A new approach to cross-bifix-free sets

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Abstract

Cross-bifix-free sets are sets of words such that no prefix of any word is a suffix of any other word. In this paper, we introduce a general constructive method for the sets of cross-bifix-free binary words of fixed length. It enables us to determine a cross-bifix-free words subset which has the property to be non-expandable.

1 Introduction

In digital communication systems, synchronization is an essential requirement to establish and maintain a connection between a transmitter and a receiver.

Analytical approaches to the synchronization acquisition process and methods for the construction of sequences with the best aperiodic autocorrelation properties have been the subject of numerous analyses in the digital transmission.

The historical engineering approach started with the introduction of bifix. It denotes a subsequence that is both a prefix and suffix of a longer observed sequence. Rather than to the bifix, much attention has been devoted to a bifix-indicator, an indicator function implying the existence of the bifix [10]. Such indicators were shown to be without equal in performing various statistical analysis, mainly concerning the search process [3, 10]

However, an analytical study of simultaneous search for a set of sequences urged the invention of cross-bifix indicators [1, 2] and, accordingly, turned attention to the sets of sequences which avoid cross-bifixes, called cross-bifix-free sets.

In [1], the author analyzes some properties of binary words that form a cross-bifix-free set, in particular, a general constructing method called the kernel method is presented. This approach leads to sets $S(n)$ of cross-bifix-free binary words, of fixed length n, having cardinality $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233$ for $n = 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15$ respectively.

This sequence forms a Fibonacci progression and satisfies the recurrence relation $|S(n)| = |S(n-1)| + |S(n-2)|$ with $|S(3)| = 1$ and $|S(4)| = 1$.

The problem of determining cross-bifix-free sets is also related to several other scientific applications, for instance in multiaccess systems, pattern matching and automata theory.

The aim of this paper is to introduce a method for the generation of sets of cross-bifix-free binary words of fixed length based upon the study of lattice paths on the Cartesian plane. This approach enables us to obtain cross-bifix-free sets having greater cardinality than the ones presented in [1].

The paper is organized as follows. In Section 2 we give some basic definitions and notation related to the notions of bifix-free word and cross-bifix-free set. In Section 3 we propose a

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method to construct particular sets of cross-bifix-free binary words of fixed length n related to the parity of n. We are not able to say if such cross-bifix-free sets have maximal cardinality on the set of bifix-free binary words of fixed length n or not.

2 Basic definitions and notations

Let A be a finite, non-empty set called *alphabet*. The elements of A are called *letters*. A (finite) sequence of letters in A is called (finite) *word*. Let A^* denote the monoid of all finite words over A where ε denotes the *empty word* and $A^+ = A^* \backslash \varepsilon$. Let ω be a word in A^*, then $|\omega|$ indicates the length of ω and $|\omega|_a$ denotes the number of occurrences of a in ω, being $a \in A$. Let $\omega = uv$ then u is called *prefix* of ω and v is called *suffix* of ω. A *bifix* of ω is a subsequence of ω that is both its prefix and suffix.

A word ω of A^+ is said to be *bifix-free* or *unbordered* [7, 11] if and only if no strict prefix of ω is also a suffix of ω. Therefore, ω is bifix-free if and only if $\omega \neq uwu$, being u any necessarily non-empty word and w any word. Obviously, a necessary condition for ω to be bifix-free is that the first and the last letters of ω must be different.

Example 2.1 *In the monoid* $\{0,1\}^*$*, the word* 111010100 *of length* $n = 9$ *is bifix-free, while the word* 101001010 *contains two bifixes,* 10 *and* 1010*.*

Let $BF_q(n)$ denote the set of all bifix-free words of length n over an alphabet of fixed size q. The following formula for the cardinality of $BF_q(n)$, denoted by $|BF_q(n)|$, is well-known [11].

$$\begin{cases} |BF_q(1)| = q \\ \\ |BF_q(2n+1)| = q|BF_q(2n)| \\ \\ |BF_q(2n)| = q|BF_q(2n-1)| - |BF_q(n)| \end{cases} \tag{2.1}$$

The number sequences related to this recurrence can be found in Sloane's database of integer sequences [12]: sequences A003000 ($q = 2$), A019308 ($q = 3$) and A019309 ($q = 4$).

