Linear extensions and order-preserving poset partitions

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Abstract

We examine the lattice of all order congruences of a finite poset from the viewpoint of combinatorial algebraic topology. We will prove that the order complex of the lattice of all nontrivial order congruences (or order-preserving partitions) of a finite n-element poset P with $n \geq 3$ is homotopy equivalent to a wedge of spheres of dimension $n-3$. If P is connected, then the number of spheres is equal to the number of linear extensions of P. In general, the number of spheres is equal to the number of cyclic extensions of P.

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1. Introduction

An order congruence of a poset P can be defined as a kernel of an order-preserving map with domain P. Even if this notion is simple and natural, the amount of papers dealing with it appears to be relatively small. The notion appears in the seventies in a series of papers by T. Sturm [8, 9, 10], the same notion with a different formulation appears in the W.T. Trotter's book [11]. A related notion in the area of ordered algebras appeared in two papers by G. Czédli a A. Lenkehegyi [2, 1]. In our approach, we will follow a recent paper by P. Körtesi, S. Radeleczki and S. Szilágyi [7].

In the present paper we will examine the lattice of all order congruences of a finite poset from the viewpoint of combinatorial algebraic topology. We will prove that the order complex of the lattice of all nontrivial order congruences (or order-preserving partitions) of a finite n-element poset P with $n \geq 3$ is homotopy equivalent to a wedge of spheres of dimension $n-3$. If P is connected, then the number of spheres is equal to the number of linear extensions of P. In general,

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the number of spheres is equal to the number of cyclic extensions (Definition 3) of P.

2. Preliminaries

2.1. Simplicial complexes, homotopy

An *n-dimensional simplex* ($n \geq -1$) is a convex closure of $n+1$ affinely independent points (called *vertices*) in a finite dimensional real space.

A *simplicial complex* is a finite set $\mathcal{K}$ of simplices such that

- any face of a simplex belonging to $\mathcal{K}$ belongs to $\mathcal{K}$,
- the intersection of any two simplices belonging to $\mathcal{K}$ is again a simplex belonging to $\mathcal{K}$.

An *abstract simplicial complex* is a finite set A together with a finite collection Δ of subsets of A such that if $X \in \Delta$ and $Y \subseteq X$, then $Y \in \Delta$. The elements of Δ are called (abstract) simplices. The union of all simplices belonging to Δ is called the *vertex set* of Δ, denoted by $V(\Delta)$.

Let $\mathcal{K}$ be a simplicial complex. Let Δ be the system of all vertex sets of all simplices that belong to K. Then Δ is an abstract simplicial complex, called *the vertex skeleton of* $\mathcal{K}$. Symmetrically, we call $\mathcal{K}$ the *geometric realization* of Δ. Any abstract simplicial complex has a geometric realization.

Let X, Y be topological spaces. We say that two continuous maps $f, g\colon X \to Y$ are *homotopic* if there exists a continuous map $F\colon X \times [0,1] \to Y$ such that $F(-,0) = f$ and $F(-,1) = g$. In that case, we write $f \simeq g$. We say that two topological spaces X and Y have the same *homotopy type* (or that they are *homotopy equivalent*) if there exist continuous maps $\phi\colon X \to Y$ and $\psi\colon Y \to X$ such that $\phi \circ \psi \simeq id_X$ and $\psi \circ \phi \simeq id_Y$.

Since any two geometric realizations of an abstract simplicial complex are homotopy equivalent, we may (and we will) extend the notion of homotopy equivalence to abstract simplicial complexes.

A *wedge of k spheres of dimension d* is a topological space constructed in the following way.

- Take k copies of d-dimensional spheres $\mathbb{S}^d$.
- On each of the spheres pick a point.
- Identify the points.

As remarked by Forman [4], wedges of spheres arise frequently in combinatorial applications of algebraic topology.

2.2. Poset terminology

A binary relation ρ on a set P is a *quasiorder* if ρ is reflexive and transitive. A transitive quasiorder is a *partial order*. A pair $(P, \leq)$, where $\leq$ is a partial order on a set P is called a *poset*.

Let P, Q be posets. A mapping $f\colon P \to Q$ is *order-preserving* if, for all $x, y \in P$, $x \leq y$ implies $f(x) \leq f(y)$. A mapping $f\colon P \to Q$ is *order-inverting* if, for all $x, y \in P$, $x \leq y$ implies $f(x) \geq f(y)$.

If P, Q are posets and $f\colon P \to Q$ is an order-preserving map, then the *kernel of f* is the equivalence relation $\sim_f$ on P given by

$$x \sim_f y :\Leftrightarrow f(x) = f(y).$$

In a poset, we say that two elements x, y are *comparable* if and only if $x \leq y$ or $y \leq x$; otherwise we say they are incomparable. The incomparability relation is denoted by $\|$. An *antichain* is a poset in which every pair of elements is incomparable. A *chain* is a poset in which every pair of elements is comparable. For a poset P, *a chain of P* is a subset of P that is a chain when equipped with the partial order inherited from P.

For elements x, y of a poset, we say that x covers y if $x \geq y$, $x \neq y$ and for every element z such that $x \geq z \geq y$ we have either $z = x$ or $z = y$. The covering relation is denoted by $\succ$, $\prec$ denotes the inverse of $\succ$.

We say that a subset A of a poset P is *lower bounded* if there is an element $a \in P$ such that, for all $x \in A$, $a \leq x$. The element a is called a *lower bound of A*. A lower bound of a A that belongs to A is called the *smallest element of A*. It is easy to check that every subset of a poset has at most one smallest element. A lower bound of P (it is necessarily the smallest element of P) is called the *bottom element of P* and is denoted by $\hat{0}$.

The dual notions are *upper bounded*, *the greatest element*, and *the top element of P*, respectively. The top element of a poset is denoted by $\hat{1}$.

An element of a poset P that covers $\hat{0}$ is an *atom of P*.

A subset of a poset that is both upper and lower bounded is called *bounded*.

We say that a poset L is a *lattice* if for every set $A = \{a_1, a_2\} \subseteq L$ the set of all upper bounds of A has the smallest element, denoted by $a_1 \vee a_2$,and the set of all lower bounds of A has the greatest element, denoted by $a_1 \wedge a_2$. Note that a finite lattice is always bounded.

A chain of a poset P is *maximal* if it cannot be extended to a bigger chain. A finite bounded poset P is *ranked* if and only if any two maximal chains of P have the same number of elements; this number minus one is then called the *height of P*. It is easy to check that a finite poset P is ranked if and only if it there is a (necessarily unique) order-preserving mapping $r\colon P \to \mathbb{N}$ such that $r(\hat{0}) = 0$ and $x \succ y$ implies $r(x) = r(y) + 1$. The mapping r is then called *the rank function of P*.

A finite lattice L is *semimodular* if L is ranked and its rank function r satisfies

$$r(x) + r(y) \geq r(x \wedge y) + r(x \vee y).$$

Let P be a finite poset. The graph with the vertex set P and the edge set given by the comparability relation is called *comparability graph* of P. The connected components of the comparability graph are called *connected components of* P. A poset with a single connected component is called *connected*.

Let P be a finite poset with n elements. A *linear extension* of P is an order-preserving bijection $f\colon P \to \{0, \dots, n-1\}$, where the codomain is ordered in the usual way. For our purposes, this definition is more appropriate than the standard one. The set of all linear extensions is denoted by $\ell(P)$. The number of linear extensions of P is denoted by $e(P)$.

