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ON THE NUMBER OF CONGRUENCE CLASSES OF PATHS

ZHICONG LIN AND JIANG ZENG

Abstract. Let P_n denote the undirected path of length $n-1$. The cardinality of the set of congruence classes induced by the graph homomorphisms from P_n onto P_k is determined. This settles an open problem of Michels and Knauer (Disc. Math., 309 (2009) 5352-5359). Our result is based on a new proven formula of the number of homomorphisms between paths.

Keywords: Graph, graph endomorphisms, graph homomorphisms, paths, lattice paths

1. Introduction

We use standard notations and terminology of graph theory in [3] or [6, Appendix]. The graphs considered here are finite and undirected without multiple edges and loops. Given a graph G, we write $V(G)$ for the vertex set and $E(G)$ for the edge set. A *homomorphism* from a graph G to a graph H is a mapping $f : V(G) \to V(H)$ such that the images of adjacent vertices are adjacent. An *endomorphism* of a graph is a homomorphism from the graph to itself. Denote by $\mathrm{Hom}(G, H)$ the set of homomorphisms from G to H and by $\mathrm{End}(G)$ the set of endomorphisms of a graph G. For any finite set X we denote by $|X|$ the cardinality of X. A *path* with n vertices is a graph whose vertices can be labeled $v_1, ..., v_n$ so that v_i and v_j are adjacent if and only if $|i-j| = 1$; let P_n denote such a graph with $v_i = i$ for $1 \le i \le n$. Every endomorphism f on G induces a partition ρ of $V(G)$, also called *the congruence classes induced by* f, with vertices in the same block if they have the same image.

Let $\mathscr{C}(P_n)$ denote the set of endomorphism-induced partitions of $V(P_n)$, and let $|\rho|$ denote the number of blocks in a partition ρ. For example, if $f \in \mathrm{End}(P_4)$ is defined by $f(1) = 3$, $f(2) = 2$, $f(3) = 1$, $f(4) = 2$, then the induced partition ρ is $\{\{1\}, \{2, 4\}, \{3\}\}$ and $|\rho| = 3$.

The problem of counting the homomorphisms from G to H is difficult in general. However, some algorithms and formulas for computing the number of homomorphisms of paths have been published recently (see [1, 2, 5]). In particular, Michels and Knauer [5] give an algorithm based on the *epispectrum* $\mathrm{Epi}(P_n)$ of a path P_n. They define $\mathrm{Epi}(P_n) = (l_1(n), ..., l_{n-1}(n))$, where

$$l_k(n) = |\{\rho \in \mathscr{C}(P_n) \ : \ |\rho| = n - k + 1\}|. \tag{1.1}$$

Here a misprint in the definition of $l_k(n)$ in [5] is corrected.

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In [5], based on the first values of $l_k(n)$, Michels and Knauer speculated the following conjecture.

Conjecture 1. *There exists a polynomial $f_k \in \mathbb{Q}[x]$ with $deg(f_k) = \lceil (k-2)/2 \rceil$ such that for a fixed n_k (most probably $n_k = 2k$) the equality $l_k(n) = f_k(n)$ holds for $n \geq n_k$.*

The aim of this paper is to confirm this conjecture by giving an explicit formula for the polynomial f_k. For this purpose, we shall prove a new formula for the number of homomorphisms from P_n to P_k, which is the content of the following theorem.

Theorem 2. *For any positive integers n and k,*

$$|\mathrm{Hom}(P_n, P_k)| = k \times 2^{n-1} - \sum_{i=0}^{n-2} 2^{n-1-i} \sum_{j \in \mathbb{Z}} \left(\binom{i}{\lceil \frac{i}{2} \rceil - j(k+1)} - \binom{i}{\lfloor \frac{i+k+1}{2} \rfloor - j(k+1)} \right). \tag{1.2}$$

From the above theorem we are able to derive the following main result.

