) = V(Q^0_{k-1}) \cup V(Q^1_{k-1}) = B_0 \cup B_1$ is a partition and (ii) $E(Q_k) = E(Q^0_{k-1}) \cup E(Q^1_{k-1}) \cup C_{k-1}$ is also a partition of the edges where

$$C_{k-1} = \{\{(\tilde{x}, 0), (\tilde{x}, 1)\} : \tilde{x} \in V(Q_{k-1})\}.$$

For $b \in \{0, 1\}$ and $\tilde{x} \in V(Q_{k-1})$, the *copy* of $(\tilde{x}, b) \in V(Q^b_{k-1})$ is the vertex $(\tilde{x}, 1-b) \in V(Q^{1-b}_{k-1})$, and these well be referred as *copies*. This decomposition of $Q_k$ will be denoted by $Q_k = Q^0_{k-1} \boxplus Q^1_{k-1}$.

## 2 Some properties of the upper boundary function

The following proposition on the *binomial representation* of an integer is stated in [16] and in [9], and a simple proof by greedy algorithm can be found in the latter citation.

**Proposition 2.1** *For $m, i \in \mathbb{N}$ there is a* unique binomial representation (UBR) *of $m$ as*

$$m = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j} \tag{2}$$

*where $n_i > n_{i-1} > \cdots > n_j \geq j \geq 1$.*

For $m, i \in \mathbb{N}$ one can use the UBR to define the *upper $i$-boundary of $m$*

$$m^{(i)} = \binom{n_i}{i+1} + \binom{n_{i-1}}{i} + \cdots + \binom{n_j}{j+1}.$$

**Proposition 2.2** *For a fixed $i \in \mathbb{N}$ the function $m \mapsto m^{(i)}$ is increasing.*

*Proof.* For $m, i \in \mathbb{N}$ consider the UBR of $m$ as in (2).

If $j \geq 2$, then

$$m + 1 = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j} + \binom{j-1}{j-1}$$

is the UBR of $m+1$ and so $(m+1)^{(i)} = m^{(i)} + \binom{j-1}{j} = m^{(i)}$.

Otherwise $j = 1$, and hence there is a largest index $\ell \in [i]$ such that $n_h = n_1 + h - 1$ for all $h \in \{1, \ldots, \ell\}$. In this case we have $n_{\ell+1} > n_\ell + 1 = n_1 + \ell$ and

$$\begin{aligned} m + 1 &= \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_{\ell+1}}{\ell+1} + \binom{n_1 + \ell - 1}{\ell} + \ldots + \binom{n_1}{1} + 1 \\ &= \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_{\ell+1}}{\ell+1} + \binom{n_1 + \ell}{\ell} \end{aligned}$$

and hence

$$(m+1)^{(i)} - m^{(i)} = \binom{n_1 + \ell}{\ell + 1} - \left[\binom{n_1 + \ell - 1}{\ell + 1} + \cdots + \binom{n_1}{2}\right] = \binom{n_1}{1} = n_1.$$

□

We see from the above proof when exactly the function $m \mapsto m^{(i)}$ is strictly increasing; namely, whenever the last binomial coefficient in the UBR of $m$ has the form $\binom{n_1}{1}$, then $(m+1)^{(i)} = m^{(i)} + n_1 > m^{(i)}$. In particular, for $i < n$ we have

$$\binom{n}{i} - 1 = \binom{n-1}{i} + \binom{n-2}{i-1} + \cdots + \binom{n-i}{1}$$

and hence the following observation.

**Observation 2.3** *For $i, n \in \mathbb{N}$ with $i < n$ then*

$$\left(\binom{n}{i} - 1\right)^{(i)} = \binom{n}{i}^{(i)} - (n-i) < \binom{n}{i}^{(i)}.$$

In this article the *$f$-vector* of a $(d-1)$-dimensional simplicial complex $\Delta$ will be given by $\tilde{f}(\Delta) = \tilde{f} = (f_{-1}, f_0, f_1, \ldots, f_{d-1}) \in \mathbb{Z}^{d+1}$ where $f_i = f_i(\Delta)$ denotes the number of $i$-dimensional faces of $\Delta$. For convenience we include the empty face $\emptyset$ in $\Delta$. Since by convention $\dim(\emptyset) = -1$ then we always have $f_{-1} = 1$. The following celebrated result proved independently by Kruskal [11], Katona [10] and Schützenberger [15], is usually called the *Kruskal-Katona Theorem*, since it was not realized at first that Schützenberger had the first proof. It is sometimes called the *KKS Theorem* for short.

**Theorem 2.4** *An integral vector $\tilde{f} = (f_{-1}, f_0, f_1, \ldots, f_{d-1}) \in \mathbb{Z}^{d+1}$ is an $f$-vector of a $(d-1)$-dimensional simplicial complex $\Delta$ if and only if $0 < f_i \leq f_{i-1}^{(i)}$ for each $i \in \{1, \ldots, d-1\}$.*

Although we will not regurgitate the proof of Theorem 2.4 here, a few comments about it will be useful for us here in this section. – Note that a simplicial complex $\Delta$ on vertices $V = \{v_1, \ldots, v_n\}$ can be viewed as an abstract simplicial complex; a collection of subsets of $[n]$ satisfying (1) $\{i\} \in \Delta$ for each $i \in [n]$, and (2) $F \subseteq G \in \Delta \Rightarrow F \in \Delta$. For each $i$ we can linearly order the $i$-element subsets of $\mathbb{N}$ in the *reverse lexicographical order*. So for $i = 3$ the order would start as follows:

$$\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,5\}, \{1,3,5\}, \{2,3,5\},$$
$$\{1,4,5\}, \{2,4,5\}, \{3,4,5\}, \{1,2,6\}, \{1,3,6\}, \{2,3,6\}, \ldots$$

For an integral vector $\tilde{f} = (f_{-1}, f_0, f_1, \ldots, f_{d-1}) \in \mathbb{Z}^{d+1}$ let $\Delta_{\tilde{f}} \subseteq 2^{\mathbb{N}}$ consist of the first $f_{i-1}$ $i$-element subsets of $\mathbb{N}$ in the reverse lexicographical ordering for each $i \in \{0, 1, \ldots, d\}$. The proof of Theorem 2.4 is based on proving the equivalence of the following three statements [16].

1. The integral vector $\tilde{f}$ is an $f$-vector of a simplicial complex $\Delta$.

2. $\Delta_{\tilde{f}}$ is a simplicial complex.

3. $f_i \leq f_{i-1}^{(i)}$ for each $i \in \{1, \ldots, d-1\}$.

The hard part of the proof is the implication $1 \Rightarrow 2$.

