Quadrant marked mesh patterns in 132-avoiding permutations I

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Abstract

This paper is a continuation of the systematic study of the distributions of quadrant marked mesh patterns initiated in [6]. Given a permutation $\sigma = \sigma_1 \cdots \sigma_n$ in the symmetric group S_n, we say that σ_i matches the quadrant marked mesh pattern $MMP(a, b, c, d)$ if there are at least a elements to the right of σ_i in σ that are greater than σ_i, at least b elements to left of σ_i in σ that are greater than σ_i, at least c elements to left of σ_i in σ that are less than σ_i, and at least d elements to the right of σ_i in σ that are less than σ_i. We study the distribution of $MMP(a, b, c, d)$ in 132-avoiding permutations. In particular, we study the distribution of $MMP(a, b, c, d)$, where only one of the parameters a, b, c, d are non-zero. In a subsequent paper [7], we will study the the distribution of $MMP(a, b, c, d)$ in 132-avoiding permutations where at least two of the parameters a, b, c, d are non-zero.

Keywords: permutation statistics, marked mesh pattern, distribution, Catalan numbers, Fibonacci numbers, Fine numbers

Contents

1 Introduction

The notion of mesh patterns was introduced by Brändén and Claesson [1] to provide explicit expansions for certain permutation statistics as (possibly infinite) linear combinations of (classical) permutation patterns. This notion was further studied in [4, 6, 10]. The present paper, as well as the upcoming paper [7], are continuations of the systematic study of distributions of quadrant marked mesh patterns on permutations initiated by Kitaev and Remmel [6].

In this paper, we study the number of occurrences of what we call *quadrant marked mesh patterns*. To start with, let $\sigma = \sigma_1 \cdots \sigma_n$ be a permutation written in one-line notation. Then we will consider the graph of σ, $G(\sigma)$, to be the set of points $\{(i, \sigma_i) : 1 \leq i \leq n\}$. For example, the graph of the permutation $\sigma = 471569283$ is pictured in Figure 1. Then if we draw a coordinate system centered at a point (i, σ_i), we will be interested in the points that lie in the four quadrants I, II, III, and IV of that coordinate system as pictured in Figure 1. For any $a, b, c, d \in \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the set of natural numbers, we say that σ_i matches the quadrant marked mesh pattern $MMP(a, b, c, d)$ in σ if, in the coordinate system centered at (i, σ_i), $G(\sigma)$ has at least a points in quadrant I, at least b points in quadrant II, at least c points in quadrant III, and at least d points in quadrant IV. For example, if $\sigma = 471569283$, then $\sigma_4 = 5$ matches $MMP(2, 1, 2, 1)$, since relative to the coordinate system with origin $(4, 5)$, $G(\sigma)$ has 3, 1, 2, and 2 points in quadrants I, II, III, and IV, respectively. Note that if a coordinate in $MMP(a, b, c, d)$ is 0, then there is no condition imposed on the points in the corresponding quadrant.

In addition, we shall consider quadrant marked mesh patterns $MMP(a, b, c, d)$ where $a, b, c, d \in \mathbb{N} \cup \{\emptyset\}$. Here, when a coordinate of $MMP(a, b, c, d)$ is $\emptyset$, there must be no points in the corresponding quadrant for σ_i to match $MMP(a, b, c, d)$ in σ. For example, if $\sigma =$

471569283, then $\sigma_3 = 1$ matches $MMP(4, 2, \emptyset, \emptyset)$, since relative to the coordinate system with origin $(3, 1)$, $G(\sigma)$ has 6, 2, 0, and 0 points in quadrants I, II, III, and IV, respectively. We let $\mathrm{mmp}^{(a,b,c,d)}(\sigma)$ denote the number of i such that σ_i matches $MMP(a, b, c, d)$ in σ.

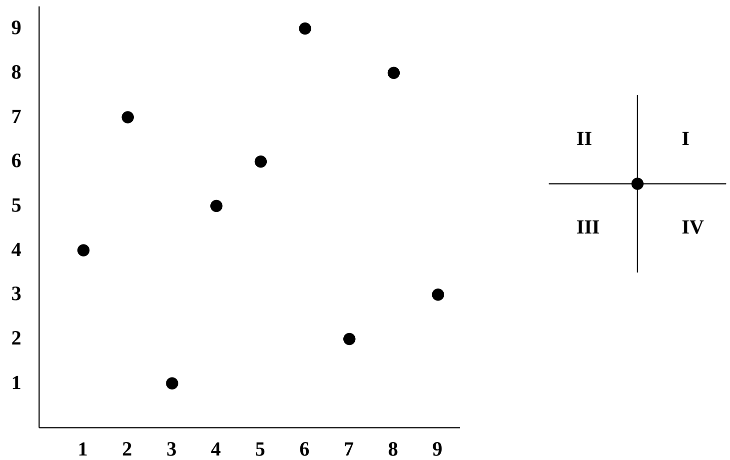
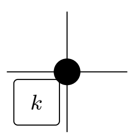
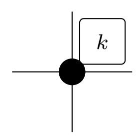
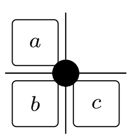
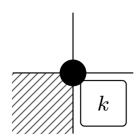


Figure 1: The graph of $\sigma = 471569283$.

Note how the (two-dimensional) notation of Úlfarsson [10] for marked mesh patterns corresponds to our (one-line) notation for quadrant marked mesh patterns. For example,

$\mathrm{MMP}(0, 0, k, 0) =$  , $\mathrm{MMP}(k, 0, 0, 0) =$  ,

$\mathrm{MMP}(0, a, b, c) =$  , and $\mathrm{MMP}(0, 0, \emptyset, k) =$  .

Given a sequence $w = w_1 \cdots w_n$ of distinct integers, let $\mathrm{red}(w)$ be the permutation found by replacing the ith largest integer that appears in w by i. For example, if $w = 2754$, then $\mathrm{red}(w) = 1432$. Given a permutation $\tau = \tau_1 \cdots \tau_j \in S_j$, we say that the pattern τ *occurs* in $\sigma \in S_n$ if there exist $1 \leq i_1 < \cdots < i_j \leq n$ such that $\mathrm{red}(\sigma_{i_1} \cdots \sigma_{i_j}) = \tau$. We say that a permutation σ *avoids* the pattern τ if τ does not occur in σ. We will let $S_n(\tau)$ denote the set of permutations in S_n that avoid τ. In the theory of permutation patterns, τ is called a *classical pattern*. See [5] for a comprehensive introduction to the area of permutation patterns.

It has been a rather popular direction of research in the literature on permutation patterns to study permutations avoiding a 3-letter pattern subject to extra restrictions (see [5, Subsection 6.1.5]). The main goal of this paper and the upcoming paper [7] is to study the generating functions

$$Q_{132}^{(a,b,c,d)}(t, x) = 1 + \sum_{n \geq 1} t^n Q_{n,132}^{(a,b,c,d)}(x), \tag{1}$$

where for any $a,b,c,d \in \mathbb{N} \cup \{\emptyset\}$,

$$Q_{n,132}^{(a,b,c,d)}(x) = \sum_{\sigma \in S_n(132)} x^{\mathrm{mmp}^{(a,b,c,d)}(\sigma)}. \tag{2}$$

More precisely, we will study the generating functions $Q_{132}^{(a,b,c,d)}(t,x)$ in all cases where exactly one of the coordinates a,b,c,d is non-zero and the remaining coordinates are 0 plus the generating functions $Q_{132}^{(k,0,\emptyset,0)}(t,x)$ and $Q_{132}^{(\emptyset,0,k,0)}(t,x)$. In [7], we will study the generating functions $Q_{132}^{(a,b,c,d)}(t,x)$ for $a,b,c,d \in \mathbb{N}$ where at least two of the parameters a,b,c,d are greater than 0.

For example, here are two tables of statistics for $S_3(132)$ that we will be interested in.

σ	$\mathrm{mmp}^{(1,0,0,0)}(\sigma)$	$\mathrm{mmp}^{(0,1,0,0)}(\sigma)$	$\mathrm{mmp}^{(0,0,1,0)}(\sigma)$	$\mathrm{mmp}^{(0,0,0,1)}(\sigma)$
123	2	0	2	0
213	2	1	1	1
231	1	1	1	2
312	1	2	1	1
321	0	2	0	2

σ	$\mathrm{mmp}^{(2,0,0,0)}(\sigma)$	$\mathrm{mmp}^{(0,2,0,0)}(\sigma)$	$\mathrm{mmp}^{(0,0,2,0)}(\sigma)$	$\mathrm{mmp}^{(0,0,0,2)}(\sigma)$
123	1	0	1	0
213	0	0	1	0
231	0	1	0	0
312	0	0	0	1
321	0	1	0	1

Note that there is one obvious symmetry in this case. That is, we have the following lemma.

Lemma 1. *For any $a,b,c,d \in \mathbb{N} \cup \{\emptyset\}$, $Q_{n,132}^{(a,b,c,d)}(x) = Q_{n,132}^{(a,d,c,b)}(x)$.*

Proof. If we start with the graph $G(\sigma)$ of a permutation $\sigma \in S_n(132)$ and reflect the graph about the line $y = x$, then we get the permutation σ^{-1}, which is also in $S_n(132)$. It is easy to see that points in quadrants I, II, III, and IV in the coordinate system with origin (i,σ_i) in $G(\sigma)$ will reflect to points in quadrants I, IV, III, and II, respectively, in the coordinate system with origin (σ_i,i) in $G(\sigma^{-1})$. It follows that the map $\sigma \to \sigma^{-1}$ shows that $Q_{n,132}^{(a,b,c,d)}(x) = Q_{n,132}^{(a,d,c,b)}(x)$. $\square$

As a matter of fact, *avoidance* of a marked mesh pattern $MMP(a,b,c,d)$ with $a,b,c,d \in \mathbb{N}$ can always be expressed in terms of multi-avoidance of (usually many) classical patterns. For example, a permutation $\sigma \in S_n$ avoids the pattern $MMP(2,0,0,0)$ if and only if it avoids both 123 and 132. Thus, among our results we will re-derive several known facts in the permutation patterns theory and get seemly new enumeration of permutations avoiding simultaneously the patterns 132 and 1234 (see the discussion right below (13)). However, our main goals are more ambitious in that we will compute the generating function for the distribution of the occurrences of the pattern in question, not just the generating function for the number of permutations that avoid the pattern.

