Automatic Solution of Richard Stanley's Amer. Math. Monthly Problem #11610 and ANY Problem of That Type

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Preamble

Suppose you toss a (fair) coin n times. If n is large, the *law of large numbers* promises you that (with high probability) you would *roughly* get as many Heads as Tails. But what is the *exact* probability that you would have **exactly** as many Heads as Tails? If n is odd, the answer is easy (**you do it!**). If n is even, then it is almost as easy, and there is a *nice*, "closed-form" formula for that probability, namely $n!/((n/2)!^22^n)$.

Richard Stanley [St1] proposed the problem of finding, a(n), the number of n-letter words in the alphabet $\{H,T\}$ where there are as many occurrences of "HT" (i.e. Head immediately followed by Tail) as there are occurrences of "TT" (two Tails in a row). He didn't give a "closed form" formula, but he gave something almost as good, an *explicit* formula as an (*algebraic*, as it turned out, in fact quadratic) formal power series for the (ordinary) generating function $P(t) := \sum_{n=0}^{\infty} a(n)t^n$.

The fact that the generating function, P(t), is an algebraic generating function is not at all surprising! This can be seen in (at least) two ways.

One way is to show that the "language" of words with as many occurrences of "HT" as "HH" is context-free (type 2) with an unambiguous grammar, and hence its weight-enumerator is algebraic. It is possible to (automatically!) generate its grammar, and then automatically generate a system of algebraic equations one of whose unknowns is the desired generating function, and solving that system would (presumbly, we didn't do it) yield Stanley's proposed expression.

A better way is to find (automatically, of course!), the rational generating function F(t; z[HT], z[TT]) that is the weight-enumerator of all words in the alphabet $\{H, T\}$ according to the weight $Weight(w) = t^{length(w)}z[HT]^{\#HT(w)}z[TT]^{\#TT(w)}$. This can be done in several ways, including the Goulden-Jackson method, beautifully surveyed in [NZ], and efficiently implemented in the Maple package

http://www.math.rutgers.edu/~zeilberg/tokhniot/DAVID_IAN

accompanying that article.

Having done that, the desired generating function, P(t), is the coefficient of s^0 (i.e. the constant term) in F(t; s, 1/s). Hillel Furstenberg[F] promises us that P(t) is an algebraic formal power series in t, and his proof implies a (rather awkward and inefficient) algorithm (using Cauchy's integral

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http://www.math.rutgers.edu/~zeilberg/. Dec. 27, 2011. Accompanied by Maple packages RPS and RPSplus downloadable from Zeilberger's website. Supported in part by the USA National Science Foundation.

formula and residues) for computing it. This should yield a second *rigorous* derivation of Stanley's proposed solution.

But since we know a priori (by "general nonsense") that the desired sequence belongs to the algebraic ansatz (see [Z1] and the **wonderful** new book by Manuel Kauers and Peter Paule[KP], that should be required reading to any mathematics student (and professional!)) a semi-rigorous derivation would be to crank out the first 40 (or even fewer) terms of Stanley's sequence (a quick way would be to find the first 40 terms in the expansion of the constant term, in s, of F(t; s, 1/s) above), and then use a guessing program, e.g. listoalg in the Maple package gfun([SaZ]) (but please enlarge the very small default values of the parameters) that now is part of Maple, or procedure Empir in my own Maple package http://www.math.rutgers.edu/~zeilberg/tokhniot/SCHUTZENBERGER.

In order to make the above semi-rigorous derivation fully rigorous (for those obtuse people who desire it), one would need to derive a priori bounds on the degree (in t and P(t)) of the defining equation F(t, P(t)) = 0 for the desired generating function P(t). Unlike the C-finite ansatz (see [Z2] and [KP]) where finding these upper bounds is trivial, I don't know how to do it in the present case. But there is **another** way to make everything fully rigorous. Via the holonomic ansatz!

Using the Continuous Almkvist-Zeilberger Algorithm[AlZ], that is implemented in procedure AZc of the Maple package http://www.math.rutgers.edu/~zeilberg/tokhniot/EKHAD, one can obtain a differential equation (and its proof (a certain certificate)), and then verify that the above "conjectured" algebraic expression for P(t) satisfies that very same differential equation, and check that the initial conditions match.

The general case

The beauty of algorithmic mathematics is that it is not much harder to write a general program to handle a whole class of problems rather than just solve one problem. The above discussion applies equally to any (finite) alphabet (not just a two-lettered one) and any two distinguished substrings, w_1 and w_2 not just HT and TT.

The Maple package RPS

Since we require procedures from four different Maple packages (DAVID_IAN, SCHUTZENBERGER, EKHAD and AsyRec), we conveniently assembled all the necessary procedures, together with new "interfacing code" needed to solve problems of the above type. The result is the Maple package RPS (named after Richard Peter Stanley), available free of charge from:

http://www.math.rutgers.edu/~zeilberg/tokhniot/RPS.

This Maple package does much more! It computes holonomic representations (or as Richard Stanley [St2] would say, *P-recursive* ones), that are used, in turn, to derive asymptotic expressions using procedures borrowed from

http://www.math.rutgers.edu/~zeilberg/tokhniot/AsyRec

Hence we have "three-quarters" of the Kauers-Paule "concrete tetrahedron": generating function, recurrence, and asymptotics. The last one "definite sum" could also be obtained, but we would (usually) get complicated and ugly *multi-sums* with many sigmas, so it would be stupid to look for these.

Out of sheer laziness we have only programmed the case where the two distinguished words, w_1 , w_2 , have the same length.

