ON A SEQUENCE OF POLYNOMIALS WITH HYPOTHETICALLY INTEGER COEFFICIENTS

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Abstract. The first author introduced a sequence of polynomials ([8], sequence A174531) defined recursively. One of the main results of this study is proof of the integrality of its coefficients.

1. Introduction

In point of fact, there are only a few examples of sequences known where the question of the integrality of the terms is a difficult problem. In 1989, Somos [9] posed a problem on the integrality of sequences depending on parameter $k \geq 4$ which are defined by the recursion

$$a_n = \frac{\sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} a_{n-j} a_{n-(k-j)}}{a_{n-k}}, \quad n \geq k \geq 4, \tag{1.1}$$

with the initial conditions $a_i = 1, \ \ i = 1, ..., k-1$.
Gale [3] proved the integrality of Somos sequences when $k = 4$ and 5, attributing a proof to Malouf [4]. Hickerson and Stanley (see [6]) independently proved the integrality of the $k = 6$ case in unpublished work and Fomin and Zelevinsky (2002) gave the first published proof. Finally, Lotto (1990) gave an unpublished proof for the $k = 7$ case. These are sequences A006720-A006723 in [8]. It is interesting that, for $k \geq 8$, the property of integrality disappears (see sequence A030127 in [8]). In connection with this, note that in the so-called Göbel's sequence ([11]) defined by the recursion

$$x_n = \frac{1}{n}(1 + \sum_{i=0}^{n-1} x_i^2), \ \ n \geq 1, \ \ x_0 = 1, \tag{1.2}$$

the first non-integer term is $x_{43} = 5.4093 \times 10^{178485291567}$.

In this paper we study the Shevelev sequence of polynomials $\{P_n(x)\}_{n \geq 1}$ that are defined by the following recursion $P_1 = 1, \ \ P_2 = 1$, and, for $n \geq 2$,

$$4(2x+n)P_{n+1}(x) = 2(x+n)P_n(x) +$$

$$(2x+n)P_n(x+1) + (4x+n)l_n(x), \ \ if \ \ n \ \ is \ \ odd, \tag{1.3}$$

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$$4P_{n+1}(x) = 4(x+n)P_n(x)+$$

$$2(2x+n+1)P_n(x+1)+(4x+n)l_{n-1}(x), \ \ if \ \ n \ \ is \ \ even, \tag{1.4}$$

where

$$l_n(x) = (x+\frac{n-1}{2})(x+\frac{n-3}{2})\cdot ... \cdot (x+1). \tag{1.5}$$

The first few polynomials are the following ([8], sequence A174531):

$$P_1 = 1,$$
$$P_2 = 1,$$
$$P_3 = 3x+4,$$
$$P_4 = 2x+4,$$
$$P_5 = 5x^2+25x+32,$$
$$P_6 = 3x^2+19x+32,$$
$$P_7 = 7x^3+77x^2+294x+384,$$
$$P_8 = 4x^3+52x^2+240x+384,$$
$$P_9 = 9x^4+174x^3+1323x^2+4614x+6144,$$
$$P_{10} = 5x^4+110x^3+\ \ 967x^2+3934x+6144,$$
$$P_{11} = 11x^5+330x^4+4169x^3+27258x^2+90992x+122880,$$
$$P_{12} = \ \ 6x^5+200x^4+2842x^3+21040x^2+79832x+122880.$$

According to our observations, the following conjectures are natural.
1) The coefficients of all the polynomials are integers. Moreover, the greatest common divisor of all coefficients is $n/rad(n)$, where $rad(n) = \prod_{p|n} p$;
2) $P_n(0) = 4^{\lfloor \frac{n-1}{2} \rfloor} \lfloor \frac{n-1}{2} \rfloor !$;
3) For even n, $P_n(1) = (2^n-1)(\frac{n}{2})!/(n+1)$, and for odd n, $P_n(1) = (2^n-1)(\frac{n-1}{2})!$;
4) $P_n(x)$ has a real rational root if and only if either $n=3$ or $n \equiv 0 \pmod 4$. In the latter case, such a unique root is $-\frac{n}{2}$;
5) Coefficients of x^k increase when k decreases;
6) If n is even, then the coefficients of P_n do not exceed the corresponding coefficients of P_{n-1} and the equality holds only for the last ones; moreover, the ratios of coefficients of x^k of polynomials P_{n-1} and P_n monotonically decrease to 1 when k decreases;
7) All coefficients of P_n, except of the last one, are multiple of n if and only if n is prime.

The main results of our paper consist of the following two theorems.

Theorem 1. (Explicit formula for $P_n(k)$) *For integer $x = k$, we have*

$$P_n(k) = \tag{1.6}$$
$$\begin{cases} \left(\binom{(n-1)/2+k-1}{k-1} / \binom{n+2k-2}{k-1}\right)\left(\frac{n-1}{2}\right)!\ T_n(k), & if \ \ n \geq 1 \ \ is \ \ odd, \\ \left(\binom{n/2+k-1}{k} / \binom{n+2k-1}{k}\right)\left(\frac{n}{2}-1\right)!\ T_n(k), & if \ \ n \geq 2 \ \ is \ \ even, \end{cases}$$

$$= 2^{-(\lfloor \frac{n}{2} \rfloor + k - 1)} \frac{(n+k-1)!}{(2\lfloor \frac{n}{2} \rfloor + 2k - 1)!!} T_n(k), \tag{1.7}$$

where

$$T_n(k) = \sum_{i=1}^{n} 2^{i-1} \binom{n+2k-i-1}{k-1}. \tag{1.8}$$

Using Theorem 1, we prove Conjectures 2)-3) and the following main result.

Theorem 2. *For $n \geq 1$, $P_n(x)$ is a polynomial of degree $\lfloor \frac{n-1}{2} \rfloor$ with integer coefficients.*

Nevertheless, the subtle second part of conjecture 1) remains open.

2. Representation of $P_n(k)$ via a polynomial in n of degree $k-1$ with integer coefficients

Theorem 3. *For integer $k \geq 1$, $n \geq 1$, the following recursion holds*

$$P_n(k) = c_n(k)\left(2^{n+k-1} - \frac{R_k(n)}{(2k-2)!!}\right), \tag{2.1}$$

where $R_k(n)$ is a polynomial in n of degree $k-1$ with integer coefficients and

$$c_n(k) = \begin{cases} \left(\frac{n-1}{2}\right)! \prod_{i=1}^{k-1} \frac{n+i}{n+2i}, & if \ \ n \ \ is \ \ odd, \\ \frac{1}{2}\left(\frac{n}{2}-1\right)! \prod_{i=0}^{k-1} \frac{n+i}{n+2i+1}, & if \ \ n \ \ is \ \ even, \end{cases} \tag{2.2}$$

Proof. Write (1.3)-(1.4) in the form

$$P_n(k+1) = -\frac{2f}{g} P_n(k) + 4P_{n+1}(k) -$$
$$\frac{h}{g}\left(\frac{n-1}{2}\right)! \binom{\frac{g-1}{2}}{k}, \ \ if \ \ n \equiv 1 \pmod 2; \tag{2.3}$$

$$P_n(k+1) = -\frac{2f}{g+1} P_n(k) + \frac{2}{g+1} P_{n+1}(k) -$$

(2.4) $$\frac{h}{2(g+1)}(\frac{n}{2}-1)!\binom{\frac{g}{2}-1}{k}, \ \ if \ \ n\equiv 0 \pmod 2,$$

where $f=n+k,\ \ g=n+2k,\ \ h=n+4k.$

Let n be odd. We use induction over k. For $k=1$, (2.1) gives

(2.5) $$R_1(n)=2^n-\frac{P_n(1)}{c_n(1)}=Const(k).$$

Thus the base of induction is valid. Suppose the theorem is true for some value of k. Then, using this supposition and (2.1)-(2.4), we have

$$P_n(k+1)=$$
$$-\frac{2f}{g}(\frac{n-1}{2})!(2^{n+k-1}-\frac{R_k(n)}{(2k-2)!!})\prod_{i=1}^{k-1}\frac{n+i}{n+2i}+$$
$$2(\frac{n-1}{2})!(2^{n+k}-\frac{R_k(n+1)}{(2k-2)!!})\prod_{i=0}^{k-1}\frac{n+i+1}{n+2i+2}-$$
$$\frac{h}{g}(\frac{n-1}{2})!\frac{\frac{g-1}{2}\frac{g-3}{2}\cdot\ldots\cdot\frac{n+1}{2}}{k!}.$$