For a finite poset $(P, \leq)$, we write $\Delta(P)$ for the abstract simplicial complex consisting of all chains of P, including the empty set. If a finite poset P has an upper or lower bound, then $\Delta(P)$ is topologically trivial, that means, it is homotopy equivalent to a point. Thus, when dealing with posets from the point of view of algebraic topology, it is usual (and useful) to remove bounds from a poset before applying Δ. If P is a poset, then $\hat{P}$ denotes the same poset minus upper or lower bounds, if it has any.

The *face poset* of a finite abstract simplicial complex Δ is the poset of all faces of Δ, ordered by inclusion. It is denoted by $\mathcal{F}(\Delta)$.

2.3. Acyclic matchings

Definition 1. Let P be a finite poset. An *acyclic matching* on P is a set $M \subseteq P \times P$ such that the following conditions are satisfied.

1. For all $(a, b) \in M$, $a \succ b$.
2. Each $a \in P$ occurs in at most one element in M; if $(a, b) \in M$ we write $a = u(b)$ and $b = d(a)$.
3. There does not exist a cycle

$$b_1 \succ d(b_1) \prec b_2 \succ d(b_2) \prec \cdots \prec b_n \succ d(b_n) \prec b_1.$$

When constructing acyclic matchings for posets, the following theorem is sometimes used to make the induction step.

Theorem 1. *([5], Theorem 11.10) Let P be a finite poset. Let $\varphi\colon P \to Q$ be an order-preserving or an order-inverting mapping and assume that we have acyclic matchings on subposets $\varphi^{-1}(q)$, for all $q \in Q$. Then the union of these acyclic matchings is itself an acyclic matching on P.*

In the context of Theorem 1, the sets $\varphi^{-1}(q)$ are called the *fibers of* φ.

In general, we cannot infer the homotopy type of a simplicial complex from the existence of an acyclic matching on the face poset of a simplicial complex. However, if the simplicial complex has a homotopy type of a wedge of spheres of constant dimension, we can use the following theorem.

Theorem 2. *([4], Theorem 6.3) Let Δ be a finite simplicial complex. Let M be an acyclic matching of the face poset of Δ such that all faces of Δ are matched by M except for n unmatched faces of dimension d. Then Δ has the homotopy type of the wedge of n spheres of dimension d.*

We remark that our wording of Theorem 2 is slightly different than the original one, since we allow the empty face of Δ to be matched.

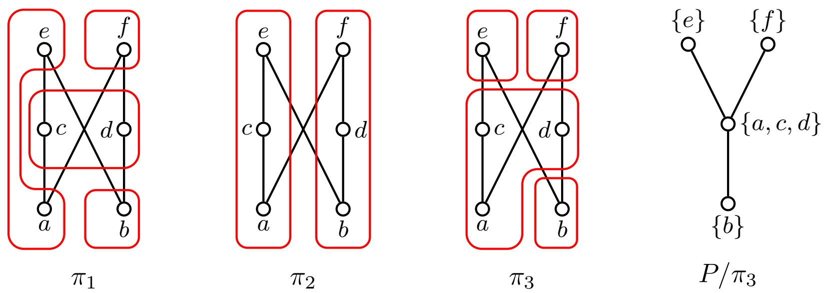


Figure 1: π_3 is order-preserving, π_1, π_2 are not.

3. Order-preserving partitions

Definition 2. [7] Let $(P, \leq)$ be a poset and let $\rho \subseteq P^2$ be an equivalence relation on it.

(i) A sequence $x_0, \ldots, x_n \in P$ is called a *ρ-sequence* if for each $i \in \{1, \ldots, n\}$ either $(x_{i-1}, x_i) \in \rho$ or $x_{i-1} < x_i$ holds. If in addition $x_0 = x_n$, then $x_0, \ldots, x_n$ is called a *ρ-circle*

(ii) ρ is called an *order-congruence* of $(P, \leq)$ if for every ρ-circle $x_0, \ldots, x_n \in P$, $\rho[x_0] = \cdots = \rho[x_n]$ is satisfied.

(iii) A partition π is called an *order-preserving partition* of $(P, \leq)$ if $\pi = (P/\rho)$ for some order congruence ρ of $(P, \leq)$. We write $\pi = \pi_\rho$ or $\rho = \rho_\pi$.

(iv) If π is an order-preserving partition we say that a sequence $x_0, \ldots, x_n$ is a *π-sequence* or a *π-cycle* if $x_0, \ldots, x_n$ is a ρ_π-sequence or a ρ_π-cycle, respectively.

Lemma 1. *[7] If ρ is an order-congruence of the a poset $(P, \leq)$, then it induces a partial order $\leq_\rho$ defined on the set P/ρ as follows:*

$\rho[x] \leq_\rho \rho[y]$ if there exists a ρ-sequence $x_0, \ldots, x_n \in P$, with $x_0 = x$ and $x_n = y$.

In view of the previous lemma, we can consider π_ρ as a poset with the partial order $\leq_\rho$ determined by $\leq$. In what follows, we write simply $\leq$ instead of $\leq_\rho$, if there is no danger of confusion.

Theorem 3. *[2] Let $(P, \leq)$ be a poset and let ρ be an equivalence on P. Then the following are equivalent.*

(i) ρ is an order-congruence of $(P, \leq)$.

(ii) There exists a poset $(Q, \leq)$ an an order-preserving map $f : P \to Q$ such that $\rho = \operatorname{Ker} f$.

(iii) $\leq$ can be extended to a quasiorder θ such that $\rho = \theta \cap \theta^{-1}$.

Example 1. Consider a 6-element poset P and its three partitions π_1, π_2, π_3 as shown in Figure 1.

The partition π_1 is not order-preserving, since a, c, e, a a π_1-cycle with $[a]_{\pi_1} \neq [c]_{\pi_1}$. In fact, it is easy to see that every block of an order-preserving partition must be order-convex.

Although π_2 has only order-convex blocks, yet it fails to be order-preserving. Indeed a, f, b, e, a is a π_2-cycle with $[a]_{\pi_2} \neq [f]_{\pi_2}$.

Finally, π_3 is an order-preserving partition, the diagram of the quotient poset P/π_3 is shown in the picture.

Let us consider the set $\mathcal{O}(P)$ of all order-preserving partitions of P equipped with a partial order $\leq$ defined as the usual refinement order of partitions: $\pi_1 \leq \pi_2$ iff every block of π_1 is a subset of a block of π_2.

The bottom element of $\mathcal{O}(P)$ is the partition consisting of singletons, the top element is the partition with a single block.

The poset $\mathcal{O}(P)$ is an algebraic lattice [10, Theorem 30]. For order-preserving partitions π_1, π_2

$$\pi_1 \wedge \pi_2 = \{B_1 \cap B_2 : B_1 \in \pi_1,\ B_2 \in \pi_2 \text{ and } B_1 \cap B_2 \neq \emptyset\}.$$

To define joins, we may proceed as follows. Let $\pi_1, \pi_2 \in \mathcal{O}(P)$ and $\lesssim$ be the transitive closure of the union of $\lesssim_{\pi_1}$ and $\lesssim_{\pi_2}$. Clearly, $\lesssim$ is a quasiorder on P. For $x, y \in P$, write $x \sim y$ iff $x \lesssim y$ and $y \lesssim x$. Then $P/\sim$ is an order-preserving partition of P and $\pi_1 \vee \pi_2 = (P/\sim)$

The covering relation in the lattice of order-preserving partitions of a finite poset is easy to describe: for a pair π_1, π_2 of order-preserving partitions of a finite poset P we have $\pi_1 \prec \pi_2$ iff π_2 arises from π_1 by merging of two blocks B_1, B_2 of π_1 such that

- either $B_1 \prec B_2$ in the poset $(\pi_1, \leq)$, or
- $B_1 \parallel B_2$ in the poset $(\pi_1, \leq)$.