Table 2.1 lists the set $BF_2(n)$, $2 \leq n \leq 6$, the last row reports the cardinality of each set.

n=2	n=3	n=4	n=5	n=6
10 01	100 001	1000 0001	10000 00001	100000 000001
	110 011	1100 0011	10100 00101	101000 000101
		1110 0111	11000 00011	101100 001101
			11100 00111	110000 000011
			11010 01011	110100 001011
			11110 01111	111000 000111
				111100 001111
				110010 010011
				111010 010111
				111110 011111
2	4	6	12	20

Table 2.1: The set $BF_2(n)$, $2 \leq n \leq 6$

Let $q > 1$ and $n > 1$ be fixed. Two distinct words $\omega, \omega' \in BF_q(n)$ are said to be *cross-bifix-free* if and only if no strict prefix of ω is also a suffix of ω' and vice-versa.

Example 2.2 *The binary words* 111010100 *and* 110101010 *in* $BF_2(9)$ *are cross-bifix-free, while the binary words* 111001100 *and* 110011010 *in* $BF_2(9)$ *have the cross-bifix* 1100*.*

A subset of $BF_q(n)$ is said to be *cross-bifix-free set* if and only if for each ω, ω', with $\omega \neq \omega'$, in this set, ω and ω' are cross-bifix-free. This set is said to be *non-expandable* on $BF_q(n)$ if and only if the set obtained by adding any other word is not a cross-bifix-free set. A non-expandable cross-bifix-free set on $BF_q(n)$ having maximal cardinality is called *maximal cross-bifix-free set* on $BF_q(n)$.

Each word $\omega \in BF_2(n)$ can be naturally represented as a lattice path on the Cartesian plane, by associating a *rise step*, defined by $(1,1)$ and denoted by x, to each 1's in $BF_2(n)$, and a *fall step*, defined by $(1,-1)$ and denoted by $\overline{x}$, to each 0's in $BF_2(n)$, running from $(0,0)$ to (n,h), $-n < h < n$.

From now on, we will refer interchangeably to words or their graphical representations on the Cartesian plane, that is paths.

The definition of bifix-free and cross-bifix-free can be easily extended to paths. Figure 2.1 shows the two paths corresponding to the cross-bifix-free words of Example 2.2.

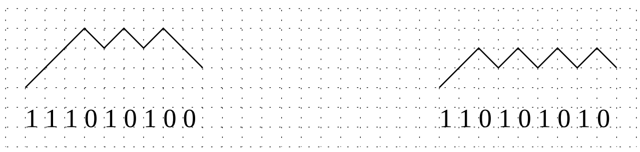

Figure 2.1: Two paths in $BF_2(9)$ which are cross-bifix-free

A lattice path on the Cartesian plane using the steps $(1,1)$ and $(1,-1)$ and running from $(0,0)$ to $(2m,0)$, with $m \geq 0$, is said to be *Grand-Dyck* or *Binomial* path (see [5] for further details). A *Dyck* path is a sequence of rise step and fall steps running from $(0,0)$ to $(2m,0)$ and remaining weakly above the x-axis (see Figure 2.2). The number of $2m$-length Dyck paths is the mth Catalan number $C_m = 1/(m+1)\binom{2m}{m}$, see [13] for further details.

