DISTRIBUTION OF EIGENVALUES OF WEIGHTED, STRUCTURED MATRIX ENSEMBLES

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ABSTRACT. The study of the limiting distribution of eigenvalues of $N \times N$ random matrices as $N \to \infty$ has many applications, including nuclear physics, number theory and network theory. One of the most studied ensembles is that of real symmetric matrices, where the limiting spectral measure converges to the semi-circle. Studies have also determined the limiting spectral measures for many structured ensembles, such as Toeplitz and circulant matrices. These systems have very different behavior; the limiting spectral measures for both have unbounded support. Given a structured ensemble, we introduce a parameter to continuously interpolate between these two behaviors. We fix a $p \in [1/2, 1]$ and study the ensemble of signed structured matrices by multiplying the $(i,j)^{\text{th}}$ and $(j,i)^{\text{th}}$ entries of a matrix by a randomly chosen $\epsilon_{ij} \in \{1, -1\}$, with $\text{Prob}(\epsilon_{ij} = 1) = p$. For $p = 1/2$, we prove that the limiting spectral measure is the semi-circle. For all other p, for many structured ensembles (including the Toeplitz and circulant) we prove the measure has unbounded support, and converges to the original ensemble as $p \to 1$.

The proofs are by Markov's Method of Moments. The analysis of the $2k^{\text{th}}$ moment for such distributions involves the pairings of $2k$ vertices on a circle. The contribution of each pairing in the signed case is weighted by a factor depending on p and the number of vertices involved in at least one crossing. These numbers are of interest in their own right, appearing in problems in combinatorics and knot theory. The number of configurations with no vertices involved in a crossing is well-studied, and are the Catalan numbers. We discover and prove similar formulas for configurations with $4, 6, 8$ and 10 vertices in at least one crossing. For higher-order moments, we prove closed-form expressions for the expected value and variance for the number of vertices in at least one crossing. As the variance converges to 4, these results allow us to deduce properties of the limiting measure.

CONTENTS

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1. INTRODUCTION

1.1. **Background.** Though Random Matrix Theory began with statistics investigations by Wishart [Wis], it was through the work of Wigner [Wig1, Wig2, Wig3, Wig4, Wig5], Dyson [Dy1, Dy2] and others that its true power and universality became apparent. Wigner's great insight was that ensembles of matrices with randomly chosen entries model well many nuclear phenomena. For example, in quantum mechanics the fundamental equation is $H\Psi_n = E_n\Psi_n$ (H is the Hamiltonian, Ψ_n the energy eigenstate with eigenvalue E_n). Though H is too complicated to diagonalize, a typical H behaves similarly to the average behavior of the ensemble of matrices where each independent entry is chosen independently from some fixed probability distribution. Depending on the physical system, the matrix H is constrained. The most common are H is real-symmetric (where the limiting spectral measure is the semi-circle) or Hermitian. In addition to physics, these matrix ensembles successfully model diverse fields from number theory [ILS, KS1, KS2, KeSn, Mon, RS] to random graphs [JMRR, MNS] to bus routes in Mexico [BBDS, KrSe].

The original ensembles studied had independent entries chosen from a fixed probability distribution with mean 0, variance 1 and finite higher moments. For such ensembles, the limiting spectral measure could often be computed, though only recently [ERSY, ESY, TV1, TV2] could the limiting spacings between normalized eigenvalues be determined for general distributions. See [Fo, Meh] for a general introduction to Random Matrix Theory, and [Dy3, FM, Hay] for a partial history.

Recently there has been much interest in studying highly structured sub-ensembles of the family of real symmetric matrices, where new limiting behavior emerges. Examples include band matrices, circulant matrices, random abelian G-circulant matrices, adjacency matrices associated to d-regular graphs, and Hankel and Toeplitz matrices, among others [BasBo1, BasBo2, BanBo, BCG, BH, BM, BDJ, GKM, HM, JMP, Kar, KKMSX, LW, MMS, McK, Me, Sch]. Two particularly interesting cases are the Toeplitz [BDJ, HM] and palindromic Toeplitz ensemble [MMS], which we now generalize (though our arguments would follow through with only minor changes for other structured ensembles). Recall a real symmetric Toeplitz matrix is constant along its diagonals, while its palindromic variant has the additional property that its first row is a palindrome. The limiting spectral measures of these ensembles have been proven to exist; it is the Gaussian in the palindromic case, and almost a Gaussian in the Toeplitz case (the limiting spectral measure has unbounded support, though the moments grow significantly slower than the Gaussian's).

As these matrices are small sub-families of the family of all real symmetric matrices, it is not surprising that new behavior is seen. A natural question to ask is whether or not there is a way to 'fatten' these ensembles and regain the behavior of the full ensemble. This is similar to what happens for the adjacency matrices of d-regular graphs. For fixed d the limiting spectral measure is Kesten's measure [McK], which converges as $d \to \infty$ to the semi-circle. We can ask similar questions about band matrices, and again see a transition in behavior as a parameter grows [Sch].

Before stating our results, we first quickly review some standard notation (see for example [HM, JMP, KKMSX, MMS]). Given an $N \times N$ matrix A, its associated eigenvalue measure is

$$\mu_A(x) \ = \ \frac{1}{N}\sum_{k=1}^{N} \delta\left(x - \frac{\lambda_k(A)}{\mathfrak{c}\sqrt{N}}\right), \tag{1.1}$$

where the $\lambda_k(A)$'s are the eigenvalues of A, and $\mathfrak{c}\sqrt{N}$ is a normalization constant (with $\mathfrak{c}$ a function of the structured ensemble). Using the Eigenvalue Trace Lemma, we find that the k^{th} moment of μ_A is

$$M_{k;N}(A) \;=\; \int_{-\infty}^{\infty} x^k \mu_A(x)dx \;=\; \frac{\text{Trace}(A^k)}{\mathfrak{c}^k N^{k/2+1}}. \tag{1.2}$$

The advantage of this formulation is that we convert what we want to study (the eigenvalues) to something we understand (the matrix entries, which are randomly chosen). We now integrate the above over the family, reducing the computation to averaging polynomials of the matrix elements over the family. Determining the answer frequently involves solving difficult combinatorial problems to count the number of configurations with a given contribution.

Many proofs of the limiting behavior (averaged over the ensemble) proceed via Markov's Method of Moments (see for example [Ta]), where one shows the average moments over the ensemble converge to the moments of a nice distribution. This, plus some control over the variance / rate of convergence (usually done through counting arguments and then appeals to Chebyshev's inequality and the Borel-Cantelli lemma), suffice to prove various types of convergence of the limiting spectral measure to a fixed distribution. See the above references for exact statements on the needed assumptions for the various types of convergence.

1.2. **Results.** We fix a $p \in [1/2, 1]$ and study the ensemble of signed structured matrices by multiplying the $(i,j)^{\text{th}}$ and $(j,i)^{\text{th}}$ entries of a matrix by a randomly chosen $\epsilon_{ij} \in \{1, -1\}$, with $\text{Prob}(\epsilon_{ij} = 1) = p$. As we vary p, we continuously interpolate between highly structured (when $p = 1$) and less structured (when $p = 1/2$) ensembles.

Unfortunately, in general it is very hard to obtain closed-form expressions for the limiting spectral measures (exceptions are the Gaussian behavior in palindromic Toeplitz [MMS], circulant ensembles [KKMSX] and d-regular graphs [McK]); however, we are still able to prove many results about the moments of our weighted, structured ensembles. Using the expansion from the Eigenvalue Trace Lemma, a degree of freedom argument shows that the elements in the trace expansions must be matched in pairs; the difficulty is figuring out the contribution of each. The odd moments trivially vanish, and for even moments, the only contribution in the limit comes from when the indices are matched in pairs with opposite orientation. We show that we may view these terms as pairings of $2k$ vertices, $(i_1, i_2), (i_2, i_3), \ldots, (i_{2k}, i_1)$, on a circle. Our main result is to show that the contribution of each pairing c in the unsigned case is weighted by $(2p-1)^{e(c)}$ in the signed case, where $e(c)$ is the number of vertices in crossing pairs in the pairing. This extends previous results. When $p = 1/2$, we are reduced almost completely to the real symmetric case, and our result implies that all crossing configurations contribute 0, and all non-crossing configurations contribute 1. This gives us a $2k^{\text{th}}$ moment equal to the k^{th} Catalan number, which is both the number of non-crossing pairings of $2k$ objects and the $2k^{\text{th}}$ moment of the semicircle density.[1] By contrast, when $p = 1$ we are reduced to the unsigned case, and indeed our theorem implies that each configuration contributes what it did in the unsigned case.

Our first result is the following (though identical arguments would work for similar distributions).