In particular, this implies that the atoms of the lattice of order-preserving partitions of a finite poset P is the set of all partitions of P that are of the form

$$\pi_{a,b} := \{\{a, b\}\} \cup \{\{x\} : x \in P - \{a, b\}\},$$

where $a, b \in P$ is such that either $a \prec b$ or $a \parallel b$. Moreover, the lattice $\mathcal{O}(P)$ is ranked. The ranking function is given by $|P| - |\pi|$.

Example 2. If A_n is an n-element antichain, then every partition of A_n is order-preserving. The lattice of order-preserving partitions is then the partition lattice of the set A_n, usually denoted by Π_n. It is well known [3, 5], that for all $n \geq 3$ the order complex of $\hat{\Pi}_n$ is homotopic to the wedge of $(n-1)!$ spheres of dimension $n - 3$.

Example 3. If C_n is an n-element chain, $n \geq 3$, then a partition π of C_n is order-preserving if and only if all blocks of π are convex subsets of C_n. It is easy to see that $\mathcal{O}(C_n)$ is a Boolean algebra B_{n-1} with $n-1$ atoms. It is well known that the order complex of $\hat{B}_{n-1}$ is homotopic to a single sphere of dimension $n - 3$.

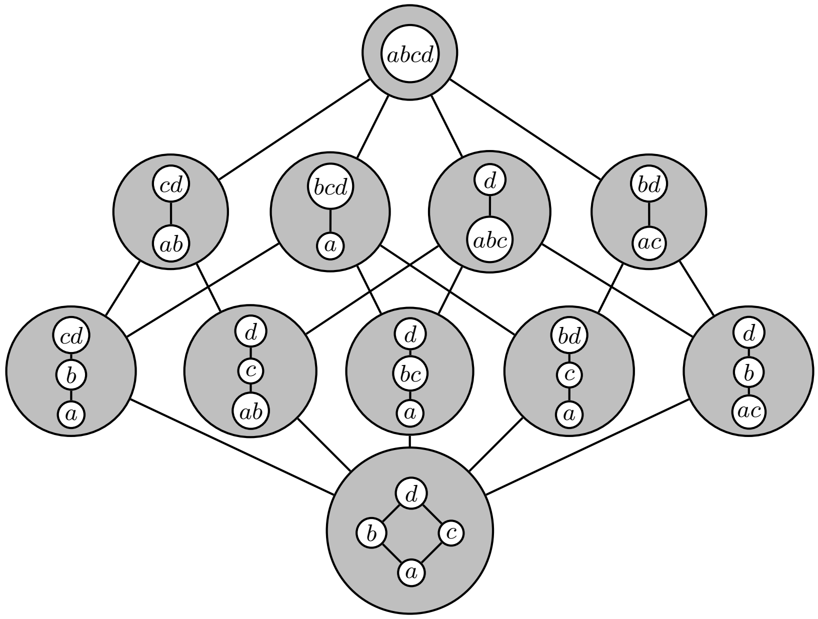


Figure 2: Order-preserving partitions of a Boolean algebra with two atoms

Example 4. To give a slightly more complicated example, let B_2 be a Boolean algebra with two atoms. The lattice of order-preserving partitions of B_2 has 11 elements; its Hasse diagram is Figure 2. Note that $\Delta(\hat{\mathcal{O}}(B_2))$ is not semimodular.

It is easy to see that $\Delta(\hat{\mathcal{O}}(B_2))$ has the homotopy type of two spheres of dimension one.

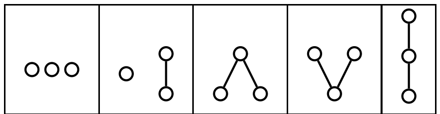
The proof of the following Theorem is inspired by the proof of Theorem 11.18 in [5], where the homotopy type of $\Delta(\hat{\Pi}_n)$ is determined.

Theorem 4. *Let P be a finite poset with n elements. Then $\Delta(\hat{\mathcal{O}}(P))$ is homotopy equivalent to a wedge of spheres of dimension $n-3$. Let a be a minimal element of P. Write $s_O(P)$ for the number of spheres in $\Delta(\hat{\mathcal{O}}(P))$. For $n > 3$, $s_O(P)$ satisfies the recurrence*

$$s_O(P) = \sum_{\pi_{a,b} \text{ order-preserving}} s_O(\pi_{a,b}).$$

Proof. There are, up to isomorphism, five posets with three elements. For each of them, the lattice $\mathcal{O}(P)$ is ranked of height two. Thus, $\hat{\mathcal{O}}(P)$ is an antichain and $\Delta(\hat{\mathcal{O}}(P))$ is a wedge of spheres of dimension 0. If P is a 3-element chain, then $s_O(P) = 1$, for the remaining four types of P we have $s_O(P) = 2$; see Table 1.

Let us assume that $n > 3$. Fix a minimal element a of P. Let P_a be a poset of all order-preserving partitions containing $\{a\}$ as a singleton class, ordered by



2	2	2	2	1

Table 1: $s_O(P)$ for 3-element posets

refinement. Let $\pi_a = \{\{a\}, P \setminus \{a\}\}$; it is clear that π_a is an order-preserving partition of P and that it is the top element of P_a. Let $\phi\colon \mathcal{F}(\Delta(\hat{\mathcal{O}}(P))) \to P_a$ be given by the following rules:

- if c is a chain consisting solely of elements of P_a, then $\phi(c) = \pi_a$,
- otherwise let π_{min} be the smallest element of c such that $\pi_{min} \notin P_a$; put $\phi(c) = \pi_{min} \wedge \pi_a$.

It is obvious that ϕ is an order-inverting mapping. We shall construct acyclic matchings on the fibers of ϕ. By Theorem 1, the union of these matchings is an acyclic matching on $\mathcal{F}(\Delta(\hat{\mathcal{O}}(P)))$.

Let $S = \phi^{-1}(\pi)$ where π is not the bottom element of P_a. Then we can construct the matching on S by either removing or adding π from each chain, depending on whether it does or does not contain π. The only unmatched chain occurs only if $\pi = \pi_a$ and the unmatched chain is $\{\pi_a\}$.

Let $S = \phi^{-1}(\hat{0})$, where $\hat{0}$ is the partition of P into singletons. This means, that for every chain $c \in S$ the top element π_{min} of c not belonging to P_a must be such that $\pi_{min} \wedge \pi_a = \hat{0}$. This implies that π_{min} has a single non-singleton class, in other words, $\pi_{min} = \pi_{a,b}$ for some b. Moreover, whenever $c \in \mathcal{F}(\Delta(\hat{\mathcal{O}}(P)))$ is such that $\pi_{a,b} \in c$, then $c \in S$. Thus S is the set of all $c \in \Delta(\hat{\mathcal{O}}(P))$ such that $\pi_{a,b} \in c$. Let us write

$$S_{a,b} = \{c \in \mathcal{F}(\Delta(\hat{\mathcal{O}}(P))) : \pi_{a,b} \in c\}.$$