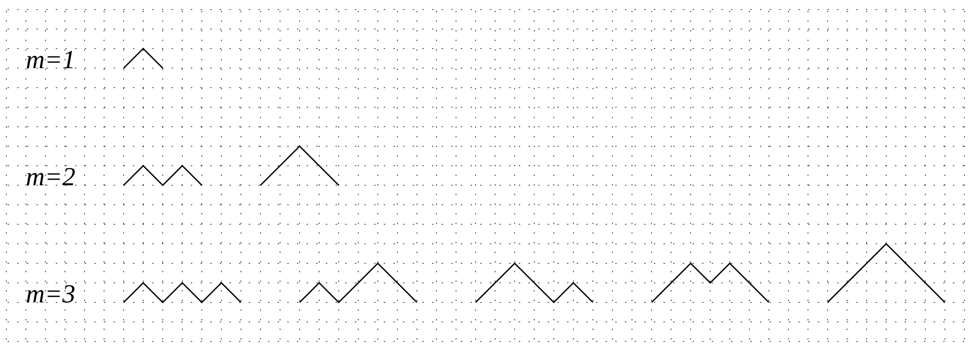

Figure 2.2: The $2m$-length Dyck paths, $1 \leq m \leq 3$

In this paper, we are interested in investigating a possible non-expandable cross-bifix-free set, that is the set $CBFS_2(n)$ of cross-bifix-free words of fixed length $n > 1$ on the monoid $\{0,1\}^*$. In order to do so, we focus on the set $\hat{BF}_2(n)$ of bifix-free binary words of fixed length n having 1 as the first letter and 0 as last letter or equivalently the set of bifix-free lattice paths on the Cartesian plane using the steps $(1,1)$ and $(1,-1)$, running from $(0,0)$ to (n,h), $-n < h < n$, beginning with a rise step and ending with a fall step. Of course $\check{BF}_2(n) = BF_2(n) \backslash \hat{BF}_2(n)$ is obtained by switching rise and fall steps.

Let $\hat{BF}_2^h(n)$ denote the set of the paths in $\hat{BF}_2(n)$ having h as the ordinate of their endpoint, $-n < h < n$.

3 On the non-expandability of $CBFS_2(n)$

In order to prove that $CBFS_2(n)$ is a non-expandable cross-bifix-free set on $BF_2(n)$ we have to distinguish the following two cases depending on the parity of n.

3.1 Non-expandable $CBFS_2(2m+1)$

Let $CBFS_2(2m+1) = \{x\alpha : \alpha \in D_{2m}\}$ that is the set of paths beginning with a rise step linked to a $2m$-length Dyck path (see Figure 3.3). Note that $CBFS_2(2m+1)$ is a subset of $\hat{BF}_2^1(2m+1)$, $m \geq 1$.

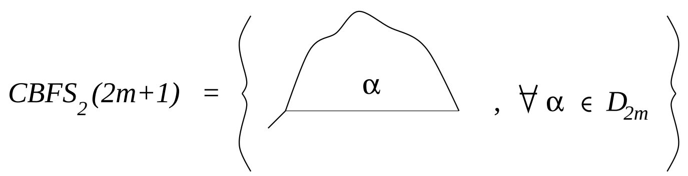

Figure 3.3: A graphical representation of $CBFS_2(2m+1)$, with $m \geq 1$

Of course $|CBFS_2(2m+1)| = C_m$, being C_m the mth Catalan number, $m \geq 1$.
Figure 3.4 shows the set $CBFS_2(7)$, with $|CBFS_2(7)| = C_3 = 5$.

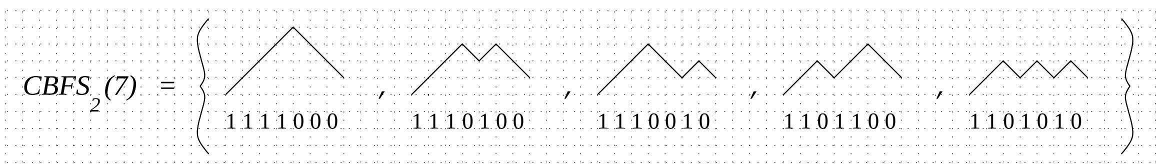

Figure 3.4: A graphical representation of $CBFS_2(7)$

Proposition 3.1 *The set $CBFS_2(2m+1)$ is a cross-bifix-free set on $BF_2(2m+1)$, $m \geq 1$.*

Proof. The proof consists of two distinguished steps. The first one proves that each $\omega \in CBFS_2(2m+1)$ is bifix-free and the second one proves that $CBFS_2(2m+1)$ is a cross-bifix-free set. Each $\omega \in CBFS_2(2m+1)$ can be written as $\omega = vwu$, being v, u any necessarily non-empty word while w can also be an empty word. For each prefix v of ω we have $|v|_1 > |v|_0$ and for each suffix u of ω we have $|u|_1 \leq |u|_0$. Therefore $v \neq u$, $\forall v, u \in \omega$ so ω is bifix-free.

The proof that, for each $\omega, \omega' \in CBFS_2(2m+1)$ then ω and ω' are cross-bifix-free, follows the logical steps described above. ■

Proposition 3.2 *The set $CBFS_2(2m+1)$ is a non-expandable cross-bifix-free set on $BF_2(2m+1)$, $m \geq 1$.*

Proof. It is sufficient to prove that the set $CBFS_2(2m+1)$ is a non-expandable cross-bifix-free set on $\hat{BF}_2(2m+1)$, as each $\omega \in CBFS_2(2m+1)$ and $\varphi \in \check{BF}_2(2m+1)$ match on the last letter of ω and the first one of φ at least.