Theorem 1.1. *Consider the ensemble of real-symmetric Toeplitz or palindromic Toeplitz matrices, where the independent entries are drawn from a distribution* $\mathfrak{p}$ *with mean 0, variance 1 and finite*

[1]The normalized semi-circular density is $f_{\text{sc}}(x) = \frac{1}{\pi}\sqrt{1 - \left(\frac{x}{2}\right)^2}$ if $|x| \le 2$ and 0 otherwise, and the even moments are the Catalan numbers.

higher moments. Consider the weighted ensemble where the $(i,j)^{\text{th}}$ and $(j,i)^{\text{th}}$ entries of these matrices are multiplied by a randomly chosen $\epsilon_{ij} \in \{1,-1\}$, with $\text{Prob}(\epsilon_{ij}=1)=p$.

For $p=1/2$, the limiting spectral measure for the signed, structured ensemble of real-symmetric Toeplitz or palindromic Toeplitz matrices is the semi-circle. For all other p, the limiting measure has unbounded support, and converges to the original ensemble's limiting measure as $p \to 1$. The convergence is weakly and in probability to their corresponding limiting spectral measures, and almost surely if additionally the density $\mathfrak{p}$ is even.

We find that the controlling factor is how many vertices are involved in a crossing (we make this precise in §3). This reduces our problem to one in combinatorics (and, in fact, our problem turns out to be related to issues in knot theory as well; see for example [KT, Kont, FN, Rio, Sto]). In the course of our investigations, we prove several interesting combinatorial results, which we isolate below.

Theorem 1.2. *Consider all $(2k-1)!!$ pairings of $2k$ vertices on a circle. Let $\text{Cr}_{2k,2m}$ denote the number of these pairings where exactly $2m$ vertices are involved in a crossing and let C_k denote the k^{th} Catalan number, $\frac{1}{k+1}\binom{2k}{k}$. For small values of m, we obtain the exact formulas for $\text{Cr}_{2k,2m}$ listed below; for large k (and thus a large range of possible m) we prove the limiting behavior of the expected value and variance of the number of vertices involved in at least one crossing.*

- *For $m \le 10$ we have*

$$\begin{aligned}
\text{Cr}_{2k,0} &= C_k \\
\text{Cr}_{2k,2} &= 0 \\
\text{Cr}_{2k,4} &= \binom{2k}{k-2} \\
\text{Cr}_{2k,6} &= 4\binom{2k}{k-3} \\
\text{Cr}_{2k,8} &= 31\binom{2k}{k-4} + \sum_{d=1}^{k-4}\binom{2k}{k-4-d}(4+d) \\
\text{Cr}_{2k,10} &= 288\binom{2k}{k-5} + 8\sum_{d=1}^{k-5}\binom{2k}{k-5-d}(5+d)\,. \qquad (1.3)
\end{aligned}$$

- *As $k \to \infty$, the expected number of vertices involved in a crossing converges to $2k-2$ (see Theorem 4.1 for the exact value in terms of hypergeometric functions), and the variance of the number of vertices involved in a crossing converges to 4.*

We review the basic framework and definitions used in studying the moments in §2. In §3 we determine formulas for the moments, and prove the first part of Theorem 1.2, completing the proof by determining the limiting behavior in §4. All that remains to prove Theorem 1.1 is to handle the convergence issues; this analysis is standard, and is quickly reviewed in §5.

2. MOMENT PRELIMINARIES

We briefly summarize the needed expansions from previous work (see [HM, JMP, KKMSX, MMS] for complete details). We use a standard method to compute the moments. For a fixed

$N \times N$ matrix A, its k^{th} moment is

$$M_{k,N}(A) = \frac{1}{N^{\frac{k}{2}+1}} \sum_{1 \le i_1,\ldots,i_k \le N} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}, \tag{2.1}$$

which when applied to our signed Toeplitz and palindromic Toeplitz matrices (where the entries of the unsigned ensemble are constant along diagonals) gives that

$$M_{k,N}(A) = \frac{1}{N^{\frac{k}{2}+1}} \sum_{1 \le i_1,\ldots,i_k \le N} \epsilon_{i_1 i_2} b_{|i_1 - i_2|} \epsilon_{i_2 i_3} b_{|i_2 - i_3|} \cdots \epsilon_{i_k i_1} b_{|i_k - i_1|}. \tag{2.2}$$

By linearity of expectation,

$$\mathbb{E}\left(M_{k,N}(A)\right) = \frac{1}{N^{\frac{k}{2}+1}} \sum_{1 \le i_1,\ldots,i_k \le N} \mathbb{E}\left(\epsilon_{i_1 i_2} b_{|i_1 - i_2|} \epsilon_{i_2 i_3} b_{|i_2 - i_3|} \cdots \epsilon_{i_k i_1} b_{|i_k - i_1|}\right). \tag{2.3}$$

Of the N^k terms in the above sum corresponding to the N^k choices of $(i_1, \ldots, i_k)$ in the above sum, we can immediately see that some contribute zero in the limit as $N \to \infty$ by using the following lemmas.

Lemma 2.1. *Any term that contributes in the limit must have each b_α in the product appearing exactly twice, and all such terms have a finite contribution.*

Proof. We first prove that any term that doesn't have every b_α appearing at least twice does not contribute. As the expected value of a product of independent variables is the product of the expected values, since each b_α is drawn from a distribution with mean zero, there is no contribution in this case. Thus each b_α occurs at least twice if the term is to contribute.

We now show that any term that has some b_α appearing more than twice cannot contribute in the limit. If each b_α appears exactly twice, then there are $k/2$ values of b_α to choose. Recall that $b_{|i_j i_{j+1}|}$ is paired with $b_{|i_k - i_{k+1}|}$ if and only if

$$i_j - i_{j+1} = \pm(i_k - i_{k+1}). \tag{2.4}$$

Once we have specified the b's and one index i_l, there are at most two values for each remaining index. Thus there are $O\left(N^{\frac{k}{2}+1}\right)$ terms where the b_α's are matched in exactly pairs. By contrast, any term that has some b_α appearing more than twice has fewer than $\frac{k}{2}+1$ degrees of freedom, and thus does not contribute in the limit as we divide by $N^{k/2+1}$.

Finally, we show that the sum of the contributions from all terms arising from matching in pairs is $O_k(1)$. Suppose there are $r \le k$ different ϵ_γ's and $s \le k$ different b_α's in the product, say $\epsilon_{\gamma_1}, \ldots, \epsilon_{\gamma_r}$ and $b_{\alpha_1}, \ldots, b_{\alpha_s}$, with each ϵ_{γ_j} occurring n_j times and each b_{α_j} occurring m_j times. Such a term contributes $\prod_{j=1}^{r} \mathbb{E}\left(\epsilon_{\gamma_j}^{n_j}\right) \prod_{j=1}^{s} \mathbb{E}\left(b_{\alpha_j}^{m_j}\right)$. Since the probability distributions of the ϵ's and b's have finite moments, this contribution is thus $O_k(1)$, and thus the sum of all such contributions is finite in the limit. □

Lemma 2.2. *The odd moments of the limiting spectral measure vanish.*

Proof. This follows directly from Lemma 2.1 (since the odd moments have an odd number of b's, they cannot be matched exactly in pairs). □

Since the odd moments vanish, we concern ourselves in the rest of the paper with the limiting behavior of the even moments, M_{2k}. Further, in the moment expansion for the even moments, we only have to consider terms in which the b_α's are matched in exactly pairs. With the next lemma,

we further reduce the number of terms we must consider by showing that only those terms where every pairing between the b's is with a minus sign in (2.4) contribute in the limit. The following proof is adapted from [HM].

Lemma 2.3. *The only terms that contribute to the $2k^{\text{th}}$ moment of the limiting spectral measure are terms where the b's are matched in exactly pairs with a minus sign in* (2.4)*.*

Proof. For each term, there are k corresponding equations of the form (2.4). We let $x_1, \dots, x_k$ be the values of the $|i_j - i_{j+1}|$ in these equations, and let $\delta_1, \dots, \delta_k$ be the choices of sign in these equations. We further let $\tilde{x}_1 = i_1 - i_2$, $\tilde{x_2} = i_2 - i_3, \dots, \tilde{x}_{2k} = i_{2k} - i_1$. Then each of the previous k equations can be written as

$$\tilde{x}_m \;=\; \delta_j \tilde{x}_n. \tag{2.5}$$

By definition, there is some $\eta_j = \pm 1$ such that $\tilde{x}_m = \eta_j x_j$. Then $\tilde{x}_n = \delta_j \eta_j x_j$, so

$$\tilde{x}_1 + \tilde{x}_2 + \cdots + \tilde{x}_{2k} \;=\; \sum_{j=1}^{k} \eta_j \left(1 + \delta_j\right) x_j. \tag{2.6}$$

Finally, notice that

$$\tilde{x}_1 + \tilde{x}_2 + \cdots + \tilde{x}_{2k} \;=\; i_1 - i_2 + i_2 - i_3 + \cdots + i_{2k} - i_1 \;=\; 0. \tag{2.7}$$

Thus

$$\sum_{j=1}^{k} \eta_j \left(1 + \delta_j\right) x_j \;=\; 0. \tag{2.8}$$

If any $\delta_j = 1$, then (2.8) gives us a linear dependence between the x_j. Recall from the proof of Lemma 2.1 that we require all x_j to be independently chosen for a pairing to contribute; otherwise, there are fewer than $k + 1$ degrees of freedom. Thus, the only terms that contribute have each $\delta_j = -1$. □

The above results motivate the following definition.