Note that

$$\frac{f}{g}\prod_{i=0}^{k-1}\frac{n+i}{n+2i}=\prod_{j=1}^{k}\frac{n+j}{n+2j}=\prod_{i=0}^{k-1}\frac{n+i+1}{n+2i+2}.$$

Therefore,

$$P_n(k+1)=(\frac{n-1}{2})!(-2^{n+k}+\frac{2R_k(n)}{(2k-2)!!}+2^{n+k+1}-$$
$$\frac{2R_k(n+1)}{(2k-2)!!}-\frac{h}{g}\frac{\frac{g-1}{2}\frac{g-3}{2}\cdot\ldots\cdot\frac{n+1}{2}}{k!}\prod_{j=1}^{k}\frac{n+2j}{n+j})\prod_{j=1}^{k}\frac{n+j}{n+2j}.$$

Here we note that

$$(g-1)(g-3)\cdot\ldots\cdot(n+1)\prod_{j=1}^{k}\frac{n+2j}{n+j}=(n+2k)_k,$$

where $(x)_k$ is a falling factorial. Hence

$$P_n(k+1)=c_n(k+1)(2^{n+k}-2\frac{R_k(n+1)-R_k(n)}{(2k-2)!!}-$$
$$\frac{4k+n}{(2k)!!}(n+2k-1)_{k-1})=c_n(k+1)(2^{n+k}-\frac{R_{k+1}(n)}{(2k)!!}),$$

where

(2.6) $$R_{k+1}(n)=4k(R_k(n+1)-R_k(n))+(4k+n)(n+2k-1)_{k-1}.$$

Since, by the inductive supposition, $R_k(n)$ is a polynomial of degree $k-1$ with integer coefficients, then, by (2.6), $R_{k+1}(n)$ is a polynomial of degree k with integer coefficients.

Note that the case of even n is considered quite analogously, obtaining the *same* formula (2.6). ■

Put in (2.1)-(2.2) $n = 1$. Then, for $k \geq 1$ we have

$$(2^k - \frac{R_k(1)}{(2k-2)!!})\frac{k!}{(2k-1)!!} = 1,$$

whence

$$R_k(1) = (k-1)!(2^{2k-1} - \binom{2k-1}{k}). \tag{2.7}$$

In particular, $R_1(1) = 1$ and, since $R_1(n)$ is of degree 0, then $R_1(n) = 1$. Further, we find polynomials $R_k(n)$ using the recursion (2.6). The first polynomials $R_k(n)$ are

$$R_1(n) = 1,$$
$$R_2(n) = n + 4,$$
$$R_3(n) = n^2 + 11n + 32,$$
$$R_4(n) = n^3 + 21n^2 + 152n + 384,$$
$$R_5(n) = n^4 + 34n^3 + 443n^2 + 2642n + 6144,$$
$$R_6(n) = n^5 + 50n^4 + 1015n^3 + 10510n^2 + 55864n + 122880.$$

3. Proof of Conjectures 2) and 3)

We start with proof of Conjecture 3) for $P_n(1)$. Note that, since $R_1(n) = 1$, then from (2.5) we find

$$P_n(1) = c_n(1)(2^n - 1). \tag{3.1}$$

Besides, by (2.2), we have

$$c_n(1) = \begin{cases} (\frac{n-1}{2})!, & if \;\; n \;\; is \;\; odd, \\ \frac{1}{2}(\frac{n}{2}-1)!\frac{n}{n+1} = (\frac{n}{2})!/(n+1), & if \;\; n \;\; is \;\; even, \end{cases} \tag{3.2}$$

and Conjecture 3) follows.

Let us prove now Conjecture 2). Note that (2.3)-(2.4), as (1.3)-(1.4), is valid for every nonnegative k. For $k = 0$ and odd $n \geq 1$, (2.3) gives

$$P_n(1) = -2P_n(0) + 4P_{n+1}(0) - (\frac{n-1}{2})!,$$

or, using (3.1)-(3.2), we have

$$4P_{n+1}(0) - 2P_n(0) = 2^n(\frac{n-1}{2})!$$

Analogously, for $k = 0$ and even $n \geq 1$, from (2.4) and (3.1)-(3.2) we find

$$P_{n+1}(0) - nP_n(0) = 2^{n-1}(\frac{n}{2})!$$

Thus

$$P_{n+1}(0) = \begin{cases} \frac{1}{2}P_n(0) + 2^{n-2}(\frac{n-1}{2})!, & if \;\; n \;\; is \;\; odd, \\ \\ nP_n(0) + 2^{n-1}(\frac{n}{2})!, & if \;\; n \;\; is \;\; even \end{cases} \tag{3.3}$$

with $P_1(0) = 1,\;\; P_2(0) = 1$. Since the difference equation

$$y(n+1) = \begin{cases} \frac{1}{2}y(n) + 2^{n-2}(\frac{n-1}{2})!, & if \;\; n \;\; is \;\; odd, \\ \\ ny(n) + 2^{n-1}(\frac{n}{2})!, & if \;\; n \;\; is \;\; even \end{cases}$$

with the initials $y(1) = 1,\;\; y(2) = 1$ has an unique solution, then it is sufficient to verify that $y(n) = P_n(0) = 4^{\lfloor \frac{n-1}{2} \rfloor} \lfloor \frac{n-1}{2} \rfloor !$ is a solution. ■

4. Explicit formula for $R_k(n)$

Since from (2.6)

$$4kR_k(n+1) = 4kR_k(n)+$$

$$R_{k+1}(n) - (4k+n)(n+2k-1)_{k-1}, \tag{4.1}$$

we have a recursion in n for $R_k(n)$ given by (2.7) and (4.1).
Our aim in this section is to find a generalization of (2.7) for an arbitrary integer $n \geq 1$. Note that we can write (2.7) in the form

$$R_k(1) = 2(k-1)!4^{k-1} - \frac{(2k-1)!}{k!}. \tag{4.2}$$

Using (4.1) and (2.7), after some transformations, we find

$$R_k(2) = 2^2(k-1)!4^{k-1} - 2\frac{(2k-1)!}{k!} - \frac{(2k)!}{(k+1)!}. \tag{4.3}$$

The regularity is fixed in the following theorem.

Theorem 4. *For integer $k \geq 1,\;\; n \geq 1$, we have*

$$R_k(n) = 2^n(k-1)!4^{k-1} - \sum_{i=1}^{n} 2^{n-i}\frac{(2k+i-2)!}{(k+i-1)!}. \tag{4.4}$$

Proof. Taking into account that $\frac{(2k+i-2)!}{(k+i-1)!} = \binom{2k+i-2}{k-1}(k-1)!$, we prove (4.4) in the following equivalent form:

$$R_k(n) = 2^n(k-1)!(4^{k-1} - \sum_{i=1}^{n} 2^{-i}\binom{2k+i-2}{k-1}). \tag{4.5}$$

We use induction over n. Suppose that (4.5) is valid for a some value of n and an arbitrary integer $k \geq 1$. Then, by (4.1), we have

$$R_k(n+1) = 2^n(k-1)!(4^{k-1} - \sum_{i=1}^{n} 2^{-i}\binom{2k+i-2}{k-1})+$$

$$2^{n-2}(k-1)!(4^k - \sum_{i=1}^{n} 2^{-i}\binom{2k+i}{k}) - \frac{4k+n}{4k}(n+2k-1)_{k-1} =$$
$$2^n(k-1)!(4^{k-1} - \sum_{i=1}^{n} 2^{-i}\binom{2k+i-2}{k-1})+$$
$$2^n(k-1)!(4^{k-1} - \sum_{i=1}^{n} 2^{-i-2}\binom{2k+i}{k})-$$
$$\frac{(n+2k-1)!}{(n+k)!} - \frac{n}{4k}\frac{(n+2k-1)!}{(n+k)!}.$$

Thus we should prove the identity

$$2^{n+1}(k-1)!4^{k-1} - 2^n(k-1)!\sum_{i=1}^{n} 2^{-i}\binom{2k+i-2}{k-1}-$$
$$2^{n-2}(k-1)!\sum_{i=1}^{n} 2^{-i}\binom{2k+i}{k} - \frac{n+4k}{4k}\frac{(n+2k-1)!}{(n+k)!} =$$
$$2^{n+1}(k-1)!(4^{k-1} - \sum_{i=1}^{n+1} 2^{-i}\binom{2k+i-2}{k-1}),$$

which is easily reduced to the identity

$$4\sum_{i=1}^{n} 2^{-i}\binom{2k+i-2}{k-1} - \sum_{i=1}^{n} 2^{-i}\binom{2k+i}{k} =$$
$$2^{-n}\frac{n+4k}{4k}\frac{(n+2k-1)!}{(n+k)!} - 4\cdot 2^{-n}\binom{2k+n-1}{k-1}.$$