Theorem 3. *If $n \geq 2k$, then*

$$l_k(n) = \binom{n-1}{\lceil \frac{k}{2} \rceil - 1} + \binom{n-1}{\lfloor \frac{k}{2} \rfloor - 1}. \tag{1.3}$$

Equivalently, the above formula can be rephrased as follows

$$l_{2k}(n) = 2\binom{n-1}{k-1}, \qquad l_{2k+1}(n) = \binom{n}{k}. \tag{1.4}$$

When $n \geq 2k$, Theorem 3 shows immediately that $l_k(n)$ is a polynomial in n of degree $\lceil (k-2)/2 \rceil$. This proves Conjecture 1. In particular, we have $l_1(n) = 1, l_2(n) = 2, l_3(n) = n, l_4(n) = 2(n-1), l_5(n) = \frac{1}{2}n(n-1)$ and $l_6(n) = (n-1)(n-2)$, which coincide with the conjectured values in [5] after shifting the index by 1.

In the next section, we first recall some basic counting results about the lattice paths and then prove Theorem 2. In Section 3 we give the proof of Theorem 3.

2. The number of homomorphisms between paths

One can enumerate homomorphisms from P_n to P_k by picking a fixed point as image of 1 and moving to vertices which are adjacent to this vertex, as

$$f \in \mathrm{Hom}(P_n, P_k) \Leftrightarrow \forall x \in \{1, \ldots, n-1\} : \{f(x), f(x+1)\} \in E(P_k).$$

Hence, one can describe all possible moves through the edge structure of the two paths.

For $1 \leq j \leq k$, let

$$\mathrm{Hom}^j(P_n, P_k) = \{f \in \mathrm{Hom}(P_n, P_k) \ : \ f(1) = j\}. \tag{2.1}$$

Obviously, we have

$$|\mathrm{Hom}^j(P_n, P_k)| = |\{f \in \mathrm{Hom}(P_n, P_k) \ : \ f(n) = j\}|. \tag{2.2}$$

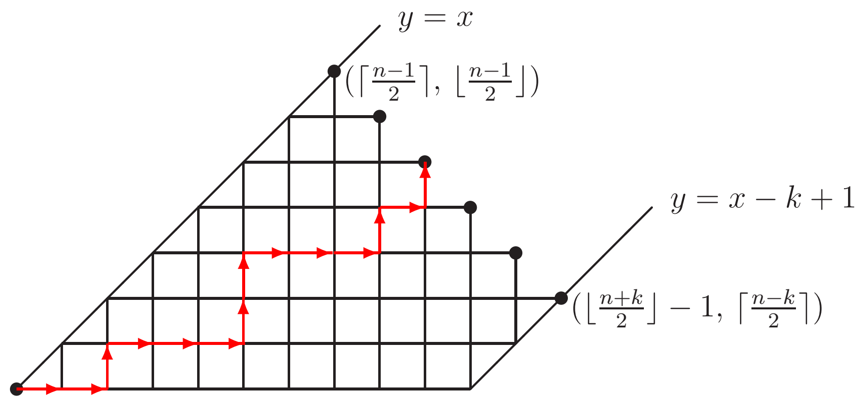

FIGURE 1. A lattice path from $(0,0)$ to $(9,5)$ that stays between lines $y = x$ and $y = x - k + 1$, where $n = 15$ and $k = 11$.

Definition 1. *A* lattice path *of length n is a sequence $(\gamma_0, \dots, \gamma_n)$ of points γ_i in the plan $\mathbb{Z} \times \mathbb{Z}$ for all $0 \le i \le n$ and such that $\gamma_{i+1} - \gamma_i = (1,0)$ (east-step) or $(0,1)$ (north-step) for $1 \le i \le n-1$.*

As shown by Arworn [1], we can encode each homomorphism $f \in \mathrm{Hom}^1(P_n, P_k)$ by a lattice path $\gamma = (\gamma_0, \dots, \gamma_{n-1})$ in $\mathbb{N} \times \mathbb{N}$ between the lines $y = x$ and $y = x - k + 1$ as follows:

- $\gamma_0 = (0,0)$, and for $j = 1, \dots, n-1$,
- $\gamma_{j+1} = \gamma_j + (1,0)$ if $f(j) > f(j-1)$,
- $\gamma_{j+1} = \gamma_j + (0,1)$ if $f(j) < f(j-1)$.

For example, if the images of successive vertices of $f \in \mathrm{Hom}(P_{15}, P_{11})$ are

$$1,\ 2,\ 3,\ 2,\ 3,\ 4,\ 5,\ 4,\ 3,\ 4,\ 5,\ 6,\ 5,\ 6,\ 5;$$

then the corresponding lattice path is given by Figure 1.