Definition 2.4 (Pairing)**.** *A pairing is a matching of the vertices $i_1, i_2, \dots, i_{2k}$ such that the vertices are matched exactly in pairs, and with a negative sign in* (2.4)*. There are $(2k-1)!!$ pairings of the $2k$ vertices. As argued above in the proof of Lemma 2.1, these pairings correspond to $O\left(N^{k+1}\right)$ terms in the sum in* (2.3) *for the $2k^{\text{th}}$ moment.*

As suggested above, we find that a good way to investigate the contribution of each potentially contributing term, i.e., each choice or tuple of $(i_1, \dots, i_{2k})$, is to associate each term with a pairing of $2k$ vertices on a circle, where the vertices are $|i_1 - i_2|, |i_2 - i_3|, \dots, |i_{2k} - i_1|$. Because what matters are not the values of the $|i_j - i_{j+1}|$'s, but rather the pattern of how they are matched, any terms associated with the same pairing of the $2k$ vertices will have the same contribution. Thus, pairings that are the same up to a rotation of the vertices contribute the same since it is not the values of i_j that matters but rather the distance between each vertex and its matching and the indices of the other pairs. Therefore, to further simplify the moment analysis, we make the following definition.

Definition 2.5 (Configuration)**.** *Two pairings $\left\{\left(i_{a_1}, i_{a_2}\right), \left(i_{a_3}, i_{a_4}\right), \dots, \left(i_{a_{2k-1}}, i_{a_{2k}}\right)\right\}$ and $\left\{\left(i_{b_1}, i_{b_2}\right), \left(i_{b_3}, i_{b_4}\right), \dots, \left(i_{b_{2k-1}}, i_{b_{2k}}\right)\right\}$ are said to be in the same configuration if they are equivalent up to a relabeling by rotating the vertices; i.e., there is some constant l such that $b_j = a_j + l \mod 2k$.*

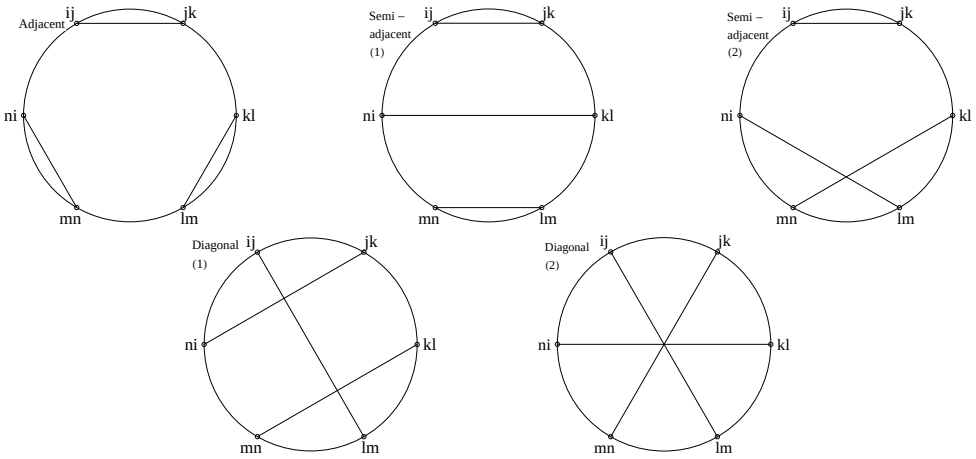

FIGURE 1. The five distinct configurations for the 6^{th} moment where vertices are matched exactly in pairs. The multiplicity under rotation of the five patterns are 2, 3, 6, 3 and 1 (for example, rotating the first pattern twice returns it to its initial configuration, while the third requires six rotations). The nomenclature is from [KKMSX], and not that relevant to our purposes here.

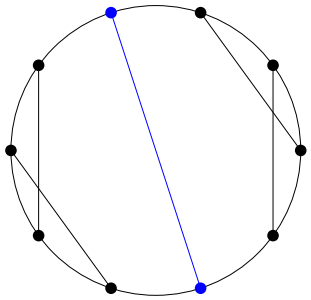

FIGURE 2. A pairing of 10 vertices with 8 crossing vertices (in two symmetric sets of 4 vertices), and 2 dividing vertices (connected by a main diagonal).

For example, we list the five distinct configurations needed for the sixth moment in Figure 1. The problem of determining the moments is thus reduced to determining for each configuration both the contribution of a pairing belonging to that configuration to the sum in (2.3) and the number of pairings belonging to that configuration.

3. DETERMINING THE MOMENTS

By Lemma 2.1, for the rest of the paper we assume the vertices are matched in exactly pairs. We distinguish between three types of vertices in these pairings.

Definition 3.1. *We say that a pair (i_a, i_b), $a < b$, is in a **crossing** if there exists a pair (i_x, i_y) such that $a < x < b$ and either $y < a$ or $y > b$. A pair (i_a, i_b) is **non-crossing** if for every pair (i_x, i_y), $a < x < y$ if and only if $a < y < b$.*

Pictorially, a pair is crossing if the line connecting its two vertices crosses another line connecting two other vertices. In Figure 1, the first two configurations have no crossing vertices, the third has

four, while all vertices are crossing for the fourth and fifth. Note the number of crossing vertices is always even and never two.

Definition 3.2. *We say that a non-crossing pair (i_a, i_b) is **dividing** if there exist at least one crossing pair (i_x, i_y), (i_w, i_z) such that $a < x, y, w, z < b$ and at least one crossing pair (i_p, i_q), (i_r, i_s) such that each index $j \in \{p, q, r, s\}$ satisfies either $j < a$ or $j > b$.*

Pictorially, a pair is dividing if it "divides" the circle into two regions of pairs (no pair can cross a dividing edge since it must be non-crossing), where each region contains at least one crossing pair. From the definition, we see that at least 10 vertices are needed for a "dividing" pair to exist. See Figure 2 for an illustration.

All other pairs will be called **non-crossing non-dividing** pairs. Note that all pairings belonging to a given configuration have the same number of crossing pairs and the same number of dividing pairs.

We show in this section that the contribution of each pairing in the unsigned case is weighted by a factor depending on the number of crossing pairs in that pairing. We then prove some combinatorial formulas that allow us to obtain closed form expressions for the number of pairings with m vertices crossing for small k. As the combinatorics becomes prohibitively difficult for large k, we determine the limiting behavior in §4.

3.1. **Weighted Contributions.** The following theorem is central to our determination of the moments. It reduces the calculations to two parts. First, we need to know the contribution of a pairing in the non-weighted case (equivalently, when $p = 1$). While this is known precisely for the palindromic Toeplitz case, where each pairing contributes 1, in the Toeplitz case we only have upper and lower bounds on the contribution of all pairing. Second, we need to determine the number of vertices involved in crossing pairs, which we do in part in §3.2.