For any a, b, c, d, we will write $Q_{n,132}^{(a,b,c,d)}(x)|_{x^k}$ for the coefficient of x^k in $Q_{n,132}^{(a,b,c,d)}(x)$. We shall also show that sequences of the form $(Q_{n,132}^{(a,b,c,d)}(x)|_{x^r})_{n\geq s}$ count a variety of combinatorial objects that appear in the *On-line Encyclopedia of Integer Sequences* (OEIS) [8]. Thus, our results will give new combinatorial interpretations of such classical sequences as the Fine numbers and the Fibonacci numbers, as well as provide certain sequences that appear in the OEIS with a combinatorial interpretation where none had existed before.

2 Connections with other combinatorial objects

It is well-known that the cardinality of $S_n(132)$ is the nth Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. There are many combinatorial interpretations of the Catalan numbers. For example, in his book [9], Stanley lists 66 different combinatorial interpretations of the Catalan numbers, and he gives many more combinatorial interpretations of the Catalan numbers on his web site. Hence, any time one has a natural bijection from $S_n(132)$ into a set of combinatorial objects O_n counted by the nth Catalan number, one can use the bijection to transfer our statistics $\mathrm{mmp}^{(a,b,c,d)}$ to corresponding statistics on the elements of O_n. In this section, we shall briefly describe some of these statistics in two of the most well-known interpretations of the Catalan numbers, namely Dyck paths and binary trees.

A Dyck path of length $2n$ is a path that starts at $(0,0)$ and ends at the point $(2n,0)$ that consists of a sequence of up-steps $(1,1)$ and down-steps $(1,-1)$ such that the path always stays on or above the x-axis. We will generally encode a Dyck path by its sequence of up-steps and down-steps. Let $\mathcal{D}_{2n}$ denote the set of Dyck paths of length $2n$. Then it is easy to construct a bijection $\phi_n : S_n(132) \to \mathcal{D}_{2n}$ by induction. To define ϕ_n, we need to define the lifting of a path $P \in \mathcal{D}_{2n}$ to a path $L(P) \in \mathcal{D}_{2n+2}$. Here $L(P)$ is constructed by simply appending an up-step at the start of P and a down-step at the end of P. That is, if $P = (p_1, \ldots, p_{2n})$, then $L(P) = ((1,1), p_1, \ldots, p_{2n}, (1,-1))$. An example of this map is pictured in Figure 2. If $P_1 \in \mathcal{D}_{2k}$ and $P_2 \in \mathcal{D}_{2n-2k}$, we let P_1P_2 denote the element of $\mathcal{D}_{2n}$ that consists of the path P_1 followed by the path P_2.

To define ϕ_n, we first let $\phi_1(1) = ((1,1),(1,-1))$. For any $n > 1$ and any $\sigma \in S_n(132)$, we define $\phi_n(\sigma)$ by cases as follows.

Case 1. $\sigma_n = n$.
Then $\phi_n(\sigma) = L(\phi_{n-1}(\sigma_1 \cdots \sigma_{n-1}))$.

Case 2. $\sigma_i = n$, where $1 \leq i < n$. In this case, $\phi_n(\sigma) = P_1P_2$, where
$P_1 = \phi_i(\mathrm{red}(\sigma_1 \cdots \sigma_i))$ and $P_2 = \phi_{n-i}(\mathrm{red}(\sigma_{i+1} \cdots \sigma_n)) = \phi_{n-i}(\sigma_{i+1} \cdots \sigma_n)$.

We have pictured this map for the first few values of n by listing the permutation σ on the left and the value of $\phi_n(\sigma)$ on the right in Figure 3.

Suppose we are given a path $P = (p_1, \ldots, p_{2n}) \in \mathcal{D}_{2n}$. Then we say that a step p_i has

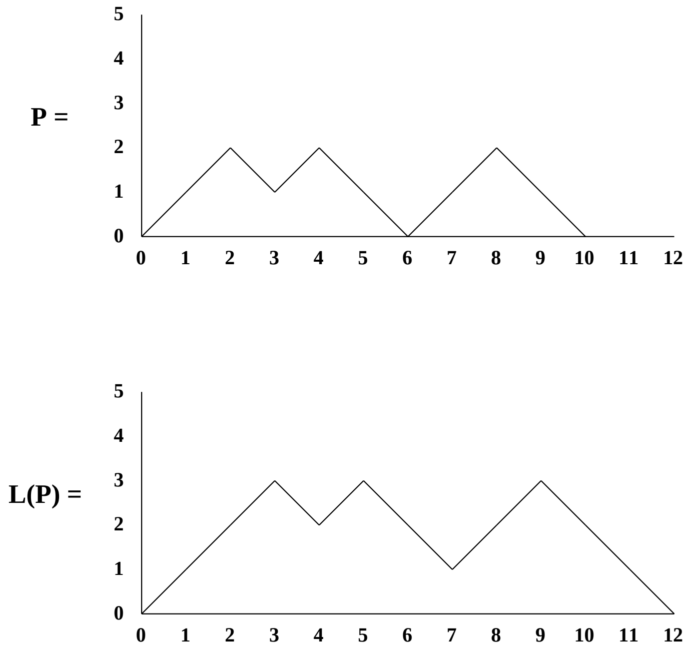


Figure 2: The lifting of a Dyck path.

height s if p_i is an up-step and the right-hand end point of p_i is (i, s) or p_i is a down-step and the left-hand end point of p_i is $(i-1, s)$. We say that $(p_i, \ldots, p_{i+2k-1})$ is an *interval of length* $2k$ if p_i is an up-step, p_{i+2k-1} is a down-step, p_i and p_{i+2k-1} have height 1, and, for all $i < j < i+2k-1$, the height of p_j is strictly greater than 1. Thus, an interval is a segment of the path that starts and ends on the x-axis but does not hit the x-axis in between. For example, if we consider the path $\phi_3(312) = (p_1, \ldots, p_6)$ pictured in Figure 3, then the heights of the steps reading from left to right are $1, 1, 1, 2, 2, 1$ and there are two intervals, one of length 2 consisting of (p_1, p_2) and one of length 4 consisting of (p_3, p_4, p_5, p_6).

The following theorem is straightforward to prove by induction.

Theorem 2. *Let* $k \geq 1$.

1. *For any* $\sigma \in S_n(132)$, $\mathrm{mmp}^{(k,0,0,0)}(\sigma)$ *is equal to the number of up-steps (equivalently, to the number of down-steps) of height* $\geq k+1$ *in* $\phi_n(\sigma)$.

2. *For any* $\sigma \in S_n(132)$, 1 *plus the maximum* k *such that* $\mathrm{mmp}^{(0,0,k,0)}(\sigma) \neq 0$ *is equal to one half the maximum length of an interval in* $\phi_n(\sigma)$.

Proof. We proceed by induction on n. Clearly the theorem is true for $n = 1$. Now suppose that $n > 1$ and the theorem is true for all $m < n$. Let $\sigma \in S_n(132)$ with $\sigma_i = n$. Then it must be the case that $\sigma_1, \ldots, \sigma_{i-1}$ are all strictly bigger than all the elements in $\{\sigma_{i+1}, \ldots, \sigma_n\}$, so $\{1, \ldots, n-i\} = \{\sigma_{i+1}, \ldots, \sigma_n\}$ and $\{n-i+1, \ldots, n\} = \{\sigma_1, \ldots, \sigma_i\}$. Now consider the two cases in the definition of ϕ_n.

Case 1. $\sigma_n = n$.

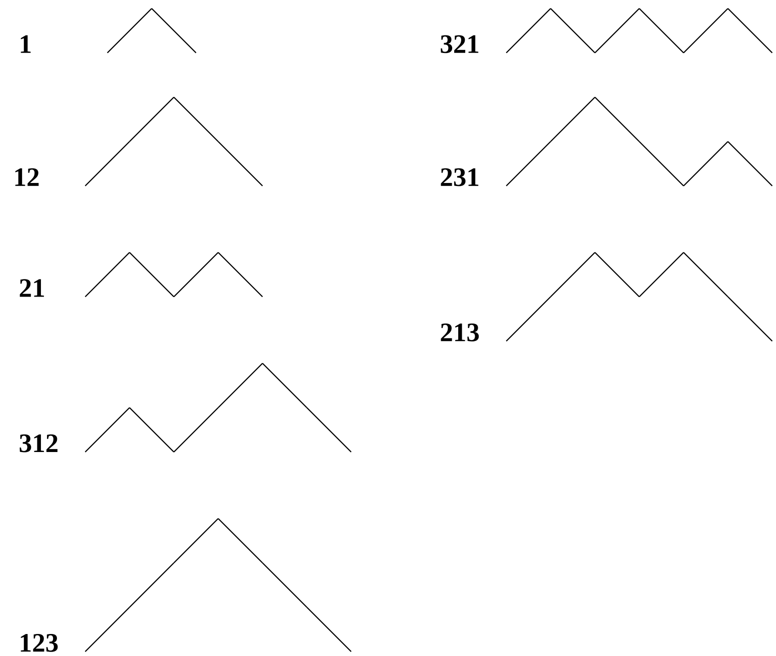


Figure 3: Some initial values of the map ϕ_n.