Note that S is the disjoint union of all these $S_{a,b}$. Moreover, there is an easy-to-see bijection between the elements of $S_{a,b}$ and the elements of $\mathcal{F}(\Delta(\hat{\mathcal{O}}(\pi_{a,b})))$. Indeed, observe that each of the $c \in S_{a,b}$ can be constructed from a simplex in $\mathcal{F}(\Delta(\hat{\mathcal{O}}(\pi_{a,b})))$ by adding $\pi_{a,b}$. Thus, we may apply induction hypothesis: the homotopy type of $\Delta(\hat{\mathcal{O}}(\pi_{a,b}))$ is a wedge of $s_O(\pi_{a,b})$ spheres of dimension $n-4$, so there is an acyclic matching on $\mathcal{F}(\Delta(\hat{\mathcal{O}}(\pi_{a,b})))$ with $s_O(\pi_{a,b})$ critical simplices of dimension $n-4$. In an obvious way, we may extend this acyclic matching to an acyclic matching on $S_{a,b}$, leaving $s_O(\pi_{a,b})$ critical simplices of dimension $n-3$. This proves the recurrence stated in the Theorem. □

The recurrence in Theorem 4 allows us to compute the number of spheres in $\Delta(\hat{\mathcal{O}}(P))$ for any relevant finite poset P. For a small poset P, this can be easily done by hand. Playing with small examples yields a hypothesis that $s_O(P) = e(P)$ – the number of spheres is equal to the number of linear extensions

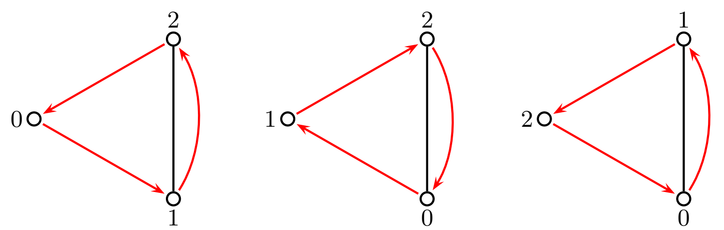


Figure 3: Actions of $\mathbb{Z}_3$ on a 3-element poset

of P. However, this is clearly not true, because for every n-element antichain A_n one has $s_O(A_n) = (n-1)!$ (Example 2) while $e(A_n) = n!$. On the other hand, it is possible to prove directly that things go well for a connected poset: whenever P is connected, $s_O(P) = e(P)$. This will be proved as a corollary of the main result (Corollary 2).

4. Cyclic extensions

Let P be a finite nonempty poset with n elements. Let $f\colon P \to [0, n-1]_{\mathbb{N}}$ be a linear extension of P. Consider the natural right action $(u, k) \mapsto u \oplus k$ of the finite n-element cyclic group $(\mathbb{Z}_n, \oplus)$ on itself. We write $\oplus_f\colon P \times \mathbb{Z}_n \to P$ for the pullback of this action by f. In other words, for all $x \in P$ and $k \in \mathbb{Z}_n$,

$$x \oplus_f k = f^{-1}(f(x) \oplus k).$$

Analogously, for $k \in \mathbb{Z}_n$, we write $x \ominus_f k := x \oplus_f (n-k)$.

Obviously, the $\oplus_f$ action of the element $1 \in \mathbb{Z}_n$ can be represented by an oriented cycle digraph. The vertices of the digraph are the elements of P, the edges are

$$\{(x, x \oplus_f 1) : x \in P\} =$$
$$\{(f^{-1}(0), f^{-1}(1)), \dots, (f^{-1}(n-2), f^{-1}(n-1)), (f^{-1}(n-1), f^{-1}(0))\}$$

We denote this digraph by $C(f, P)$. As $\mathbb{Z}_n$ is cyclic, the action of 1, and thus the digraph, determines the action of $\mathbb{Z}_n$ on the set P.

Definition 3. Let P be a finite poset, let f, g be linear extensions of P. We say that f, g are *cyclically equivalent*, in symbols $f \sim g$, if $\oplus_f = \oplus g$. An equivalence class of $\sim$ is called a *cyclic extension of* P. The number of cyclic extensions of P is denoted by $e_C(P)$.

Example 5. Consider the disjoint sum of a chain of height 1 and a one-element poset (Figure 3). This poset has 3 linear extensions giving rise to 2 cyclic extensions.

As we can see from the Example 5, it may well happen that two distinct linear extensions of a finite poset determine the same action. In this case, the

$\sim$ relation is nontrivial and the number of cyclic extensions is smaller than the number of linear extensions, $e_C(P) < e(P)$. In the remaining part of this section, we shall prove that this phenomenon occurs if and only if the finite poset in question is disconnected.

Proposition 1. *Let P be an n-element poset. Let f, g be linear extensions of P. The following are equivalent.*

(a) There is $k \in \mathbb{Z}_n$ such that for all $x \in P$, $f(x) = g(x) \oplus k$.
(b) $\oplus_f = \oplus_g$.

Proof. (a) $\Longrightarrow$ (b): We shall apply (a) twice. Let $y \in P$. Put $x = y \oplus_g 1$ in (a) to obtain

$$f(y \oplus_g 1) = g(y \oplus_g 1) \oplus k = g(y) \oplus k \oplus 1.$$

Let us use (a) second time, this time with $x = y$ to obtain

$$g(y) \oplus k \oplus 1 = f(y) \oplus 1,$$

so that

$$f(y \oplus_g 1) = f(y) \oplus 1.$$

It remains to apply f^{-1} to both sides of the last equality to obtain $y \oplus_g 1 = y \oplus_f 1$, which means (b).

(b) $\Longrightarrow$ (a): Let us write, for all $x \in P$, $s(x) = x \oplus_f 1 = x \oplus_g 1$. We shall prove that, for all $x \in P$, $f(x) \ominus g(x) = f(s(x)) \ominus g(s(x))$. Clearly, this implies that $f(x) \ominus g(x)$ is the same for all $x \in P$, that means, (a).

$$f(s(x)) \ominus g(s(x)) = f(x \oplus_f 1) \ominus g(x \oplus_g 1) = (f(x) \oplus 1) \ominus (g(x) \oplus 1) = f(x) \ominus g(x)$$

□

Proposition 2. *Let P be a finite n-element poset, let $k \in \mathbb{Z}_n$. Let g be a linear extension of P. The following are equivalent.*

(a) For every $x, y \in P$ such that $x \leq y$, $g(x) + k \geq n$ iff $g(y) + k \geq n$.
(b) For every connected component Q of P and for every $x, y \in Q$, $g(x) + k \geq n$ iff $g(y) + k \geq n$.
(c) $f(x) := g(x) \oplus k$ is a linear extension of P.

Proof.

(a) $\Longrightarrow$ (b): The proof is a trivial induction with respect to the distance of x and y in the comparability graph of P and is thus omitted.

(b) $\Longrightarrow$ (c): Clearly, $f\colon P \to [0, n-1]_{\mathbb{N}}$ is a bijection. It remains to prove that f is order-preserving. Let $x, y \in P$, $x \leq y$. Since x, y are comparable, they belong to the same connected component Q of P, hence $g(x) + k \geq n$ iff $g(y) + k \geq n$. As g is a linear extension of P, $g(x) \leq g(y)$.

Assume that $g(x) + k < n$. Then $g(y) + k < n$ and

$$f(x) = g(x) \oplus k = g(x) + k \leq g(y) + k = g(y) \oplus k = f(y).$$

Assume that $g(x) + k \geq n$. Then $g(y) + k \geq n$ and Thus,

$$f(x) = g(x) \oplus k = g(x) + k - n \leq g(y) + k - n = g(y) \oplus k = f(y).$$

(c) $\implies$ (a): Let $x, y \in P$ be such that $x \leq y$. As both f and g are linear extensions, $f(x) \leq f(y)$ and $g(x) \leq g(y)$. We prove the implications in (a) indirectly.