Theorem 3.3. *For each choice of a pairing c of $(i_1, \dots, i_{2k})$, let $x(c)$ denote the contribution of this tuple in the unsigned case. Then the contribution in the signed case is $x(c)(2p-1)^{e(c)}$, where $e(c)$ represents the number of vertices in crossing pairs in the configuration corresponding to c.*

Proof. Recall that the contribution from any choice of $(i_1, \dots, i_{2k})$ is

$$\begin{aligned}\mathbb{E}(\epsilon_{i_1 i_2} b_{|i_1 - i_2|} \epsilon_{i_2 i_3} b_{|i_2 - i_3|} \cdots \epsilon_{i_{2k} i_1} b_{|i_{2k} - i_1|}) &= \mathbb{E}(\epsilon_{i_1 i_2} \epsilon_{i_2 i_3} \cdots \epsilon_{i_{2k} i_1}) \mathbb{E}(b_{|i_1 - i_2|} \cdots b_{|i_k - i_1|}) \\ &= \mathbb{E}(\epsilon_{i_1 i_2} \epsilon_{i_2 i_3} \cdots \epsilon_{i_{2k} i_1}) x(c). \end{aligned} \tag{3.1}$$

Thus, we want to show that $\mathbb{E}(\epsilon_{i_1 i_2} \epsilon_{i_2 i_3} \cdots \epsilon_{i_{2k} i_1}) = (2p-1)^{e(c)}$. We do this by showing that for each pair $(i_j, i_{j+1}), (i_k, i_{k+1})$ where $b_{|i_j - i_{j+1}|} = b_{|i_k - i_{k+1}|}$,

$$\mathbb{E}\left(\epsilon_{i_j i_{j+1}} \epsilon_{i_k i_{k+1}}\right) = \begin{cases} (2p-1)^2 & \text{if } (i_j, i_{j+1}), (i_k, i_{k+1}) \text{ are a crossing pair} \\ 1 & \text{otherwise.} \end{cases} \tag{3.2}$$

Notice that

$$\mathbb{E}(\epsilon_\alpha) = 1 \cdot p + (-1) \cdot (1-p) = 2p - 1, \quad \mathbb{E}\left(\epsilon_\alpha^2\right) = 1. \tag{3.3}$$

Therefore, if m epsilons are chosen independently, the expected value of their product is $(2p-1)^m$.

Our first step is proving $\mathbb{E}(\epsilon_{i_1 i_2} \epsilon_{i_2 i_3} \cdots \epsilon_{i_{2k} i_1}) \geq (2p-1)^{e(c)}$ by showing that pairs not in a crossing contribute 1. Consider a non-crossing pair $(i_r, i_{r+1}), (i_p, i_{p+1})$, with $r < p$. For each (i_q, i_{q+1}) paired with $(i_{q'}, i_{q'+1})$, we have $r < q < p$ if and only if $r < q' < p$. Recall from (2.4) and Lemma 2.3 that

$$i_q - i_{q+1} = -(i_{q'} - i_{q'+1}). \tag{3.4}$$

Thus

$$\sum_{k=r}^{p}(i_k - i_{k+1}) \;=\; 0 \tag{3.5}$$

because each difference in the sum is paired with its additive inverse, which is also in the sum. As

$$\sum_{k=r}^{p}(i_k - i_{k+1}) \;=\; (i_r - i_{r+1}) + (i_{r+1} - i_{r+1}) + \cdots + (i_p - i_{p+1}) \;=\; i_r - i_{p+1}, \tag{3.6}$$

we must have $i_r = i_{p+1}$, and for the differences to be equal, $i_{r+1} = i_p$. Therefore $\epsilon_{i_r i_{r+1}} = \epsilon_{i_p i_{p+1}}$, and hence $\mathbb{E}(\epsilon_{i_r i_{r+1}}\epsilon_{i_p i_{p+1}}) = 1$.

Now we show $\mathbb{E}(\epsilon_{i_1 i_2}\epsilon_{i_2 i_3}\cdots\epsilon_{i_{2k} i_1}) \leq (2p-1)^{e(c)}$ by showing that if $\epsilon_{i_a i_{a+1}} = \epsilon_{i_b i_{b+1}}$, $a < b$, then $(i_a, i_{a+1}), (i_b, i_{b+1})$ are non-crossing. This suffices to prove the result since we know that the only dependency between the ϵ's arises from the requirement that the matrix is real symmetric, and thus we have a dependency between $\epsilon_{i_s i_{s+1}}$ and $\epsilon_{i_p i_{p+1}}$ if and only if we know they are equal. In showing that a dependency between ϵ's implies the corresponding vertex pair must be non-crossing, we show that crossing pairs imply independent ϵ's and thus contribute $(2p-1)^2$.

If $\epsilon_{i_a i_{a+1}} = \epsilon_{i_b i_{b+1}}$ then it must be true that the unordered sets $\{i_a, i_{a+1}\}$ and $\{i_b, i_{b+1}\}$ are equal. Notice that this then implies that $|i_a - i_{a+1}| = |i_b - i_{b+1}|$, so $(i_a, i_{a+1}), (i_b, i_{b+1})$ must be paired on the circle. Since the only contributing terms are when they are paired in opposite orientation, we then know that $i_a = i_{b+1}$, so

$$\sum_{k=a}^{b}(i_k - i_{k+1}) \;=\; i_a - i_{b+1} \;=\; 0. \tag{3.7}$$

We can rewrite this sum as

$$\sum_{k=b}^{d}\delta_k |i_k - i_{k+1}| \;=\; 0, \tag{3.8}$$

where δ_k is ± 1 if the vertex k is paired with is less than a or greater than b, and 0 if and only if the vertex k is paired with is between a and b. However, a linear dependence among the differences is impossible, as we need to have N^{k+1} degrees of freedom for each configuration (see the proof of Lemma 2.1). So each $\delta_k = 0$, and each vertex between vertices a and b is paired with something else between a and b. Thus, no edges cross the edge between vertices a and b.

We have shown that an epsilon is unmatched if and only if its edge is in a crossing. Thus, an epsilon is not paired if and only if its edge is not in a crossing. Therefore the contribution is weighted by $\mathbb{E}(\epsilon_{i_1 i_2}\epsilon_{i_2 i_3}\cdots\epsilon_{i_{2k} i_1}) = (2p-1)^{e(c)}$, completing the proof. □

3.2. **Counting Crossing Configurations.** Theorem 3.3 reduced the determination of the moments to counting the number of pairings with a given contribution $x(c)$, and then weighting those by $(2p-1)^{e(c)}$, where $e(c)$ is the number of vertices involved in crossings in the configuration. As remarked above, in the palindromic Toeplitz case each $x(c) = 1$, while in the general Toeplitz case we only have bounds on the $x(c)$'s, and thus must leave these as parameters in the final answer (though any specific $x(c)$ may be computed by brute force, we do not have a closed form expression in general).

In this section we turn to computing the $e(c)$'s for various configurations. As previously mentioned, these and similar numbers have also been studied in knot theory where these chord diagrams

are used in the study of Vassiliev invariants (see [KT, Kont, FN, Rio, Sto]). While we cannot determine exact formulas in general, we are able to solve many special cases, which we now describe.

Definition 3.4 ($\mathrm{Cr}_{2k,2m}$)**.** *Let $\mathrm{Cr}_{2k,2m}$ denote the number of pairings involving $2k$ vertices where exactly $2m$ vertices are involved in a crossing.*

Let $C_k = \frac{1}{k+1}\binom{2k}{k}$ denote the k^{th} Catalan number (see [AGZ] for statements and proofs of their needed properties). One of its many definitions is as the number of ways to match $2k$ objects on a circle in pairs without any crossings; this interpretation is the reason why Wigner's Semi-Circle Law holds. Thus, we immediately deduce the following.

Lemma 3.5. *We have $\mathrm{Cr}_{2k,0} = C_k$.*

We use this result to prove the following theorem, which is instrumental in the counting we need to do.

Theorem 3.6. *Consider $2k$ vertices on the circle, with a partial pairing on a subset of $2v$ vertices. The number of ways to place the remaining $2k-2v$ vertices in non-crossing, non-dividing pairs is $\binom{2k}{k-v}$.*

Proof. Let $\mathcal{W}$ denote the desired quantity. Notice that each of the remaining $2k-2v$ vertices must be placed between two of the $2v$ already paired vertices on the circle. These $2v$ vertices have created $2v$ regions. A necessary and sufficient condition for these $2k-2v$ vertices to be in non-crossing, non-dividing pairs is that the vertices in each of these $2v$ regions pair only with other vertices in that region in a non-crossing configuration.

Thus, if there are $2s$ vertices in one of these regions, by Lemma 3.5 the number of valid ways they can pair is C_{2s}. As the number of valid matchings in each region depends only on the number of vertices in that region and not on the matchings in the other regions, we obtain a factor of $C_{2s_1}C_{2s_2}\cdots C_{2s_{2v}}$, where $2s_1+2s_2+\cdots+2s_{2v} = 2k-2v$.