For a fixed $i$ we have a well-defined function $m \mapsto m^{(i)}$ which we will refer to as the *upper boundary function*[1] or the *UB* function for short. The remainder of this section will be devoted to

[1] as a function this has been called the "pseudo power function" [12]. It is similar to the "upper boundary operator" [6].

the derivation of some properties of the UB function. We will, in part, use the above equivalence to prove these properties.

For one such property of the UB function, let $i, m_1, m_2, N \in \mathbb{N}$ be such that $m_1, m_2 \le \binom{N}{i}$ and consider two integral vectors in $\mathbb{Z}^{i+2}$

$$\begin{aligned}\tilde{f}_1 &= \left(\binom{N}{0}, \ldots, \binom{N}{i-1}, m_1, m_1^{(i)}\right), \\ \tilde{f}_2 &= \left(\binom{N}{0}, \ldots, \binom{N}{i-1}, m_2, m_2^{(i)}\right).\end{aligned}$$

By Theorem 2.4 both $\Delta_{\tilde{f}_1}$ and $\Delta_{\tilde{f}_1}$ are simplicial complexes. Assume we have disjoint representations $\Delta_{\tilde{f}_1} \subseteq 2^{[N]}$ and $\Delta_{\tilde{f}_2} \subseteq 2^{[2N]\setminus[N]}$ (where in the latter representation $N$ has been added to each element of each set in $\Delta_{\tilde{f}_2}$) and let $\Delta := \Delta_{\tilde{f}_1} \cup \Delta_{\tilde{f}_2}$ be their union. By definition $\Delta$ is clearly a simplicial complex with $f_{i-1}(\Delta) = m_1 + m_2$ and $f_i(\Delta) = m_1^{(i)} + m_2^{(i)}$. By Theorem 2.4 we have the following.

**Claim 2.5** *$(m_1 + m_2)^{(i)} \ge m_1^{(i)} + m_2^{(i)}$.*

Let $\mathcal{F}_i(N)$ denote the first $N$ elements of of $\binom{\mathbb{N}}{i}$. We clearly have then (i) $|\mathcal{F}_i(N)| = N$ for $i \ge 1$, (ii) $\mathcal{F}_i(N_1) \subseteq \mathcal{F}_i(N_2)$ iff $N_1 \le N_2$, (iii) $\mathcal{F}_i\left(\binom{k}{i}\right) = \binom{[k]}{i}$, and by definition of $\Delta_{\tilde{f}}$ here above we have for an $f$-vector $\tilde{f}$ of a simplicial complex that (iv)

$$\Delta_{\tilde{f}} = \mathcal{F}_0(f_{-1}) \cup \mathcal{F}_1(f_0) \cup \cdots \cup \mathcal{F}_d(f_{d-1}).$$

(v) Finally note that if $|X| = i$ and $|Y| = i+1$, then by Theorem 2.4 we have that $X \subseteq Y \in \mathcal{F}_{i+1}(N^{(i)})$ implies that $X \in \mathcal{F}_i(N)$.

For $N \in \mathbb{N}$, $m_1 \le \binom{N}{i}$ and $m_2 \le \binom{N}{i-1}$ let $\mu = \min(m_2^{(i-1)}, m_1)$ and consider two integral vectors

$$\begin{aligned}\tilde{f}_1 &= \left(\binom{N}{0}, \binom{N}{1} \ldots, \binom{N}{i-1}, m_1, m_1^{(i)}\right) \in \mathbb{Z}^{i+2}, \\ \tilde{f}_2 &= \left(\binom{N}{0}, \ldots, \binom{N}{i-2}, m_2, \mu\right) \in \mathbb{Z}^{i+1}.\end{aligned}$$

By the Theorem 2.4 both $\Delta_{\tilde{f}_1}$ and $\Delta_{\tilde{f}_2}$ are simplicial complexes of dimensions $i$ and $i-1$ respectively. Assume we have abstract representations $\Delta_{\tilde{f}_1} \subseteq 2^{[N]}$ and $\Delta_{\tilde{f}_2} \subseteq 2^{[N]}$ and let

$$\Delta = \Delta_{\tilde{f}_1} \cup (\Delta_{\tilde{f}_2} \uplus \{N+1\}) \subseteq 2^{[N+1]}. \tag{3}$$

**Claim 2.6** *$\Delta$ from (3) is a simplicial complex of dimension $i$.*

*Proof.* For $\ell \in \{0, 1, \ldots, i-1\}$ we have $\binom{[N+1]}{\ell} \subseteq \Delta$ and

$$\begin{aligned}\Delta \cap \binom{[N+1]}{i} &= \mathcal{F}_i(m_1) \cup (\mathcal{F}_{i-1}(m_2) \uplus \{N+1\}) \\ \Delta \cap \binom{[N+1]}{i+1} &= \mathcal{F}_{i+1}(m_1^{(i)}) \cup (\mathcal{F}_i(\mu) \uplus \{N+1\})\end{aligned}$$

We only need to check $F \subseteq G \in \Delta \Rightarrow F \in \Delta$ for $F \in \Delta \cap \binom{[N+1]}{i}$ and $G \in \Delta \cap \binom{[N+1]}{i+1}$. Here there are three cases to consider.

(a) $N+1 \notin F, G$: Here we have $F \subseteq G \in \mathcal{F}_{i+1}(m_1^{(i)})$ and hence $F \in \mathcal{F}_i(m_1) \subseteq \Delta$.

(b) $N+1 \in F, G$: Here $G \in \mathcal{F}_i(\mu) \uplus \{N+1\}$ and hence $G \setminus \{N+1\} \in \mathcal{F}_i(\mu) \subseteq \mathcal{F}_i(m_2^{(i-1)})$ Since $F \setminus \{N+1\} \subseteq G \setminus \{N+1\} \in \mathcal{F}_i(m_2^{(i-1)})$, we have $F \setminus \{N+1\} \in \mathcal{F}_{i-1}(m_2)$ and hence $F \in \mathcal{F}_{i-1}(m_2) \uplus \{N+1\} \subseteq \Delta \cap \binom{[N+1]}{i}$ and so $F \in \Delta$.

(c) $N+1 \notin F$ and $N+1 \in G$: As in (b) we have $G \in \mathcal{F}_i(\mu) \uplus \{N+1\}$ and hence $F = G \setminus \{N+1\} \in \mathcal{F}_i(\mu) \subseteq \mathcal{F}_i(m_1) \subseteq \Delta \cap \binom{[N+1]}{i}$ and hence $F \in \Delta$.