Suppose that $g(x)+k \geq n$ and that $g(y)+k < n$. Then $g(y)+k < g(x)+k$, which contradicts $g(x) \geq g(y)$.

Suppose that $g(x) + k < n$ and that $g(y) + k \geq n$. As $g(y) + k \geq n$, $f(y) = g(y) \oplus k = g(y) + k - n$. As $g(x) + k < n$, $f(x) = g(x) \oplus k = g(x) + k$. Since $f(x) \leq f(y)$,

$$g(x) + k \leq g(y) + k - n.$$

This implies that $g(x) \leq g(y) - n < 0$, which is a contradiction. □

Proposition 3. *Let P be a finite poset. The following are equivalent.*

(a) P is connected.

(b) For all linear extensions f, g of P, $\oplus_f = \oplus_g$ implies that $f = g$.

Proof. (a) $\implies$ (b): Let P be connected and let f, g be linear extensions of P such that $\oplus_f = \oplus_g$. By Proposition 1, there is $k \in \mathbb{Z}_n$ such that, for all $x \in P$, $f(x) = g(x) \oplus k$. By Proposition 2, this implies that for all $x, y \in P$, $g(x)+k \geq n$ iff $g(y) + k \geq n$.

Suppose that $f \neq g$, that means $k > 0$. Put $x = g^{-1}(n-1)$ and $y = g^{-1}(0)$. Then $g(x) + k \geq n$ and $g(y) + k = 0 + k < n$. This contradicts Proposition 2 (b), hence $k = 0$ and $f = g$.

(b) $\implies$ (a): Suppose that P is disconnected. We will construct a pair f, g of linear extensions such that $\oplus_f = \oplus_g$ and $f \neq g$. Let $P_1, \ldots, P_m$ be the components of P ordered according to cardinality, so that $|P_1| \geq \cdots \geq |P_m|$. Let f be a linear extension of P such that, for $i \in [1, m]_{\mathbb{N}}$,

$$f(P_i) = \big[|P_1| + \cdots + |P_{i-1}|, |P_1| + \cdots + |P_i|\big)_{\mathbb{N}}.$$

Put $k := |P_m|$ and let $g(x) = f(x) \oplus k$, in other words,

$$g(x) = \begin{cases} f(x) + k & \text{for } x \in P_1 \cup \cdots \cup P_{m-1} \\ f(x) + k - n & \text{for } x \in P_m. \end{cases}$$

Then g is a linear extension of P and, by Proposition 1, $\oplus_f = \oplus_g$. □

Corollary 1. *A finite poset P is connected if and only if $e(P) = e_C(P)$.*

5. Combinatorics of $e(P)$ and $e_C(P)$

In this section, we shall determine the connection between the counts $e_C(P)$ and $e(P)$ for a certain type of posets. Let P be an n-element poset with connected components $P_1, \ldots, P_m$. The structure of every linear extension $g\colon P \to [0, n-1]_{\mathbb{N}}$ naturally breaks down into structure of the individual restrictions $g\restriction_{P_i}$. Every such a restriction represents, up to a monotone transformation, a linear extension of the corresponding connected component. In the other way round, every linear extension of P_i together with the set $g(P_i)$ determines the restriction $g\restriction_{P_i}$ completely. Information about the sets $g(P_i)$ is uniquely represented by a mapping $w\colon [0, n-1]_{\mathbb{N}} \to [1, m]_{\mathbb{N}}$ via the correspondence is $w^{-1}(\{i\}) = g(P_i)$. Since the mappings w can be seen as permutations of the multiset $\{1^{|P_i|}, \ldots, m^{|P_m|}\}$, the number of linear extensions of P is

$$e(P) = \binom{n}{|P_1|, \ldots, |P_m|} \prod_{i=1}^{m} e(P_i)\,,$$

the multinomial coefficient being the number of such permutations.

In order to derive a similar relationship for the number of cyclic extensions $e_C(P)$ we will consider the mappings w as words. Let us call them *P-words*. Two generic words u and v are said to be *letter-disjoint* if the sets of letters in u and v are disjoint. Let $L = (l_1, l_2, \ldots, l_p)$ be a *composition* of n – that is a tuple of positive integers that add up to n. We say that a word w is *L-detangled* (alternatively, that L is a *detanglement* of w) if w can be written as a concatenation $w = u_1 \cdot u_2 \cdots u_p$ of pairwise letter-disjoint words u_j with lengths $|u_j| = l_j$.

Example 6. Consider the multiset $A = \{1^2, 2^3, 3^4\}$ and some words that arise as permutations of A. For example, the word 112223333 admits the detanglements $(9), (2, 7), (5, 4)$, and $(2, 3, 4)$, since

$$112223333 = 11 \cdot 2223333 = 11222 \cdot 3333 = 11 \cdot 222 \cdot 3333$$

are all concatenations of letter-disjoint words. The word 122123333 admits only two detanglements: (9) and $(5, 4)$.

Let us denote $\mathrm{Comp}\,(n)$ the set of all compositions of n. There exists a bijetive correspondence $\eta\colon \mathrm{Comp}\,(n) \to \mathcal{O}(C_n)$, between the compositions of n and order-congruences of an n-element chain C_n; since members of $\mathcal{O}(C_n)$ are exactly the partitions of C_n into intervals (compare with Example 3) we can define $\eta(L)$ to be the partition of C_n into intervals of lengths given by the entries of L in the consecutive order. Let us write $\sqsubset$ for the pull-back of the standard refinement order of partitions in $\mathcal{O}(C_n)$ by η. For L_1, L_2 in $\mathrm{Comp}\,(n)$, we say that L_1 is *finer* than L_2 (or, that it *refines* L_2), if $L_1 \sqsubset L_2$. Dually, we say that L_2 is *coarser* than L_1. By Example 3, the poset $(\mathrm{Comp}\,(n)\,, \sqsubset)$ is isomorphic to a Boolean algebra with $n-1$ atoms. The bottom element is the trivial composition of n into n consecutive ones, the top element is the trivial composition of n into one single n. Given a fixed P-word w, the detanglements of w form a filter in $(\mathrm{Comp}(n), \sqsubset)$. Indeed, every P-word is detangled by the trivial composition (n), meaning that the set of detanglements is non-empty. Given two detanglements of w, their coarsest common refinement is a detanglement of w as well, meaning that the set of detanglements is downwards directed. Finally, if w admits a detanglement L_1 which is a refinement of the composition L_2, then L_2 is also a detanglement of w, meaning that the set of detanglements is an upset. Since the lattice of compositions of n is finite, the ideal of detanglements of w has the finest composition L'. This finest composition is unique and, hence, an inherent property of w. Let us say, that L' is the *finest detanglement* of w.

A word w of length n is said to be *entangled* if the trivial composition (n) is its finest detanglement. Notice that this is equivalent to the fact, that w cannot be expressed as a concatenation of two nonempty, letter-disjoint words. If L is the finest detanglement of w and $w = u_1 \cdot u_2 \cdots u_p$ is its letter-disjoint decomposition given by L, then each u_j is an entangled word. Indeed, were some u_i's not entangled, the composition would admit a proper refinement that detangles w, which cotradicts the assumption.

Since $(\mathrm{Comp}(n), \sqsubset)$ is essentially a Boolean algebra, it is ranked; we will denote its ranking function $r_{\sqsubset}$. If $L = (l_1, l_2, \ldots, l_p)$ is a composition of n we have $r_{\sqsubset}(L) = n - p$. Let w be a word and let L be its finest detanglement. We will refer to the number $n - r_{\sqsubset}(L)$ as the *detanglement index* of w and will denote it $\mathrm{di}(w)$. The detanglement index of a word can be seen as the maximal number of non-empty pairwise letter-disjoint words from which w can be obtained by concatenation. Since the detanglements of a fixed word w form a filter in a boolean algebra, the value $\mathrm{di}(w) - 1$ is also the number of distinct co-atomic detanglements of w.