We need only determine how many pairings this factor corresponds to. First we notice that by specifying one index and $(s_1, s_2, \ldots, s_{2v})$, we have completely specified a pairing of the $2k$ vertices. However, as we are pairing on a circle, this specification does not uniquely determine a pairing since the labelling of $(s_1, s_2, \ldots, s_{2v})$ is arbitrary. Each pairing can in fact be written as any of the $2v$ circular permutations of some choice of $(s_1, s_2, \ldots, s_{2v})$ and one index. Thus the quantity we are interested in is

$$\mathcal{W} \;=\; \frac{2k}{2v}\sum_{2s_1+2s_2+\cdots+2s_{2v}=2k-2v} C_{s_1}C_{s_2}\cdots C_{s_{2v}} \tag{3.9}$$

To evaluate this expression, we use the k-fold self-convolution identity of Catalan numbers [Reg], which states

$$\sum_{i_1+\cdots+i_r=n} C_{i_r-1}\cdots C_{i_r-1} \;=\; \frac{r}{2n-r}\binom{2n-r}{n}. \tag{3.10}$$

Setting $i_j = s_j+1$, $r=2v$ and $n=k+v$, we obtain

$$\sum_{s_1+s_2+\cdots+s_{2v}+2v=k+v} C_{s_1}C_{s_2}\cdots C_{s_{2v}} \;=\; \frac{2v}{2k}\binom{2k}{k+v}. \tag{3.11}$$

We may rewrite this as

$$\frac{2k}{2v}\sum_{2s_1+2s_2+\cdots+2s_{2v}=2k-2v} C_{s_1}C_{s_2}\cdots C_{s_{2v}} \;=\; \binom{2k}{k-v}, \tag{3.12}$$

which completes the proof as the left hand side is just (3.9). □

Given Theorem 3.6, our ability to find formulas for $\mathrm{Cr}_{2k,2m}$ rests on our ability to find the number of ways to pair $2v$ vertices where $2m$ vertices are crossing and $2v-2m$ vertices are dividing. We are able to do this for small values of m, but for large m, the combinatorics becomes very involved.

Definition 3.7 ($\mathrm{P}_{2k,2m,i}$, partitions)**.** *Let $\mathrm{P}_{2k,2m,i}$ represent the number of pairings of $2k$ vertices with $2m$ crossing vertices in i partitions. We define a partition to be a set of crossing vertices separated from all other sets of crossing vertices by at least one dividing edge.*

It takes a minimum of 4 vertices to form a partition, so the maximum number of partitions possible is $\lfloor 2m/4 \rfloor$. Our method of counting involves writing

$$\mathrm{Cr}_{2k,2m} \;=\; \sum_{i=1}^{\lfloor 2m/4 \rfloor} \mathrm{P}_{2k,2m,i}. \tag{3.13}$$

Our first combinatorial result is the following.

Lemma 3.8. *We have*

$$\mathrm{P}_{2k,2m,1} \;=\; \mathrm{Cr}_{2m,2m}\binom{2k}{k-m}. \tag{3.14}$$

Proof. The proof follows immediately from Theorem 3.6. If there is only one partition, then there can be no dividing edges. Therefore, we simply multiply the number of ways we can choose $2k-2m$ non-crossing non-dividing pairs by the number of ways to then choose how the $2m$ crossing vertices are paired. □

Our next result is

Lemma 3.9. *We have*

$$\mathrm{P}_{2k,2m,2} \;=\; \sum_{d=1}^{k-m}\binom{2k}{k-m-d}(m+d)\left(\sum_{0<a<m}\mathrm{Cr}_{2a,2a}\mathrm{Cr}_{2m-2a,2m-2a}\right). \tag{3.15}$$

Proof. We let d be the number of dividing edges. In order to have two partitions, at least one of the $k-m$ non-crossing edges must be a dividing edge. We thus sum over d from 1 to $k-m$. Given d, we know that we can pair and place the non-crossing non-dividing edges in $\binom{2k}{k-m-d}$ ways from Theorem 3.6. We then choose a way to pair the $2m$ crossing vertices into 2 partitions, one with $2a$ vertices, the other with $2b$ vertices. If $a=b$, there are $m+d$ distinct spots where we may place the dividing edge. If $a \neq b$, there are $2m+2d$ spots. Since each choice of $a \neq b$ appears twice in the above sum, the result follows. □

Determining $\mathrm{P}_{2k,2m,3}$ requires the analysis of several more cases, and we were unable to find a nice way to generalize the results of Lemmas 3.8 and 3.9. However, these two results do allow us to write down the following formulas.

Lemma 3.10. *We have*

$$\begin{aligned}
\mathrm{Cr}_{2k,4} &= \binom{2k}{k-2} \\
\mathrm{Cr}_{2k,6} &= 4\binom{2k}{k-3} \\
\mathrm{Cr}_{2k,8} &= 31\binom{2k}{k-4} + \sum_{d=1}^{k-4}\binom{2k}{k-4-d}(4+d) \\
\mathrm{Cr}_{2k,10} &= 288\binom{2k}{k-5} + 8\sum_{d=1}^{k-5}\binom{2k}{k-5-d}(5+d)\,. \qquad (3.16)
\end{aligned}$$

Proof. We recall that

$$\begin{aligned}
\mathrm{Cr}_{2k,0} &= C_k \\
\mathrm{Cr}_{2k,2} &= 0, \qquad (3.17)
\end{aligned}$$

where the second equation follows from the fact that at least 4 vertices are needed for a crossing. From (3.13) and (3.8) we find

$$\mathrm{Cr}_{2k,4} \;=\; \mathrm{P}_{2k,4,1} \;=\; \mathrm{Cr}_{4,4}\binom{2k}{k-2}. \qquad (3.18)$$

We can calculate $\mathrm{Cr}_{4,4}$ by using (3.17) and the fact that

$$\sum_{m=0}^{k}\mathrm{Cr}_{2k,2m} \;=\; (2k-1)!!. \qquad (3.19)$$

This follows because the number of ways to match $2k$ objects in pairs of 2 with order not mattering is $(2k-1)!!$, and thus the sum of all our matchings in pairs must equal this. Note that this number is also the $2k^{\text{th}}$ moment of the standard normal; this is the reason the palindromic Toeplitz have a limiting spectral measure that is normal, as each contribution contributes fully. We thus find

$$\mathrm{Cr}_{4,4} \;=\; (2k-1)!! - \mathrm{Cr}_{4,2} - \mathrm{Cr}_{4,0} \;=\; 3-2 \;=\; 1. \qquad (3.20)$$

This completes the proof of the first formula: $\mathrm{Cr}_{2k,4} = \binom{2k}{k-2}$.

The other coefficients are calculated in a similar recursive fashion – essentially, once we have values for $\mathrm{Cr}_{2k,2l}$ for $l = 0, 1, 2, \ldots, m-1$, we can find $\mathrm{Cr}_{2m,2m}$ by using (3.19), which allows us to write the general formulas above for $\mathrm{Cr}_{2k,2m}$. We show the calculations below. We have

$$\begin{aligned}
\mathrm{Cr}_{6,6} &= (6-1)!! - \mathrm{Cr}_{6,4} - \mathrm{Cr}_{6,2} - \mathrm{Cr}_{6,0} \\
&= 5!! - \binom{6}{1} - 0 - C_3 \;=\; 15 - 6 - 0 - 5 \;=\; 4 \qquad (3.21)
\end{aligned}$$

so $\mathrm{Cr}_{2k,6} = 4\binom{2k}{k-3}$, and thus

$$\begin{aligned}
\mathrm{Cr}_{8,8} &= (8-1)!! - \mathrm{Cr}_{8,6} - \mathrm{Cr}_{8,4} - \mathrm{Cr}_{8,2} - \mathrm{Cr}_{8,0} \\
&= 7!! - 4\binom{8}{1} - \binom{8}{2} - 0 - C_4 \;=\; 105 - 32 - 28 - 14 \;=\; 31. \qquad (3.22)
\end{aligned}$$

To finish the calculation for $\mathrm{Cr}_{2k,8}$ we need to compute:

$$\sum_{0<a<4} \mathrm{Cr}_{2a,2a}\mathrm{Cr}_{8-2a,8-2a} \;=\; \mathrm{Cr}_{2,2}\mathrm{Cr}_{6,6} + \mathrm{Cr}_{4,4}\mathrm{Cr}_{4,4} + \mathrm{Cr}_{6,6}\mathrm{Cr} \;=\; 0+1+0 \;=\; 1. \tag{3.23}$$

so that we get $\mathrm{Cr}_{2k,8} \;=\; 31\binom{2k}{k-4} + \sum_{d=1}^{k-4}\binom{2k}{k-4-d}(4+d)$.