In this case, $\phi_n(\sigma) = L(P)$, where $P = \phi_{n-1}(\sigma_1 \cdots \sigma_{n-1})$. Thus, for $k \geq 2$, the number of up-steps of height $> k$ in $\phi_n(\sigma)$ equals the number of up-steps of height $\geq k$ in $\phi_{n-1}(\sigma_1 \cdots \sigma_{n-1})$, which equals $\mathrm{mmp}^{(k-1,0,0,0)}(\sigma_1 \cdots \sigma_{n-1})$ by induction. But since $\sigma_n = n$, it is clear that for $k \geq 2$, $\mathrm{mmp}^{(k-1,0,0,0)}(\sigma_1 \cdots \sigma_{n-1}) = \mathrm{mmp}^{(k,0,0,0)}(\sigma)$. Thus, $\mathrm{mmp}^{(k,0,0,0)}(\sigma)$ equals the number of up-steps of height $> k$ in $\phi_n(\sigma)$. Finally, $\mathrm{mmp}^{(1,0,0,0)}(\sigma) = n-1$, and there are $n-1$ up-steps of height ≥ 2 in $\phi_n(\sigma)$.

In this case, the maximum length of an interval in $\phi_n(\sigma)$ equals $2n$ and $\sigma_n = n$ shows that $\mathrm{mmp}^{(0,0,n-1,0)}(\sigma) = 1$, so one half of the maximum length interval in $\phi_n(\sigma)$ equals 1 plus the maximum k such that $\mathrm{mmp}^{(0,0,k,0)}(\sigma) \neq 0$.

Case 2. $\sigma_i = n$, where $1 \leq i \leq n-1$.
In this case, $\phi_n(\sigma) = P_1 P_2$, where $P_1 = \phi_i(\mathrm{red}(\sigma_1 \cdots \sigma_i))$ and $P_2 = \phi_{n-i}(\sigma_{i+1} \cdots \sigma_n)$. It follows that for any $k \geq 1$, the number of up-steps of height $> k$ in $\phi_n(\sigma)$ equals the number of up-steps of height $> k$ in P_1 plus the number of up-steps of height $> k$ in P_2, which by induction is equal to

$$\mathrm{mmp}^{(k,0,0,0)}(\mathrm{red}(\sigma_1 \cdots \sigma_i)) + \mathrm{mmp}^{(k,0,0,0)}(\sigma_{i+1} \cdots \sigma_n).$$

But clearly

$$\mathrm{mmp}^{(k,0,0,0)}(\sigma) = \mathrm{mmp}^{(k,0,0,0)}(\mathrm{red}(\sigma_1 \cdots \sigma_i)) + \mathrm{mmp}^{(k,0,0,0)}(\sigma_{i+1} \cdots \sigma_n),$$

so $\mathrm{mmp}^{(k,0,0,0)}(\sigma)$ is equal to the number of up-steps of height $> k$ in $\phi_n(\sigma)$.

Finally, the maximum length of an interval in $\phi_n(\sigma)$ is the maximum of the maximum length intervals in P_1 and P_2. On the other hand, the maximum k such that

$\mathrm{mmp}^{(0,0,k,0)}(\sigma) \neq 0$ is the maximum k such that $\mathrm{mmp}^{(0,0,k,0)}(\mathrm{red}(\sigma_1 \cdots \sigma_i)) \neq 0$ or $\mathrm{mmp}^{(0,0,k,0)}(\sigma_{i+1} \cdots \sigma_n) \neq 0$. Thus, it follows from the induction hypothesis that one half of the maximum length of an interval in $\phi_n(\sigma)$ is 1 plus the maximum k such that $\mathrm{mmp}^{(0,0,k,0)}(\sigma) \neq 0$. ☐

We have the following corollary to Theorem 2.

Corollary 1. *Let* $k \geq 1$.

1. *The number of permutations* $\sigma \in S_n(132)$ *such that* $\mathrm{mmp}^{(k,0,0,0)}(\sigma) = 0$ *equals the number of Dyck paths* $P \in \mathcal{D}_{2n}$ *such that all steps have height* $\leq k$.

2. *The number of permutations* $\sigma \in S_n(132)$ *such that* $\mathrm{mmp}^{(0,0,k,0)}(\sigma) = 0$ *equals the number of Dyck paths* $P \in \mathcal{D}_{2n}$ *such that the maximum length of an interval is* $\leq 2k$.

Another set counted by the Catalan numbers is the set of rooted binary trees on n nodes where each node is either a leaf, a node with a left child, a node with a right child, or a node with both a right and a left child. Let $\mathcal{B}_n$ denote the set of rooted binary trees with n nodes. Then it is well-known that $|\mathcal{B}_n| = C_n$. In this paper, we shall draw binary trees with their root at the bottom and the tree growing upward. Again it is easy to define a bijection $\theta_n : S_n(132) \rightarrow \mathcal{B}_n$ by induction. Start with a single node, denoted the root, and let i be such that $\sigma_i = n$. Then, if $i > 1$, the root will have a left child, and the subtree above that child is $\theta_{i-1}(\mathrm{red}(\sigma_1 \cdots \sigma_{i-1}))$. If $i < n$, then the root will have a right child, and the subtree above that child is $\theta_{n-i}(\sigma_{i+1} \cdots \sigma_n)$. We have pictured the first few values of this map by listing a permutation σ on the left and the value of $\theta_n(\sigma)$ on the right in Figure 4.

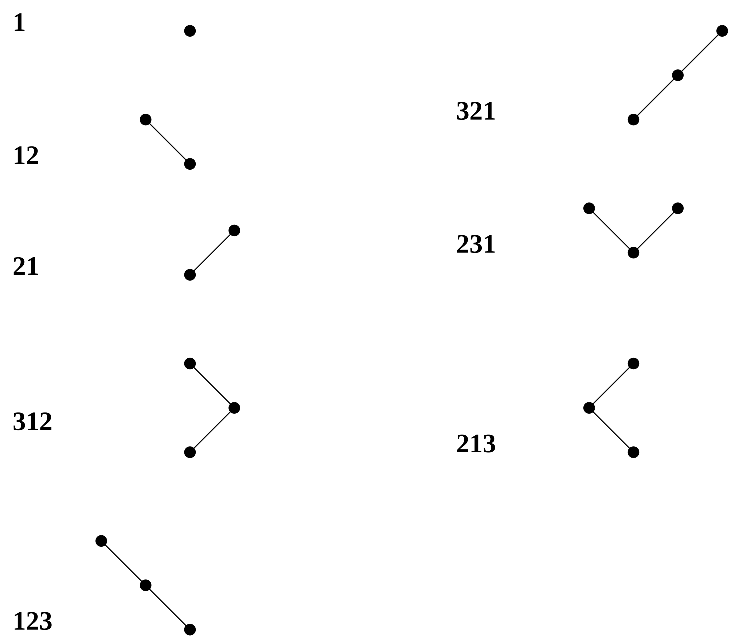


Figure 4: Some initial values of the map θ_n.

If $T \in \mathcal{B}_n$ and η is a node of T, then the left subtree of η is the subtree of T whose root is the left child of η and the right subtree of η is the subtree of T whose root is the right child of η. The edge that connects η to its left child will be called a *left edge* and the edge that connects η to its right child will be called a *right edge*.

The following theorem is straightforward to prove by induction.

Theorem 3. *Let $k \geq 1$.*

1. *For any $\sigma \in S_n(132)$, $\mathrm{mmp}^{(k,0,0,0)}(\sigma)$ is equal to the number of nodes η in $\theta_n(\sigma)$ such that there are $\geq k$ left edges on the path from η to the root of $\theta_n(\sigma)$.*

2. *For any $\sigma \in S_n(132)$, $\mathrm{mmp}^{(0,0,k,0)}(\sigma)$ is the number of nodes η in $\theta_n(\sigma)$ whose left subtree has size $\geq k$.*

Proof. We proceed by induction on n. Clearly the theorem is true for $n = 1$. Now suppose that $n > 1$ and the theorem is true for all $m < n$. Let $\sigma \in S_n(132)$ with $\sigma_i = n$, let r be the root of $\theta_n(\sigma)$, and let η be a node in $\theta_n(\sigma)$.

If η is in r's left subtree, then η has $\geq k$ left edges on the path to r if and only if it has $\geq k-1$ left edges on the path to the root of the left subtree of r. If η is in r's right subtree, then η has $\geq k$ left edges on the path to r if and only if it has $\geq k$ left edges on the path to the root of the right subtree of r. Therefore, by the induction hypothesis the number of nodes with $\geq k$ left edges on the path to the root is $\mathrm{mmp}^{(k-1,0,0,0)}(\mathrm{red}(\sigma_1 \cdots \sigma_{i-1})) + \mathrm{mmp}^{(k,0,0,0)}(\sigma_{i+1} \cdots \sigma_n)$, regarding each term as 0 if there is no corresponding subtree. However, since each term in $\sigma_1 \cdots \sigma_{i-1}$ has n to the right of it and n never matches $MMP(k,0,0,0)$, we see that $\mathrm{mmp}^{(k-1,0,0,0)}(\mathrm{red}(\sigma_1 \cdots \sigma_{i-1})) = \mathrm{mmp}^{(k,0,0,0)}(\mathrm{red}(\sigma_1 \cdots \sigma_i))$. Thus, the number of nodes with $\geq k$ left edges on the path to the root is $\mathrm{mmp}^{(k,0,0,0)}(\mathrm{red}(\sigma_1 \cdots \sigma_i)) + \mathrm{mmp}^{(k,0,0,0)}(\sigma_{i+1} \cdots \sigma_n) = \mathrm{mmp}^{(k,0,0,0)}(\sigma)$.

It is clear that the number of nodes with left subtrees of size $\geq k$ is equal to the sum of those from each subtree of the root, possibly plus one for the root itself. In other words, if $\chi(\text{statement})$ equals 1 if the statement is true and 0 otherwise, then by the induction hypothesis, the number of such nodes is $\mathrm{mmp}^{(0,0,k,0)}(\mathrm{red}(\sigma_1 \cdots \sigma_{i-1})) + \mathrm{mmp}^{(0,0,k,0)}(\sigma_{i+1} \cdots \sigma_n) + \chi(i > k)$, again regarding each term as 0 if there is no corresponding subtree. However, since n does not affect whether any other point matches $MMP(0,0,k,0)$ and itself matches whenever $i > k$, we see this number of nodes is precisely equal to $\mathrm{mmp}^{(0,0,k,0)}(\sigma)$. $\square$

Thus, we have the following corollary.