Example 7. Consider the same multiset $A = \{1^2, 2^3, 3^4\}$ as in the previous example. The finest detanglement of 112223333 is $(2, 3, 4)$, meaning that the word is not entangled. Also $\mathrm{di}(112223333) = 3$ and, indeed, there are $3 - 1 = 2$ co-atomic detanglements of this word: $(2, 7)$ and $(5, 4)$. Example of an entagled word would be 221231333 since the only detanglement of this word is the trivial composition (9); the detanglement index of this word is 1.

By Proposition 1 two linear extensions f and g of P are cyclically equivalent if and only if there exists $k \in \mathbb{Z}_n$ such that $f(x) = g(x) \oplus k$ for every $x \in P$. Further, by Proposition 2.b, given a linear extension g and a number $k \in \mathbb{Z}_n$, the mapping $f(x) = g(x) \oplus k$ is a linear extension if and only if for every connected component P_i of P one has either $g(P_i) < n - k$ or $g(P_i) \geq n - k$. Let w be the P-word induced by g. The latter property, translated into the language of detanglements, reads: either $k = 0$ or w is $(n - k, k)$-detangled. Since the detanglements of type $(n - k, k)$ are co-atomic, there are $\mathrm{di}(w) - 1$ of them; including also the case $k = 0$, there are $\mathrm{di}(w)$ different k's that satisfy the latter condition. Hence the number of different linear extensions that are cyclically equivalent with g is $\mathrm{di}(w)$. As a consequence, the number of cyclic extensions of P is

$$e_C(P) = \left(\sum_{t=1}^{m} \frac{U(P,t)}{t}\right) \prod_{i=1}^{m} e(P_i)$$

where $U(P,t)$ stands for the number of distinct P-words w with $\mathrm{di}(w) = t$.

In the sequel of the present section we will elaborate the combinatorial count $U(P,t)$ for the special case when all the connected components $P_1, P_2, \ldots, P_m$ of P are of the same size s, that is $n = ms$. For such posets, detanglements of any P-word are compositions $L = (l_1, l_2, \ldots, l_p)$ where every l_i is a multiple of s. The set of all such compositions forms a sublattice of $\mathrm{Comp}\,(n)$ isomorphic with $\mathrm{Comp}\,(m)$ via the correspondence $L \mapsto (1/s)L$ where the multiplication of a tuple by a number is defined componentwise. On the other hand, for every $L \in \mathrm{Comp}\,(m)$ there exists a P-word w detangled by sL. Hence $(\mathrm{Comp}\,(m), \sqsubset)$ is the lattice of representations of all detanglements of all P-words. Given $L \in \mathrm{Comp}\,(m)$, let us denote by $\mathrm{dw}\,(P,L)$ the set of all sL-detangled P-words. For the combinatorial count $|\,\mathrm{dw}\,(P,L)\,|$ we have

$$|\,\mathrm{dw}\,(P,L)\,| = m! \prod_{i=1}^{|L|} \frac{1}{l_i!} \binom{sl_i}{s, s, \ldots, s} = \binom{m}{l_1, l_2, \ldots, l_p} \prod_{i=1}^{|L|} \binom{sl_i}{s, s, \ldots, s}.$$

In order to establish the first equality, we can view the multinomial coefficient under the product as the number of distinct words over the alphabet $\{1^s, 2^s, \ldots, l_i^s\}$. Dividing this count by $l_i!$ we obtain the number of distinct word-patterns of such words. Hence the overall product counts the distinct patterns of P-words which are detangled by sL. Finally, every such a pattern represents $m!$ different words, which explains the leading multiplicative term.

Let us denote $\mathrm{fdw}\,(P,L)$ the set of all P-words for which sL is their finest detanglement. For L' ranging over $\mathrm{Comp}\,(m)$ such that $L' \sqsubset L$ the sets $\mathrm{fdw}\,(P,L')$ form a partition of $\mathrm{dw}\,(P,L)$. Therefore

$$|\,\mathrm{dw}\,(P,L)\,| = \sum_{\substack{L' \in \mathrm{Comp}(m) \\ L' \sqsubset L}} |\,\mathrm{fdw}\,(P,L')\,|.$$

and the count $|\,\mathrm{fdw}\,(P,L)\,|$ can be obtained by Möbius inversion of $|\,\mathrm{dw}\,(P,L)\,|$ over the poset $(\mathrm{Comp}\,(m), \sqsubset)$. Knowing that the poset is is essentially a Boolean algebra, the Möbius inversion boils down to the standard inclusion-exclusion principle and yields

$$|\,\mathrm{fdw}\,(P,L)\,| = \sum_{\substack{L' \in \mathrm{Comp}(m) \\ L' \sqsubset L}} (-1)^{(r_{\sqsubset}(L) - r_{\sqsubset}(L'))} |\,\mathrm{dw}\,(P,L')\,|.$$

Example 8. Let us compute the count $|\,\mathrm{fdw}\,(P,(m))\,|$ of the entangled P-words. Clearly, the count is a function of m and s. The latter combinatorial identity allows us to evaluate its values for small m and s.

	s			
m	1	2	3	4
1	1	1	1	1
2	0	4	18	68
3	0	60	1566	34236
4	0	1776	354456	62758896
5	0	84720	163932120	304863598320
6	0	5876640	134973740880	3242854167461280
7	0	556466400	180430456454640	66429116436728636640
8	0	68882446080	366311352681348480	2389384600126093124110080

Notice, that the second row of this table conincides with the OEIS sequence `A115112` [6]. To our best knowledge, no other feature of the table is present in the OEIS database (as of Dec. 2011).

Our main aim, however, is the count $U(P,t)$ of all P-words w with $\mathrm{di}(w) = t$. Knowing the values $|\,\mathrm{fdw}\,(P,L)\,|$, computation of this count is fairly simple. In view of the Möbius inversion used above and knowing the precise structure of the poset $(\mathrm{Comp}\,(m)\,, \sqsubset)$, we can express the count also in terms of $|\,\mathrm{dw}\,(P,L)\,|$ as follows

$$\begin{aligned} U(P,t) &= \sum_{\substack{L\in\mathrm{Comp}(m)\\ r_{\sqsubset}(L)=n-t}} |\,\mathrm{fdw}\,(P,L)\,| \\ &= \sum_{\substack{L\in\mathrm{Comp}(m)\\ r_{\sqsubset}(L)\leq n-t}} \sum_{\substack{L'\in\mathrm{Comp}(m)\\ L'\sqsubset L}} (-1)^{(r_{\sqsubset}(L)-r_{\sqsubset}(L'))}|\,\mathrm{dw}\,(P,L')\,| \\ &= \sum_{\substack{L\in\mathrm{Comp}(m)\\ r_{\sqsubset}(L)\leq n-t}} \binom{n-t}{r_{\sqsubset}(L)} (-1)^{(n-t-r_{\sqsubset}(L))}|\,\mathrm{dw}\,(P,L)\,|. \end{aligned}$$

6. Main result

Theorem 5. *Let P be a finite poset with n elements, $n \geq 3$. Then $\Delta(\hat{\mathcal{O}}(P))$ is homotopy equivalent to a wedge of $e_C(P)$ spheres of dimension $n-3$.*

Our goal is to show that the number of cyclic extensions is the same as the number of spheres in $\Delta(\hat{\mathcal{O}}(P))$. To do this, we prove that the recurrence for $s_O(P)$ from Theorem 4 holds for $e_C(P)$ as well. Since it is easy to check that $s_O(P) = e_C(P)$ for any 3-element poset P, the quantities must be equal.