For the formula for $\mathrm{Cr}_{2k,10}$,

$$\begin{aligned}
\mathrm{Cr}_{10,10} &= (10-1)!! - \mathrm{Cr}_{10,8} - \mathrm{Cr}_{10,6} - \mathrm{Cr}_{10,4} - \mathrm{Cr}_{10,2} - \mathrm{Cr}_{10,0} \\
&= 9!! - \left(31\binom{10}{1} + \sum_{d=1}^{1}\binom{10}{1-d}(4+d)\right) - 4\binom{10}{2} - \binom{10}{3} - 0 - C_5 \\
&= 945 - (310+5) - 4\,(45) - 120 - 0 - 42 \;=\; 288
\end{aligned} \tag{3.24}$$

and finally

$$\begin{aligned}
\sum_{0<a<5} \mathrm{Cr}_{2a,2a}\mathrm{Cr}_{10-2a,10-2a} &= \mathrm{Cr}_{2,2}\mathrm{Cr}_{8,8} + \mathrm{Cr}_{4,4}\mathrm{Cr}_{6,6} + \mathrm{Cr}_{6,6}\mathrm{Cr}_{4,4} + \mathrm{Cr}_{8,8}\mathrm{Cr}_{2,2} \\
&= 0+4+4+0 \;=\; 8.
\end{aligned} \tag{3.25}$$

so $\mathrm{Cr}_{2k,10} \;=\; 288\binom{2k}{k-5} + 8\sum_{d=1}^{k-5}\binom{2k}{k-5-d}(5+d)$.

□

Notice that by using the formulas in Lemma 3.10 to calculate the number of terms with each of the possible contributions given in Theorem 3.3, we are able to calculate up to the 12^{th} moment exactly (where for the 12^{th} moment we use the same recursive procedure as in the proof of Lemma 3.10 to calculate $\mathrm{Cr}_{12,12}$).

Remark 3.11. *The coefficients in front of the binomial coefficient of the leading term of* $\mathrm{Cr}_{2k,2m}$ *is sequence A081054 from the OEIS* [Kl].

4. Limiting Behavior of the Moments

As we are unable to find exact expressions for the number of pairings with exactly $2m$ crossing vertices for all m, we determine the expected value and variance of the number of vertices in a crossing. Such expressions, and the limiting behavior of these expressions, are useful for obtaining bounds for the moments. To find these, we make frequent use of arguments about the probabilities of certain pairings, recognizing that since all configurations are equally likely, the probability that a vertex i pairs with a vertex j is just $\frac{1}{2k-1}$.

Theorem 4.1. *The expected number of vertices involved in a crossing is*

$$\frac{2k}{2k-1}\left(2k-2-\frac{{}_2F_1(1,3/2,5/2-k;-1)}{2k-3} - (2k-1)\,{}_2F_1(1,1/2+k,3/2;-1)\right), \tag{4.1}$$

which is

$$2k-2-\frac{2}{k}+O\left(\frac{1}{k^2}\right) \tag{4.2}$$

as $k\to\infty$.

Proof. In our main applications (such as computing the asymptotic behavior of the mean and the variance), we only need the asymptotic expression (4.2), which we prove elementarily below. We give the proof of (4.1) in Appendix 4.1, which involves converting the expansions below to differences of hypergeometric series.

For a given pairing of $2k$ vertices, let $X_i = 1$ if vertex i is involved in a crossing and 0 otherwise. Then $Y_{2k} = \sum_{i=1}^{2k} X_i$ is the number of vertices involved in a crossing in this pairing. By linearity of expectation,

$$\mathbb{E}\left(Y_{2k}\right) \;=\; \mathbb{E}\left(\sum_{i=1}^{2k} X_i\right) \;=\; 2k\mathbb{E}\left(X_i\right) \;=\; 2kp_{\text{cross}}, \tag{4.3}$$

where p_{cross} is the probability that a given vertex is in a crossing as, by symmetry, this is the same for all vertices. Thus, without loss of generality, we may think of p_{cross} as the probability that vertex 1 is in a crossing. We notice that

(1) If vertex 1 is matched with another odd indexed vertex, which happens with probability $\frac{k-1}{2k-1}$, then it must be involved in a crossing, since there are an odd number of vertices in the two regions created by the matching, meaning that the regions cannot only pair by themselves.

(2) If vertex 1 is matched with an even indexed vertex, then it is involved in a crossing if and only if it does not partition the remaining vertices into two parts that pair exclusively with themselves. Suppose it is matched with vertex $2m$ (which happens with probability $\frac{1}{2k-1}$). Then its edge divides the vertices into a region of $2m-2$ and a region of $2k-2m$ vertices. As the number of ways to match 2ℓ objects in pairs with order immaterial is $(2\ell-1)!! = (2\ell-1)\,(2\ell-3)\cdots 3\cdot 1$, the probability that each region pairs only with itself is

$$\frac{(2m-3)!!\,(2k-2m-1)!!}{(2k-3)!!}. \tag{4.4}$$

Thus, the probability that vertex 1 is involved in a crossing is

$$\begin{aligned}
p_{\text{cross}} &= \frac{k-1}{2k-1} + \sum_{m=2}^{k-1} \frac{1}{2k-1}\left(1 - \frac{(2m-3)!!\,(2k-2m-1)!!}{(2k-3)!!}\right) \\
&= \frac{2k-3}{2k-1} - \frac{1}{2k-1}\sum_{m=2}^{k-1} \frac{(2m-3)!!\,(2k-2m-1)!!}{(2k-3)!!} \\
&= \frac{2k-3}{2k-1} - \frac{1}{2k-1}\sum_{m=2}^{k-1} \frac{(2m-3)!\,(2k-2m)!\,(2k-4)!!}{(2m-4)!!\,(2k-2m)!!\,(2k-3)!} \\
&= \frac{2k-3}{2k-1} - \frac{1}{2k-1}\sum_{m=2}^{k-1} \frac{(2m-3)!\,(2k-2m)!2^{k-2}\,(k-2)!}{2^{m-2}\,(m-2)!2^{k-m}\,(k-m)!\,(2k-3)!} \\
&= \frac{2k-3}{2k-1} - \frac{1}{2k-1}\sum_{m=2}^{k-1} \frac{\binom{k-2}{m-2}}{\binom{2k-3}{2m-3}}.
\end{aligned} \tag{4.5}$$

Therefore

$$\mathbb{E}\left(Y_{2k}\right) \;=\; 2kp_{\text{cross}} \;=\; (2k)\,\frac{2k-3}{2k-1} - (2k)\,\frac{1}{2k-1}\sum_{m=2}^{k-1} \frac{\binom{k-2}{m-2}}{\binom{2k-3}{2m-3}}. \tag{4.6}$$

In the above sum, the first and last terms are both $\frac{1}{2k-3}$, as for $m = 2$ we have

$$\frac{\binom{k-2}{0}}{\binom{2k-3}{1}} = \frac{1}{2k-3}, \tag{4.7}$$

and for $m = k-1$ we have

$$\frac{\binom{k-2}{k-3}}{\binom{2k-3}{2k-5}} = \frac{\binom{k-2}{1}}{\binom{2k-3}{2}} = \frac{2(k-2)}{(2k-3)(2k-4)} = \frac{1}{2k-3}. \tag{4.8}$$

Looking at the ratio of subsequent terms, straightforward algebra shows

$$\frac{\binom{k-2}{m-1}/\binom{2k-3}{2m-1}}{\binom{k-2}{m-2}/\binom{2k-3}{2m-3}} = \frac{2m-1}{2k-2m-1}. \tag{4.9}$$

Thus for m up to the halfway point, each term in the sum is less than the previous. In particular, the $m = 3$ term is $5/(2k-7)$ times the $m = 2$ term, and hence all of these terms are $O(1/k^2)$. Similarly, working from $m = k-2$ to the middle we find all of these terms are also $O(1/k^2)$, and thus the sum in (4.6) can be rewritten, giving

$$\begin{aligned} \mathbb{E}(Y_{2k}) &= (2k)\frac{2k-3}{2k-1} - (2k)\frac{1}{2k-1}\left(\frac{2}{2k-3} + O\left(\frac{1}{k^2}\right)\right) \\ &= 2k - 2 - \frac{2}{k} + O\left(\frac{1}{k^2}\right). \end{aligned} \tag{4.10}$$

□

Theorem 4.2. *The variance of the number of vertices involved in a crossing approaches* 4 *as* $k \to \infty$.

Proof. We need to calculate $\mathrm{Var}(Y_{2k}) = \mathbb{E}(Y_{2k}^2) - \mathbb{E}(Y_{2k})^2$. As we know the second term by Theorem 4.1, we concentrate on the first term:

$$\mathbb{E}\left(Y_{2k}^2\right) = \sum_{i,j \in \{1,\ldots,2k\}} \mathbb{E}(X_i X_j). \tag{4.11}$$

The above sum has $4k^2$ terms.

For $2k$ of those terms, $i = j$ so $\mathbb{E}(X_i X_j) = \mathbb{E}(X_i^2) = \mathbb{E}(X_i) = p_{\text{cross}}$ as the X_ℓ's are binary indicator variables with probability of success p_{cross}. For another $2k$ terms, we have i and j are paired on the same edge, so $\mathbb{E}(X_i X_j) = \mathbb{E}(X_i) = p_{\text{cross}}$ as before.