Therefore we have $F \subseteq G \in \Delta \Rightarrow F \in \Delta$ for all $F$ and $G$, and this completes the proof of the claim. □

For the $f$-vector of $\Delta$ in Claim 2.6 we have

$$f_{i-1}(\Delta) = |\mathcal{F}_i(m_1)| + |(\mathcal{F}_{i-1}(m_2) \uplus \{N+1\})| = m_1 + m_2,$$

and

$$f_i(\Delta) = |\mathcal{F}_i(m_1^{(i)})| + |(\mathcal{F}_{i-1}(\mu) \uplus \{N+1\})| = m_1^{(i)} + \min(m_2^{(i-1)}, m_1).$$

By Theorem 2.4 we obtain the following lemma as a corollary.

**Lemma 2.7** *For $m_1, m_2, i \in \mathbb{N}$ we have*

$$(m_1 + m_2)^{(i)} \geq m_1^{(i)} + \min(m_2^{(i-1)}, m_1).$$

Let $m_1, m_2, i, N \in \mathbb{N}$ be such that $m_1 + m_2 \leq \binom{N}{i-1}$. By Lemma 2.7 we get

$$\left(m_1 + m_2 + \binom{N}{i}\right)^{(i)} \geq \binom{N}{i+1} + \min\left((m_1+m_2)^{(i-1)}, \binom{N}{i}\right).$$

By assumption and Proposition 2.2 we have $(m_1+m_2)^{(i-1)} \leq \binom{N}{i}$ and hence

$$\min\left((m_1+m_2)^{(i-1)}, \binom{N}{i}\right) = (m_1+m_2)^{(i-1)}.$$

By Claim 2.5 we therefore have the following.

**Lemma 2.8** *For $m_1, m_2, i, N \in \mathbb{N}$ with $m_1 + m_2 \leq \binom{N}{i-1}$ we have*

$$\left(m_1 + m_2 + \binom{N}{i}\right)^{(i)} \geq m_1^{(i-1)} + m_2^{(i-1)} + \binom{N}{i+1}.$$

Our final objective in this section is to prove the following

**Lemma 2.9** *If $0 \leq m, m_1, m_2 \leq \binom{N}{i}$ and $m_1 + m_2 = m + \binom{N}{i}$ then*

$$m_1^{(i)} + m_2^{(i)} \leq m^{(i)} + \binom{N}{i+1}.$$

To prove Lemma 2.9 we let $\mathbf{P}(i)$ be the statement of Lemma 2.9 for a fixed $i \in \mathbb{N}$.

$\mathbf{P}(i)$ : For all nonnegative integers $m, m_1, m_2, N$ that satisfy $0 \leq m, m_1, m_2 \leq \binom{N}{i}$ and $m_1 + m_2 = m + \binom{N}{i}$ we have $m_1^{(i)} + m_2^{(i)} \leq m^{(i)} + \binom{N}{i+1}$.

We let $\mathbf{Q}(i)$ be the following seemingly weaker statement for a fixed $i \in \mathbb{N}$.

$\mathbf{Q}(i)$ : For all nonnegative integers $m, m_1, m_2, N$ that satisfy $0 \leq m \leq m_1 \leq m_2 < \binom{N}{i}$ and $m_1 + m_2 = m + \binom{N}{i}$ we have $m_1^{(i)} + m_2^{(i)} \leq x^{(i)} + \binom{y_i+1}{i+1}$, where

$$m_1 = \binom{x_i}{i} + \cdots + \binom{x_p}{p}, \quad m_2 = \binom{y_i}{i} + \cdots + \binom{y_q}{q} \tag{4}$$

are their UBR, and $m_1 + m_2 = x + \binom{y_i+1}{i}$ where $x \geq 0$.

We now briefly argue the equivalence of $\mathbf{P}(i)$ and $\mathbf{Q}(i)$.

$\mathbf{P}(i) \Rightarrow \mathbf{Q}(i)$: Let $i \in \mathbb{N}$ be given. Suppose $0 \leq m \leq m_1 \leq m_2 < \binom{N}{i}$ and $m_1 + m_2 = m + \binom{N}{i}$, and the UBR of $m_1$ and $m_2$ are given as in (4), then by assumption we have $m_2 < \binom{y_i+1}{i} \leq \binom{N}{i}$. Hence $0 \leq m_1, m_2 \leq \binom{y_i+1}{i}$ and $x \geq 0$. By $\mathbf{P}(i)$ we then obtain $m_1^{(i)} + m_2^{(i)} \leq x^{(i)} + \binom{y_i+1}{i+1}$, so we have $\mathbf{Q}(i)$.

$\mathbf{Q}(i) \Rightarrow \mathbf{P}(i)$: Let $i \in \mathbb{N}$ be given. Suppose $0 \leq m, m_1, m_2 \leq \binom{N}{i}$ and $m_1 + m_2 = m + \binom{N}{i}$. If $m_2 = \binom{N}{i}$, then $\mathbf{P}(i)$ is trivially true. Also, by symmetry we may assume that $m_1 \leq m_2$, and so we may assume $0 \leq m \leq m_1 \leq m_2 < \binom{N}{i}$ and therefore we can apply $\mathbf{Q}(i)$. Repeated use of $\mathbf{Q}(i)$, say $j \geq 1$ times, will eventually yield

$$m_1^{(i)} + m_2^{(i)} \leq x^{(i)} + \binom{y_i + j}{i+1}$$

where $y_i + j = N$ and $x = m \geq 0$, which is $\mathbf{P}(i)$.

Therefore for each $i \in \mathbb{N}$ the statements $\mathbf{P}(i)$ and $\mathbf{Q}(i)$ are equivalent.

*Proof.* [Lemma 2.9] We will use induction and prove $\mathbf{P}(i)$ for all $i \in \mathbb{N}$. For $i = 1$ we have $x^{(1)} = \binom{x}{2}$ for any integer $x$, so proving $\mathbf{P}(1)$ amounts to showing that $m_1^2 + m_2^2 \leq m^2 + N^2$ when $m, m_1, m_2 \leq N$ and $m_1 + m_2 = m + N$ which is easily established.

We proceed by induction on $i$, assuming the equivalent statements $\mathbf{P}(i-1)$ and $\mathbf{Q}(i-1)$, and prove $\mathbf{Q}(i)$. Assume $0 \leq m \leq m_1 \leq m_2 < \binom{N}{i}$. Let the UBR of $m_1$ and $m_2$ be as stated in (4). By assumption we have $x_i \leq y_i$.