Note that, the right hand part is $\frac{n}{k2^n}\binom{2k+n-1}{k-1}$. Therefore, it is left to prove the identity

$$4\sum_{i=1}^{n} 2^{-i}\binom{2k+i-2}{k-1} - \sum_{i=1}^{n} 2^{-i}\binom{2k+i}{k} = \frac{n}{k2^n}\binom{2k+n-1}{k-1}. \tag{4.6}$$

Since this is trivially satisfied for $n = 0$, then it is sufficient to verify the equality of the first differences of the left and the right hand parts, which is reduced to the identity

$$2(n+2k-1)\binom{2k+n-2}{k-1} = n\binom{2k+n-1}{k-1} + k\binom{2k+n}{k},$$

which is verified directly. ■

5. Proof of Theorem 1

Now we are able to prove Theorem 1. According to (1.7), we have

$$T_n(k)=\sum_{i=1}^{n}2^{i-1}\binom{n+2k-i-1}{k-1}=$$

$$\sum_{j=1}^{n}2^{n-j}\binom{2k+j-2}{k-1}. \tag{5.1}$$

Hence, by (4.5), we find

$$R_k(n)=2^n(k-1)!(4^{k-1}-2^{-n}T_n(k))=$$

$$(k-1)!(2^{n+2k-2}-T_n(k)). \tag{5.2}$$

Now from (2.1) and (5.2) we have

$$P_n(k)=2^{-(k-1)}c_n(k)T_n(k). \tag{5.3}$$

Let n be odd. Note that, by (2.2),

$$2^{-(k-1)}c_n(k)=$$

$$2^{-(k-1)}(\frac{n-1}{2})!\frac{(n+k-1)(n+k-2)...(n+1)}{(n+2k-2)(n+2k-4)...(n+2)}=$$

$$2^{-(k-1)}(\frac{n-1}{2})!\frac{(n+k-1)!n!!}{n!(n+2k-2)!!}. \tag{5.4}$$

Taking into account that

$$n!!=\frac{n!}{(n-1)!!}=\frac{n!}{2^{\frac{n-1}{2}}(\frac{n-1}{2})!}, \tag{5.5}$$

we find from (5.4)

$$2^{-(k-1)}c_n(k)=\frac{(n+k-1)!(\frac{n-1}{2}+k-1)!}{(n+2k-2)!}=$$

$$\frac{(\frac{n-1}{2}+k-1)!}{(k-1)!\binom{n+2k-2}{k-1}}=\frac{\binom{\frac{n-1}{2}+k-1}{k-1}}{\binom{n+2k-2}{k-1}}(\frac{n-1}{2})!$$

and (1.6) follows from (5.3). Furthermore, since by (5.5) $\frac{n!!(\frac{n-1}{2})!}{n!}=2^{-\frac{n-1}{2}}$, then from (5.3)-(5.4) we find

$$P_n(k)=2^{-(\frac{n-1}{2}+k-1)}\frac{(n+k-1)!}{(n+2k-2)!!}T_n(k)$$

that corresponds to (1.7) in the case of odd n. The case of even n is considered quite analogously. ■

6. Bisection of sequence $\{P_n(x)\}$

Note that $T_n(k)$ (1.8) has rather a simple structure, which allows us to find different relations for it. Using (1.6), this, in turn, allows us to find recursion relations for $P_n(x)$ which are simpler than the basis recursion (1.3)-(1.4). We start with the following simple recursions for $T_n(k)$.

Lemma 1.

$$T_n(k) - 2T_{n-1}(k) = \binom{n+2k-2}{k-1}, \quad k \geq 1; \tag{6.1}$$

$$T_n(k) - 4T_{n-2}(k) = \binom{n+2k-2}{k-1} + 2\binom{n+2k-3}{k-1}, \quad k \geq 2. \tag{6.2}$$

Proof. By (1.8), we have

$$T_n(k) - 2T_{n-1}(k) =$$
$$\sum_{i=1}^{n} 2^{i-1}\binom{n+2k-i-1}{k-1} - \sum_{j=1}^{n-1} 2^{j}\binom{n+2k-j-2}{k-1} =$$
$$\sum_{i=1}^{n} 2^{i-1}\binom{n+2k-i-1}{k-1} - \sum_{i=2}^{n} 2^{i-1}\binom{n+2k-i-1}{k-1}$$

and (6.1) follows; (6.2) is a simple corollary of (6.1). ■

Theorem 5. *(Bisection) If $n \geq 3$ is odd, then*

$$(2x+n-2)P_n(x) = 2(x+n-1)(x+n-2)P_{n-2}(x)+$$
$$(4x+3n-4)(x+\frac{n-1}{2}-1)(x+\frac{n-1}{2}-2)\cdot \ldots \cdot x; \tag{6.3}$$

if $n \geq 4$ is even, then

$$(2x+n-1)P_n(x) = 2(x+n-1)(x+n-2)P_{n-2}(x)+$$
$$\frac{1}{2}(4x+3n-4)(x+\frac{n-2}{2}-1)(x+\frac{n-2}{2}-2)\cdot \ldots \cdot x. \tag{6.4}$$

Proof. According to (1.6), we have

$$T_n(k) = \begin{cases} \binom{n+2k-2}{k-1}/(\binom{(n-1)/2+k-1}{k-1}(\frac{n-1}{2})!)\ P_n(k), & if\ \ n\ \ is\ \ odd, \\ \binom{n+2k-1}{k}/(\binom{n/2+k-1}{k}(\frac{n}{2}-1)!)\ P_n(k), & if\ \ n\ \ is\ \ even. \end{cases} \tag{6.5}$$

Substituting this to (6.2), after simple transformations, we obtain (6.3)-(6.4), where k is replaced by arbitrary x. ■

Note that from (6.3)-(6.4), using a simple induction, we conclude that, for even $n \geq 4$, $P_n(x)$ is a polynomial of degree $\frac{n-2}{2}$, while, for odd $n \geq 3$, $P_n(x)$ is a polynomial of degree $\frac{n-1}{2}$. However, a structure of formulas (6.3)-(6.4) does not allow us to prove that all coefficients of $P_n(x)$ are integer.

This will be done in the following section by the discovery of the special relationships with the required structure.

7. Proof of Theorem 2

Lemma 2. *For $n \geq 1$, we have*

$$T_n(k) - T_{n-2}(k+1) = \binom{n+2k-1}{k}. \tag{7.1}$$

Proof. By (5.1), we should prove that

$$\binom{2k+n-1}{k} = T_n(k) - T_{n-2}(k+1) =$$

$$\sum_{j=1}^{n} 2^{n-j}\binom{2k+j-2}{k-1} - \sum_{j=1}^{n-2} 2^{n-j-2}\binom{2k+j}{k} =$$

$$\sum_{j=1}^{n} 2^{n-j}\binom{2k+j-2}{k-1} - \sum_{i=1}^{n} 2^{n-i}\binom{2k+i-2}{k} +$$

$$2^{n-1}\binom{2k-1}{k} + 2^{n-2}\binom{2k}{k},$$

or

$$\sum_{j=1}^{n} 2^{-j}(\binom{2k+j-2}{k-1} - \binom{2k+j-2}{k}) =$$

$$2^{-n}\binom{2k+n-1}{k} - \frac{1}{2}\binom{2k-1}{k} - \frac{1}{4}\binom{2k}{k}. \tag{7.2}$$

It is verified directly that (7.2) is valid for $n = 1$. Therefore, it is sufficient to verify that the first differences over n of the left hand side and the right hand side coincide. The corresponding identity

$$2^{-n}(\binom{2k+n-2}{k-1} - \binom{2k+n-2}{k}) =$$

$$2^{-n}\binom{2k+n-1}{k} - 2^{-n+1}\binom{2k+n-2}{k}$$

reduces to the equality $\binom{2k+n-2}{k-1} + \binom{2k+n-2}{k} = \binom{2k+n-1}{k}$.■

Now we are able to complete proof of Theorem 2. Considering even $n \geq 4$, by (6.5), we obtain the following relation for $P_n(k)$ corresponding to (7.1):

$$P_n(x) = (n+x-1)P_{n-2}(x+1)+$$

$$(x + \frac{n}{2} - 1)(x + \frac{n}{2} - 2)...(x+1). \tag{7.3}$$

On the other hand, using (6.1), for odd $n \geq 3$, we obtain the following relation

$$P_n(x) = 2(x+n-1)P_{n-1}(x)+$$

$$(x+\frac{n-1}{2}-1)(x+\frac{n-1}{2}-2)\cdot ...\cdot x. \tag{7.4}$$

From (7.3), by a simple induction, we see that, for even $n \geq 4$, $P_n(x)$ is a polynomial with integer coefficients. Then from (7.4) we find that $P_n(x)$, for odd n, is a polynomial with integer coefficients as well. ■

8. Other relations

Together with (6.3)-(6.4), (7.3)-(7.4) there exist many other relations for $P_n(x)$. All of them are corollaries of the corresponding relations for $T_n(k)$. Below we give a few pairs of some such relations.