Precomputed Output of the Maple package RPS

The "front" of this article, the webpage

http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/rps.html

contains links to several "webbooks" that systematically states (proved!) (algebraic) generating functions, recurrence equations, and asymptotics for analogs of Stanley's problem for *all* possible pairs of words (up to trivial images under permutations of the letters) of the same length (let's call it k) for an m-letter alphabet for the following cases.

- m=2, k=2: http://www.math.rutgers.edu/~zeilberg/tokhniot/oRPS22 (containing 3 propositions)
- m=2, k=3: http://www.math.rutgers.edu/~zeilberg/tokhniot/oRPS23 (containing 11 propositions)
- m=2, k=4: http://www.math.rutgers.edu/~zeilberg/tokhniot/oRPS24 (containing 38 propositions)
- m=3, k=2: http://www.math.rutgers.edu/~zeilberg/tokhniot/oRPS32 (containing 6 propositions)
- m=3, k=3: http://www.math.rutgers.edu/~zeilberg/tokhniot/oRPS32 (containing 40 propositions)
- m=4, k=2: http://www.math.rutgers.edu/~zeilberg/tokhniot/oRPS42 (containing 7 propositions)
- m=4, k=3: http://www.math.rutgers.edu/~zeilberg/tokhniot/oRPS43 (containing 63 propositions)
- m=5, k=2: http://www.math.rutgers.edu/~zeilberg/tokhniot/oRPS52 (containing 7 propositions)
- m=5, k=3: http://www.math.rutgers.edu/~zeilberg/tokhniot/oRPS53 (containing 69 propositions)

- m=6, k=2: http://www.math.rutgers.edu/~zeilberg/tokhniot/oRPS62 (containing 7 propositions)
- m = 6, k = 3: http://www.math.rutgers.edu/~zeilberg/tokhniot/oRPS63 (containing 70 propositions)

For Neil Sloane's sake, we have also computed the first 50 terms of each of the considered enumerating sequences. All the sequences for m=2 and k=2,3,4 have already been entered to OEIS by R.H. Hardin, for example http://oeis.org/A164147. Some of the pages for these sequences come with conjectured recurrences. The present webbooks supply rigorous proofs to all them, and supplies proved recurrences for the remaining ones.

The Maple Package RPSplus

With hardly any more (programming) effort, one can consider the enumerating sequences of words for which, for three given positive integers a_1, a_2 and r,

" a_1 times [the numer of occurrences of w_1]" **minus** a_2 times "[the numer of occurrences of w_2] "**equals** r.

Once again the generating functions are guaranteed to be algebraic and everything goes through. See the procedures listed in ezraG(); in the more general Maple package RPSplus, available from

http://www.math.rutgers.edu/~zeilberg/tokhniot/RPSplus

Readers are welcome to generate their own output.

What About Several Distinguished Words?

The more general case where one has a finite alphabet of, say, m letters, and s, say, distinguished words w_1, \ldots, w_s , and v diophantine affine linear relations between the quantities "number of occurrences of w_i ", then we leave the *algebraic ansatz* and enter the *holonomic* ansatz. By WZ theory we are *guaranteed* that the enumerating sequence, in each case, is holonomic (alias P-recursive), and we are justified, semi-rigorously, just to *guess* the holonomic description, using gfun's listtorec, or procedure Findrec in the Maple package RPS.

For those obtuse people who insist on a rigorous proof, they are welcome to use the Maple package

http://www.math.rutgers.edu/~zeilberg/tokhniot/MultiAlmkvistZeilberger

that is one of the Maple packages accompanying the seminal article [ApZ]. Alas, it may take them quite some time, and frankly, for us, a *semi-rigorous* proof suffices. But so far, we ran out of steam, and we do not even have an implementation of the semi-rigorous, pure guessing, version.

References

[AlZ] Gert Almkvist and Doron Zeilberger, The Method of Differentiating Under The Integral Sign, J. Symbolic Computation 10(1990), 571-591.

[ApZ] Moa Apagodu and Doron Zeilberger, Multi-Variable Zeilberger and Almkvist-Zeilberger Algorithms and the Sharpening of Wilf-Zeilberger Theory, Adv. Appl. Math. 37(2006), (Special issue in honor of Amitai Regev), 139-152.

- [F] Hillel Furstenberg, Algebraic Functions over finite fields, J. Algebra 7 (1967). 271-277.
- [KP] Manuel Kauers and Peter Paule, "The Concrete Tetrahedron", Springer, 2011.
- [NZ] John Noonan and Doron Zeilberger, *The Goulden-Jackson Cluster Method: Extensions, Applications, and Implementations*, J. Difference Eq. Appl. 5(1999), 355-377. http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/gj.html
- [SaZ] Bruno Salvy and Paul Zimmermann, Gfun: a Maple package for the manipulation of generating and holonomic functions in one variable, ACM Transactions on Mathematical Software, 20 (1994), 163-177.
- [St1] Richard P. Stanley, *Problem #11610*, Amer. Math. Monthly **118(10)** (Dec. 2011), 937.
- [St2] Richard P. Stanley, Differentiably finite power series, Europ. J. Combinatorics 1 (1980), 175-188.
- [Z1] Doron Zeilberger, An Enquiry Concerning Human (and Computer!) [Mathematical] Understanding, in: "RANDOMNESS AND COMPLEXITY, FROM LEIBNIZ TO CHAITIN", Cristian C. Calude, ed.,

http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/enquiry.html

[Z2] Doron Zeilberger, *The C-finite Ansatz*, submitted, http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/cfinite.html.