If $x' = m_1 - \binom{x_i}{i}$ and $y' = m_2 - \binom{y_i}{i}$, assume for a moment that $x' \geq y'$. Then by the UBR of $m_1$ and $m_2$ we have $y_i \geq x_i > x_{i-1} \geq y_{i-1}$. If now $m_1' = \binom{x_i}{i} + y'$ and $m_2' = \binom{y_i}{i} + x'$, then $m_1 + m_2 = m_1' + m_2'$ and $m_1' \leq m_1 \leq m_2 \leq m_2'$. Since $x_i > y_{i-1}$ and $y_i > x_{i-1}$ we have

$$\begin{aligned} m_1^{(i)} + m_2^{(i)} &= \binom{x_i}{i+1} + x'^{(i-1)} + \binom{y_i}{i+1} + y'^{(i-1)} \\ &= \left(\binom{x_i}{i+1} + y'^{(i-1)}\right) + \left(\binom{y_i}{i+1} + x'^{(i-1)}\right) \\ &= m_1'^{(i)} + m_2'^{(i)}. \end{aligned}$$

Therefore we may further assume that $x' \leq y'$. We now consider two cases.

First case $x' + y' \geq \binom{y_i}{i-1}$: In this case we have $x' + y' = x'' + \binom{y_i}{i-1}$ for some $x'' \geq 0$. Since $x' \leq y'$ we have $x_{i-1} < x_i \leq y_i$ and $y_{i-1} < y_i$ and hence from the UBR of $x'$ and $y'$ we have $x'' < x' \leq y' < \binom{y_i}{i-1}$. By induction hypothesis $\mathbf{Q}(i-1)$ we then have $x'^{(i-1)} + y'^{(i-1)} \leq x''^{(i-1)} + \binom{y_i}{i}$ and hence

$$\begin{aligned} m_1^{(i)} + m_2^{(i)} &= \left(\binom{x_i}{i} + x'\right)^{(i)} + \left(\binom{y_i}{i} + y'\right)^{(i)} \\ &= \binom{x_i}{i+1} + x'^{(i-1)} + \binom{y_i}{i+1} + y'^{(i-1)} \\ &\leq \binom{x_i}{i+1} + x''^{(i-1)} + \binom{y_i+1}{i+1}. \end{aligned}$$

If $x = \binom{x_i}{i} + x''$, then since $x'' < x'$, we have $x^{(i)} = \binom{x_i}{i+1} + x''^{(i-1)}$ and $x + \binom{y_i+1}{i} = m_1 + m_2$. From above we then have $m_1^{(i)} + m_2^{(i)} \leq x^{(i)} + \binom{y_i+1}{i+1}$, thereby obtaining $\mathbf{Q}(i)$ in this case.

Second case $x' + y' < \binom{y_i}{i-1}$: Note that for every $k \in \{1, \ldots, x_i\}$ we have

$$\binom{x_i}{i} = \binom{x_i - k}{i} + \sum_{\ell=1}^{k} \binom{x_i - \ell}{i-1}.$$

By assumption of $\mathbf{Q}(i)$ we have $m_2 < \binom{N}{i}$ and $m_1 + m_2 \geq \binom{N}{i}$ and hence by the UBR of $m_2$ we have $m_1 + m_2 \geq \binom{y_i+1}{i}$, or $\binom{x_i}{i} + x' + y' \geq \binom{y_i}{i-1}$. Therefore there is a unique $k \in \{1, \ldots, x_i\}$ such that

$$\sum_{\ell=1}^{k} \binom{x_i - \ell}{i-1} + x' + y' \geq \binom{y_i}{i-1} > \sum_{\ell=1}^{k-1} \binom{x_i - \ell}{i-1} + x' + y'.$$

Hence $\sum_{\ell=1}^{k} \binom{x_i-\ell}{i-1} + x' + y' = \delta + \binom{y_i}{i-1}$ where $0 \leq \delta < \binom{x_i-k}{i-1}$. Since $y_i \geq x_i > x_i - k$ we have further $\binom{y_i}{i-1} > \binom{x_i-k}{i-1}$ and hence

$$0 \leq \delta, \ \binom{x_i - k}{i-1}, \ \sum_{\ell=1}^{k-1} \binom{x_i - \ell}{i-1} + x' + y' < \binom{y_i}{i-1}.$$

By Claim 2.5 and then by induction hypothesis $\mathbf{P}(i-1)$ we have

$$\begin{aligned} \sum_{\ell=1}^{k} \binom{x_i - \ell}{i} + x'^{(i-1)} + y'^{(i-1)} &= \sum_{\ell=1}^{k} \binom{x_i - \ell}{i-1}^{(i-1)} + x'^{(i-1)} + y'^{(i-1)} \\ &\leq \left(\sum_{\ell=1}^{k-1} \binom{x_i - \ell}{i-1} + x' + y'\right)^{(i-1)} + \binom{x_i - k}{i-1}^{(i-1)} \\ &\leq \delta^{(i-1)} + \binom{y_i}{i-1}^{(i-1)} \\ &= \delta^{(i-1)} + \binom{y_i}{i}. \end{aligned}$$

Note that by definition of $\delta$ and its range, we have $m_1 + m_2 = x + \binom{y_i+1}{i}$ where $x = \binom{x_i-k}{i} + \delta$ and also

$$x^{(i)} + \binom{y_i+1}{i}^{(i)} = \binom{x_i-k}{i+1} + \delta^{(i-1)} + \binom{y_i+1}{i+1}.$$

Since

$$\binom{x_i}{i+1} = \binom{x_i-k}{i+1} + \sum_{\ell=1}^{k} \binom{x_i-\ell}{i}$$

we then finally get

$$\begin{aligned} m_1^{(i)} + m_2^{(i)} &= \binom{x_i}{i+1} + x'^{(i-1)} + \binom{y_i}{i+1} + y'^{(i-1)} \\ &= \binom{x_i-k}{i+1} + \sum_{\ell=1}^{k}\binom{x_i-\ell}{i} + x'^{(i-1)} + \binom{y_i}{i+1} + y'^{(i-1)} \\ &\leq \binom{x_i-k}{i+1} + \delta^{(i-1)} + \binom{y_i}{i} + \binom{y_i}{i+1} \\ &= \binom{x_i-k}{i+1} + \delta^{(i-1)} + \binom{y_i+1}{i+1} \\ &= x^{(i)} + \binom{y_i+1}{i}^{(i)} \end{aligned}$$

which is $\mathbf{Q}(i)$. This completes the inductive proof that $\mathbf{P}(i-1)$ and $\mathbf{Q}(i-1)$ imply $\mathbf{Q}(i)$, and so this completes the proof of of Lemma 2.9. □

# 3 The main theorem

In this section we use results from previous section to prove our main result of this article Theorem 3.2 here below.