Definition 2. *For nonnegative integers n, m, t, s, Let $\mathcal{L}(n,m)$ be the set of all the lattice paths from the origin to (n,m) and $\mathcal{L}(n,m;t,s)$ the set of lattice paths in $\mathcal{L}(n,m)$ that stay between the lines $y = x + t$ and $y = x - s$ (being allowed to touch them), where $n + t \ge m \ge n - s$.*

Lemma 4. *Let $K = \min(\lfloor \frac{n+k}{2} \rfloor, n)$, then*

$$|\mathrm{Hom}^1(P_n, P_k)| = \sum_{l=\lceil \frac{n-1}{2} \rceil}^{K-1} |\mathcal{L}(l, n-1-l; 0, k-1)|. \tag{2.3}$$

Proof. It follows from the above correspondence that each homomorphism from P_n to P_k is encoded by a lattice path in some $\mathcal{L}(\#E, \#N; 0, k-1)$, where $\#E$ is the number of

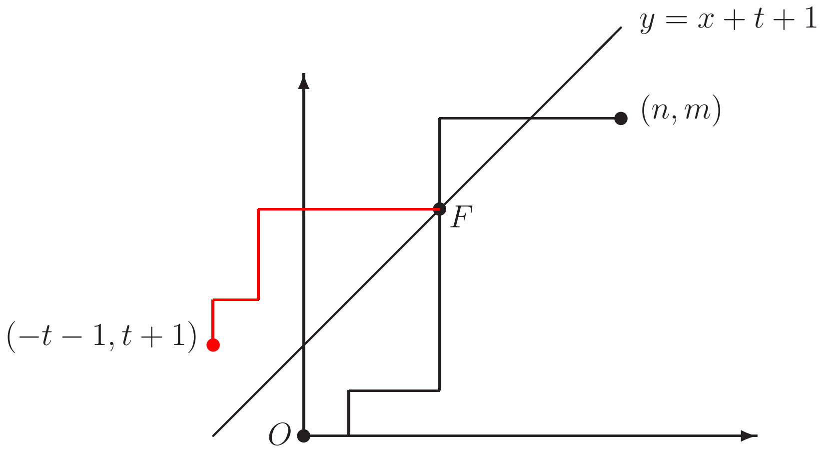

Figure 2. Reflection of the segment of the path from O to the first reaching point F with respect to the line $y = x + t + 1$.

east-steps and $\#N$ the number of north-steps. The path structures require that

$$\#E + \#N = n - 1, \quad \#E - \#N \leq k - 1, \quad \#E - \#N \geq 0.$$

Therefore, we must have $\#E \geq (n-1)/2$, $\#E \leq n - 1$ and $\#E \leq (k + n - 2)/2$. □

To evaluate the sum in (2.3), we need a formula for the cardinality of $\mathcal{L}(n, m; t, s)$. First of all, each lattice path in $\mathcal{L}(n, m)$ can be encoded by a word of length $n+m$ on the alphabet $\{A, B\}$ with n letters A and m letters B. So, the cardinality of $\mathcal{L}(n, m)$ is given by the binomial coefficient $\binom{n+m}{n}$. Next, each lattice path in $\mathcal{L}(n, m)$ which passes above the line $y = x + t$ (or reaching the line $y = x + t + 1$) can be mapped to a lattice path from $(-t-1, t+1)$ to (n, m) by the *reflection* with respect to the line $y = x + t + 1$ (see Figure 2). Hence, there are $\binom{n+m}{n+t+1}$ such lattice paths. Therefore, the number of lattice paths in $\mathcal{L}(n, m)$ which do not pass above the line $y = x + t$ (or not reaching the line $y = x + t + 1$), where $m \leq n + t$, is given by

$$\binom{n+m}{n} - \binom{n+m}{n+t+1}.$$

By a similar reasoning, we can prove the following known result (see [4, Lemma 4A], for example). For the reader's convenience, we provide a sketch of the proof.

Lemma 5. *The cardinality of $\mathcal{L}(n, m; t, s)$ is given by*

$$|\mathcal{L}(n, m; t, s)| = \sum_{k \in \mathbb{Z}} \left(\binom{n+m}{n - k(t+s+2)} - \binom{n+m}{n - k(t+s+2) + t + 1} \right), \tag{2.4}$$

where $\binom{n}{k} = 0$ if $k > n$ or $k < 0$.