Let $m \geq 1$ be fixed, we can prove that $CBFS_2(2m+1)$ is a non-expandable cross-bifix-free set on $\hat{BF}_2^h(2m+1)$ by distinguishing $h > 0$ from $h < 0$.

- $h > 0$: a path γ in $\hat{BF}_2^h(2m+1) \backslash CBFS_2(2m+1)$ can be written as $\gamma = \phi x \alpha_1 x \alpha_2 x \ldots x \alpha_r$ (see Figure 3.5, where $n = 2m+1$), being ϕ a Grand-Dyck path beginning with a rise step, x a rise step, α_l Dyck paths, $1 \leq l \leq r-1$, and α_r a necessarily non-empty Dyck path. Therefore, we can find paths in $CBFS_2(2m+1)$ having a prefix which matches with a suffix of γ. It is sufficient to consider the path $\omega = x\alpha_r\alpha_s$, being α_s a Dyck path of appropriate length.

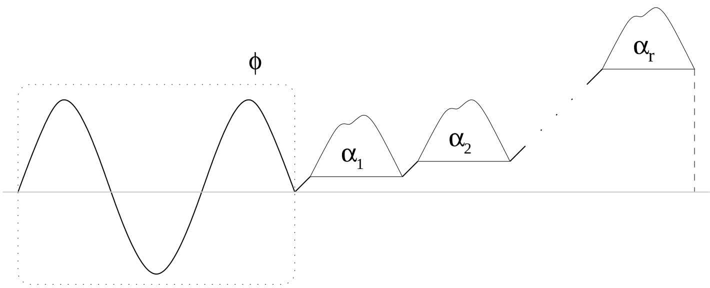

Figure 3.5: A graphical representation of a path γ in $\hat{BF}_2^h(n), h > 0$

- $h < 0$: a path γ in $\hat{BF}_2^h(2m+1)$ can be written as $\gamma = \alpha_r \overline{x} \alpha_{r-1} \overline{x} \ldots \overline{x} \alpha_1 \overline{x} \phi$ (see Figure 3.6, where $n = 2m+1$), being α_r a necessarily non-empty Dyck path, $\overline{x}$ a fall step, α_l Dyck paths, $1 \leq l \leq r-1$, and ϕ a Grand-Dyck path. Therefore, we can find paths in $CBFS_2(2m+1)$ having a suffix which matches with a prefix of γ. It is sufficient to consider the path $\omega = x\alpha_s\alpha_r$, being α_s a Dyck path of appropriate length.

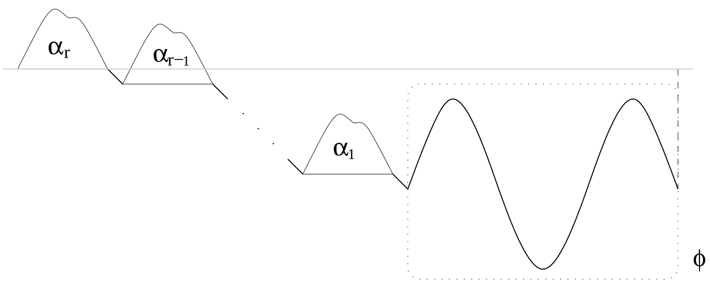

Figure 3.6: A graphical representation of a path γ in $\hat{BF}_2^h(n), h < 0$

■

3.2 Non-expandable $CBFS_2(2m+2)$

In this case we have to distinguish two further subcases depending on the parity of $m > 0$.

If m is an even number then $CBFS_2(2m+2) = \{\alpha x \beta \overline{x} : \alpha \in D_{2i}\ ,\ \beta \in D_{2(m-i)}\ ,\ 0 \leq i \leq \frac{m}{2}\}$, that is the set of paths consisting of the following consecutive sub-paths: a $2i$-length Dyck path, a rise step, a $2(m-i)$-length Dyck path, a fall step, where $0 \le i \le \frac{m}{2}$ (see Figure 3.7). Note that $CBFS_2(2m+2)$ is a subset of $\hat{BF}^0_2(2m+2)$, for any even number $m > 1$.