Corollary 2. *Let $k \geq 1$.*

1. *The number of permutations $\sigma \in S_n(132)$ such that $\mathrm{mmp}^{(k,0,0,0)}(\sigma) = 0$ equals the number of rooted binary trees $T \in \mathcal{B}_n$ that have no nodes η with $\geq k$ left edges on the path from η to the root of T.*

2. *The number of permutations $\sigma \in S_n(132)$ such that $\mathrm{mmp}^{(0,0,k,0)}(\sigma) = 0$ equals the number of rooted binary trees $T \in \mathcal{B}_n$ such that there is no node η of T whose left subtree has size $\geq k$.*

3 The function $Q_{132}^{(k,0,0,0)}(t,x)$

In this section, we shall study the generating function $Q_{132}^{(k,0,0,0)}(t,x)$ for $k \geq 0$.

Throughout this paper, we shall classify the 132-avoiding permutations $\sigma = \sigma_1 \cdots \sigma_n$ by the position of n in σ. Let $S_n^{(i)}(132)$ denote the set of $\sigma \in S_n(132)$ such that $\sigma_i = n$.

Clearly the graph $G(\sigma)$ of each $\sigma \in S_n^{(i)}(132)$ has the structure pictured in Figure 5. That is, in $G(\sigma)$, the elements to the left of n, $A_i(\sigma)$, have the structure of a 132-avoiding permutation, the elements to the right of n, $B_i(\sigma)$, have the structure of a 132-avoiding permutation, and all the elements in $A_i(\sigma)$ lie above all the elements in $B_i(\sigma)$. As mentioned above, $|S_n(132)| = C_n = \frac{1}{n+1}\binom{2n}{n}$. The generating function for these numbers is given by

$$C(t) = \sum_{n\geq 0} C_n t^n = \frac{1-\sqrt{1-4t}}{2t} = \frac{2}{1+\sqrt{1-4t}}. \tag{3}$$

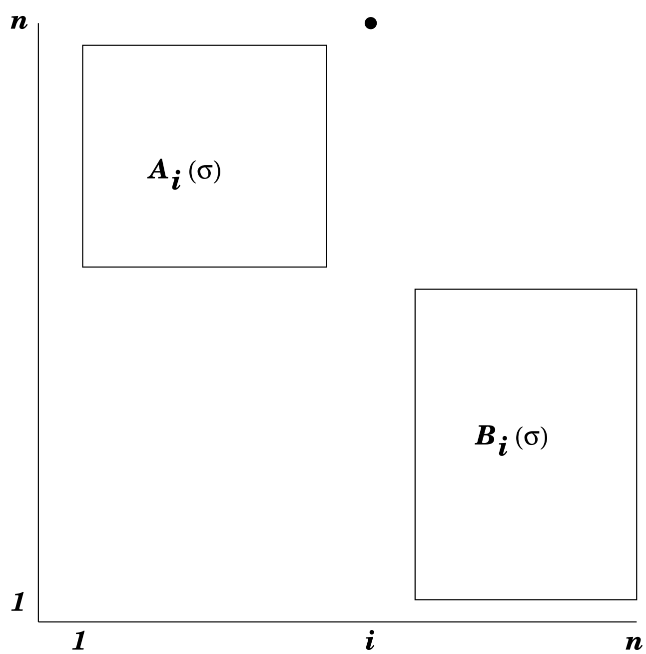


Figure 5: The structure of 132-avoiding permutations.

Clearly,

$$Q_{132}^{(0,0,0,0)}(t,x) = \sum_{n\geq 0} C_n x^n t^n = C(xt) = \frac{1-\sqrt{1-4xt}}{2xt}.$$

Next we consider $Q_{132}^{(k,0,0,0)}(t,x)$ for $k \geq 1$. It is easy to see that $A_i(\sigma)$ will contribute $\mathrm{mmp}^{(k-1,0,0,0)}(\mathrm{red}(A_i(\sigma)))$ to $\mathrm{mmp}^{(k,0,0,0)}(\sigma)$, since each of the elements to the left of n will match the pattern $MMP(k,0,0,0)$ in σ if and only if it matches the pattern $MMP(k-1,0,0,0)$ in the graph of $A_i(\sigma)$. Similarly, $B_i(\sigma)$ will contribute $\mathrm{mmp}^{(k,0,0,0)}(\mathrm{red}(B_i(\sigma)))$

to $\mathrm{mmp}^{(k,0,0,0)}(\sigma)$ because the elements to the left of $B_i(\sigma)$ have no effect on whether an element in $B_i(\sigma)$ matches the pattern $MMP(k,0,0,0)$ in σ. It follows that

$$Q_{n,132}^{(k,0,0,0)}(x) = \sum_{i=1}^{n} Q_{i-1,132}^{(k-1,0,0,0)}(x) Q_{n-i,132}^{(k,0,0,0)}(x). \tag{4}$$

Multiplying both sides of (4) by t^n and summing for $n \geq 1$, we see that

$$-1 + Q_{132}^{(k,0,0,0)}(t,x) = tQ_{132}^{(k-1,0,0,0)}(t,x)\ Q_{132}^{(k,0,0,0)}(t,x).$$

Hence for $k \geq 1$,

$$Q_{132}^{(k,0,0,0)}(t,x) = \frac{1}{1 - tQ_{132}^{(k-1,0,0,0)}(t,x)}.$$

Thus, we have the following theorem.

Theorem 4.

$$Q_{132}^{(0,0,0,0)}(t,x) = C(xt) = \frac{1-\sqrt{1-4xt}}{2xt} \tag{5}$$

and, for $k \geq 1$,

$$Q_{132}^{(k,0,0,0)}(t,x) = \frac{1}{1 - tQ_{132}^{(k-1,0,0,0)}(t,x)}. \tag{6}$$

Theorem 4 immediately implies the following corollary.

Corollary 3.

$$Q_{132}^{(1,0,0,0)}(t,0) = \frac{1}{1-t} \tag{7}$$

and, for $k \geq 2$,

$$Q_{132}^{(k,0,0,0)}(t,0) = \frac{1}{1 - tQ_{132}^{(k-1,0,0,0)}(t,0)}. \tag{8}$$

3.1 Explicit formulas for $Q_{n,132}^{(k,0,0,0)}(x)|_{x^r}$

First we shall consider the problem of computing $Q_{n,132}^{(k,0,0,0)}(x)|_{x^0}$. That is, we shall be interested in the generating functions $Q_{132}^{(k,0,0,0)}(t,0)$. Using Corollary 3, one can easily compute that

$$
\begin{aligned}
Q_{132}^{(2,0,0,0)}(t,0) &= \frac{1-t}{1-2t}, \\
Q_{132}^{(3,0,0,0)}(t,0) &= \frac{1-2t}{1-3t+t^2}, \\
Q_{132}^{(4,0,0,0)}(t,0) &= \frac{1-3t+t^2}{1-4t+3t^2}, \\
Q_{132}^{(5,0,0,0)}(t,0) &= \frac{1-4t+3t^2}{1-5t+6t^2-t^3}, \\
Q_{132}^{(6,0,0,0)}(t,0) &= \frac{1-5t+6t^2-t^3}{1-6t+10t^2-4t^3}, \text{and} \\
Q_{132}^{(7,0,0,0)}(t,0) &= \frac{1-6t+10t^3-4t^3}{1-7t+15t^2-10t^3+t^4}.
\end{aligned}
$$

By Corollary 1, $Q_{132}^{(k,0,0,0)}(t,0)$ is also the generating function for the number of Dyck paths whose maximum height is less than or equal to k. For example, this interpretation is given to sequence A080937 in the OEIS, which is the sequence $(Q_{n,132}^{(5,0,0,0)}(0))_{n\geq 0}$, and to sequence A080938 in the OEIS, which is the sequence $(Q_{n,132}^{(7,0,0,0)}(0))_{n\geq 0}$. However, similar interpretations are not given to $(Q_{n,132}^{(k,0,0,0)}(0))_{n\geq 0}$, where $k \notin \{5,7\}$. For example, such an interpretation is not found for $(Q_{n,132}^{(2,0,0,0)}(0))_{n\geq 0}$, $(Q_{n,132}^{(3,0,0,0)}(0))_{n\geq 0}$, $(Q_{n,132}^{(4,0,0,0)}(0))_{n\geq 0}$, or $(Q_{n,132}^{(6,0,0,0)}(0))_{n\geq 0}$, which are sequences A011782, A001519, A124302, and A024175 in the OEIS, respectively. Similarly, by Corollary 2, the generating function $Q_{132}^{(k,0,0,0)}(t,0)$ is the generating function for the number of rooted binary trees T that have no nodes η such that there are $\geq k$ left edges on the path from η to the root of T.

We can easily compute the first few terms of $Q_{132}^{(k,0,0,0)}(t,x)$ for small k using Mathematica. For example, we have computed the following.

$$
\begin{aligned}
&Q_{132}^{(1,0,0,0)}(t,x) = 1+t+(1+x)t^2+\left(1+2x+2x^2\right)t^3+\left(1+3x+5x^2+5x^3\right)t^4 \\
&\left(1+4x+9x^2+14x^3+14x^4\right)t^5+\left(1+5x+14x^2+28x^3+42x^4+42x^5\right)t^6+ \\
&\left(1+6x+20x^2+48x^3+90x^4+132x^5+132x^6\right)t^7+ \\
&\left(1+7x+27x^2+75x^3+165x^4+297x^5+429x^6+429x^7\right)t^8+ \\
&\left(1+8x+35x^2+110x^3+275x^4+572x^5+1001x^6+1430x^7+1430x^8\right)t^9+\cdots.
\end{aligned}
$$

In this case, it is quite easy to explain some of the coefficients that appear in the polynomials $Q_{n,132}^{(1,0,0,0)}(x)$. Some of these explanations are given in the following theorem.