Let $k \in \mathbb{N}$ and $S \subseteq V(Q_k)$. Call a vertex/binary string of a subgraph $G = Q_k[S]$ of the $k$-dimensional hypercube $Q_k$ *full* if its degree is $k$ in $G$. For $n \in [2^k]$ Let $\phi_k(n)$ denote the maximum number of full vertices of an induced subgraph of $Q_k$ on $n$ vertices:

$$\phi_k(n) = \max_{S\subseteq V(Q_k),|S|=n} |\{\tilde{x} \in S : d_{Q_k[S]}(\tilde{x}) = k\}.$$

Clearly $\phi_k(2^k) = 2^k$ as every vertex of $Q_k$ is full. If $n < 2^k$ and $S \subseteq V(Q_k)$ contains $n$ vertices and induces $\phi_k(n)$ full vertices in $Q_k$, then we can by symmetry of $Q_k$ (or relabeling of the vertices) assume that the vertex corresponding to the binary string consisting of $k$ 1's is not in $S$. In this case a vertex/string in $S$ with the maximum number of 1's is not full in $Q_k[S]$. In particular we have $\phi_k(n) < n$ for each $n < 2^k$.

**Observation 3.1** *For $k \in \mathbb{N}$ we have:*

1. *If $n < 2^k$ then $\phi_k(n) < n$.*
2. *The function $\phi_k : [2^k] \to [2^k]$ is increasing.*

Remark: By Observation 3.1 we see that $\phi_k$ cannot be strictly increasing.

Note that every $n \in [2^k]$ has a unique *hypercube representation (HCR)* as $n = \sum_{\ell=0}^{i} \binom{k}{\ell} + m$ where $0 \le m < \binom{k}{i+1}$. The main result of this section is the following.

**Theorem 3.2** *For $k \in \mathbb{N}$ and $n \in [2^k]$ with HCR $n = \sum_{\ell=0}^{i} \binom{k}{\ell} + m$, then*

$$\phi_k(n) = \sum_{\ell=0}^{i-1} \binom{k}{\ell} + m^{(k-i-1)}.$$

We will prove Theorem 3.2 by induction on $k$. In order to do that, we will first derive a recursive upper bound for $\phi_k(n)$.

Let $S \subseteq V(Q_k)$ be a set of $n$ vertices/binary strings, and let $F_k(S) \subset S$ be the vertices of $S$ that are full in $Q_k[S]$. Looking at the decomposition $Q_k = Q_{k-1}^0 \boxplus Q_{k-1}^1$ let $S_b = S \cap V(Q_{k-1}^b)$ for $b = 0, 1$ and $n_b = |S_b|$. Clearly $S = S_0 \cup S_1$ is a partition and we have $n_0 + n_1 = n$. Note that for $b \in \{0, 1\}$, a vertex in $S_b$ is full in $Q_k[S]$ iff (i) it is full in $Q_{k-1}^b[S_b]$, and (ii) its copy is contained in $S_{1-b}$. By (i) and (ii) the number of vertices in $S_b$ that are full in $Q_k[S]$ is at most $\min(\phi_{k-1}(n_b), n_{1-b})$, that is $|F_k(S) \cap S_b)| \le \min(\phi_{k-1}(n_b), n_{1-b})$. Since $F_k(S) = (F_k(S) \cap S_0) \cup (F_k(S) \cap S_1)$ is a partition we then have

$$|F_k(S)| = |F_k(S) \cap S_0| + |F_k(S) \cap S_1| \le \min(\phi_{k-1}(n_0), n_1) + \min(\phi_{k-1}(n_1), n_0).$$

By definition we then have the following recursive max-min upper bound

$$\phi_k(n) \le \max_{n_0+n_1=n} (\min(\phi_{k-1}(n_0), n_1) + \min(\phi_{k-1}(n_1), n_0)). \tag{5}$$

Note that is impossible to have $n_b < \phi_{k-1}(n_{1-b})$ for both $b = 0, 1$, since then $n_0 < \phi_{k-1}(n_1) < n_1 < \phi_{k-1}(n_0) < n_0$, a blatant contradiction. From this we see that (5) can be written as

$$\phi_k(n) \le \max_{n_0+n_1=n} (\min(\phi_{k-1}(n_0) + \phi_{k-1}(n_1), n_0 + \phi_{k-1}(n_0), n_1 + \phi_{k-1}(n_1))). \tag{6}$$

Further, by symmetry the maximum in (6) is attained when $n_0 \ge n_1$, in which case we have $n_1 + \phi_{k-1}(n_1) \le n_0 + \phi_{k-1}(n_0)$. Hence we obtain

$$\phi_k(n) \le \max_{n_0+n_1=n, n_0 \ge n_1} (\min(\phi_{k-1}(n_0) + \phi_{k-1}(n_1), n_1 + \phi_{k-1}(n_1))). \tag{7}$$

Let $f_k(n)$ be the function on the right in the displayed formula in the above Theorem 3.2

$$f_k(n) := \sum_{\ell=0}^{i-1} \binom{k}{\ell} + m^{(k-i-1)}$$

where $n = \sum_{\ell=0}^{i} \binom{k}{\ell} + m$ is its HCR. We first show that $\phi_k(n) \ge f_k(n)$ by explicitly show that an induced subgraph on $n$ vertices of $Q_k$ can have $f_k(n)$ full vertices. Then we will show that $f_k(n)$ satisfies

$$f_k(n) = \max_{n_0+n_1=n, n_0 \ge n_1} (\min(f_{k-1}(n_0) + f_{k-1}(n_1), n_1 + f_{k-1}(n_1))), \tag{8}$$

which by (7) shows that $f_k(n) \ge \phi_k(n)$.

For $k \in \mathbb{N}$ let $n \in [2^k]$ with HCR $n = \sum_{\ell=0}^{i} \binom{k}{\ell} + m$. To show that $\phi_k(n) \geq f_k(n)$ we construct an induced subgraph of $Q_k$ on vertices with $f_k(n)$ full vertices as follows. Let $S \subseteq V(Q_k)$ be the set of $n$ vertices containing all $\sum_{\ell=0}^{i} \binom{k}{\ell}$ binary strings having at most $i$ 1's in their representation, and the first $m$ binary strings with exactly $i+1$ 1's in their representation in the lexicographical order. *Note!* Here a binary string represents the *opposite* subset of $[k]$; where the $j$-th bit is 0 indicates that $j$ is included in the subset. In this way the binary strings are ordered as their corresponding subsets of $[k]$ in the reverse lexicographical order. Clearly every vertex in the induced graph $Q_k[S] \subseteq Q_k$ with at most $i-1$ 1's in their representation is full, these amount to $\sum_{\ell=0}^{i-1} \binom{k}{\ell}$ full vertices. Also note that none of the $m$ vertices with exactly $i+1$ 1's in their representation is full, as they are not connected to any vertex with $i+2$ 1's in $Q_k[S]$. Among the $\binom{k}{i}$ binary strings in $S$ containing exactly $i$ 1's, we briefly argue that $m^{(k-i-1)}$ of them are full in the following way.