To prove the recurrence for $e_C(P)$, we need to link cyclic extensions of the poset P with the cyclic extensions of the posets $\pi_{a,b}$, where $\pi_{a,b}$ is an order-preserving partition P.

Let us outline the schema of the proof of Theorem 5.

1. We prove that, for a fixed minimal element a, there is a mapping S_a from the set of all linear extensions of P to the disjoint union of sets of all linear extensions of all $\pi_{a,b}$, where $\pi_{a,b}$ is order-preserving (Lemmas 2 and 3).

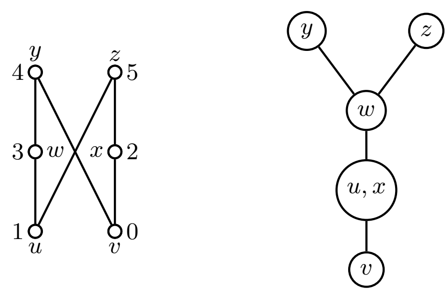


Figure 4:

2. We prove that this mapping is surjective (Lemma 4).
3. We prove that two linear extensions f, g of P are cyclically equivalent if and only if their images $S_a(f), S_a(g)$ are cyclically equivalent (Lemma 5).
4. These facts imply that S_a determines a bijection from the set of all cyclic extensions of P to the disjoint union of sets of all cyclic extensions of all $\pi_{a,b}$, where $\pi_{a,b}$ is an order-preserving partition of P.
5. This implies that the $s_O(P)$ and $e_C(P)$ satisfy the same recurrence. Since s_O and e_C are equal for 3-element posets, they are equal for any poset with at least 3 elements.

Lemma 2. *Let P be a finite poset with n elements, $n \geq 2$. Let f be a linear extension of P, let a be a minimal element of P. Then $\pi_{a,a\oplus_f 1}$ is an order-preserving partition of P.*

Proof. If $f(a) < n-1$, then $f\big(a \oplus_f 1\big) = f(a)+1$, hence $a \not\geq a \oplus_f 1$. Therefore, either $a \leq a \oplus_f 1$ or $a \parallel a \oplus_f 1$. If $a \parallel a \oplus_f 1$, the $\pi_{a,a\oplus_f 1}$ is order-preserving. If $a \leq a \oplus_f 1$ then $\pi_{a,a\oplus_f 1}$ is order-preserving iff $a \prec a \oplus_f 1$. Suppose that $a < b < a \oplus_f 1$. Then $f(a) < f(b) < f\big(a \oplus_f 1\big)$, which contradicts $f\big(a \oplus_f 1\big) = f(a)+1$.

If $f(a) = n-1$ (or, equivalently, $f(a \oplus_f 1) = 0$), then a is maximal. Since we assume that a is minimal, this implies that a is an isolated element, hence a and $a \oplus_f 1$ are incomparable. This implies that $\pi_{a,a\oplus_f 1}$ is order-preserving. □

For a finite poset P with $n \geq 2$ elements, a linear extension f of P, and a minimal element a of P, let us define a mapping $f_a \colon \pi_{a,a\oplus_f 1} \to [0, n-2]_{\mathbb{N}}$ by the rule

$$f_a(B) = \begin{cases} f(x) & \text{if } B = \{x\} \text{ and } f(x) < f(a), \\ \min\big(f(a), f\big(a \oplus_f 1\big)\big) & \text{if } B = \{a, a \oplus_f 1\}, \\ f(x) - 1 & \text{if } B = \{x\} \text{ and } f(x) > f(a) + 1. \end{cases}$$

Example 9. Consider the 6-element poset P from the left-hand side of Figure 4. Let g be a linear extension given by the number in the picture. Then the order-preserving partition $\pi_{u,u\oplus_f 1}$ is equal to $\pi_{u,x}$, see the right hand side of Figure 4.

The values of the mapping $f_u \colon \pi_{u,x} \to [0,4]$ are computed as follows.

- Since $0 = f(v) < f(u) = 1$, $f_u(\{v\}) = f(v) = 0$.
- $f(\{u, x\}) = \min(f(u), f(x)) = 1$.
- Since $3 = f(w) > f(u) + 1 = 2$, $f_u(\{w\}) = f(w) - 1 = 2$.
- Similarly, $f(\{y\}) = 3$ and $f(\{z\}) = 4$.

Lemma 3. *Let P be a finite poset with $n \geq 2$ elements, let f be a linear extension of P, a be a minimal element of P. Then f_a is a linear extension of the poset $(\pi_{a,a\oplus_f 1}, \leq)$.*

Proof. It is obvious that f_a is a bijection. It remains to prove that f_a is order-preserving. Let B_1, B_2 be blocks of $\pi_{a,a\oplus_f 1}$ such that $B_1 \leq B_2$.

(Case 1) If both B_1 and B_2 are singletons, say $B_1 = \{x_1\}$ and $B_2 = \{x_2\}$, then $x_1 \leq x_2$.

If $f(x_1) \leq f(x_2) < f(a)$, then $f_a(B_1) = f(x_1)$ and $f_a(B_2) = f(x_2)$, so $f_a(B_1) \leq f_a(B_2)$.

The case $f(a) + 1 < f(x_1) \leq f(x_2)$ can be handled in a similar way.

If $f(x_1) < f(a)$ and $f(a) + 1 < f(x_2)$, then $f_a(B_1) = f(x_1) < f(a)$ and $f_a(B_2) = f(x_2) - 1 > f(a)$. This implies $f_a(B_1) < f_a(B_2)$.

(Case 2) Suppose that $B_1 = \{x_1\}$ is a singleton and that B_2 is a non-singleton, that means $B_2 = \{a, a \oplus_f 1\}$. As $B_1 \leq B_2$, $x_1 \leq a$ or $x_1 \leq a \oplus_f 1$. However, a is minimal. Since it is clear that $x_1 \neq a$, we see that $x_1 \leq a \oplus_f 1$.

If $f(a) < n - 1$, then $f(a \oplus_f 1) = f(a) + 1$ and hence

$$f_a(B_2) = \min\bigl(f(a), f\bigl(a \oplus_f 1\bigr)\bigr) = f(a).$$

Thus, $f_a(B_1) = f(x_1) < f(a) = f_a(B_2)$.

If $f(a) = n + 1$, then $f(a \oplus_f 1) = 0$. This implies that $a \oplus_f 1$ is minimal. However, $x_1 \leq a \oplus_f 1$ implies $x_1 = a \oplus_f 1$, which is not true.

(Case 3) Suppose that $B_1 = \{a, a \oplus_f 1\}$ is a non-singleton and that $B_2 = \{x_2\}$ is a singleton.

If $f(a) = n + 1$, then $f_a(B_1) = 0$ and it is clear that $f_a(B_1) \leq f_a(B_2)$.

If $f(a) < n - 1$ then $f_a(B_1) = f(a)$. Since $B_1 \leq B_2$, $a \leq x_2$ or $a \oplus_f 1 \leq x_2$. If $a \leq x_2$, then

$$f_a(B_1) = f(a) \leq f(x_2) = f_a(B_2).$$

If $a \oplus_f 1 \leq x_2$, then

$$f_a(B_1) = f(a) < f(a) + 1 = f(a \oplus_f 1) \leq f(x_2) = f_a(B_2).$$

□

Let a be a minimal element of a finite poset P. By the previous two propositions, there is a mapping

$$S_a \colon \ell(P) \to \bigcup \{\ell(\pi_{a,b}) : \pi_{a,b} \text{ is order-preserving}\}$$

given by $S_a(f) := f_a$. In fact, this mapping is surjective, as shown by the following lemma.