Theorem 5. *1.* $Q_{n,132}^{(1,0,0,0)}(0) = 1$ *for* $n \geq 1$,

2. $Q_{n,132}^{(1,0,0,0)}(x)|_x = n-1$ *for* $n \geq 2$,

3. $Q_{n,132}^{(1,0,0,0)}(x)|_{x^2} = \binom{n}{2}-1$ *for* $n \geq 3$,

4. $Q_{n,132}^{(1,0,0,0)}(x)|_{x^{n-1}} = C_{n-1}$ *for* $n \geq 1$*, and*

5. $Q_{n,132}^{(1,0,0,0)}(x)|_{x^{n-2}} = C_{n-1}$ *for* $n \geq 2$.

Proof. There is only one permutation $\sigma \in S_n$ with $\mathrm{mmp}^{(1,0,0,0)}(\sigma) = 0$, namely, $\sigma = n(n-1)\cdots 1$. Thus, the constant term in $Q_{n,132}^{(1,0,0,0)}(x)$ is always 1. Also the only way to get a permutation $\sigma \in S_n$ that has $\mathrm{mmp}^{(1,0,0,0)}(\sigma) = n-1$ is to have $\sigma_n = n$. It follows that the coefficient of x^{n-1} in $Q_{n,132}^{(1,0,0,0)}(x)$ is the number of permutations $\sigma \in S_n(132)$ such that $\sigma_n = n$, which is clearly C_{n-1}. It is also easy to see that the only permutations $\sigma \in S_n(132)$ with $\mathrm{mmp}^{(1,0,0,0)}(\sigma) = 1$ are the permutations of the form

$$\sigma = n(n-1)\cdots(i+1)(i-1)i(i-2)\cdots 21.$$

Thus, the coefficient of x in $Q_{n,132}^{(1,0,0,0)}(x)$ is always $n-1$.

For (3), note that we have $Q_{3,132}^{(1,0,0,0)}(x)|_{x^2} = 2 = \binom{3}{2} - 1$. For $n \geq 4$, let $a(n)$ denote the coefficient of x^2 in $Q_{n,132}^{(1,0,0,0)}(x)$. The permutations $\sigma \in S_n(132)$ such that $\mathrm{mmp}^{(1,0,0,0)}(\sigma) = 2$ must have either $\sigma_1 = n$, $\sigma_2 = n$, or $\sigma_3 = n$. If $\sigma_3 = n$, it must be the case that $\{\sigma_1, \sigma_2\} = \{n-1, n-2\}$ and that $\mathrm{mmp}^{(1,0,0,0)}(\sigma_4 \cdots \sigma_n) = 0$. Thus, $\sigma_4 \cdots \sigma_n$ must be decreasing, so there are exactly two permutations $\sigma \in S_n(132)$ such that $\sigma_3 = n$ and $\mathrm{mmp}^{(1,0,0,0)}(\sigma) = 2$. If $\sigma_2 = n$, it must be the case that $\sigma_1 = n-1$ and that $\mathrm{mmp}^{(1,0,0,0)}(\sigma_3 \cdots \sigma_n) = 1$. In that case, we know that there are $n-3$ choices for $\sigma_3 \cdots \sigma_n$, so there are $n-3$ permutations $\sigma \in S_n(132)$ such that $\sigma_2 = n$ and $\mathrm{mmp}^{(1,0,0,0)}(\sigma) = 2$. Finally, it is clear that if $\sigma_1 = n$, then we must have that $\mathrm{mmp}^{(1,0,0,0)}(\sigma_2 \cdots \sigma_n) = 2$, so there are $a(n-1)$ permutations $\sigma \in S_n(132)$ such that $\sigma_1 = n$ and $\mathrm{mmp}^{(1,0,0,0)}(\sigma) = 2$. Thus, we have shown that $a(n) = a(n-1) + n - 1$ from which it easily follows by induction that $a(n) = \binom{n}{2} - 1$.

Finally, for (5), let $\sigma = \sigma_1 \cdots \sigma_n \in S_n(132)$ be such that $\mathrm{mmp}^{(1,0,0,0)}(\sigma) = n-2$. We clearly cannot have $\sigma_n = n$, so n and σ_n must be the two elements of σ that do not match the pattern $MMP(1,0,0,0)$ in σ. Now if $\sigma_i = n$, then $B_i(\sigma)$ consists of the elements $1, \ldots, n-i$. But then it must be the case that $\sigma_n = n-i$. Note that this implies that σ_n can be removed from σ in a completely reversible way. That is, $\sigma \to \mathrm{red}(\sigma_1 \cdots \sigma_{n-1})$ is a bijection onto $S_{n-1}(132)$. Hence there are C_{n-1} such σ. $\square$

We have computed that

$$\begin{aligned}
&Q_{132}^{(2,0,0,0)}(t,x) = 1 + t + 2t^2 + (4+x)t^3 + \left(8 + 4x + 2x^2\right)t^4 + \\
&\left(16 + 12x + 9x^2 + 5x^3\right)t^5 + \left(32 + 32x + 30x^2 + 24x^3 + 14x^4\right)t^6 + \\
&\left(64 + 80x + 88x^2 + 85x^3 + 70x^4 + 42x^5\right)t^7 + \\
&\left(128 + 192x + 240x^2 + 264x^3 + 258x^4 + 216x^5 + 132x^6\right)t^8 + \\
&\left(256 + 448x + 624x^2 + 760x^3 + 833x^4 + 819x^5 + 693x^6 + 429x^7\right)t^9 + \cdots.
\end{aligned}$$

Again it is easy to explain some of these coefficients. That is, we have the following theorem.

Theorem 6. *1.* $Q_{n,132}^{(2,0,0,0)}(0) = 2^{n-1}$ *if* $n \geq 3$,

2. *for $n \geq 3$, the highest power of x that appears in $Q_{n,132}^{(2,0,0,0)}(x)$ is x^{n-2}, with $Q_{n,132}^{(2,0,0,0)}(x)|_{x^{n-2}} = C_{n-2}$, and*

3. *$Q_{n,132}^{(2,0,0,0)}(x)|_x = (n-2)2^{n-3}$ for $n \geq 3$.*

Proof. It is easy to see that the only $\sigma \in S_n(132)$ that have $\mathrm{mmp}^{(2,0,0,0)}(\sigma) = n-2$ must have $\sigma_{n-1} = n-1$ and $\sigma_n = n$. Note that if $\sigma_{n-1} = n$ and $\sigma_n = n-1$ then we have an occurrence of 132 for $n \geq 3$. Thus, the coefficient of x^{n-2} in $Q_{n,132}^{(2,0,0,0)}(x)$ is C_{n-2} if $n \geq 3$.

The fact that $Q_{n,132}^{(2,0,0,0)}(0) = 2^{n-1}$ for $n \geq 1$ is an immediate consequence of the fact that $Q_{132}^{(2,0,0,0)}(t,0) = \frac{1-t}{1-2t}$. In fact, this is a known result, since avoidance of the pattern $MMP(2,0,0,0)$ is equivalent to avoiding simultaneously the (classical) patterns 132 and 123 (see [5, p. 224]). One can also give a simple combinatorial proof of this fact. Clearly it is true for $n = 1$. For $n \geq 2$, note that σ_1 must be either n or $n-1$. Also, $\mathrm{red}(\sigma_2 \cdots \sigma_n)$ must avoid the pattern $MMP(2,0,0,0)$. Since every permutation $\mathrm{red}(\sigma_2 \cdots \sigma_n)$ avoiding $MMP(2,0,0,0)$ can be obtained in this manner in exactly two ways, once with $\sigma_1 = n$ and once with $\sigma_n = n-1$, we see that there are $2 \cdot 2^{n-2} = 2^{n-1}$ such σ.

The initial terms of the sequence $(Q_{132}^{(2,0,0,0)}(t,x)|_x)_{n \geq 3}$ are

$$1, 4, 12, 32, 80, 192, 448, \ldots,$$

which are the initial terms of sequence A001787 in OEIS whose n-th term is $a_n = n2^{n-1}$. Now a_n has many combinatorial interpretations including the number of edges in the n-dimensional hypercube and the number of permutations in $S_{n+2}(132)$ with exactly one occurrence of the pattern 123. The ordinary generating function of the sequence is $\frac{x}{(1-2x)^2}$, which implies that

$$Q_{132}^{(2,0,0,0)}(t,x)|_x = \frac{t^3}{(1-2t)^2}.$$

This can be proved in two different ways. That is, for any $k \geq 2$,

$$\begin{aligned}
Q_{132}^{(k,0,0,0)}(t,x)|_x &= \left(\frac{1}{1-tQ_{132}^{(k-1,0,0,0)}(t,x)}\right)|_x \\
&= \left(1 + \sum_{n\geq 1} t^n (Q_{132}^{(k-1,0,0,0)}(t,x))^n\right)|_x \\
&= \sum_{n\geq 1} n t^n (Q_{132}^{(k-1,0,0,0)}(t,0))^{n-1} Q_{132}^{(k-1,0,0,0)}(t,x)|_x \\
&= Q_{132}^{(k-1,0,0,0)}(t,x)|_x \sum_{n\geq 1} n t^n (Q_{132}^{(k-1,0,0,0)}(t,0))^{n-1}. \qquad (9)
\end{aligned}$$

However,

$$\begin{aligned}
\frac{d}{dt} Q_{132}^{(k,0,0,0)}(t,0) &= \frac{d}{dt}\left(\frac{1}{1-tQ_{132}^{(k-1,0,0,0)}(t,0)}\right) \\
&= \sum_{n\geq 1} n (tQ_{132}^{(k-1,0,0,0)}(t,0))^{n-1} \frac{d}{dt}\left(tQ_{132}^{(k-1,0,0,0)}(t,0)\right),
\end{aligned}$$

so

$$\frac{t\frac{d}{dt}Q_{132}^{(k,0,0,0)}(t,0)}{\frac{d}{dt}\left(tQ_{132}^{(k-1,0,0,0)}(t,0)\right)} = \sum_{n\geq 1} nt^n (Q_{132}^{(k-1,0,0,0)}(t,0))^{n-1}. \tag{10}$$

Combining (9) and (10), we obtain the following recursion.