Consider the $(k-i-1)$-dimensional simplicial complex $\Delta_{\tilde{f}}$ where

$$\tilde{f} = \left( \binom{k}{0}, \ldots, \binom{k}{k-i-2}, m, m^{(k-i-1)} \right) \in \mathbb{Z}^{k-i+1}.$$

Note that $\Delta_{\tilde{f}} \cap \left( \binom{k}{k-i} \cup \binom{k}{k-i-1} \right)$ is represented by the bipartite subgraph $G$ of $Q_k[S]$ induced by the binary strings containing exactly $i$ or $i+1$ 1's, where two stings are adjacent in $G$ iff for their opposite sets the smaller one, with $k-i-1$ elements, is contained in the other one with $k-i$ elements.

Since each of the $m^{(k-i-1)}$ subsets from $\binom{k}{k-i} \cap \Delta_{\tilde{f}}$ has all of its $k-i-1$ subsets among the $m$ subsets from $\Delta_{\tilde{f}} \cap \binom{k}{k-i-1}$, then the representing $m^{(k-i-1)}$ opposite binary strings in $G$, containing exactly $i$ 1's, are each connected to all the $k-i$ opposite binary strings among the $m$ ones in $G$, that contain exactly $i+1$ 1's. Since each binary string in $G \subseteq Q_k[S]$ with $i$ 1's is clearly connected to all $i$ binary strings with $i-1$ 1's in $Q_k[S]$, we see that each of the mentioned $m^{(k-i-1)}$ opposite binary strings in $G \subseteq Q_k[S]$ are full. This shows that $Q_k[S]$ is an induced subgraph of $Q_k$ with $n$ vertices and at least $f_k(n)$ full vertices. Therefore we have $\phi_k(n) \geq f_k(n)$.

To complete the proof of Theorem 3.2 we show that $f_k(n)$ satisfies (8), which by (7) then implies that $\phi_k(n) \leq f_k(n)$, and hence $\phi_k(n) = f_k(n)$. This will occupy the remainder of this section. To show (8), we will show that $f_k(n) \geq \min(f_{k-1}(n_0) + f_{k-1}(n_1), n_1 + f_{k-1}(n_1))$, whenever $n_0 + n_1 = n$ and $n_0 \geq n_1$. There are all together six cases we will consider to verify this inequality; the first case (A) has two sub-cases (A1) and (A2), the second case (B) has four sub-cases (B11), (B12), (B21) and (B22).

Case (A) $f_{k-1}(n_0) \geq n_1$: Here we want to show that $f_k(n) \geq n_1 + f_{k-1}(n_1)$. By definition of $f_k(n)$ we have here that $n_0 > f_{k-1}(n_0) \geq n_1$. Since $f_{k-1}$ is increasing there is a critical pair $(n_0^*, n_1^*)$ summing up to $n$ such that (i) $f_{k-1}(n_0^*) \geq n_1^*$, and (ii) $f_{k-1}(n_0^* - 1) < n_1^* + 1$. Clearly we have $n_0 \geq n_0^*$ and $n_1 \leq n_1^*$, and so $n_1 + f_{k-1}(n_1) \leq n_1^* + f_{k-1}(n_1^*)$. It therefore suffices to show that $f_k(n) \geq n_1^* + f_{k-1}(n_1^*)$. Let $n = \sum_{\ell=0}^{i} \binom{k}{\ell} + m$ be its HCR. Since $0 \leq m < \binom{k}{i+1} = \binom{k-1}{i+1} + \binom{k-1}{i}$, we consider two sub-cases.

Sub-case (A1) $0 \leq m < \binom{k-1}{i}$: Here in this case we have a bipartition $n = n_0' + n_1'$ where

$$n_0' = \sum_{\ell=0}^{i} \binom{k-1}{\ell}, \quad n_1' = \sum_{\ell=0}^{i-1} \binom{k-1}{\ell} + m \tag{9}$$

for which

$$f_{k-1}(n_0') = \sum_{\ell=0}^{i-1} \binom{k-1}{\ell} \le n_1' \tag{10}$$

and hence, by definition of $n_0^*$ and $n_1^*$, we have $n_0^* \ge n_0'$, $n_1^* \le n_1'$ and so

$$n_0^* = \sum_{\ell=0}^{i} \binom{k-1}{\ell} + m_0^*, \quad n_1^* = \sum_{\ell=0}^{i-1} \binom{k-1}{\ell} + m_1^*,$$

where $m_0^*, m_1^* \ge 0$ are integers, $m_0^* + m_1^* = m$, and (i) $m_0^{*(k-i-2)} \ge m_1^*$ and (ii) $(m_0^* - 1)^{(k-i-2)} < m_1^* + 1$. Now note that $f_k(n) \ge n_1^* + f_{k-1}(n_1^*)$ is, by definition of $f_{k-1}$, equivalent to $m^{(k-i-1)} \ge m_1^* + m_1^{*(k-i-1)}$, which is implied by $m^{(k-i-1)} \ge \min(m_1^*, m_0^{*(k-i-2)}) + m_1^{*(k-i-1)}$, which holds by Lemma 2.7 since $m_0^* + m_1^* = m$.

Sub-case (A2) $\binom{k-1}{i} \le m < \binom{k}{i+1}$: Similarly to Sub-case (A1) we have here in this case a bipartition $n = n_0' + n_1'$ where

$$n_0' = \sum_{\ell=0}^{i} \binom{k-1}{\ell} + m', \quad n_1' = \sum_{\ell=0}^{i} \binom{k-1}{\ell} \tag{11}$$

where $m' = m - \binom{k-1}{i}$ for which

$$f_{k-1}(n_0') = \sum_{\ell=0}^{i-1} \binom{k-1}{\ell} + m'^{(k-i-2)} \le n_1' \tag{12}$$

and hence again, by definition of $n_0^*$ and $n_1^*$, we have $n_0^* \ge n_0'$, $n_1^* \le n_1'$ and so

$$n_0^* = \sum_{\ell=0}^{i} \binom{k-1}{\ell} + m_0^*, \quad n_1^* = \sum_{\ell=0}^{i-1} \binom{k-1}{\ell} + m_1^*,$$

where $m_0^* \ge m'$ and $m_1 < \binom{k-1}{i}$ are integers, $m_0^* + m_1^* = m$, and (i) $m_0^{*(k-i-2)} \ge m_1^*$ and (ii) $(m_0^* - 1)^{(k-i-2)} < m_1^* + 1$. Exactly as in the previous case (A1), we note that $f_k(n) \ge n_1^* + f_{k-1}(n_1^*)$ is by definition of $f_{k-1}$, equivalent to $m^{(k-i-1)} \ge m_1^* + m_1^{*(k-i-1)}$, which is implied by $m^{(k-i-1)} \ge \min(m_1^*, m_0^{*(k-i-2)}) + m_1^{*(k-i-1)}$, which again holds by Lemma 2.7 since $m_0^* + m_1^* = m$.