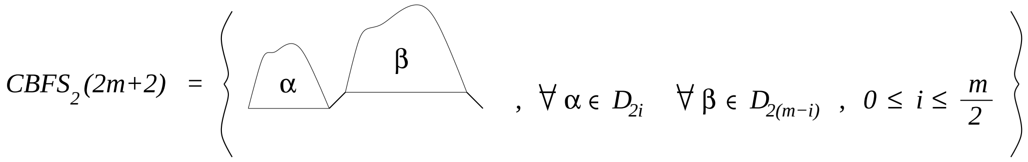

Figure 3.7: A graphical representation of $CBFS_2(2m+2)$, for any even number $m > 1$

Of course $|CBFS_2(2m+2)| = \sum_{i=0}^{m/2} C_i C_{m-i}$, C_m is the mth Catalan Number, for any even number $m > 1$. Figure 3.8 shows the set $CBFS_2(10)$, with $|CBFS_2(10)| = C_4 + C_1C_3 + C_2C_2 = 23$.

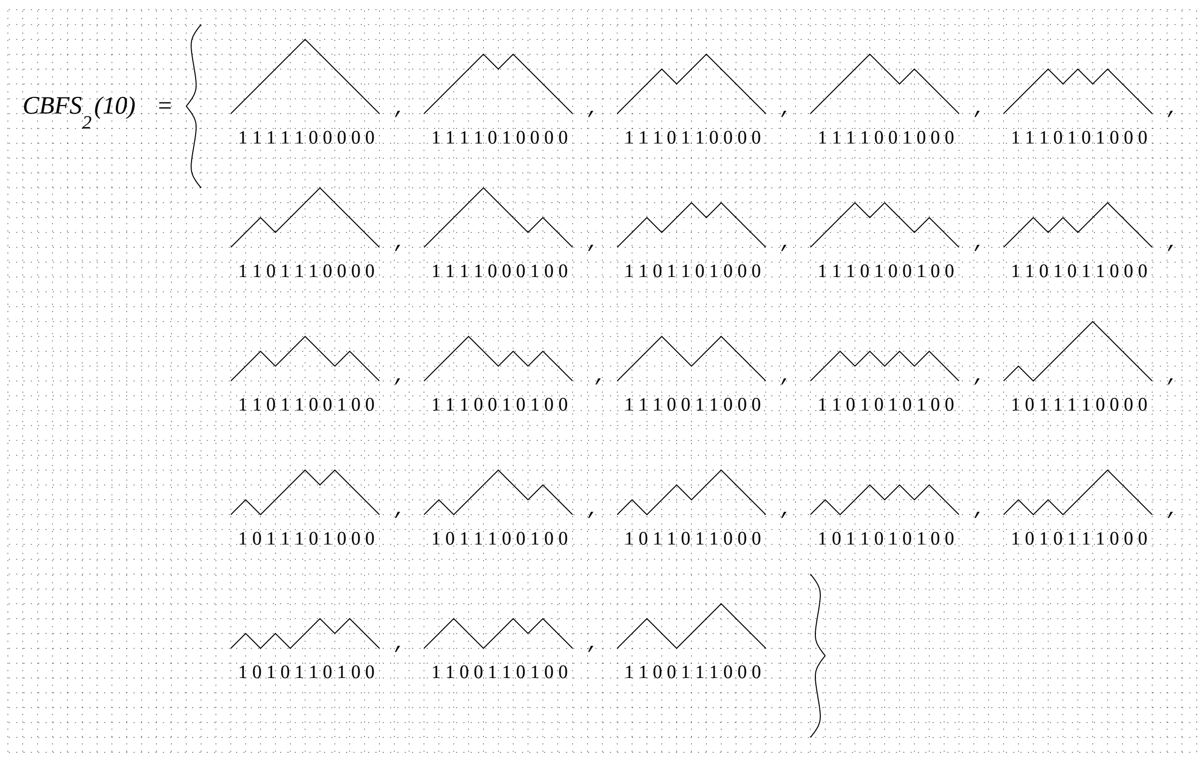

Figure 3.8: A graphical representation of $CBFS_2(10)$

Proposition 3.3 *The set $CBFS_2(2m+2)$ is a cross-bifix-free set on $BF_2(2m+2)$, for any even number $m > 1$.*

Proof. The proof consists of two distinguished steps. The first one proves that each $\omega \in CBFS_2(2m+2)$ is bifix-free and the second one proves that $CBFS_2(2m+2)$ is a cross-bifix-free set. Each $\omega \in CBFS_2(2m+2)$ can be written as $\omega = vwu$, being v, u any necessarily non-empty word while w can also be an empty word. Let $m > 1$ be fixed, we have to take into consideration two different cases: in the first one $i = 0$ and in the second one $0 < i \le \frac{m}{2}$.