Theorem 7. *For $k \geq 1$,*

$$Q_{132}^{(k,0,0,0)}(t,x)|_x = Q_{132}^{(k-1,0,0,0)}(t,x)|_x \frac{t\frac{d}{dt}Q_{132}^{(k,0,0,0)}(t,0)}{\frac{d}{dt}\left(tQ_{132}^{(k-1,0,0,0)}(t,0)\right)}. \tag{11}$$

We know that

$$Q_{132}^{(1,0,0,0)}(t,x)|_x = \sum_{n\geq 2}(n-1)t^n = \frac{t^2}{(1-t)^2}$$

and

$$Q_{132}^{(1,0,0,0)}(t,0) = \frac{1}{1-t} \text{ and } Q_{132}^{(2,0,0,0)}(t,0) = \frac{1-t}{1-2t}.$$

Thus,

$$\begin{aligned} Q_{132}^{(2,0,0,0)}(t,x)|_x &= Q_{132}^{(1,0,0,0)}(t,x)|_x \frac{t\frac{d}{dt}Q_{132}^{(2,0,0,0)}(t,0)}{\frac{d}{dt}\left(tQ_{132}^{(1,0,0,0)}(t,0)\right)} \\ &= \frac{t^2}{(1-t)^2}\frac{t\frac{d}{dt}\left(\frac{1-t}{1-2t}\right)}{\frac{d}{dt}\frac{t}{1-t}} \\ &= \frac{t^3}{(1-2t)^2}. \end{aligned}$$

We can also give a direct proof of this result. That is, we can give a direct proof of the fact that for $n \geq 3$, $b(n) = Q_{n,132}^{(2,0,0,0)}(x)|_x = (n-2)2^{n-3}$. Note that $b(3) = 1 = (3-2)2^{3-3}$ and $b(4) = (4-2)2^{4-3} = 4$, so our claim holds for $n = 3, 4$. Then let $n \geq 5$ and assume by induction that $b(k) = (k-2)2^{k-3}$ for $3 \leq k < n$. Now suppose that $\sigma \in S_n^{(i)}(132)$ and $\text{mmp}^{(2,0,0,0)} = 1$. If the element of σ that matches $MMP(2,0,0,0)$ occurs in $A_i(\sigma)$, then it must be the case that $\text{mmp}^{(1,0,0,0)}(A_i(\sigma)) = 1$ and $\text{mmp}^{(2,0,0,0)}(B_i(\sigma)) = 0$. By our previous results, we have $(i-2)$ choices for $A_i(\sigma)$ and $a(n-i) = 2^{n-i-1}$ choices for $B_i(\sigma)$. Note that this can happen only for $3 \leq i \leq n-1$, so such permutations contribute

$$\sum_{i=3}^{n-1}(i-2)2^{n-i-1} = \sum_{j=1}^{n-3} j2^{n-3-j} = \sum_{k=0}^{n-4}(n-3-k)2^k$$

to $b(n)$. If the element of σ that matches $MMP(2,0,0,0)$ occurs in $B_i(\sigma)$, then we have $\text{mmp}^{(1,0,0,0)}(A_i(\sigma)) = 0$, which means that $A_i(\sigma)$ is decreasing and $\text{mmp}^{(2,0,0,0)}(B_i(\sigma)) = 1$. This can happen only for $1 \leq i \leq n-3$. Thus, such permutations will contribute

$$b(3) + \cdots + b(n-1) = \sum_{i=3}^{n-1}(i-2)2^{(i-3)} = \sum_{k=0}^{n-4}(k+1)2^k$$

to $b(n)$. The only permutations that we have not accounted for are the permutations $\sigma = \sigma_1 \cdots \sigma_n \in S_n(132)$ where $\sigma_n = n$ and $\text{mmp}^{(1,0,0,0)}(\sigma_1 \cdots \sigma_{n-1}) = 1$, and there are $n-2$ such permutations. Thus,

$$\begin{aligned} b(n) &= (n-2) + \sum_{k=0}^{n-4} 2^k (n-3-k+k+1) \\ &= (n-2)\left(1 + \sum_{k=0}^{n-4} 2^k\right) \\ &= (n-2)(1+2^{n-3}-1) = (n-2)2^{n-3}. \end{aligned}$$

□

We can also explain the coefficient of second highest power of x that appears in $Q_{n,132}^{(k,0,0,0)}(x)$ for $k \geq 2$.

Theorem 8. *For all $k \geq 2$ and $n \geq k+2$,*

$$Q_{n,132}^{(k,0,0,0)}(x)|_{x^{n-k-1}} = C_{n-k} + 2(k-1)C_{n-k-1}. \tag{12}$$

Proof. We first consider the case $k = 2$. That is, we must compute $Q_{n,132}^{(2,0,0,0)}(x)|_{x^{n-3}}$. In this case,

$$Q_{n,132}^{(2,0,0,0)}(x) = \sum_{i=1}^{n} Q_{i-1,132}^{(1,0,0,0)}(x) Q_{n-i,132}^{(2,0,0,0)}(x).$$

We have shown that for $n \geq 1$, the highest power of x that occurs in $Q_{n,132}^{(1,0,0,0)}(x)$ is x^{n-1} and, for $n \geq 2$, the highest power of x that occurs in $Q_{n,132}^{(2,0,0,0)}(x)$ is x^{n-2}. It follows that for $i = 2, \ldots, n-2$, the highest power of x which occurs in $Q_{i-1,132}^{(1,0,0,0)}(x) Q_{n-i,132}^{(2,0,0,0)}(x)$ is less than $n-3$ so that we have only three cases to consider.

Case 1. $i = 1$.
In this case, $Q_{i-1,132}^{(1,0,0,0)}(x) Q_{n-i,132}^{(2,0,0,0)}(x) = Q_{n-1,132}^{(2,0,0,0)}(x)$ and we know that

$$Q_{n-1,132}^{(2,0,0,0)}(x)|_{x^{n-3}} = C_{n-3} \text{ for } n \geq 4.$$

Case 2. $i = n-1$.
In this case, $Q_{i-1,132}^{(1,0,0,0)}(x) Q_{n-i,132}^{(2,0,0,0)}(x) = Q_{n-2,132}^{(1,0,0,0)}(x)$ and we know that

$$Q_{n-2,132}^{(2,0,0,0)}(x)|_{x^{n-3}} = C_{n-3} \text{ for } n \geq 4.$$

Case 3. $i = n$.
In this case, $Q_{i-1,132}^{(1,0,0,0)}(x) Q_{n-i,132}^{(2,0,0,0)}(x) = Q_{n-1,132}^{(1,0,0,0)}(x)$ and we know that

$$Q_{n-1,132}^{(1,0,0,0)}(x)|_{x^{n-3}} = C_{n-2} \text{ for } n \geq 4.$$

Thus for $n \geq 4$, $Q_{n,132}^{(2,0,0,0)}(x)|_{x^{n-3}} = C_{n-2} + 2C_{n-3}$.

Now suppose that $k \geq 3$ and we have proved by induction that $Q_{n,132}^{(k-1,0,0,0)}(x)|_{x^{n-k}} = C_{n-k+1} + 2(k-2)C_{n-k}$ for $n \geq k+1$. In this case,

$$Q_{n,132}^{(k,0,0,0)}(x) = \sum_{i=1}^{n} Q_{i-1,132}^{(k-1,0,0,0)}(x) Q_{n-i,132}^{(k,0,0,0)}(x).$$

We have shown that for $n \geq k$, the highest power of x that occurs in $Q_{n,132}^{(k-1,0,0,0)}(x)$ is x^{n-k+1} and, for $n \geq k+1$, the highest power of x that occurs in $Q_{n,132}^{(k,0,0,0)}(x)$ is x^{n-k}. It is easy to check that for $i = 2, \ldots, n-2$, the highest power of x which occurs in $Q_{i-1,132}^{(1,0,0,0)}(x) Q_{n-i,132}^{(2,0,0,0)}(x)$ is less that $n-k-1$ so that we have only three cases to consider.

Case 1. $i = 1$.
In this case, $Q_{i-1,132}^{(k-1,0,0,0)}(x) Q_{n-i,132}^{(k,0,0,0)}(x) = Q_{n-1,132}^{(k,0,0,0)}(x)$ and we know that

$$Q_{n-1,132}^{(k,0,0,0)}(x)|_{x^{n-k-1}} = C_{n-1-k} \text{ for } n \geq k+2.$$

Case 2. $i = n-1$.
In this case, $Q_{i-1,132}^{(k-1,0,0,0)}(x) Q_{n-i,132}^{(k,0,0,0)}(x) = Q_{n-2,132}^{(k-1,0,0,0)}(x)$ and we know that

$$Q_{n-2,132}^{(k-1,0,0,0)}(x)|_{x^{n-k-1}} = C_{n-k-1} \text{ for } n \geq k+2.$$

Case 3. $i = n$.
In this case, $Q_{i-1,132}^{(k-1,0,0,0)}(x) Q_{n-i,132}^{(k,0,0,0)}(x) = Q_{n-1,132}^{(k-1,0,0,0)}(x)$ and we know by induction that

$$Q_{n-1,132}^{(k-1,0,0,0)}(x)|_{x^{n-k-1}} = C_{n-k} + 2(k-2)C_{n-k-1} \text{ for } n \geq k+2.$$

Thus for $n \geq k+2$, $Q_{n,132}^{(k,0,0,0)}(x)|_{x^{n-k-1}} = C_{n-k} + 2(k-1)C_{n-k-1}$.