Case (B) $f_{k-1}(n_0) < n_1$: Here we want to show that $f_k(n) \ge f_{k-1}(n_0) + f_{k-1}(n_1)$. By definition of $f_k(n)$ we have here that $n_0 \ge n_1 > f_{k-1}(n_0)$. Let $n = \sum_{\ell=0}^{i} \binom{k}{\ell} + m$ be its HCR. Since $0 \le m < \binom{k}{i+1} = \binom{k-1}{i+1} + \binom{k-1}{i}$, we consider the two cases of whether $0 \le m < \binom{k-1}{i}$ or $\binom{k-1}{i} \le m < \binom{k}{i+1}$.

Sub-case (B1) $0 \le m < \binom{k-1}{i}$: As in case (A1), we have a partition $n = n_0' + n_1'$ given by (9) such that we have (10). The two sub-cases here, (B11) and (B12), depend on whether $n_0 \ge n_0'$ or $n_0 \le n_0'$.

Sub-sub-case (B11) $n_0 \ge n_0'$ in (9): Considering the critical pair $(n_0^*, n_1^*)$ from Case (A), we have here that $n_0' \le n_0 < n_0^*$ and hence

$$n_0 = \sum_{\ell=0}^{i} \binom{k-1}{\ell} + m_0, \quad n_1 = \sum_{\ell=0}^{i-1} \binom{k-1}{\ell} + m_1$$

where $0 \leq m_0 < m_0*$, $m_1^* < m_1 \leq m$, and $m_0 + m_1 = m$. Now note that $f_k(n) \geq f_{k-1}(n_0) + f_{k-1}(n_1)$ is by definition of $f_{k-1}$, equivalent to

$$m^{(k-i-1)} \geq m_0^{(k-i-2)} + m_1^{(k-i-1)}. \tag{13}$$

By definition of $m_0^*$ we have $m_0^{(k-i-2)} \leq m_0^{*(k-i-2)} \leq m_1^* < m_1$ and hence (13) is equivalent to $(m_1 + m_0)^{(k-i-1)} \geq m_1^{(k-i-1)} + \min(m_0^{(k-i-2)}, m_1)$, which is implied by Lemma 2.7.

Sub-sub-case (B12) $n_0 \leq n_0'$ in (9): Here we then have $n/2 \leq n_0 \leq n_0'$ and $n_1' \leq n_1 \leq n/2$, and hence

$$n_0 = \sum_{\ell=0}^{i-1} \binom{k-1}{\ell} + m_0, \quad n_1 = \sum_{\ell=0}^{i-1} \binom{k-1}{\ell} + m_1$$

where $0 \leq m_1 \leq m_0$ and $m_0 + m_1 = m + \binom{k-1}{i}$, and hence $\left(m + \binom{k-1}{i}\right)/2 \leq m_1 \leq m_0 < \binom{k-1}{i}$. Here $f_k(n) \geq f_{k-1}(n_0) + f_{k-1}(n_1)$ is by definition of $f_{k-1}$, equivalent to $\binom{k-1}{i-1} + m^{(k-i-1)} \leq m_0^{(k-i-1)} + m_0^{(k-i-1)}$, which holds by Lemma 2.9.

Sub-case (B2) $\binom{k-1}{i} \leq m < \binom{k}{i+1}$: As in case (A2), we have a partition $n = n_0' + n_1'$ given by (11) such that we have (12). As in the case (B1), the two sub-cases here, (B21) and (B22), depend on whether $n_0 \geq n_0'$ or $n_0 \leq n_0'$.

Sub-sub-case (B21) $n_0 \geq n_0'$ in (11): Considering the critical pair $(n_0^*, n_1^*)$ from Case (A), we have here that $n_0' \leq n_0 < n_0^*$ and hence

$$n_0 = \sum_{\ell=0}^{i} \binom{k-1}{\ell} + m_0, \quad n_1 = \sum_{\ell=0}^{i-1} \binom{k-1}{\ell} + m_1$$

where $m' := m - \binom{k-1}{i} \leq m_0 < m_0*$, $m_1^* < m_1 \leq \binom{k-1}{i}$, and $m_0 + m_1 = m$. Now note that $f_k(n) \geq f_{k-1}(n_0) + f_{k-1}(n_1)$ is by definition of $f_{k-1}$, equivalent to

$$m^{(k-i-1)} \geq m_0^{(k-i-2)} + m_1^{(k-i-1)}. \tag{14}$$

Since $m_0 \leq m_0^* - 1$ we have by definition of $m_0^*$ that $m_0^{(k-i-2)} \leq (m_0^* - 1)^{(k-i-2)} < m_1^* + 1 \leq m_1$ and hence (14) is equivalent to $(m_1 + m_0)^{(k-i-1)} \geq m_1^{(k-i-1)} + \min(m_0^{(k-i-2)}, m_1)$, which is implied by Lemma 2.7.

Sub-sub-case (B22) $n_0 \leq n_0'$ in (11): Here we then have $n/2 \leq n_0 \leq n_0'$ and $n_1' \leq n_1 \leq n/2$, and hence

$$n_0 = \sum_{\ell=0}^{i} \binom{k-1}{\ell} + m_0, \quad n_1 = \sum_{\ell=0}^{i} \binom{k-1}{\ell} + m_1$$

where $m'/2 \leq m_0 \leq m'$ and $0 \leq m_1 \leq m'/2$, and $m_0 + m_1 = m' = m - \binom{k-1}{i}$. Here $f_k(n) \geq f_{k-1}(n_0) + f_{k-1}(n_1)$ is by definition of $f_{k-1}$, equivalent to $m^{(k-i-1)} \geq m_0^{(k-i-2)} + m_0^{(k-i-2)} + \binom{k-1}{i-1}$. Since $m = m_0 + m_1 + \binom{k-1}{k-i-1}$ this follows from Lemma 2.8.