Sketch of proof. Let T and S be the lines $y = x+t+1$ and $y = x-s-1$, respectively. Let A_1 denote the set of lattice paths in $\mathcal{L}(n,m)$ reaching T at least once, regardless of what happens at any other step, and let A_2 denote the set of lattice paths in $\mathcal{L}(n,m)$ reaching T, S at least once in the order specified. Generally, let A_i denote the set of lattice paths in $\mathcal{L}(n,m)$ reaching $T, S, \ldots$, alternatively (i times) at least once in the specified order. Let B_i be the set defined in the same way as A_i with S, T interchanged. A standard Inclusive-Exclusive principle argument yields:

$$|\mathcal{L}(n,m;t,s)| = \binom{n+m}{n} + \sum_{i\geq 1}(-1)^i(|A_i| + |B_i|). \tag{2.5}$$

As the symmetric point of (a,b) with respect to the line $y = x+c$ is $(b-c, a+c)$, by repeatedly applying the reflection principle argument, we obtain

$$|A_{2j}| = \binom{n+m}{n+j(t+s+2)}, \quad |A_{2j+1}| = \binom{n+m}{n-j(t+s+2)-(t+1)},$$

and

$$|B_{2j}| = \binom{n+m}{n-j(t+s+2)}, \quad |B_{2j+1}| = \binom{n+m}{n+j(t+s+2)-(s+1)}.$$

Substituting this into (2.5) leads to (2.4). □

Lemma 6. *For each positive integers n and k,*

$$|\mathrm{Hom}^1(P_n,P_k)| = \sum_{j\in\mathbb{Z}}\left(\binom{n-1}{\lceil\frac{n-1}{2}\rceil - j(k+1)} - \binom{n-1}{\lfloor\frac{n+k}{2}\rfloor - j(k+1)}\right). \tag{2.6}$$

Proof. Substituting (2.4) into (2.3) and exchanging the order of the summations,

$$\begin{aligned}|\mathrm{Hom}^1(P_n,P_k)| &= \sum_{j\in\mathbb{Z}}\sum_{l=\lceil\frac{n-1}{2}\rceil}^{K-1}\left(\binom{n-1}{l-j(k+1)} - \binom{n-1}{l+1-j(k+1)}\right)\\ &= \sum_{j\in\mathbb{Z}}\left(\binom{n-1}{\lceil\frac{n-1}{2}\rceil - j(k+1)} - \binom{n-1}{K-j(k+1)}\right). \end{aligned}\tag{2.7}$$

Now, if $n \geq k$, then $K = \lfloor\frac{n+k}{2}\rfloor$,

$$\binom{n-1}{K-j(k+1)} = \binom{n-1}{\lfloor\frac{n+k}{2}\rfloor - j(k+1)}, \tag{2.8}$$

if $k > n$, then $K = n$, since

$$\binom{n-1}{n-j(k+1)} = \binom{n-1}{\lfloor\frac{n+k}{2}\rfloor - j(k+1)} = 0,$$

the equation (2.8) is also valid. Hence (2.7) and (2.6) are equal. □

Proof of Theorem 1. For $f \in \mathrm{Hom}(P_{i+1}, P_k)$ with $i = 1, \ldots, n-1$, consider the following three cases:

(i) if $f(i) = 1$, then $f(i+1) = 2$ and there are $|\mathrm{Hom}^1(P_i, P_k)|$ such homomorphisms.
(ii) if $f(i) = k$, then $f(i+1) = k-1$ and there are $|\mathrm{Hom}^k(P_i, P_k)|$ such homomorphisms.
(iii) if $f(i) = j$ with $j \in \{2, 3, ..., k-1\}$, then $f(i+1) = j-1$ or $j+1$ and there are $2|\mathrm{Hom}^j(P_i, P_k)|$ such homomorphisms.