□

We note that the sequence $(Q_{n,132}^{(2,0,0,0)}(x)|_{x^{n-3}})_{n\geq 4}$ is sequence A038629 in the OEIS which previously had no combinatorial interpretation. The sequences $(Q_{n,132}^{(3,0,0,0)}(x)|_{x^{n-4}})_{n\geq 5}$ and $(Q_{n,132}^{(4,0,0,0)}(x)|_{x^{n-5}})_{n\geq 6}$ do not appear in the OEIS.

We have computed that

$$\begin{aligned}
&Q_{132}^{(3,0,0,0)}(t,x) = 1 + t + 2t^2 + 5t^3 + (13+x)t^4 + \left(34 + 6x + 2x^2\right) t^5 + \\
&\left(89 + 25x + 13x^2 + 5x^3\right) t^6 + \left(233 + 90x + 58x^2 + 34x^3 + 14x^4\right) t^7 + \\
&\left(610 + 300x + 222x^2 + 158x^3 + 98x^4 + 42x^5\right) t^8 + \\
&\left(1597 + 954x + 783x^2 + 628x^3 + 468x^4 + 300x^5 + 132x^6\right) t^9 + \cdots .
\end{aligned}$$

The sequence $(Q_{n,132}^{(3,0,0,0)}(0))_{n\geq 0}$ is sequence A001519 in the OEIS whose terms satisfy the recursion $a(n) = 3a(n-1) - a(n-2)$ with $a(0) = a(1) = 1$. That is, since $Q_{132}^{(3,0,0,0)}(t,0) = \frac{1-2t}{1-3t+t^2}$, it is easy to see that for $n \geq 2$,

$$Q_{n,132}^{(3,0,0,0)}(0) = 3Q_{n-1,132}^{(3,0,0,0)}(0) - Q_{n-2,132}^{(3,0,0,0)}(0) \tag{13}$$

with $Q_{0,132}^{(3,0,0,0)}(0) = Q_{1,132}^{(3,0,0,0)}(0) = 1$.

Avoidance of $MMP(3,0,0,0)$ is equivalent to avoiding the six (classical) patterns of length 4 beginning with 1 as well as the pattern 132, which, in turn, is equivalent to avoidance of 132 and 1234 simultaneously. Even though A001519 in the OEIS gives a relevant combinatorial interpretation as the number of permutations $\sigma \in S_{n+1}$ that avoid the patterns 321 and 3412 simultaneously, our enumeration of permutations avoiding at the same time 132 and 1234 seems to be new thus extending the results in Table 6.3 in [5].

Problem 1. Can one give a combinatorial proof of (13)?

Problem 2. Do any of the known bijections between $S_n(132)$ and $S_n(321)$ (see [5, Chapter 4]) send $(132, 1234)$-avoiding permutations to $(321, 3412)$-avoiding permutations? If not, find such a bijection.

The sequence $(Q_{n,132}^{(3,0,0,0)}(x)|_x)_{n\geq 4}$ is sequence A001871 in the OEIS, which has the generating function $\frac{1}{(1-3x+x^2)^2}$. The nth term of this sequence counts the number of 3412-avoiding permutations containing exactly one occurrence of the pattern 321. We can use the recursion (11) to prove that these sequences are the same. That is,

$$\begin{aligned}
Q_{132}^{(3,0,0,0)}(t,x)|_x &= Q_{132}^{(2,0,0,0)}(t,x)|_x \frac{t\frac{d}{dt}Q_{132}^{(3,0,0,0)}(t,0)}{\frac{d}{dt}\left(tQ_{132}^{(2,0,0,0)}(t,0)\right)} \\
&= \frac{t^3}{(1-2t)^2} \cdot \frac{t\frac{d}{dt}\left(\frac{1-2t}{1-3t+t^2}\right)}{\frac{d}{dt}\frac{t(1-t)}{1-2t}} \\
&= \frac{t^4}{(1-3t-t^2)^2}.
\end{aligned}$$

We have computed that

$$\begin{aligned}
&Q_{132}^{(4,0,0,0)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + (41+x)t^5 + \left(122 + 8x + 2x^2\right)t^6 + \\
&\left(365 + 42x + 17x^2 + 5x^3\right)t^7 + \left(1094 + 184x + 94x^2 + 44x^3 + 14x^4\right)t^8 + \\
&\left(3281 + 731x + 431x^2 + 251x^3 + 126x^4 + 42x^5\right)t^9 + \cdots.
\end{aligned}$$

The sequence $(Q_{132}^{(4,0,0,0)}(t,0))_{n\geq 1}$ is A007051 in the OEIS. It is easy to compute that

$$\begin{aligned}
Q_{132}^{(4,0,0,0)}(t,0) &= \frac{1-3t+t^2}{1-4t+3t^2} \\
&= \frac{1-3t+t^2}{(1-t)(1-3t)} \\
&= 1 + \sum_{n\geq 1} \frac{3^{n-1}+1}{2} t^n.
\end{aligned}$$

Thus, for $n \geq 1$, $Q_{n,132}^{(4,0,0,0)}(0) = \frac{3^{n-1}+1}{2}$, which also counts the number of ordered trees with $n-1$ edges and height at most 4.

The sequence $(Q_{132}^{(4,0,0,0)}(t,x)|_x)_{n\geq 5}$, whose initial terms are

$$1, 8, 42, 184, 731, \ldots,$$

does not appear in the OEIS. However, we can use the recursion (11) to find its generating function. That is,

$$\begin{aligned} Q_{132}^{(4,0,0,0)}(t,x)|_x &= Q_{132}^{(3,0,0,0)}(t,x)|_x \frac{t\frac{d}{dt}Q_{132}^{(4,0,0,0)}(t,0)}{\frac{d}{dt}\left(tQ_{132}^{(3,0,0,0)}(t,0)\right)} \\ &= \frac{t^4}{(1-3t+t^2)^2}\frac{t\frac{d}{dt}\left(\frac{1-3t+t^2}{1-4t+3t^2}\right)}{\frac{d}{dt}\frac{t(1-2t)}{1-3t+t^2}} \\ &= \frac{t^5}{(1-4t+3t^2)^2}. \end{aligned}$$

4 The function $Q_{132}^{(0,0,k,0)}(t,x)$

In this section, we shall study the generating function $Q_{132}^{(0,0,k,0)}(t,x)$ for $k \geq 1$. Fix $k \geq 1$. It is easy to see that $A_i(\sigma)$ will contribute $\mathrm{mmp}^{(0,0,k,0)}(\mathrm{red}(A_i(\sigma)))$ to $\mathrm{mmp}^{(0,0,k,0)}(\sigma)$, since neither n nor any of the elements to the right of n have any effect on whether an element in $A_i(\sigma)$ matches the pattern $MMP(0,0,k,0)$ in σ. Similarly, $B_i(\sigma)$ will contribute $\mathrm{mmp}^{(0,0,k,0)}(\mathrm{red}(B_i(\sigma)))$ to $\mathrm{mmp}^{(0,0,k,0)}(\sigma)$, since neither n nor any of the elements to the left of n have any effect on whether an element in $B_i(\sigma)$ matches the pattern $MMP(0,0,k,0)$ in σ. Note that n will contribute 1 to $\mathrm{mmp}^{(0,0,k,0)}$ if and only if $k < i$.

It follows that

$$Q_{n,132}^{(0,0,k,0)}(x) = \sum_{i=1}^{k} Q_{i-1,132}^{(0,0,k,0)}(x)Q_{n-i,132}^{(0,0,k,0)}(x) + x\sum_{i=k+1}^{n} Q_{i-1,132}^{(0,0,k,0)}(x)Q_{n-i,132}^{(0,0,k,0)}(x). \tag{14}$$

Note that if $i \leq k$, $Q_{i-1,132}^{(0,0,k,0)}(x) = C_{i-1}$. Thus,

$$Q_{n,132}^{(0,0,k,0)}(x) = \sum_{i=1}^{k} C_{i-1}Q_{n-i,132}^{(0,0,k,0)}(x) + x\sum_{i=k+1}^{n} Q_{i-1,132}^{(0,0,k,0)}(x)Q_{n-i,132}^{(0,0,k,0)}(x). \tag{15}$$

Multiplying both sides of (15) by t^n and summing for $n \geq 1$ shows that

$$\begin{aligned} &-1 + Q_{132}^{(0,0,k,0)}(t,x) = \\ &\qquad t(C_0 + C_1t + \cdots + C_{k-1}t^{k-1})Q_{132}^{(0,0,k,0)}(t,x) + \\ &\qquad\quad txQ_{132}^{(0,0,k,0)}(t,x)(Q_{132}^{(0,0,k,0)}(t,x) - (C_0 + C_1t + \cdots + C_{k-1}t^{k-1})). \end{aligned}$$

Thus, we obtain the quadratic equation

$$0 = 1-(-1+(t-tx)(C_0+C_1t+\cdots+C_{k-1}t^{k-1}))Q_{132}^{(0,0,k,0)}(t,x)+tx(Q_{132}^{(0,0,k,0)}(t,x))^2. \quad (16)$$

This implies the following theorem.

Theorem 9. *For $k \geq 1$,*

$$\begin{aligned} Q_{132}^{(0,0,k,0)}(t,x) &= \frac{1+(tx-t)(\sum_{j=0}^{k-1} C_j t^j) - \sqrt{(1+(tx-t)(\sum_{j=0}^{k-1} C_j t^j))^2-4tx}}{2tx} \quad (17)\\ &= \frac{2}{1+(tx-t)(\sum_{j=0}^{k-1} C_j t^j) + \sqrt{(1+(tx-t)(\sum_{j=0}^{k-1} C_j t^j))^2-4tx}} \end{aligned}$$

and

$$Q_{132}^{(0,0,k,0)}(t,0) = \frac{1}{1-t(C_0+C_1t+\cdots+C_{k-1}t^{k-1})}. \quad (18)$$

By Corollary 1, $Q_{132}^{(0,0,k,0)}(t,0)$ is also the generating function of the number of Dyck paths that have no interval of length $\geq 2k$ and the generating function of the number of rooted binary trees T such that T has no node η whose left subtree has size $\geq k$.