In all the above six cases (A1), (A2), and (B11), (B12), (B21) and (B22), we have that $f_k(n) \geq \min(f_{k-1}(n_0) + f_{k-1}(n_1), n_1 + f_{k-1}(n_1))$ whenever $n_0 + n_1 = n$ and $n_0 \geq n_1$. This shows that $f_k(n)$ satisfies (8) and therefore that $\phi_k(n) \leq f_k(n)$, which completes the proof of Theorem 3.2.

## 4 An application

In this section we apply the main result of the previous section, Theorem 3.2, to determine the value $\min(\max(|V(H_1)|, |V(H_2)|))$ where (i) $H_1$ and $H_2$ are induced subgraphs of $Q_k$, and (ii) together $H_1$ and $H_2$ cover all the edges of $Q_k$. The main (and the only) theorem in this section is the following.

**Theorem 4.1** *For $k \in \mathbb{N}$ we have*

$$\min_{E(H_1)\cup E(H_2)=E(Q_k)} (\max(|V(H_1)|, |V(H_2)|)) = \sum_{\ell=0}^{\lfloor k/2 \rfloor} \binom{k}{\ell} + (k \bmod 2)\binom{k-1}{\lfloor k/2 \rfloor}.$$

The rest of this final section will be devoted to prove Theorem 4.1.

Assume $k$ is even and that $|V(H_1)| < \sum_{\ell=0}^{k/2} \binom{k}{\ell}$. In this case we have

$$|V(H_1)| \le \sum_{\ell=0}^{k/2} \binom{k}{\ell} - 1 = \sum_{\ell=0}^{k/2-1} \binom{k}{\ell} + \left(\binom{k}{k/2} - 1\right).$$

By Observation 3.1, Theorem 4.1 and Observation 2.3 we obtain the following.

$$\begin{aligned} \phi_k(|V(H_1)|) &\le \phi_k\left(\sum_{\ell=0}^{k/2-1} \binom{k}{\ell} + \left(\binom{k}{k/2} - 1\right)\right) \\ &= \sum_{\ell=0}^{k/2-2} \binom{k}{\ell} + \left(\binom{k}{k/2} - 1\right)^{(k/2)} \\ &= \sum_{\ell=0}^{k/2-2} \binom{k}{\ell} + \binom{k}{k/2+1} - k/2 \\ &= \sum_{\ell=0}^{k/2-1} \binom{k}{\ell} - k/2 \\ &< \sum_{\ell=0}^{k/2-1} \binom{k}{\ell}. \end{aligned}$$

Since every vertex in $Q_k$ that is not full in $H_1$ is incident to an edge in $H_2$ and is therefore a vertex in $H_2$ we have that

$$|V(H_2)| \ge |Q_k| - \phi_k(|V(H_1)|) > 2^k - \sum_{\ell=0}^{k/2-1} \binom{k}{\ell} = \sum_{\ell=0}^{k/2} \binom{k}{\ell}$$

and hence $\max(|V(H_1)|, |V(H_2)|) > \sum_{\ell=0}^{k/2} \binom{k}{\ell}$. On the other hand, if $H_1$ and $H_2$ are the subgraph of $Q_k$ induced by binary strings of length $k$ with at most $k/2$ 0's and with at most $k/2$ 1s respectively, then $|V(H_1)| = |V(H_2)| = \sum_{\ell=0}^{k/2} \binom{k}{\ell}$ and hence $\max(|V(H_1)|, |V(H_2)|) = \sum_{\ell=0}^{k/2} \binom{k}{\ell}$. Hence, as $H_1$ and $H_2$ cover all the edges of $Q_k$, then Theorem 4.1 is valid for even $k$.

Assume $k$ is odd and that

$$|V(H_1)| < \sum_{\ell=0}^{\lfloor k/2 \rfloor} \binom{k}{\ell} + \binom{k-1}{\lfloor k/2 \rfloor},$$

and hence

$$|V(H_1)| \leq \sum_{\ell=0}^{(k-1)/2} \binom{k}{\ell} + \left( \binom{k-1}{\frac{k-1}{2}} - 1 \right).$$

As in the even case, we obtain here by Observation 3.1, Theorem 4.1 and Observation 2.3 that

$$\begin{aligned} \phi_k(|V(H_1)|) &\leq \phi_k \left( \sum_{\ell=0}^{(k-1)/2} \binom{k}{\ell} + \left( \binom{k-1}{\frac{k-1}{2}} - 1 \right) \right) \\ &= \sum_{\ell=0}^{(k-3)/2} \binom{k}{\ell} + \left( \binom{k-1}{\frac{k-1}{2}} - 1 \right)^{\left(\frac{k-1}{2}\right)} \\ &= \sum_{\ell=0}^{(k-3)/2} \binom{k}{\ell} + \binom{k-1}{\frac{k+1}{2}} - \frac{k-1}{2} \\ &< \sum_{\ell=0}^{(k-3)/2} \binom{k}{\ell} + \binom{k-1}{\frac{k+1}{2}}. \end{aligned}$$

Again, since every vertex in $Q_k$ that is not full in $H_1$ is incident to an edge in $H_2$ and is therefore a vertex in $H_2$ we have that

$$|V(H_2)| \geq |Q_k| - \phi_k(|V(H_1)|) > 2^k - \sum_{\ell=0}^{(k-3)/2} \binom{k}{\ell} - \binom{k-1}{\frac{k+1}{2}} = \sum_{\ell=0}^{(k-1)/2} \binom{k}{\ell} + \binom{k-1}{\frac{k-1}{2}}$$

and hence

$$\max(|V(H_1)|, |V(H_2)|) > \sum_{\ell=0}^{\lfloor k/2 \rfloor} \binom{k}{\ell} + \binom{k-1}{\lfloor k/2 \rfloor}.$$

On the other hand, considering the subgraphs $H_1$ and $H_2$ of $Q_k$ induced by binary strings of length $k$, where $H_1$ is induced by the strings with at most $(k-1)/2$ 1's among the first $k-1$ bits, and $H_2$ is induced by the strings with at most $(k-1)/2$ 0's among the first $k-1$ bits, we have that $|V(H_1)| = |V(H_2)| = 2\left(\sum_{\ell=0}^{(k-1)/2} \binom{k-1}{\ell}\right)$ and hence

$$\max(|V(H_1)|, |V(H_2)|) = 2 \left( \sum_{\ell=0}^{(k-1)/2} \binom{k-1}{\ell} \right) = \sum_{\ell=0}^{\lfloor k/2 \rfloor} \binom{k}{\ell} + \binom{k-1}{\lfloor k/2 \rfloor}.$$

Hence, as $H_1$ and $H_2$ cover all the edges of $Q_k$, then Theorem 4.1 is valid for odd $k$. This completes the proof of Theorem 4.1.

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