For the remaining $4k^2 - 4k$ terms, i and j are on different edges, and we must find the probability that both those edges are in crossings. We separate this probability into two disjoint probabilities, the probability p_a that they cross each other, and the probability that they don't cross each other but are each crossed by at least one other pairing. We denote this second probability by $(1-p_a)\,p_b$, where p_b is the conditional probability they are each crossing given that they don't cross each other. We will find these probabilities by taking sums over the placements of k, m, p, q above as appropriate and calculating for each the probability of observing one of our desired configurations. We have shown

$$\mathbb{E}\left(Y_{2k}^2\right) = 4kp_{\text{cross}} + \left(4k^2 - 4k\right)\left(p_a + (1-p_a)\,p_b\right), \tag{4.12}$$

thus reducing the problem to the determination of p_a and p_b.

Without loss of generality, we label our edges as $\{1, m\}$ and $\{p, q\}$. They cross each other if and only if one of $\{p, q\}$ is one of the $m-2$ vertices between 1 and m, and the other is one of the $2k-m$ vertices between m and $2k$. Thus

$$\begin{aligned} p_a &= \sum_{m=2}^{2k} \frac{1}{2k-1} \cdot 2 \cdot \frac{m-2}{2k-2} \cdot \frac{2k-m}{2k-3} \\ &= \frac{2}{(2k-1)(2k-2)(2k-3)} \left[\sum_{m=2}^{2k} -4k - \sum_{m=2}^{2k} m^2 + (2k+2) \sum_{m=2}^{2k} m \right]. \end{aligned} \tag{4.13}$$

By using the formulas for the sum of the first n integers and the first n squares, we simplify the second factor to

$$(2k-1)(-4k) - \left(\frac{2k(2k+1)(4k+1)}{6} - 1 \right) + (2k+2) \left(\frac{2k(2k+1)}{2} - 1 \right), \tag{4.14}$$

which gives

$$p_a = \frac{2}{(2k-1)(2k-2)(2k-3)} \frac{(2k-1)(2k-2)(2k-3)}{6} = \frac{1}{3}. \tag{4.15}$$

We now calculate p_b, the probability that $\{1, m\}$ and $\{p, q\}$ are both involved in crossings given they don't cross each other. We must place $\{1, m\}, \{p, q\}$. Relabeling if necessary, we may assume $1 < m < p < q$ (Note that such a labelling is possible if and only if $\{1, m\}$ and $\{p, q\}$ do not cross each other. We compute the complement of our desired probability by finding the number of configurations where one or less of $\{1, m\}$ and $\{p, q\}$ is in a crossing. We denote the number of such configurations by $N_{k,m,p,q}$ and can thus write

$$p_b = 1 - \sum_{m=2}^{2k-2} \sum_{p=m+1}^{2k-1} \sum_{q=p+1}^{2k} \frac{N_{k,m,p,q}}{(2k-5)!!}. \tag{4.16}$$

Since there are $\binom{2k-1}{3}$ terms in the above sum (corresponding to the $\binom{2k-1}{3}$ possible choices of m, p, q since we have specified the location of vertex 1 and the order of m, p, q), we can rewrite (4.16) as

$$p_b = 1 - \frac{\sum_{m=2}^{2k-2} \sum_{p=m+1}^{2k-1} \sum_{q=p+1}^{2k} N_{k,m,p,q}}{\binom{2k-1}{3} (2k-5)!!}. \tag{4.17}$$

All that remains to be done is to evaluate the sum in the above expression. To do so, we first define the following function $P(k)$, which counts the number of ways k vertices can be paired with each other:

$$P(x) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ 1 & \text{if } k = 0 \\ (k-1)!! & \text{otherwise.} \end{cases} \tag{4.18}$$

Next we think of these two edges as dividing the remaining vertices into three regions: those between $\{1, m\}$ and $\{p, q\}$, of which there are $M = p - m - 1 + 2k - q$, those on the side of $\{1, m\}$, of which there are $L = m - 2$, and those on the side of $\{p, q\}$, of which there are $R = q - p - 1$. We know that $\{1, m\}$ will not be crossed if the L vertices between 1 and m pair exclusively with each other. Likewise, $\{p, q\}$ will not be crossed if the vertices between p and q pair

exclusively with each other. Our desired quantity is thus the union of these two events less their intersection:

$$P(L+M)P(R)+P(R+M)P(L)-P(L)P(M)P(R). \tag{4.19}$$

Notice that if L or R is 0, one of $\{1,m\},\{p,q\}$ is an adjacent edge, and so will not be crossing. Thus

$$N_{k,m,p,q} = \begin{cases} (2k-5)!! & \text{if } L \text{ or } R \text{ is } 0 \\ P(L+M)P(R)+P(R+M)P(L)-P(L)P(M)P(R) & \text{otherwise.} \end{cases} \tag{4.20}$$

We now investigate the limiting behavior of p_b (given in (4.16)) by using the cases in (4.20).

- For the first case, we have L or R is zero, and thus $N_{k,m,p,q}=(2k-5)!!$. We are reduced to counting the number of terms with L or R zero. Note that $L=0$ when $m=2$, and $R=0$ when $q=p+1$. Each of these events happens in $\binom{2k-2}{2}$ pairings (we have fixed either m or q, and the other 2 vertices are chosen from the remaining $2k-2$ vertices), and their intersection is $\binom{2k-3}{1}$ (p is the only free index) pairings. In the limit, this case contributes

$$\frac{\left(2\binom{2k-2}{2}-\binom{2k-3}{1}\right)(2k-5)!!}{\binom{2k-1}{3}(2k-5)!!} = \frac{3}{k}+O\left(\frac{1}{k^3}\right). \tag{4.21}$$

- For the second case, L and R are non-zero. We first evaluate the contribution of the first two terms (notice that they will contribute the same in the sum since you can simply relabel $\{1,m\}$ and $\{p,q\}$) and then the third term, recalling that we only have to look for terms that are at least $O\left(\frac{1}{k^2}\right)$ since we can see in (4.12) that any other terms will not contribute in the limit as $k\to\infty$.
  - For $P(L+M)P(R)$, the largest terms are from when either $L+M=2$, or when $R=2$. In these cases, $N_{k,m,p,q}=(2k-7)!!$. If $R=2$ then $q=p+3$ and m,p are free so there are $\binom{2k-4}{2}$ such terms corresponding to the $\binom{2k-4}{2}$ choices of m and p. If $L+M=2$ and $L\neq 0$ then there are only two possible terms: either $L=1,M=1,R=2k-6$ or $L=2,M=0,R=2k-6$. Including the symmetric terms for $P(R+M)P(L)$, these terms thus have a combined contribution of

$$\frac{2\left(\binom{2k-4}{2}+2\right)(2k-7)!!}{\binom{2k-1}{3}(2k-5)!!} = \frac{3}{2k^2}+O\left(\frac{1}{k^3}\right). \tag{4.22}$$

  - For the third term, -$P(L)P(M)P(R)$, the largest contributions will be when two regions combine for exactly 2 vertices which will give a contribution of $(2k-7)!!$. If we disregard the requirement that L and R are nonzero in order to obtain an upper bound on the magnitude of this contribution, there are 3 possible terms. The next largest contribution will be when two regions combine for exactly 4 vertices which will give a contribution of $(2k-9)!!$. Proceeding with these diagonal terms, we know that the third term will thus contribute at most in magnitude:

$$3\frac{(2k-7)!!}{\binom{2k-1}{3}(2k-5)!!}+6\frac{(2k-9)!!}{\binom{2k-1}{3}(2k-5)!!}+9\frac{(2k-11)!!}{\binom{2k-1}{3}(2k-5)!!}+\cdots = O\left(\frac{1}{k^3}\right), \tag{4.23}$$

    so they in fact do not contribute in the limit.

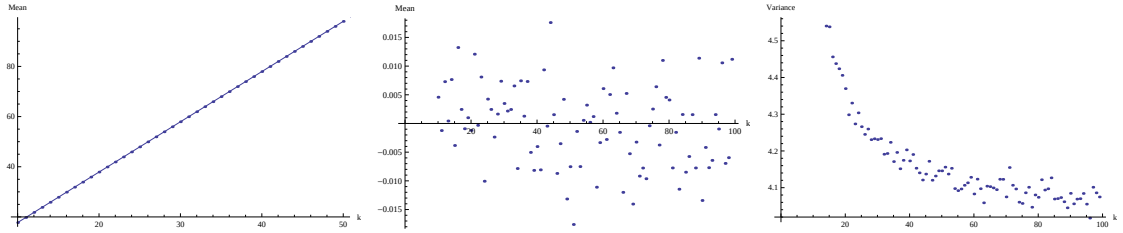

FIGURE 3. Numerical confirmation of formulas for the expected value and variance of vertices involved in crossing. The first plot is the expected value for $2k$ vertices (solid line is theory) versus k, the second plot is a plot of the deviations from theory, and the third plot is the observed variance; all plots are from 100,000 randomly chosen matchings of $2k$ vertices in pairs.