Summarizing, we get

$$|\mathrm{Hom}(P_{i+1}, P_k)| = |\mathrm{Hom}^1(P_i, P_k)| + 2\sum_{j=2}^{k-1} |\mathrm{Hom}^j(P_i, P_k)| + |\mathrm{Hom}^k(P_i, P_k)|.$$

Since $|\mathrm{Hom}(P_i, P_k)| = \sum\limits_{j=1}^{k} |\mathrm{Hom}^j(P_i, P_k)|$ and $|\mathrm{Hom}^1(P_i, P_k)| = |\mathrm{Hom}^k(P_i, P_k)|$, it follows that

$$|\mathrm{Hom}(P_{i+1}, P_k)| = 2|\mathrm{Hom}(P_i, P_k)| - 2|\mathrm{Hom}^1(P_i, P_k)|.$$

By iteration, we derive

$$\begin{aligned} |\mathrm{Hom}(P_n, P_k)| &= 2^{n-1}|\mathrm{Hom}(P_1, P_k)| - \sum_{i=1}^{n-1} 2^{n-i}|\mathrm{Hom}^1(P_i, P_k)| \\ &= k \times 2^{n-1} - \sum_{i=1}^{n-1} 2^{n-i}|\mathrm{Hom}^1(P_i, P_k)|. \end{aligned} \tag{2.9}$$

Plugging (2.6) into (2.9), we obtain (1.2) . □

Remark. The key point in the above proof is to reduce the counting problem of $|\mathrm{Hom}(P_n, P_k)|$ to $|\mathrm{Hom}^1(P_i, P_k)|$ for $i = 1, \ldots, n-1$. Arworn and Wojtylak [2] give a formula for $|\mathrm{Hom}(P_n, P_k)| = \sum_{j=1}^{k} |\mathrm{Hom}^j(P_n, P_k)|$ without using this reduction. Moreover, their expression for $|\mathrm{Hom}^j(P_n, P_k)|$ depends on the parity of $n-j$:

$$|\mathrm{Hom}^j(P_n, P_k)| = \begin{cases} \sum_{t=-n+1}^{n-1} (-1)^t \sum_{u=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{n-1}{\frac{n-j-1}{2} + u + \lceil \frac{(k+1)t}{2} \rceil} & \text{if } n-j \text{ is odd,} \\ \sum_{t=-n+1}^{n-1} (-1)^t \sum_{u=0}^{\lceil \frac{k-1}{2} \rceil} \binom{n-1}{\lfloor \frac{n-j-1}{2} \rfloor + u + \lfloor \frac{(k+1)t}{2} \rfloor} & \text{if } n-j \text{ is even.} \end{cases} \tag{2.10}$$

Note that Lemma 2.6 unifies the two cases in (2.10) when $j = 1$.

When $k = n$, we can deduce a simple formula for the number of endomorphisms of P_n (see `http://oeis.org/A102699`) by applying two binomial coefficient identities.

Lemma 7. *For $m \geq 1$, the following identities hold*

$$\sum_{k=0}^{m-1} \binom{2k}{k} 2^{2m-1-2k} = m\binom{2\,m}{m}, \tag{2.11}$$

$$\sum_{k=0}^{m-1} \binom{2k+1}{k} 2^{2m-1-2k} = (m+1)\binom{2\,m+1}{m} - 2^{2\,m}. \tag{2.12}$$

Proof. We prove (2.11) by induction on m. Clearly (2.11) is true for $m = 1$. If it is true for $m \geq 1$, then for $m+1$, the left-hand side after cutting out the last term, can be written as

$$\begin{aligned} 2^2 \sum_{k=0}^{m-1} \binom{2k}{k} 2^{2m-1-2k} + 2\binom{2m}{m} &= 4m\binom{2\,m}{m} + 2\binom{2\,m}{m} \\ &= (m+1)\binom{2m+2}{m+1}. \end{aligned}$$

Thus (2.11) is proved. Similarly we can prove (2.12). □

Proposition 8. *For $n \geq 1$,*

$$|\mathrm{End}(P_n)| = \begin{cases} (n+1)2^{n-1} - (2n-1)\binom{n-1}{(n-1)/2} & \textit{if } n \textit{ is odd,} \\ (n+1)2^{n-1} - n\binom{n}{n/2} & \textit{if } n \textit{ is even.} \end{cases} \tag{2.13}$$

Proof. When $k = n$, Theorem 2 becomes

$$|\mathrm{End}(P_n)| = n \times 2^{n-1} - \sum_{i=0}^{n-2} 2^{n-1-i} \times \binom{i}{\lceil \frac{i}{2} \rceil}. \tag{2.14}$$