As we saw, for odd $n \geq 3$, (7.4) follows from (6.1). Let us consider even $n \geq 4$. Then we obtain the second component of the following recursion

$$P_n(x) = \begin{cases} 2(x+n-1)P_{n-1}(x)+ \\ \\ ((x+n-1)P_{n-1}(x)+ \end{cases} \tag{8.1}$$

$$\begin{cases} (x+\frac{n-1}{2}-1)(x+\frac{n-1}{2}-2)\cdot ...\cdot x, \quad if \ \ n\geq 3 \ \ is \ \ odd, \\ \\ (x+\frac{n}{2}-1)(x+\frac{n}{2}-2)\cdot ...\cdot x)/(2x+n-1), \quad if \ \ n \geq 4 \ \ is \ \ even. \end{cases}$$

Lemma 3. *For $n \geq 1, \ \ k \geq 1$, we have*

$$T_n(k+1) = 4T_n(k) - \frac{n}{k}\binom{n+2k-1}{k-1}. \tag{8.2}$$

Proof. By (7.1),(6.2), we have

$$T_n(k+1) = T_{n+2}(k) - \binom{n+2k+1}{k} =$$

$$4T_n(k) + \binom{n+2k}{k-1} + 2\binom{n+2k-1}{k-1} - \binom{n+2k+1}{k}.$$

It is left to note that

$$\binom{n+2k}{k-1} + 2\binom{n+2k-1}{k-1} - \binom{n+2k+1}{k} = -\frac{n}{k}\binom{n+2k-1}{k-1}.$$

■

From Lemma 3 and (6.5) we find the following recursion

$$\begin{cases} (2x+n)P_n(x+1) = 2(x+n)P_n(x)- \\ \\ (2x+n+1)P_n(x+1) = 2(x+n)P_n(x)- \end{cases} \tag{8.3}$$

$$\begin{cases} n(x+\frac{n-1}{2})(x+\frac{n-1}{2}-1)\cdot ...\cdot (x+1), \quad if \ \ n\geq 3 \ \ is \ \ odd, \\ \\ \frac{n}{2}(x+\frac{n}{2}-1)(x+\frac{n}{2}-2)\cdot ...\cdot (x+1), \quad if \ \ n\geq 4 \ \ is \ \ even. \end{cases}$$

Lemma 4. *For $n \geq 2, \ \ k \geq 1,$ we have*

$$(n+k-1)(T_n(k)-4T_n(k-1)) = n(T_{n-1}(k)-2T_n(k-1)). \tag{8.4}$$

Proof. By (8.2),

$$T_n(k)-4T_n(k-1) = -\frac{n}{k-1}\binom{n+2k-3}{k-2}. \tag{8.5}$$

By (6.1),

$$T_n(k-1) = 2T_{n-1}(k-1)+\binom{n+2k-4}{k-2}.$$

Therefore,

$$T_{n-1}(k)-2T_n(k-1) = T_{n-1}(k)-4T_{n-1}(k-1)-2\binom{n+2k-4}{k-2}.$$

Using again (8.2), we find

$$T_{n-1}(k)-2T_n(k-1) = -(\frac{n-1}{k-1}+2)\binom{n+2k-4}{k-2}. \tag{8.6}$$

Now the lemma follows from (8.5)-(8.6) since $(n+k-1)\binom{n+2k-3}{k-2} = (n+2k-3)\binom{n+2k-4}{k-2}$.■

The passage from (8.4) to the corresponding formula for $P_n(x)$ in the case of odd $n \geq 3$ unexpectedly leads to a very simple homogeneous relation

$$P_n(x) = P_n(x-1)+nP_{n-1}(x) \tag{8.7}$$

which we use in Sections 9 and 12. The corresponding relation for even $n \geq 4$ is

$$(2x+n-1)P_n(x) = (2x+n-2)P_n(x-1)+\frac{n}{2}P_{n-1}(x). \tag{8.8}$$

Lemma 5. *For $n \geq 1, \ \ k \geq 2,$ we have*

$$2T_n(k)-T_{n-1}(k+1) = \binom{n+2k-1}{k}. \tag{8.9}$$

Proof. By (6.1), we have

$$2T_n(k)-T_{n-1}(k+1) =$$
$$4T_{n-1}(k)+2\binom{n+2k-2}{k-1}-T_{n-1}(k+1).$$

Furthermore, by (7.1),

$$T_{n-1}(k+1) = T_{n+1}(k)-\binom{n+2k}{k}.$$

Hence,

$$2T_n(k)-T_{n-1}(k+1) =$$
$$4T_{n-1}(k)-T_{n+1}(k)+2\binom{n+2k-2}{k-1}+\binom{n+2k}{k}. \tag{8.10}$$

Finally, by (6.2),

$$T_{n+1}(k)-4T_{n-1}(k)=\binom{n+2k-1}{k-1}+2\binom{n+2k-2}{k-1}$$

and the lemma follows from (8.10). ■

Using Lemma 5 and (6.5), for even $n\geq 4$, we find

$$2P_n(x)=P_{n-1}(x+1)+(x+\frac{n}{2}-1)(x+\frac{n}{2}-2)\cdot\ldots\cdot(x+1), \tag{8.11}$$

while, for odd $n\geq 3$,

$$P_n(x)=(2x+n)P_{n-1}(x+1)+(x+\frac{n-1}{2})(x+\frac{n-1}{2}-1)\cdot\ldots\cdot(x+1). \tag{8.12}$$

Proposition 1. *For odd $n\geq 3$, we have*

$$P_n(k)\equiv P_n(0)\pmod{n}. \tag{8.13}$$

Proof. From (8.7) we find

$$\sum_{i=1}^{k}P_{n-1}(i)=(P_n(k)-P_n(0))/n, \tag{8.14}$$

and the proposition follows. ■

9. On coefficients of $P_n(x)$

Using formulas (6.3)-(6.4), we give a recursion for calculation of the coefficients of $P_n(x)$ with a fixed parity of n. Let

$$P_n(x)=a_0(n)x^m+a_1(n)x^{m-1}+\ldots+a_{m-1}(n)x+a_m(n), \tag{9.1}$$

where $m=\lfloor\frac{n-1}{2}\rfloor$. We prove the following.

Theorem 6. *For $n\geq 1$, we have*

$$a_0(n)=\begin{cases}n, & if\ \ n\ \ is\ \ odd,\\ \frac{n}{2}, & if\ \ n\ \ is\ \ even;\end{cases} \tag{9.2}$$

$$a_1(n)=\begin{cases}\frac{1}{24}(7n^3-12n^2+5n)\\ \frac{1}{48}(7n^3-18n^2+8n)\end{cases}=$$

$$\begin{cases}\frac{1}{24}n(n-1)(7n-5), & if\ \ n\ \ is\ \ odd,\\ \frac{1}{48}n(n-2)(7n-4), & if\ \ n\ \ is\ \ even.\end{cases} \tag{9.3}$$

In general, for a fixed i, $a_i(n)=U_i(n)$, if n is odd, and $a_i(n)=V_i(n)$, if n is even, where $U_i,\ V_i$ are polynomials in n of degree $2i+1$.