Thus, we have that as $k \to \infty$,

$$p_b \;=\; 1 - \frac{3}{k} - \frac{3}{2k^2} + O\left(\frac{1}{k^3}\right). \tag{4.24}$$

Therefore if we substitute for p_a and p_b in (4.12) we find

$$\mathbb{E}\left(Y_{2k}^2\right) \;=\; 4k - 4 + \left(4k^2 - 4k\right)\left(\frac{1}{3} + \frac{2}{3}\left(1 - \frac{3}{k} - \frac{3}{2k^2}\right)\right) \tag{4.25}$$

$$\;=\; 4k^2 - 8k + O\left(\frac{1}{k}\right). \tag{4.26}$$

Using (4.10), we also have that

$$\mathbb{E}\left(Y_{2k}\right)^2 \;=\; \left(2k - 2 - \frac{2}{k} + O\left(\frac{1}{k^2}\right)\right)^2 \;=\; 4k^2 - 8k - 4 + O\left(\frac{1}{k}\right). \tag{4.27}$$

The variance as $k \to \infty$ is thus $\mathbb{E}\left(Y_{2k}^2\right) - \mathbb{E}\left(Y_{2k}\right)^2 = \left(4k^2 - 8k\right) - \left(4k^2 - 8k - 4\right) = 4$. □

Figure 3 provides a numerical verification of the above formulas for the expected values and variances.

5. LIMITING SPECTRAL MEASURE

We now complete the proof of Theorem 1.1 by showing convergence and determining the support.

Proof of Theorem 1.1. The proof of the claimed convergence is standard, and follows immediately from similar arguments in [HM, MMS, JMP, KKMSX]. Those arguments rely only on degree of freedom counting arguments, and are thus applicable here as well. We are left with determining the limiting spectral measures.

- $p = 1/2$: If $p = 1/2$, we know from (3.3) that only those configurations with no crossings contribute. The claim follows directly from recalling that the number of non-crossing configurations are simply the Catalan numbers, which are also the moments of the semicircle distribution.

- $p > 1/2$: To show that the limiting spectral measure has unbounded support it suffices to show that the moments of our distribution grow faster than any exponential bound, i.e., that for all B there exists some k such that $M_{2k} > B^{2k}$. The moments of the unweighted ensemble grow faster than exponentially (see [HM, KKMSX]). We prove that our distribution similarly has unbounded support using this fact and by considering the "worst-case" scenario allowed for under 3.3. Namely, we suppose that each term contributes $(2p-1)^{2k}$, which gives us the smallest moment possible. In this case, M_{2k} is decreased from the unweighted case by a factor of $(2p-1)^{2k}$, and thus the growth is still faster than any exponential bound. □

APPENDIX A. EXACT FORMULA FOR MEAN NUMBER OF CROSSINGS

To prove (4.1), it suffices to simplify the sum in the expansion of p_{cross} in (4.5). We first extend the m sum to include $m = k$; this adds 1 to the sum which must then be subtracted from the term outside. For notational convenience, set $n = k - 2$. We re-index and let m run from 0 to n, and are thus reduced to analyzing

$$S(n) \;=\; \sum_{m=0}^{n} \frac{\binom{n}{m}}{\binom{2n+1}{2m+1}}. \tag{A.1}$$

The following notation and properties are standard (see for example [GR]). The Pochhammer symbol $(x)_m$ is defined for $m \geq 0$ by

$$(x)_m \;=\; \frac{\Gamma(x+m)}{\Gamma(x)} \;=\; x(x+1)\cdots(x+m-1), \tag{A.2}$$

and the hypergeometric function ${}_2F_1$ by

$${}_2F_1(a,b,c;z) \;=\; \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{z^m}{m!}, \tag{A.3}$$

which converges for all $|z| < 1$ so long as c is not a negative integer.

For ease of exposition, we work backwards from the answer.[2] Using $\Gamma(1+z) = z\Gamma(z)$ and $\Gamma(1+\ell) = \ell!$ (for integral ℓ), we find

$$\begin{aligned} {}_2F_1(1,3/2,1/2-n,-1) &= \sum_{m=0}^{\infty} \frac{(1)_m (3/2)_m}{(1/2-n)_m} \frac{(-1)^m}{m!} \\ &= \sum_{m=0}^{\infty} \frac{\Gamma(1+m)}{\Gamma(1)} \frac{\Gamma(3/2+m)}{\Gamma(3/2)} \frac{\Gamma(1/2-n)}{\Gamma(1/2-n+m)} \frac{(-1)^m}{m!} \\ &=: T_1(n) + T_2(n), \end{aligned} \tag{A.4}$$

where $T_1(n)$ is the sum over $m \leq n$ and $T_2(n)$ is the sum over $m > n$. From the functional equation of the Gamma function and using $\ell!! = \ell(\ell-2)(\ell-4)\cdots$ down to 2 or 1, we find

$$\begin{aligned} \Gamma(3/2+m) &= 2^m (2m+1)!! \Gamma(3/2) \\ \Gamma(1/2-n+m) &= (-1)^m 2^m (2n-1)(2n-3)\cdots(2n-2m+1)\Gamma(1/2-n). \end{aligned} \tag{A.5}$$

[2] Mathematica is able to evaluate such sums and suggest the correct hypergeometric combinations. One has to be a little careful, though, as Mathematica incorrectly evaluated $S(n)$, incorrectly stating that there was zero contribution if we extend the sum to all m. In other words, it thought $S(n) = T_1(n) = T_1(n) + T_2(n)$ in the notation introduced below.

Substituting, we find

$$\begin{aligned} T_1(n) &= \sum_{m=0}^{n} \frac{(2m+1)!!(2n-2m-1)!!}{(2n-1)!!} \\ &= \sum_{m=0}^{n} \frac{(2m+1)!(2n-2m-1)!}{(2n-1)!2n} \frac{2n(2n-2)!!}{(2m)!!}(2n-2m-2)!! \\ &= \sum_{m=0}^{n} \frac{(2m+1)!(2n-2m)!}{(2n+1)!} \cdot (2n+1) \cdot \frac{2^n n!}{(2n-2m)2^{n-1}m!(n-m-1)!} \\ &= (2n+1)\sum_{m=0}^{n} \frac{\binom{n}{m}}{\binom{2n+1}{2m+1}}; \end{aligned} \tag{A.6}$$

note this is our desired sum. Thus

$$\sum_{m=0}^{n} \frac{\binom{n}{m}}{\binom{2n+1}{2m+1}} = \frac{{}_2F_1(1,3/2,1/2-n,-1) - T_2(n)}{2n+1}, \tag{A.7}$$

and the proof is completed by analyzing $T_2(n)$. To determine this term's contribution, we re-index. Writing $m = n+1+u$, we find

$$\begin{aligned} T_2(n) &= \sum_{u=0}^{\infty} \frac{\Gamma(1+n+1+u)}{\Gamma(1)} \frac{\Gamma(3/2+n+1+u)}{\Gamma(3/2)} \frac{\Gamma(1/2-n)}{\Gamma(1/2-n+n+1+u)} \frac{(-1)^{n+1+u}}{(n+1+u)!} \frac{u!}{u!} \\ &= \sum_{u=0}^{\infty} \frac{\Gamma(1+u)}{\Gamma(1)} \frac{\Gamma(5/2+n+u)}{\Gamma(3/2)} \frac{\Gamma(1/2-n)}{\Gamma(3/2+u)} \frac{(-1)^{n+1}(-1)^u}{u!} \\ &= \frac{(-1)^{n+1}\Gamma(1/2-n)\Gamma(5/2+n)}{\Gamma(3/2)^2} \sum_{u=0}^{\infty} \frac{\Gamma(1+u)}{\Gamma(1)} \frac{\Gamma(5/2+n+u)}{\Gamma(5/2+n)} \frac{\Gamma(3/2)}{\Gamma(3/2+u)} \frac{(-1)^u}{u!} \\ &= -(2n+3)(2n+1)\ {}_2F_1(1,1/2+k,3/2,-1), \end{aligned} \tag{A.8}$$

where we used $\Gamma(1-z)\Gamma(z) = \pi/\sin(\pi z)$ with $z = n+\frac{1}{2}$ to simplify the Gamma factors depending only on n. Combining the above proves (4.1).

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