By Lemma 7, if n is even, say $n = 2m$, then

$$\begin{aligned} \sum_{i=0}^{n-2} 2^{n-1-i} \times \binom{i}{\lceil \frac{i}{2} \rceil} &= \sum_{k=0}^{m-2} \binom{2k+1}{k} 2^{2m-2-2k} + \sum_{k=0}^{m-1} \binom{2k}{k} 2^{2m-1-2k} \\ &= 2m\binom{2\,m-1}{m-1} - 2^{2\,m-1} + m\binom{2\,m}{m}; \end{aligned}$$

if n is odd, say $n = 2m+1$, then

$$\begin{aligned} \sum_{i=0}^{n-2} 2^{n-1-i} \times \binom{i}{\lceil \frac{i}{2} \rceil} &= \sum_{k=0}^{m-1} \binom{2k+1}{k} 2^{2m-1-2k} + \sum_{k=0}^{m-1} \binom{2k}{k} 2^{2m-2k} \\ &= (m+1)\binom{2\,m+1}{m} - 2^{2\,m} + 2m\binom{2\,m}{m}. \end{aligned}$$

Substituting these into (2.14) we obtain the desired result. □

3. Proof of theorem 3

We first establish three lemmas. For any $n \geq 1$, let $[n] = \{1, \ldots, n\}$, which is $V(P_n)$. Denote by $\mathfrak{S}_n$ the set of permutations of $[n]$. For $1 \leq k \leq n$, denote by $\mathrm{Epi}(P_n, P_k)$ the set of epimorphisms from P_n to P_k, namely,

$$\mathrm{Epi}(P_n, P_k) = \{f \in \mathrm{Hom}(P_n, P_k) \,:\, f([n]) = [k]\}. \tag{3.1}$$

Lemma 9. *For* $1 \leq k \leq n-1$,

$$l_k(n) = |\mathrm{Epi}(P_n, P_{n-k+1})|/2. \tag{3.2}$$

Proof. Let $r = n-k+1$. Denote by $\mathrm{End}_r(P_n)$ the subset of endomorphisms in $\mathrm{End}(P_n)$ such that $|f([n])| = r$ and $\mathcal{L}_k(n)$ the set of partitions induced by endomorphisms in $\mathrm{End}_r(P_n)$. By definition (see (1.1)), the integer $l_k(n)$ is the cardinality of $\mathcal{L}_k(n)$.

For each $f \in \mathrm{End}_r(P_n)$, if $f([n]) = \{a, a+1, \ldots, a+r-1\}$ for some integer $a \in [n-r+1]$, we define $\bar{f} \in \mathrm{Epi}(P_n, P_r)$ by $\bar{f}(x) = f(x) - a + 1$. Then f and $\bar{f}$ induce the same partition in $\mathcal{L}_k(n)$. Hence, we can consider $\mathcal{L}_k(n)$ as the set of partitions induced by epimorphisms in $\mathrm{Epi}(P_n, P_r)$.

If $\{A_1, \ldots, A_r\}$ is a partition of $[n]$ induced by an $f \in \mathrm{Epi}(P_n, P_r)$, then, we can assume that $\min(A_1) \leq \min(A_2) \leq \ldots \leq \min(A_r)$. Hence, we can identify f with a permutation $\sigma \in \mathfrak{S}_r$ by $f(A_{\sigma(i)}) = i$ for $i \in [r]$. Moreover, two blocks A_i and A_j are adjacent in the arrangement $A_{\sigma(1)} \ldots A_{\sigma(r)}$ if and only if there are two consecutive integers α and β such that $\alpha \in A_i$ and $\beta \in A_j$. We show that there are exactly two such permutations for a given induced partition.