Proof. 1) Let n be even. Then, using (6.4), for integer x and $m=\frac{n-2}{2}$, we have

$$(2x+n-1)(a_0(n)x^m+a_1(n)x^{m-1}+...)=$$
$$2(x+n-1)(x+n-2)(a_0(n-2)x^{m-1}+a_1(n-2)x^{m-2}+...)+$$
$$\frac{1}{2}(\frac{n-2}{2})!(4x+3n-4)\binom{x-1+\frac{n-2}{2}}{\frac{n-2}{2}}. \tag{9.4}$$

Comparing the coefficient of x^{m+1} in both hand sides, we find

$$a_0(n)=a_0(n-2)+1,\ \ n\geq 4,\ \ a_0(4)=2.$$

Thus $a_0(6)=3, a_0(8)=4, ..., a_0(n)=n/2$.
Furthermore, comparing the coefficient of x^m in both hand sides in (9.4), we have

$$2a_1(n)+(n-1)a_0(n)=2a_1(n-2)+2(2n-3)a_0(n-2)+$$
$$Coef[x^m](\frac{1}{2}(4x+3n-4)(x+\frac{n-4}{2})(x+\frac{n-6}{2})\cdot...\cdot(x+1)x). \tag{9.5}$$

Note that

$$Coef[x^m](\frac{1}{2}(4x+3n-4)(x+\frac{n-4}{2})(x+\frac{n-6}{2})\cdot...\cdot(x+1)x)=$$
$$\frac{3n-4}{2}+2(\frac{n-4}{2}+\frac{n-6}{2}+...+1)=$$
$$\frac{3n-4}{2}+\sum_{i=2}^{m}(n-2i)=\frac{n^2}{4}.$$

Therefore, by (9.5),

$$a_1(n)-a_1(n-2)=\frac{(2n-3)(n-2)}{2}-\frac{(n-1)n}{4}$$
$$+\frac{n^2}{8}=\frac{7n^2-26n+24}{8}.$$

Hence

$$a_1(n)=\sum_{i=4,6,...,n}(a_1(i)-a_1(i-2))=\frac{1}{8}\sum_{i=4,6,...,n}(7i^2-26i+24)=$$
$$\frac{1}{2}\sum_{j=2}^{n/2}(7j^2-13j+6)=\frac{1}{48}(7n^3-18n^2+8n).$$

■

Finally, comparing the coefficient of x^{m-i} in both hand sides of (9.4), we find

$$2a_{i+1}(n)+(n-1)a_i(n)=2a_{i+1}(n-2)+$$
$$2(2n-3)a_i(n-2)+2(n-1)(n-2)a_{i-1}(n-2)+$$
$$\frac{1}{2}Coef[x^{m-i}]((4x+3n-4)(x+\frac{n-4}{2})(x+\frac{n-6}{2})\cdot...\cdot(x+1)(x)). \tag{9.6}$$

Note that, polynomial $(4x+3n-4)(x+\frac{n-4}{2})(x+\frac{n-6}{2})\cdot...\cdot(x+1)x$ has degree $m+1$. Therefore, in order to calculate $Coef[x^{m-i}]$ in (9.6), we should choose, by all possible ways, in $m-i$ brackets (from $m+1$ ones) $x's$, and in other $i+1$ brackets we choose linear forms of n. Thus $\frac{1}{2}Coef[x^{m-i}]$ in (9.6) is a polynomial $r_i(n)$ of degree $i+1$. Further we use induction over i with the formulas (9.2)-(9.3) as the inductive base. Write (9.6) in the form

$$2(a_{i+1}(n)-a_{i+1}(n-2))=$$
$$2(2n-3)a_i(n-2)-(n-1)a_i(n)+$$
$$2(n-1)(n-2)a_{i-1}(n-2)+r_i(n). \tag{9.7}$$

By the inductive supposition, $a_{i-1}(n)$, $a_i(n)$ are polynomials of degree $2i-1$ and $2i+1$ respectively. Thus $a_{i+1}(n)-a_{i+1}(n-2)$ is a polynomial of degree $2i+2$. This means that a_{i+1} is a polynomial of degree $2i+3$.
2) Let n be odd. By (6.3), for integer x and $m=\frac{n-1}{2}$, we have

$$(2x+n-2)(a_0(n)x^m+a_1(n)x^{m-1}+...)=$$
$$2(x+n-1)(x+n-2)(a_0(n-2)x^{m-1}+a_1(n-2)x^{m-2}+...)+$$
$$(\frac{n-1}{2})!(4x+3n-4)\binom{x+\frac{n-3}{2}}{\frac{n-1}{2}}. \tag{9.8}$$

Hence, comparing the coefficient of x^{m+1} in both hand sides, we find

$$a_0(n)=a_0(n-2)+2,\ \ n\geq 3,\ \ a_0(1)=1.$$

Thus $a_0(3)=3, a_0(5)=5, ..., a_0(n)=n$.
Furthermore, comparing the coefficient of x^m in both hand sides in (9.8), using the same arguments as in 1), we have

$$a_1(n)=a_1(n-2)+\frac{7n^2-22n+19}{4},\ \ n\geq 3,\ \ a_1(1)=0.$$

Since $a_1(n)=\sum_{i=3,5,\ldots,n}(a_1(i)-a_1(i-2))$, then we find

$$a_1(n)=\frac{1}{4}\sum_{i=3,5,\ldots,n}(7i^2-22i+19)=\frac{1}{24}(7n^3-12n^2+5n).$$

Finally, comparing the coefficient of x^{m-i} in both hand sides of (9.8), we find

$$2(a_{i+1}(n)-a_{i+1}(n-2))=$$
$$2(2n-3)a_i(n-2)-(n-2)a_i(n)+$$
$$2(n-1)(n-2)a_{i-1}(n-2)+s_i(n), \tag{9.9}$$

where

$$s_i(n)=Coef[x^{m-i}]((4x+3n-4)(x+\frac{n-3}{2})(x+\frac{n-5}{2})\cdot...\cdot(x+1)x)$$

and, as in 1), the statement is proved by induction over i.■

A few such polynomials are the following:
For odd n :

$$U_0(n) = n,$$
$$U_1(n) = \frac{1}{24}(n-1)n(7n-5),$$
$$U_2(n) = \frac{1}{640}(n-3)(n-1)n(29n^2-44n+7),$$
$$U_3(n) = \frac{1}{322560}(n-5)(n-3)(n-1)n(1581n^3-3775n^2+1587n+223);$$

For even n :

$$V_0(n) = \frac{1}{2}n,$$
$$V_1(n) = \frac{1}{48}(n-2)n(7n-4),$$
$$V_2(n) = \frac{1}{3840}(n-4)(n-2)n(87n^2-98n+16),$$
$$V_3(n) = \frac{1}{645120}(n-6)(n-4)(n-2)n(1581n^3-2686n^2+936n+64).$$

Proposition 2.

$$a_i(n) \equiv \begin{cases} r_i(n), & if \ \ n \ \ is \ \ even, \\ \\ s_i(n), & if \ \ n \ \ is \ \ odd. \end{cases} \pmod 2 \tag{9.10}$$

Proof. The proposition follows from (9.7), (9.9) and Theorem 2. ■

Finally, note that, from (8.7)-(8.8) follow the following homogeneous recursions for the coefficients of $P_n(x)$.

Theorem 7. *For odd* $n \geq 3$ *and* $i \geq 0$,

$$(m-i)a_i(n) = na_i(n-1) + \sum_{j=0}^{i-1}(-1)^{i-j+1}\binom{m-j}{m-i-1}a_j(n). \tag{9.11}$$

For even $n \geq 4$ *and* $i \geq 0$,

$$(n-2i-1)a_i(n) = \frac{n}{2}a_i(n-1)+$$

$$2\sum_{j=0}^{i-1}(-1)^{i-j+1}(m\binom{m-j}{m-i} - \binom{m-j}{m-i-1})a_j(n). \tag{9.12}$$

10. Arithmetic proof of the integrality $P_n(x)$ in integer points

From Theorem 2 we conclude that the polynomial $P_n(x)$ takes integers values for integer $x = k$. Here we give an independent arithmetic proof of this fact, using the explicit expression (1.6). It is well known (cf. [5], Section 8, Problem 87) that, if a polynomial $P(x)$ of degree m takes integer values for $x = 0, 1, ..., m$, then it takes integer values for every integer x. Since, as we proved at the end of Section 6, deg $P_n(k) = \lfloor\frac{n-1}{2}\rfloor$, then we suppose that $0 \le k \le \lfloor\frac{n-1}{2}\rfloor$. Moreover, from the results of Section 3, $P_n(0)$ and $P_n(1)$ are integers (in the case when $n+1$ is an odd prime, $P_n(1) = (2^n-1)(\frac{n}{2})!/(n+1)$ is integer, since $2^n-1 \equiv 0 \pmod{n+1}$, while in the case when $n+1$ is an odd composite number, no divisor exceeds $\frac{n+1}{3}$, therefore, $(\frac{n}{2})! \equiv 0 \pmod{n+1}$). Thus we can suppose that

$$2 \le k \le \lfloor\frac{n-1}{2}\rfloor. \tag{10.1}$$

Suppose that n is *even* (the case of odd n is considered quite analogously). Let p be a prime. Denote the maximal power of p dividing n by $[n]_p$. We say that, for integer $l,\ h$, the fraction $\frac{l}{h}$ is *p-integer,* if $[l]_p - [h]_p \ge 0$.