If $i = 0$ then $\omega \in \{x\beta\overline{x} : \beta \in D_{2m}\}$, and for each prefix v of ω we have $|v|_1 > |v|_0$ and for each suffix u of ω we have $|u|_1 < |u|_0$. Therefore $v \neq u$, $\forall v, u \in \omega$ and ω is bifix-free.

Otherwise, $\omega \in \{\alpha x \beta \overline{x} : \alpha \in D_{2i} \ , \ \beta \in D_{2(m-i)} \ , \ 0 < i \leq \frac{m}{2}\}$, then for each prefix v of ω we have $|v|_1 \geq |v|_0$ and for each suffix u of ω we have $|u|_1 \leq |u|_0$. If $|v|_1 > |v|_0$ then $v \neq u$, $\forall v, u \in \omega$ and therefore ω is bifix-free. Let i be fixed, if $|v|_1 = |v|_0$ then the path v is a $2k$-length Dyck path, $1 \leq k \leq i$. In this case, both $u = \mu\overline{x}$, where μ is any suffix of β, and $u = \mu' x \beta \overline{x}$, where μ' is any suffix of $\alpha \backslash v$. If $u = \mu\overline{x}$ then $|u|_1 < |u|_0$, therefore $v \neq u$, $\forall v, u \in \omega$ and therefore ω is bifix-free. If $u = \mu' x \beta \overline{x}$ then v does not match with $x\beta\overline{x}$, therefore $v \neq u$, $\forall v, u \in \omega$ and therefore ω is bifix-free.

The proof that, for each $\omega, \omega' \in CBFS_2(2m+2)$ then ω and ω' are cross-bifix-free, follows the logical steps described above. ∎

Proposition 3.4 *The set $CBFS_2(2m+2)$ is a non-expandable cross-bifix-free set on $BF_2(2m+2)$, for any even number $m > 1$.*

Proof. It is sufficient to prove that the set $CBFS_2(2m+2)$ is a non-expandable cross-bifix-free set on $\hat{BF}_2(2m+2)$, as each $\omega \in CBFS_2(2m+2)$ and $\varphi \in \check{BF}_2(2m+2)$ match on the last letter of ω and the first one of φ at least.

Let $m > 1$ be fixed, we have to take into consideration three different cases: in the first one we prove that $CBFS_2(2m+2)$ is a non-expandable cross-bifix-free set on $\hat{BF}_2^h(2m+2)$, $h > 0$, in the second one we prove that $CBFS_2(2m+2)$ is a non-expandable cross-bifix-free set on $\hat{BF}_2^h(2m+2)$, $h < 0$, and in the last one we prove that $CBFS_2(2m+2)$ is a non-expandable cross-bifix-free set on $\hat{BF}_2^0(2m+2)$.

- $h > 0$: a path γ in $\hat{BF}_2^h(2m+2)$ can be written as $\gamma = \phi x \alpha_1 x \alpha_2 x \ldots x \alpha_r$ (see Figure 3.5, where $n = 2m+2$), being ϕ a Grand-Dyck path beginning with a rise step, x a rise step, α_l Dyck paths, $1 \leq l \leq r-1$, and α_r a necessarily non-empty Dyck path. Therefore, we can find paths in $CBFS_2(2m+2)$ having a prefix which matches with a suffix of γ. It is sufficient to consider the path $\omega = x\alpha_r\alpha_s\overline{x}$, being α_s a Dyck path of appropriate length.

- $h < 0$: a path γ in $\hat{BF}_2^h(2m+2)$ can be written as $\gamma = \alpha_r \overline{x} \alpha_{r-1} \overline{x} \ldots \overline{x} \alpha_1 \overline{x} \phi$ (see Figure 3.6, where $n = 2m+2$), being α_r a necessarily non-empty Dyck path, $\overline{x}$ a fall step, α_l Dyck paths, $1 \leq l \leq r-1$, and ϕ a Grand-Dyck path. Therefore, we can find paths in $CBFS_2(2m+2)$ having a suffix which matches with a prefix of γ. It is sufficient to consider the path $\omega = x\alpha_s\alpha_r\overline{x}$, being α_s a Dyck path of appropriate length.