Starting from a partition $\{A_1, \ldots, A_r\}$ of $[n]$ induced by an $f \in \mathrm{Epi}(P_n, P_r)$, we arrange step by step the blocks $A_1, \ldots, A_i$ for $2 \leq i \leq r$ such that A_i is adjacent to the block A_j containing $\min(A_i) - 1$ and $j < i$. Since $\min(A_1) = 1$ and $\min(A_2) = 2$, there are two ways to arrange A_1 and A_2: A_1A_2 or A_2A_1. Suppose that the first i (≥ 2) blocks have been arranged as $W_i := A_{\sigma_i(1)} \ldots A_{\sigma_i(i)}$ with $\sigma_i \in \mathfrak{S}_i$, then $\min(A_{i+1}) - 1$ must belong to $A_{\sigma_i(1)}$ or $A_{\sigma_i(i)}$ because any two adjacent blocks in W_i should stay adjacent in all the W_j for $i \leq j \leq r$. Hence there is only one way to insert A_{i+1} in W_i: at the left of W_i (resp. right of W_i) if $\min(A_{i+1}) - 1 \in A_{\sigma_i(1)}$ (resp. $A_{\sigma_i(i)}$) for $i \geq 2$. As there are two possibilities for $i = 2$ we have thus proved that there are exactly two corresponding epimorphisms in $\mathrm{Epi}(P_n, P_r)$ for a given induced partition with r blocks. For example, starting from the induced partition $\{\{1,3,5,9\}, \{2,4,10\}, \{6,8\}, \{7\}, \{11\}\}$ of $V(P_{11})$, we obtain the two corresponding arrangements:

$$\{7\}\{6,8\}\{1,3,5,9\}\{2,4,10\}\{11\} \quad \text{and} \quad \{11\}\{2,4,10\}\{1,3,5,9\}\{6,8\}\{7\}.$$

This is the desired result. □

Lemma 10. *For* $1 \leq k \leq n$,

$$l_k(n) = \frac{1}{2}|\mathrm{Hom}(P_n, P_{n-k+1})| - |\mathrm{Hom}(P_n, P_{n-k})| + \frac{1}{2}|\mathrm{Hom}(P_n, P_{n-k-1})|.$$

Proof. By definition we have $\mathrm{Hom}(P_n,P_k)\setminus \mathrm{Epi}(P_n,P_k)=A\cup B$, where

$$A=\{f\in \mathrm{Hom}(P_n,P_k)\,:\,f([n])\subseteq [k-1]\},$$
$$B=\{f\in \mathrm{Hom}(P_n,P_k)\,:\,f([n])\subseteq [k]\setminus[1]\}.$$

Hence

$$|\mathrm{Hom}(P_n,P_k)|-|\mathrm{Epi}(P_n,P_k)|=|A|+|B|-|A\cap B|. \tag{3.3}$$

Since $|A|=|B|=|\mathrm{Hom}(P_n,P_{k-1})|$, and

$$|A\cap B|=|\{f\in \mathrm{Hom}(P_n,P_k)\,:\,f([n])\subseteq [k-1]\setminus[1]|=|\mathrm{Hom}(P_n,P_{k-2})|,$$

we derive from (3.3) that

$$|\mathrm{Epi}(P_n,P_k)|=|\mathrm{Hom}(P_n,P_k)|-2|\mathrm{Hom}(P_n,P_{k-1})|+|\mathrm{Hom}(P_n,P_{k-2})|.$$

The result follows then by applying Lemma 9. □

It follows from Lemma 10 and Theorem 2 that

$$l_k(n)=\sum_{i=0}^{n-2}2^{n-i-2}\sum_{j\in\mathbb{Z}}(-A_{i,j}+2B_{i,j}-C_{i,j}), \tag{3.4}$$

where

$$A_{i,j}=A^+_{i,j}-A^-_{i,j},\quad B_{i,j}=B^+_{i,j}-B^-_{i,j},\quad C_{i,j}=C^+_{i,j}-C^-_{i,j},$$

with

$$A^+_{i,j}=\binom{i}{\lceil\frac{i}{2}\rceil-j(n-k+2)},\quad A^-_{i,j}=\binom{i}{\lfloor\frac{i+n-k}{2}\rfloor+1-j(n-k+2)},$$
$$B^+_{i,j}=\binom{i}{\lceil\frac{i}{2}\rceil-j(n-k+1)},\quad B^-_{i,j}=\binom{i}{\lfloor\frac{i+n-k-1}{2}\rfloor+1-j(n-k+1)},$$
$$C^+_{i,j}=\binom{i}{\lceil\frac{i}{2}\rceil-j(n-k)},\quad C^-_{i,j}=\binom{i}{\lfloor\frac{i+n-k-2}{2}\rfloor+1-j(n-k)}.$$