A) Firstly, we show that, for $n \ge 4$, $P_n(k)$ is 2-integer. Indeed, $2k+n-1$ is odd, while $4k+3n-4$ is even. Therefore, by (6.4), using a trivial induction, we see that $P_n(k)$ is 2-integer.

Further we use the explicit formula (1.6) of Theorem 1.

B) Let p be an odd prime divisor of $\binom{n+2k-2}{k-1}$ which does not coincide with any factor of the product $(n+2k-1)(n+2k-2)...(n+k)$. Thus p could divide one or several *composite* factors of this product. Therefore, the following condition holds

$$3 \le p \le \frac{n+2k-1}{3}. \tag{10.2}$$

Let us show that

$$a(n;k) := \frac{\binom{\frac{n}{2}+k-1}{k}}{\binom{n+2k-1}{k}}(\frac{n-2}{2})! =$$

$$2^{-k}\frac{(n+2k-2)(n+2k-4)\cdot ... \cdot n}{(n+2k-1)(n+2k-2)...(n+k)}(\frac{n-2}{2})! \tag{10.3}$$

is p-integer and, consequently, $P_n(k)$ is p-integer.

Let $k \ge 3$ be even. Then, after a simplification, we have

$$2^k a(n;k) = \frac{(n+k-2)(n+k-4)\cdot ... \cdot n}{(n+2k-1)(n+2k-3)\cdot ... \cdot (n+k+1)}(\frac{n-2}{2})!,$$

or

$$2^{\frac{k}{2}} a(n;k) = \frac{(\frac{n+k-2}{2})!}{(n+2k-1)(n+2k-3)...(n+k+1)} \tag{10.4}$$

We distinguish several cases.

Case a) For $t \geq 2$, let p^t divide at least one factor of the denominator. Then $p \leq (n+2k-1)^{\frac{1}{t}}$ Let us show that $p \leq \frac{n+k-2}{2t}$. We should show that $n+2k-1 \leq (\frac{n+k-2}{2t})^t$, or, since, by (10.1), $k \leq \frac{n-2}{2}$, it is sufficient to show that $\frac{3}{2}(n+k-2) \leq (\frac{n+k-2}{2t})^t$, or $(2t)^{\frac{t}{t-1}} \leq (\frac{2}{3})^{\frac{1}{t-1}}(n+k-2)$. Since $(\frac{2}{3})^{\frac{1}{t-1}} \geq \frac{2}{3}$, it is sufficient to prove that $(2t)^{\frac{t}{t-1}} \leq \frac{2}{3}(n+k-2)$. Note that $e^t < p^t \leq n+2k-2,\ t \leq \ln(n+2k-2)$. Therefore we find $(2t)^{\frac{t}{t-1}} \leq (2\ln(n+2k-2))^2$. Furthermore, note that, if $n \geq 152$, then $\ln^2 n < \frac{n}{6}$. Thus $(2t)^{\frac{t}{t-1}} \leq \frac{2}{3}(n+k-2)$. It is left to add that up to $n = 161$ we verified that the polynomials $P_n(k)$ have integer coefficients and, consequently, is integer-valued.

Case b) Let p divide only one factor of the denominator. Then, in view of (10.1) and (10.2), $p \leq \frac{n+2k-1}{3} \leq \frac{n+k-2}{2}$ and, by (10.4), $a(n;k)$ is p-integer.

Case c) Let p divide exactly l factors of the denominator. Then $p \leq \frac{(n+2k-1)-(n+k+1)}{l} = \frac{k-2}{l}$, and, since, by (10.1), $n \geq 2k+2$, we conclude that $\frac{n+k-2}{2} \geq \frac{3k}{2} \geq k-2 \geq lp$. Hence, by (10.4), $a(n;k)$ is p-integer.

It is left to notice that the case of odd k is considered quite analogously.

C) Suppose that, as in B), $k \geq 2$ is even. Let p be an odd prime divisor of $\binom{n+2k-1}{k}$ which coincides with some factor of the product $(n+2k-1)(n+2k-3)...(n+k+1)$. In this case the fraction (10.4) is not integer. Thus in order to prove that $P_n(k)$ is p-integer, we should prove that $T_n(k)$ (5.1) is p-integer. By the condition, p has form

$$p = n+2k-1-2r,\ \ 0 \leq r \leq \frac{k-2}{2}. \tag{10.5}$$

According to (5.1) and (10.5), we should prove that

$$\sum_{j=0}^{n-1} 2^j \binom{n+2k-j-2}{k-1} =$$

$$\sum_{j=0}^{n-1} 2^j \binom{p+2r-1-j}{k-1} \equiv 0 \pmod{p}, \tag{10.6}$$

or

$$A(n,r,k) :=$$

$$\sum_{j=0}^{n-1} 2^j (j-(2r-1))(j+1-(2r-1))...(j+k-2-(2r-1)) \equiv 0 \pmod{p}.$$

Note that, since $n - 2r = p - 2k + 1$, then we have

$$\sum_{j=0}^{n-1} x^{j+k-2-(2r-1)} =$$

$$(x^{n+k-2r-1} - x^{k-2r-1})(x-1)^{-1} = (x^{p-k} - x^{k-2r-1})(x-1)^{-1}.$$

Therefore,

$$A(n,r,k)=2^{2r}\sum_{j=0}^{n-1}(x^{j+k-2-(2r-1)})^{(k-1)}\mid_{x=2}=$$
$$2^{2r}((x^{p-k}-x^{k-2r-1})(x-1)^{-1})^{(k-1)}\mid_{x=2}.$$

Thus we should prove that

$$((x^{p-k}-x^{k-2r-1})(x-1)^{-1})^{(k-1)}\mid_{x=2}\equiv 0 \pmod{p}, \tag{10.7}$$

or, using the Leibnitz formula,

$$\sum_{j=0}^{k-1}(-1)^{k-j-1}\binom{k-1}{j}(k-j-1)!(p-k)(p-k-1)...(p-k-j+1)2^{p-k-j}\equiv$$
$$\sum_{j=0}^{k-1}(-1)^{k-j-1}\binom{k-1}{j}(k-j-1)!(k-2r-1)(k-2r-2)...(k-2r-j)2^{k-2r-j-1} \pmod{p}.$$

Since $2^{p-1}\equiv 1 \pmod{p}$, then we should prove the identity

$$\sum_{j=0}^{k-1}(-1)^{k-j-1}\binom{k-1}{k-j-1}(k-j-1)!(p-k)(p-k-1)...(p-k-j+1)\mid_{p=0}2^{-k-j+1}=$$
$$\sum_{j=0}^{k-1}(-1)^{k-j-1}\binom{k-1}{k-j-1}(k-j-1)!(k-2r-1)(k-2r-2)...(k-2r-j)2^{k-2r-j-1},$$

or, after simple transformations, the identity

$$\sum_{j=0}^{k-1}\binom{k+j-1}{j}2^{-j}=2^{2k-2r-2}\sum_{j=0}^{k-1}(-1)^j\binom{k-2r-1}{j}2^{-j}. \tag{10.8}$$