- $h = 0$: a path γ in $\hat{BF}_2^0(2m+2) \backslash CBFS_2(2m+2)$ either never falls below the x-axis or crosses the x-axis. In the first case, it can be written as $\gamma = \alpha_1 x \beta_1 \overline{x}$, where α_1 is a necessarily non-empty $2k$-length Dyck path and β_1 is a $2(m-k)$-length Dyck path, with $\frac{m}{2}+1 \leq k \leq m$, see Figure 3.9 a). Therefore, we can find paths in $CBFS_2(2m+2)$ having a prefix which matches with a suffix of γ. It is sufficient to consider the path $\omega = x\beta_1\overline{x}x\beta\overline{x}$, since $x\beta_1\overline{x} \in D_{2i}$ being $i = m-k+1$.

  If a path γ in $\hat{BF}_2^0(2m+2) \backslash CBFS_2(2m+2)$ crosses the x-axis then it can be written as $\gamma = \alpha_1\phi$ where α_1 is a necessarily non-empty $2k$-length Dyck path, $1 \leq k \leq m$, and ϕ is a necessarily non-empty Grand-Dyck beginning with a fall step, see Figure 3.9 b). Therefore, we can find paths in $CBFS_2(2m+2)$ having a suffix which matches with a prefix of γ. It is sufficient to consider the path $\omega = x\alpha_s\alpha_1\overline{x}$, being α_s a Dyck path of appropriate length.

Figure 3.9: The two possible configurations for a path γ in $\hat{BF}_2^0(2m+2)\backslash CBFS_2(2m+2)$, for any even number $m > 1$

■

If m is an odd number then $CBFS_2(2m+2) = \{\alpha x \beta \overline{x} : \alpha \in D_{2i}\ ,\ \beta \in D_{2(m-i)}\ ,\ 0 \leq i \leq \frac{m+1}{2}\}\backslash\{x\alpha'\overline{x}x\beta'\overline{x} : \alpha', \beta' \in D_{m-1}\}$, that is the set of paths consisting of the following consecutive sub-paths: a $2i$-length Dyck path, a rise step, a $2(m-i)$-length Dych path, a fall step, where $0 \leq i \leq \frac{m+1}{2}$, and excluding those consisting of the following consecutive sub-paths: a rise step, a $(m-1)$-length Dyck path, a fall step followed by a rise step, a $(m-1)$-length Dyck path, a fall step (see Figure 3.10). In other words, the paths which result from the concatenation of two elevated Dyck paths of the same length must be excluded.

In particular, if $\alpha' = \beta'$ then the excluded paths are not bifix-free, otherwise if $\alpha' \neq \beta'$ then the excluded paths match with the paths $\{\alpha x \beta \overline{x} : \alpha \in D_{m+1}\ ,\ \beta \in D_{2(m-1)}\}$ in $CBFS_2(2m+2)$. Note that $CBFS_2(2m+2)$ is a subset of $\hat{BF}_2^0(2m+2)$, for any odd number $m \geq 1$.

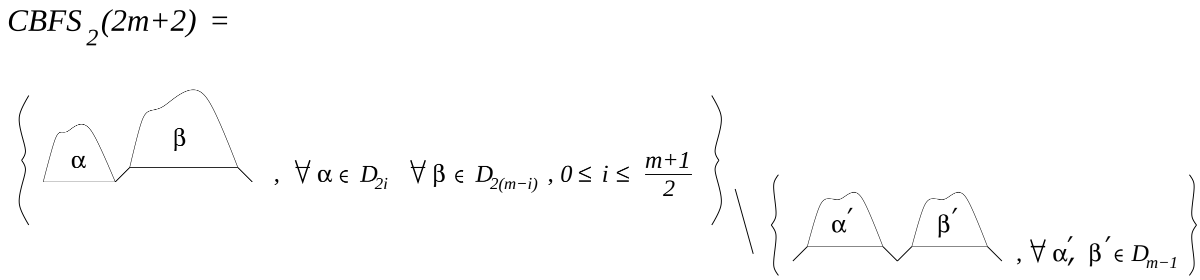

Figure 3.10: A graphical representation of $CBFS_2(2m+2)$, for any odd number $m \geq 1$

Of course $|CBFS_2(2m+2)| = (\sum_{i=0}^{\frac{m+1}{2}} C_i C_{m-i}) - (C_{\frac{m-1}{2}})^2$, C_m is the mth Catalan Number, for any odd number $m \geq 1$. Figure 3.11 shows the set $CBFS_2(8)$, with $|CBFS_2(8)| = (C_3 + C_1C_2 + C_2C_1) - (C_1)^2 = 8$.