Lemma 11. *For $n\geq 2k$,*

$$\begin{aligned}\sum_{j\in\mathbb{Z}}(-A_{i,j}+2B_{i,j}-C_{i,j})=&\left\{\binom{i+1}{\lfloor\frac{i+n-k}{2}\rfloor+1}+\binom{i+1}{\lfloor\frac{i+n-k}{2}\rfloor-n+k}\right\}\\&-2\left\{\binom{i}{\lfloor\frac{i+n-k-1}{2}\rfloor+1}+\binom{i}{\lfloor\frac{i+n-k-1}{2}\rfloor-n+k}\right\}.\end{aligned} \tag{3.5}$$

Proof. Since $0\leq k\leq\frac{n}{2}$, we have $\frac{n}{2}\leq n-k\leq n-1$. Therefore,

(1) if $j<0$, then $\lceil\frac{i}{2}\rceil-j(n-k)\geq\lceil\frac{i}{2}\rceil+n-k\geq\lceil\frac{i}{2}\rceil+\frac{n}{2}\geq\lceil\frac{i}{2}\rceil+\frac{i}{2}+1\geq i+1$ because $i\leq n-2$. Similarly we have $\lfloor\frac{i+n-k-2}{2}\rfloor+1-j(n-k)\geq i+1$. Hence, all the summands $A_{i,j}$, $B_{i,j}$ and $C_{i,j}$ vanish;

(2) if $j > 0$, then $\lceil \frac{i}{2} \rceil - j(n-k) \leq \lceil \frac{i}{2} \rceil - (n-k) \leq \lceil \frac{i}{2} \rceil - \lceil \frac{n}{2} \rceil \leq -1$ because $i \leq n-2$. Hence, all $A^+_{i,j}$, $B^+_{i,j}$ and $C^+_{i,j}$ vanish;

(3) if $j \geq 2$, then $\lfloor \frac{i+n-k}{2} \rfloor + 1 - j(n-k+2) \leq \lfloor \frac{n-2+n-k}{2} \rfloor + 1 - 2(n-k+2) \leq \frac{3}{2}k - n - 5 \leq -1$. Similarly we have $\lfloor \frac{i+n-k-1}{2} \rfloor + 1 - j(n-k+1) \leq -1$ and $\lfloor \frac{i+n-k-2}{2} \rfloor + 1 - j(n-k) \leq -1$, so all $A^-_{i,j}$, $B^-_{i,j}$ and $C^-_{i,j}$ vanish.

It follows that the summation over $j \in \mathbb{Z}$ in (3.5) reduces to

$$-A^-_{i,0} + 2B^-_{i,0} - C^-_{i,0} - A^-_{i,1} + 2B^-_{i,1} - C^-_{i,1}.$$

Using $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ to combine $A^-_{i,0}$ with $C^-_{i,0}$ and $A^-_{i,1}$ with $C^-_{i,1}$, respectively, we derive the desired formula. □

Now, we are in position to prove Theorem 1. When $n \geq 2k$, by Lemma 11, the summands in (3.4) can be written as

$$2^{n-i-2} \sum_{j \in \mathbb{Z}} (-A_{i,j} + 2B_{i,j} - C_{i,j}) = D_{i+1} - D_i,$$

where

$$D_i = 2^{n-i-1} \left\{ \binom{i}{\lfloor \frac{i+n-k-1}{2} \rfloor + 1} + \binom{i}{\lfloor \frac{i+n-k-1}{2} \rfloor - n + k} \right\}.$$

Substituting this into (3.4) we obtain

$$l_k(n) = \sum_{i=0}^{n-2} (D_{i+1} - D_i) = D_{n-1},$$

which is clearly equivalent to (1.3).

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(Zhicong Lin) Department of Mathematics and Statistics, Lanzhou University, China, and Université de Lyon; Université Lyon 1; Institut Camille Jordan; UMR 5208 du CNRS; 43, boulevard du 11 novembre 1918, F-69622 Villeurbanne Cedex, France

E-mail address: `lin@math.univ-lyon1.fr`

(Jiang Zeng) Université de Lyon; Université Lyon 1; Institut Camille Jordan; UMR 5208 du CNRS; 43, boulevard du 11 novembre 1918, F-69622 Villeurbanne Cedex, France

E-mail address: `zeng@math.univ-lyon1.fr`