It is known (([7], Ch.1, problem 7), that

$$\sum_{i=0}^{n}\binom{2n-i}{n}2^{i-n}=2^n.$$

Putting $n-i=j$, we have

$$\sum_{j=0}^{n}\binom{n+j}{n}2^{-j}=\sum_{j=0}^{n}\binom{n+j}{j}2^{-j}=2^n.$$

Therefore, the left hand side in (10.8) is 2^{k-1} and it is left to prove that

$$\sum_{j=0}^{k-1}(-1)^j\binom{k-2r-1}{j}2^{k-j}=2^{2r+1}.$$

We have

$$\sum_{j=0}^{k-1}(-1)^j\binom{k-2r-1}{j}2^{k-j}=\sum_{j=0}^{k-2r-1}(-1)^j\binom{k-2r-1}{j}2^{k-j}=$$
$$2^{2r+1}\sum_{j=0}^{k-2r-1}(-1)^j\binom{k-2r-1}{j}2^{k-2r-1-j}=2^{2r+1}(2-1)^{k-2r-1}=2^{2r+1}$$

and we are done. The case of odd $k\geq 3$ is considered quite analogously. So, formulas (10.4)-(10.5) take the form

$$2^{\frac{k-1}{2}}a(n;k)=\frac{(\frac{n+k-1}{2})!}{(n+2k-1)(n+2k-3)\cdot\ldots\cdot(n+k)},$$

$$p=n+2k-2r-1,\ \ 0\leq r\leq\frac{k-1}{2}, \tag{10.9}$$

and, for odd k, the proof reduces to the same congruence (10.6). ∎

11. Representation of $P_n(x)$ in basis $\{\binom{x}{i}\}$

The structure of explicit formula (1.6) allows us to conjecture that the coefficients of $P_n(x)$ in basis $\{\binom{x}{i}\}$ possess simpler properties. A process of expansion of a polynomial $P(x)$ in the binomial basis is indicated in [5] in a solution of Problem 85: "Functions $1,\ \ x,\ \ x^2,...,x^n$ one can consecutively express in the form of linear combinations with the constant coefficients of $1,\ \frac{x}{1},\ \ \frac{x(x-1)}{2},...,\ \frac{x(x-1)...(x-n+1)}{n!}$." Therefore,

$$P(x)=b_0\binom{x}{m}+b_1\binom{x}{m-1}+...+b_{m-1}\binom{x}{1}+b_m,$$

where $b_0,\ \ b_1,...,b_m$ are defined from the equations

$$P(0)=b_m,$$

$$P(1)=b_m+\binom{1}{1}b_{m-1},$$

$$P(2)=b_m+\binom{2}{1}b_{m-1}+\binom{2}{2}b_{m-2},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$P(m)=b_m+\binom{m}{1}b_{m-1}+...+\binom{m}{m}b_0. \tag{11.1}$$

This process one can simplify in the following way. In the identity

$$n^x=(1+(n-1))^x=$$

$$1+(n-1)\binom{x}{1}+(n-1)^2\binom{x}{2}+...+(n-1)^x\binom{x}{x}=$$

$$n^0+(n-n^0)\binom{x}{1}+(n-n^0)^2\binom{x}{2}+...+(n-n^0)^x\binom{x}{x}$$

we can evidently replace powers $n^j,\ \ j=0,...,x$, by the arbitrary numbers $a_j,\ \ j=0,...,x$. Thus we have a general identity

$$a_x=a_0+(a_1-a_0)\binom{x}{1}+$$

$$(a_2-2a_1+a_0)\binom{x}{2}+(a_3-3a_2+3a_1-a_0)\binom{x}{3}+...+$$

$$(a_x - \binom{x}{1} a_{x-1} + \binom{x}{2} a_{x-2} - ... + (-1)^x \binom{x}{x} a_0) \binom{x}{x}. \tag{11.2}$$

Essentially, we quickly obtained a special case of the so-called "Newton's forward difference formula" (cf. [10]). Here, put $a_j = P(j),\ \ j = 0, ..., m,$ and, firstly, consider values $0 \le x \le m$. Since $\binom{x}{l} = 0$ for $l > m$, then we obtain the required representation under the condition $0 \le x \le m:$

$$P(x) = P(0) + (P(1) - P(0)) \binom{x}{1} +$$

$$(P(2) - 2P(1) + P(0)) \binom{x}{2} + ... + (P(m) - \binom{m}{1} P(m-1) +$$

$$\binom{m}{2} P(m-2) - ... + (-1)^m \binom{m}{m} P(0)) \binom{x}{m}. \tag{11.3}$$

It is left to note that, since a polynomial of degree m is fully defined by its values in $m+1$ points $0, 1, ..., m$, then (11.3) is the required representation for all x.

So, for the considered polynomials $\{P_n(x)\}$, we have

$$P_1 = 1,$$

$$P_2 = 1,$$

$$P_3 = 3\binom{x}{1} + 4,$$

$$P_4 = 2\binom{x}{1} + 4,$$

$$P_5 = 10\binom{x}{2} + 30\binom{x}{1} + 32,$$

$$P_6 = 6\binom{x}{2} + 22\binom{x}{1} + 32,$$

$$P_7 = 42\binom{x}{3} + 196\binom{x}{2} + 378\binom{x}{1} + 384,$$

$$P_8 = 24\binom{x}{3} + 128\binom{x}{2} + 296\binom{x}{1} + 384,$$

$$P_9 = 216\binom{x}{4} + 1368\binom{x}{3} + 3816\binom{x}{2} + 6120\binom{x}{1} + 6144,$$

$$P_{10} = 120\binom{x}{4} + 840\binom{x}{3} + 2664\binom{x}{2} + 5016\binom{x}{1} + 6144,$$

$$P_{11} = 1320\binom{x}{5} + 10560\binom{x}{4} + 38544\binom{x}{3} + 84480\binom{x}{2} + 122760\binom{x}{1} + 122880,$$

$$P_{12} = 760\binom{x}{5} + 6240\binom{x}{4} + 25152\binom{x}{3} + 62112\binom{x}{2} + 103920\binom{x}{1} + 122880.$$

12. On coefficients of $P_n(x)$ in basis $\{\binom{x}{i}\}$

Let

$$(12.1)\quad P_n(x) = b_0(n)\binom{x}{m} + b_1(n)\binom{x}{m-1} + ... + b_{m-1}(n)\binom{x}{1} + b_m(n),$$

where $m = \lfloor \frac{n-1}{2} \rfloor$.

Since, for integer k, we have the explicit formula for $P_n(k)$ (1.6), then, according to (11.3), we have the following explicit formula for $b_i(n),\ i = 0, ..., m$:

$$(12.2)\qquad b_i(n) = \sum_{k=0}^{m-i} (-1)^{m-i-k} \binom{m-i}{k} P_n(k).$$

Let

$$P_n(x) = \sum_{j=0}^{m} a_j(n) x^{m-j}.$$

Then

$$(12.3)\qquad b_i(n) = \sum_{j=0}^{m} a_j(n) \sum_{k=0}^{m-i} (-1)^{m-i-k} k^{m-j} \binom{m-i}{k}.$$

Since the l-th difference of $f(x)$ is (cf. [1], formula 25.1.1)

$$\Delta^l f(x) = \sum_{k=0}^{l} (-1)^{l-k} \binom{l}{k} f(x+k),$$

then one can write (12.3)in the form

$$b_i(n) = \sum_{j=0}^{m} a_j(n) \Delta^{m-i} x^{m-j} \mid_{x=0} .$$

Here the summands corresponding to $j > i$, evidently, equal 0. Therefore, we have

$$(12.4)\qquad b_i(n) = \sum_{j=0}^{i} a_j(n) \Delta^{m-i} x^{m-j} \mid_{x=0} .$$

Theorem 8. *For $n \geq 1$, we have*

$$(12.5)\qquad b_0(n) = \begin{cases} n(\frac{n-1}{2})!, & if\ \ n\ \ is\ \ odd, \\ (\frac{n}{2})!, & if\ \ n\ \ is\ \ even; \end{cases}$$

$$(12.6)\qquad b_1(n) = \begin{cases} \frac{1}{6}n(5n-7)(\frac{n-1}{2})!, & if\ \ n\ \ is\ \ odd, \\ \frac{1}{6}(5n-8)(\frac{n}{2})!, & if\ \ n\ \ is\ \ even. \end{cases}$$

In general, for a fixed i, $b_i(n) = (m-i)!Y_i(n)$, if n is odd, and $b_i(n) = (m-i)!Z_i(n)$, if n is even, where $Y_i,\ Z_i$ are polynomials in n of degree $2i+1$.