Proposition 3.5 *The set $CBFS_2(2m+2)$ is a cross-bifix-free set on $BF_2(2m+2)$, for any odd number $m \geq 1$.*

Proposition 3.6 *The set $CBFS_2(2m+2)$ is a non-expandable cross-bifix-free set on $BF_2(2m+2)$, for any odd number $m \geq 1$.*

The proof of Proposition 3.5 follows the logical steps as far Proposition 3.3 and the proof of Proposition 3.6 follows the logical steps as far Proposition 3.4.

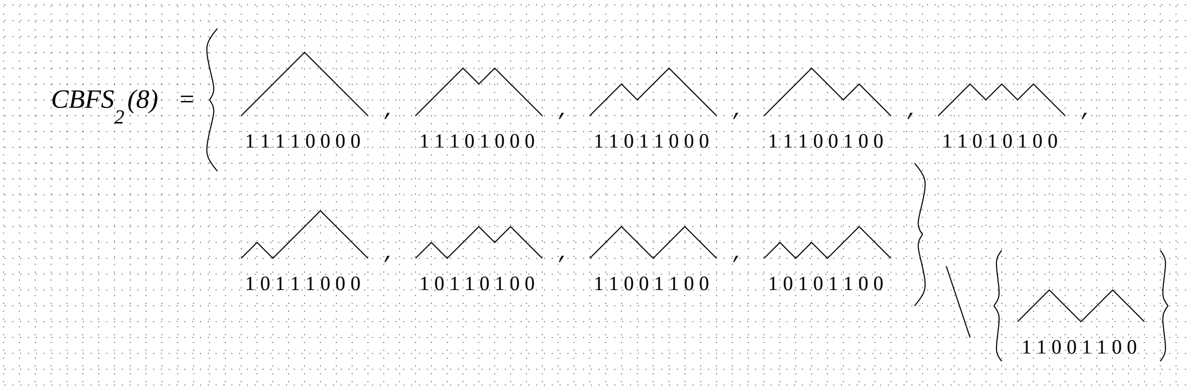

Figure 3.11: A graphical representation of the set $CBFS_2(8)$

Therefore, the presented constructing method gives sets $CBFS_2(n)$ of cross-bifix-free binary words, of fixed length n, having cardinality $1, 1, 2, 3, 5, 8, 14, 23, 42, 72, 132, 227, 429$ for $n = 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15$ respectively.

4 Conclusions and further developments

In this paper, we introduce a general constructing method for the sets of cross-bifix-free binary words of fixed length n based upon the study of lattice paths on the Cartesian plane. This approach enables us to obtain the cross-bifix-free set $CBFS_2(n)$ having greater cardinality than the ones presented in [1] based upon the kernel method.

Moreover, we prove that $CBFS_2(n)$ is a non-expandable cross-bifix-free set on $BF_2(n)$, i.e. $CBFS_2(n) \cup \gamma$ is not a cross-bifix-free set on $BF_2(n)$, for any $\gamma \in BF_2(n) \backslash CBFS_2(n)$. The non-expandable property is obviously a necessary condition to obtain a maximal cross-bifix-free set on $BF_2(n)$, anyway we are not able to find and prove a sufficient condition.

Further studies to prove that could investigate both the nontrivial subsets of $BF_2(n)$ in which $CBFS_2(n)$ is a maximal cross-bifix-free set, and the study of other non-expandable cross-bifix-free sets on $BF_2(n)$.

Another approach to reach the goal could be to find a different characterization of bifix-free words which could be obtained through bijective methods between particular bifix-free subsets and other well-known discrete structures.

Successive studies should take into consideration the general study of cross-bifix-free sets on $BF_q(n)$, where q is grater than 2.

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