Lemma 4. *Let P be a finite poset with $n \geq 2$ elements. Let a be a minimal element of P. Let $b \in P$ be such that $\pi_{a,b}$ is an order-preserving partition. For every linear extension g of $\pi_{a,b}$ there is a linear extension f of P such that $a \oplus_f 1 = b$ and $f_a = g$.*

Proof. The mapping $f\colon P \to [0, n-1]$ is given as follows:

$$f(x) = \begin{cases} g(\{x\}) & \text{If } g(\{x\}) < g(\{a,b\}), \\ g(\{a,b\}) & \text{if } x = a, \\ g(\{a,b\}) + 1 & \text{if } x = b, \\ g(\{x\}) + 1 & \text{if } g(\{x\}) > g(\{a,b\}). \end{cases}$$

Obviously, f is a bijection. We shall prove that f is order-preserving. Let $x, y \in P$ be such that $x \leq y$.

(Case 1) If $\{x,y\} \cap \{a,b\} = \emptyset$, then $x \leq y$ in P is equivalent to $\{x\} \leq \{y\}$ in $\pi_{a,b}$. Therefore $g(\{x\}) \leq g(\{y\})$. There are three subcases determined by the position of $g(\{a,b\})$ with respect to $g(\{x\})$ and $g(\{y\})$.

(Case 1.1) If $g(\{x\}) \leq g(\{y\}) < g(\{a,b\})$, then $f(x) = g(\{x\}) \leq g(\{y\}) = f(y)$.

(Case 1.2) If $g(\{x\}) < g(\{a,b\}) < g(\{y\})$, then

$$f(x) < f(x) + 1 = g(\{x\}) + 1 < g(\{y\}) + 1 = f(y).$$

(Case 1.3) If $g(\{a,b\}) < g(\{x\}) \leq g(\{y\})$, then $f(x) = g(\{x\}) + 1 \leq g(\{y\}) + 1 = f(y)$.

(Case 2) Suppose that $x \in \{a,b\}$, $y \notin \{a,b\}$. Then $x \leq y$ in P implies $\{a,b\} < \{y\}$ in $\pi_{a,b}$, hence $g(\{a,b\}) < g(\{y\})$ and $f(y) = g(\{y\}) + 1$. Therefore,

$$f(x) \leq g(\{a,b\}) + 1 < g(\{y\}) + 1 = f(y).$$

(Case 3) Suppose that $x \notin \{a,b\}$ and $y \in \{a,b\}$. As $x \leq y$ in P, $\{x\} < \{a,b\}$ in $\pi_{a,b}$. This implies that $g(\{x\}) < g(\{a,b\}$ and that $f(x) = g(\{x\})$. Since $y \in \{a,b\}$, $f(y) \leq g(\{a,b\}) + 1$. Therefore,

$$f(x) = g(\{x\}) < g(\{a,b\}) \leq f(y).$$

(Case 4) Suppose that $x, y \in \{a,b\}$. If $x = y$, there is nothing to prove. Suppose that $x < y$. Since a is minimal, $x = a$ and $y = b$. Thus,

$$f(x) = g(\{a,b\}) < g(\{a,b\}) + 1 = f(y).$$

Thus, f is a linear extension of P.

Clearly,

$$a \oplus_f 1 = f^{-1}(f(a) \oplus 1) = f^{-1}(g(\{a,b\}) + 1) = f^{-1}(f(b)) = b.$$

Let us prove that $f_a = g$. Let $B \in \pi_{a,b} = \pi_{a,a\oplus_f 1}$. Let $B \in \pi_{a,b}$, we shall prove that $f_a(B) = g(B)$.

If $B = \{x\}$ and $f(x) < f(a)$ then $f_a(B) = f(x)$. As $f(a) = g(\{a, b\})$, $f(x) < g(\{a, b\})$ and it is easy to see that $f(x) = g(\{x\})$. Hence, $f_a(B) = g(\{x\}) = g(B)$.

If $B = \{a, b\}$, then

$$f_a(B) = \min(f(a), f(a \oplus_f 1)) = \min(f(a), f(b)) = g(\{a, b\}) = g(B).$$

If $B = \{x\}$ and $f(x) > f(a)+1$ then $f_a(B) = f(x)-1$. As $f(a) = g(\{a, b\})$, $f(x) > g(\{a, b\}) + 1$ and it is easy to see that $f(x) = g(x) + 1$. Hence, $f_a(B) = f(x) - 1 = g(\{x\}) = g(B)$.

□

Lemma 5. *Let P be a finite poset with $n \geq 2$ elements. Let f, g be linear extensions of P, let a be a minimal element of P. Then $\oplus_f = \oplus_g$ if and only if $\oplus_{f_a} = \oplus_{g_a}$.*

Proof. Suppose that $\oplus_f = \oplus_g$. This implies that $C(f, P) = C(g, P)$. The mapping $f \mapsto f_a$, $g \mapsto g_a$ corresponds to the contraction of the same edge $(a, a \oplus_f 1) = (a, a \oplus_g 1)$. Thus, $C(f_a, \pi_{a,a\oplus_f 1}) = C(f_b, \pi_{a,a\oplus_g 1})$ and this implies that $\oplus_{f_a} = \oplus_{g_a}$.

Suppose that $\oplus_{f_a} = \oplus_{g_a}$. The domains of equal maps must be the same, so $\pi_{a,a\oplus_f 1} = \pi_{a,a\oplus_g 1}$. Hence, $C(f_a, \pi_{a,a\oplus_f 1}) = C(g_a, \pi_{a,a\oplus_g 1})$. The digraph $C(f, P)$ arises from $C(f_a, \pi_{a,a\oplus_f 1})$ by an expansion of the vertex $\{a, a \oplus_f 1\}$. Principally, there are two possible orientations of the new edge between $a, a\oplus_f 1$. However, only one of them gives us an oriented cycle. Therefore, $C(f, P)$ is determined by $C(f_a, \pi_{a,a\oplus_f 1})$. Similarly, $C(g, P)$ is determined by $C(g_a, \pi_{a,a\oplus_g 1})$.

□

Proof of the main result. It is easy to check that for any 3-element poset P, $e_C(P) = s_O(P)$.

Let a be a minimal element of a finite poset P, $|P| > 3$. Then Lemma 5 implies that the mapping S_a factors through the mapping $f \mapsto [f]_\sim$. By Lemma 4, S_a is surjective. This implies that S_a determines a bijection

$$S_a^\sim \colon (\ell(P)/\sim) \to \bigcup\{\ell(\pi_{a,b})/\sim \colon \pi_{a,b} \text{ is order-preserving}\}$$

given by $[f]_\sim \mapsto [f_a]_\sim$. Since the union of the right-hand side is clearly disjoint, this gives us the following recurrence

$$e_C(P) = \sum_{\pi_{a,b} \text{ is order-preserving}} e_C(\pi_{a,b}).$$

Therefore, for any finite P with $|P| > 3$, $s_O(P) = e_C(P)$. □

Corollary 2. *Let P be a finite connected poset with n elements, $n \geq 3$. Then $\Delta(\hat{\mathcal{O}}(P))$ is homotopy equivalent to a wedge of $e(P)$ spheres of dimension $n-3$.*

Proof. By Theorem 5 and Corollary 1. □

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