Proof. Note that the Stirling number of the second kind $S(n,m)$ is connected with the m-th difference of $\Delta^m x^n \mid_{x=0}$ in the following way (see [1], formulas 24.1.4)

$$S(n,m)m! = \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}k^n = \Delta^m x^n \mid_{x=0} . \tag{12.7}$$

In particular, since $S(m,m)=1$, $S(m+1,m)=\binom{m+1}{2}$, then

$$\Delta^m x^m \mid_{x=0} = m!$$

and

$$\Delta^m x^{m+1} \mid_{x=0} = \frac{m}{2}(m+1)! \tag{12.8}$$

Therefore, by (12.4),

$$b_0(n) = m!a_0(n),$$

$$b_1(n) = \frac{m-1}{2}m!a_0(n) + (m-1)!a_1(n),$$

and, by (9.2)-(9.3) (where $m = \lfloor\frac{n-1}{2}\rfloor$), we find formulas (12.5)-(12.6).■ Further we need lemma.

Lemma 6. *$S(n+k,n)$ is a polynomial in n of degree $2k$.*

Proof. For $k \geq 1$, denote

$$Q_k(n) = S(n+k,n). \tag{12.9}$$

Note that, since $S(n,n)=1$, then $Q_0(n)=1$. Further, since $S(n,0)=\delta_{n,0}$, then, for $k \geq 1$, $Q_k(0)=0$. From the main recursion for $S(n,m)$ which is $S(n,m) = mS(n-1,m) + S(n-1,m-1)$, we have

$$Q_k(n) - Q_k(n-1) = nQ_{k-1}(n). \tag{12.10}$$

and, in view of $Q_k(0)=0$, we find the recursion

$$Q_0(n) = 1,\ \ Q_k(n) = \sum_{i=1}^{n} iQ_{k-1}(i). \tag{12.11}$$

Using a simple induction, from (12.11) we obtain the lemma. ■

Remark 1. *The list of polynomials $\{Q_k(n)\}$*

$$Q_0 = 1,$$
$$Q_1 = \frac{1}{2}n(n+1),$$
$$Q_2 = \frac{1}{24}n(n+1)(n+2)(3n+1),$$
$$Q_3 = \frac{1}{48}n^2(n+1)^2(n+2)(n+3),$$

$$Q_4 = \frac{1}{5760} n(n+1)(n+2)(n+3)(n+4)(15n^3+30n^2+5n-2), etc.$$

It could be proven that the sequence of denominators coincides with A053657 [8], such that the denominator of $Q_k(n)$ *is* $\prod p^{\sum_{j\geq 0}\lfloor \frac{k}{(p-1)p^j}\rfloor}$, *where the product is over all primes.*

Note that from (12.4) and (12.7) we find

$$(12.12) \qquad b_i(n) = (m-i)!\sum_{j=0}^{i} a_j(n) S(m-j, m-i), \ \ m = \lfloor \frac{n-1}{2} \rfloor.$$

Since, by Lemma 5, $S(m-j, m-i)$ is a polynomial in n of degree $2((m-j)-(m-i)) = 2(i-j)$, while, by Theorem 6, $a_j(n)$ is a polynomial of degree $2j+1$, then $a_j(n)S(m-j, m-i)$ is a polynomial of degree $2i+1$. Thus $\sum_{j=0}^{i} a_j(n)S(m-j, m-i)$ is a polynomial of degree $2i+1$. This completes the proof. ■

The first polynomials $Y_i(n)$, $Z_i(n)$ are

$$Y_0 = n,$$
$$Y_1 = \frac{1}{12}(n-1)n(5n-7),$$
$$Y_2 = \frac{1}{480}(n-3)(n-1)n(43n^2-168n+149),$$
$$Y_3 = \frac{1}{13440}(n-5)(n-3)(n-1)n(177n^3-1319n^2+3063n-2161);$$
$$Z_0 = \frac{n}{2},$$
$$Z_1 = \frac{1}{24}(n-2)n(5n-8),$$
$$Z_2 = \frac{1}{960}(n-4)(n-2)n(43n^2-182n+184),$$
$$Z_3 = \frac{1}{26880}(n-6)(n-4)(n-2)n(3n-8)(59n^2-306n+352)).$$

Finally, we prove the following attractive result.

Theorem 9. 1) *For odd* n, $b_j(n)/n, j = 0, ..., m-1$, *are integer. Moreover, for* $n \geq 3$,

$$(12.13) \qquad b_i(n) = n(b_i(n-1) + b_{i-1}(n-1)), \ \ i = 1, ..., m-1.$$

2) *For even* $n \geq 4$,

$$(12.14) \qquad 2b_i(n) = b_i(n-1) + b_{i-1}(n-1) + m!\binom{m}{i}, \ \ i = 1, ..., m-1.$$

Proof. 1) According to (12.2), we should prove that for odd $n \geq 3$,

$$\sum_{k=0}^{m-i}(-1)^{m-i-k}\binom{m-i}{k}P_n(k) =$$

$$n(\sum_{k=0}^{m-i}(-1)^{m-i-k}\binom{m-i}{k}P_{n-1}(k)+$$

$$\sum_{k=0}^{m-i-1}(-1)^{m-i-k-1}\binom{m-i-1}{k}P_{n-1}(k)),\ \ i=1,2,...,m-1,$$

or, putting $m-i=t$,

$$\sum_{k=0}^{t}(-1)^k\binom{t}{k}P_n(k) = n(\sum_{k=0}^{t}(-1)^k\binom{t}{k}P_{n-1}(k)-$$

$$\sum_{k=0}^{t}(-1)^k\binom{t-1}{k}P_{n-1}(k)),\ \ t=1,2,...,m-1,$$

or, finally, for $t=1,...,\frac{n-3}{2}$,

$$\sum_{k=1}^{t}(-1)^{k-1}(\binom{t}{k}P_n(k)-\binom{t-1}{k-1}nP_{n-1}(k)) = P_n(0). \tag{12.15}$$

To prove (12.15), note that , by (8.7), $nP_{n-1}(k) = P_n(k)-P_n(k-1)$. Hence,

$$\binom{t}{k}P_n(k)-\binom{t-1}{k-1}nP_{n-1}(k) =$$

$$P_n(k)(\binom{t}{k}-\binom{t-1}{k-1})+\binom{t-1}{k-1}P_n(k-1) =$$

$$\binom{t-1}{k}P_n(k)+\binom{t-1}{k-1}P_n(k-1).$$

Thus the summands of (12.15) are

$$(-1)^{k-1}(\binom{t}{k}P_n(k)-\binom{t-1}{k-1}nP_{n-1}(k)) =$$

$$(-1)^{k-1}\binom{t-1}{k}P_n(k)-(-1)^{k-2}\binom{t-1}{k-1}P_n(k-1),$$

and the summing gives

$$\sum_{k=1}^{t}(-1)^{k-1}(\binom{t}{k}P_n(k)-\binom{t-1}{k-1}nP_{n-1}(k)) =$$

$$(-1)^{k-1}\binom{t-1}{k}P_n(k)\mid_{k=t}-(-1)^{k-2}\binom{t-1}{k-1}P_n(k-1)\mid_{k=1}= P_n(0).$$

2) Analogously, the proof of (12.14) reduces to proof of the following equality for $t=1,2,...,m-1:$

$$\sum_{k=0}^{t+1}(-1)^k(2\binom{t}{k}P_n(k)+\binom{t}{k-1}P_{n-1}(k)) = (-1)^t m!\binom{m}{t}. \tag{12.16}$$

Note that, by (8.11),

$$(-1)^k(2\binom{t}{k}P_n(k)+\binom{t}{k-1}P_{n-1}(k))=$$

$$(-1)^k\binom{t}{k}P_{n-1}(k+1)-$$

$$(-1)^{k-1}\binom{t}{k-1}P_{n-1}(k)+(-1)^k\binom{t}{k}\binom{k+m}{m}m! \tag{12.17}$$

Since

$$\sum_{k=0}^{t+1}((-1)^k\binom{t}{k}P_{n-1}(k+1)-(-1)^{k-1}\binom{t}{k-1}P_{n-1}(k))=$$

$$(-1)^k\binom{t}{k}P_{n-1}(k)\mid_{k=t+1}-(-1)^{k-1}\binom{t}{k-1}P_{n-1}(k)\mid_{k=0}=0,$$

then, by (12.16)-(12), the proof reduces to the known combinatorial identity

$$\sum_{k=0}^{t}(-1)^k\binom{t}{k}\binom{k+m}{m}=(-1)^t\binom{m}{t},\ \ t=1,...,m-1$$

(see [7], Ch.1, formula (8) with $p=0$ up to the notations). ■

In conclusion, note that we have an interesting voyage from the, until now, unproved conjecture 7), when the primes in 7) are replaced by the odd numbers in Theorem 9.

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