4.1 Explicit formulas for $Q_{n,132}^{(0,0,k,0)}(x)|_{x^r}$

It is easy to explain the highest power and the second highest power of x that occurs in $Q_{n,132}^{(0,0,k,0)}(x)$ for any $k \geq 1$. The case is $k = 1$ is special and will be handled in the theorem following our next theorem which handles the cases where $k \geq 2$.

Theorem 10. *For all $k \geq 2$ and $n > k$,*

1. *the highest power of x that occurs in $Q_{n,132}^{(0,0,k,0)}(x)$ is x^{n-k}, with $Q_{n,132}^{(0,0,k,0)}(x)|_{x^{n-k}} = C_k$, and*

2. *$Q_{n,132}^{(0,0,k,0)}(x)|_{x^{n-k-1}} = C_{k+1} - C_k + 2(n-k-1)C_k$.*

Proof. For (1), it is easy to see that, for any $k \geq 1$, the maximum number of $MMP(0,0,k,0)$-matches occurs in a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n(132)$ only when $\sigma_1 \cdots \sigma_k \in S_k(132)$ and $\sigma_{k+1} \cdots \sigma_n = (k+1)(k+2)\cdots n$. Thus, $Q_{n,132}^{(0,0,k,0)}(x)|_{x^{n-k}} = C_k$ for $n \geq k+1$.

For (2), suppose that $k \geq 3$, and define $a_{n,k} = Q_{n,132}^{(0,0,k,0)}(x)|_{x^{n-k-1}}$, where $n > k+1$. Then, suppose that $\sigma = \sigma_1 \cdots \sigma_{n+1} \in S_{n+1}(132)$ is such that $\mathrm{mmp}^{(0,0,k,0)}(\sigma) = n-k$. By definition, the number of such σ is $a_{n+1,k}$. Then, if $\sigma_{n+1} = n+1$, we must have $\mathrm{mmp}^{(0,0,k,0)}(\sigma_1 \cdots \sigma_n) = n-k-1$, so we have $a_{n,k}$ choices for $\sigma_1 \cdots \sigma_n$. If $\sigma_1 = n+1$, then $\mathrm{mmp}^{(0,0,k,0)}(\sigma_2 \cdots \sigma_{n+1}) = n-k$, so we have C_k choices for $\sigma_2 \cdots \sigma_{n+1}$. If $\sigma_n = n+1$, then $\sigma_{n+1} = 1$ and $\mathrm{mmp}^{(0,0,k,0)}(\sigma_1 \cdots \sigma_{n-1}) = n-k-1$, so we have C_k choices for $\sigma_1 \cdots \sigma_{n-1}$. If $\sigma_i = n+1$, where $2 \leq i \leq k$, then $\sigma_1 \cdots \sigma_i$ cannot contribute to $\mathrm{mmp}^{(0,0,k,0)}(\sigma)$, so

$\text{mmp}^{(0,0,k,0)}(\sigma) = \text{mmp}^{(0,0,k,0)}(\sigma_{k+1}\cdots\sigma_{n+1}) \leq n-i-k < n-k-1$. If $\sigma_i = n+1$, where $n-k+1 \leq i \leq n-1$, then $\sigma_{i+1}\cdots\sigma_{n+1}$ cannot contribute to $\text{mmp}^{(0,0,k,0)}(\sigma)$, so $\text{mmp}^{(0,0,k,0)}(\sigma) = \text{mmp}^{(0,0,k,0)}(\sigma_1\cdots\sigma_i) \leq i-k \leq n-k-1$. Finally if $\sigma_i = n+1$, where $k+1 \leq i \leq n-k$, then

$$\begin{aligned}\text{mmp}^{(0,0,k,0)}(\sigma) &= \text{mmp}^{(0,0,k,0)}(\text{red}(\sigma_1\cdots\sigma_i)) + \text{mmp}^{(0,0,k,0)}(\sigma_{i+1}\cdots\sigma_{n+1}) \\ &\leq i-k+(n+1-i-k) = n+1-2k < n-k-1.\end{aligned}$$

Thus, it follows that for $n \geq k+1$, $a_{n,k}$ satisfies the recursion

$$a_{n+1,k} = a_{n,k} + 2C_k. \tag{19}$$

In general, if $n = k+1$, then there are $C_{k+1} - C_k$ permutations in $S_n(132)$ avoiding $MMP(0,0,k,0)$, namely, those that do not have $\sigma_{k+1} = k+1$. Using this as the base case, we may solve recursion (19) to obtain $a_{n,k} = C_{k+1} - C_k + 2(n-k-1)C_k$. ☐

Again, we can easily use Mathematica to compute some initial terms of the generating function $Q_{132}^{(0,0,k,0)}(t,x)$ for small k. For example, we have computed that

$$\begin{aligned}&Q_{132}^{(0,0,1,0)}(t,x) = 1+t+(1+x)t^2+\left(1+3x+x^2\right)t^3+\left(1+6x+6x^2+x^3\right)t^4+ \\ &\left(1+10x+20x^2+10x^3+x^4\right)t^5+\left(1+15x+50x^2+50x^3+15x^4+x^5\right)t^6+ \\ &\left(1+21x+105x^2+175x^3+105x^4+21x^5+x^6\right)t^7+ \\ &\left(1+28x+196x^2+490x^3+490x^4+196x^5+28x^6+x^7\right)t^8+ \\ &\left(1+36x+336x^2+1176x^3+1764x^4+1176x^5+336x^6+36x^7+x^8\right)t^9+\cdots.\end{aligned}$$

It is easy to explain several of the coefficients of $Q_{n,132}^{(0,0,1,0)}(x)$. That is, the following hold.

Theorem 11.

1. *$Q_{n,132}^{(0,0,1,0)}(0) = 1$ for $n \geq 1$,*

2. *$Q_{n,132}^{(0,0,1,0)}(x)|_{x^{n-1}} = 1$ for $n \geq 2$,*

3. *$Q_{n,132}^{(0,0,1,0)}(x)|_{x} = \binom{n}{2}$ for $n \geq 2$, and*

4. *$Q_{n,132}^{(0,0,1,0)}(x)|_{x^{n-2}} = \binom{n}{2}$ for $n \geq 3$.*

Proof. It is easy to see that $n(n-1)\cdots 1$ is the only permutation $\sigma \in S_n(132)$ such that $\text{mmp}^{(0,0,1,0)}(\sigma) = 0$. Thus, $Q_{n,132}^{(0,0,1,0)}(0) = 1$ for all $n \geq 1$. Similarly, for $n \geq 2$, $\sigma = 12\cdots(n-1)n$ is the only permutation in $S_n(132)$ with $\text{mmp}^{(0,0,1,0)}(\sigma) = n-1$ so that $Q_{n,132}^{(0,0,1,0)}(x)|_{x^{n-1}} = 1$ for $n \geq 2$.

To prove (3), let $\sigma^{(i,j)} = n(n-1)\cdots(j+1)(j-1)\cdots ij(i-1)\cdots 1$ for any $1 \leq i < j \leq n$. It is easy to see that $\text{mmp}^{(0,0,1,0)}(\sigma^{(i,j)}) = 1$ and that these are the only permutations σ in $S_n(132)$ such that $\text{mmp}^{(0,0,1,0)}(\sigma) = 1$. Thus, $Q_{n,132}^{(0,0,1,0)}(x)|_{x} = \binom{n}{2}$ for $n \geq 2$.

For (4), we prove by induction that $Q_{n,132}^{(0,0,1,0)}(x)|_{x^{n-2}} = \binom{n}{2}$ for $n \geq 3$. The theorem holds for $n = 3, 4$. Now suppose that $n \geq 5$ and $\sigma \in S_n(132)$ and $\mathrm{mmp}^{(0,0,1,0)}(\sigma) = n-2$. Then if $\sigma_n = n$, it must be the case that $\mathrm{mmp}^{(0,0,1,0)}(\sigma_1 \cdots \sigma_{n-1}) = n-3$, so by induction we have $\binom{n-1}{2}$ choices for $\sigma_1 \cdots \sigma_{n-1}$. If $\sigma_i = n$, where $1 \leq i \leq n-1$, then it must be the case that $\sigma = (n-k+1) \cdots (n-1)n12 \cdots (n-k)$, so there are $n-1$ such permutations where $\sigma_n \neq n$. Thus, we have a total of $\binom{n}{2}$ with $\mathrm{mmp}^{(0,0,1,0)}(\sigma) = n-2$. ☐

More generally, one can observe that the coefficients of x^j and x^{n-j-1} in $Q_{n,132}^{(0,0,1,0)}(x)$ are the same. This can be proved directly from its generating function. That is, by Theorem 9,

$$Q_{132}^{(0,0,1,0)}(t,x) = \frac{1+t(x-1) - \sqrt{(1+t(x-1))^2 - 4xt}}{2xt}.$$

Further, define

$$R_{132}^{(0,0,1,0)}(t,x) = \frac{Q_{132}^{(0,0,1,0)}(t,x) - 1}{t} = \frac{1-t(x+1) - \sqrt{(1+t(x-1))^2 - 4xt}}{2xt^2}.$$

The observed symmetry is then just the statement that $R_{132}^{(0,0,1,0)}(t,x) = R_{132}^{(0,0,1,0)}(tx, 1/x)$, which can be easily checked. We shall give a combinatorial proof of this symmetry in Section 6; see the discussion of (35).

We have computed that

$$\begin{aligned}
&Q_{132}^{(0,0,2,0)}(t,x) = 1 + t + 2t^2 + (3+2x)t^3 + \left(5+7x+2x^2\right)t^4 + \\
&\left(8+21x+11x^2+2x^3\right)t^5 + \left(13+53x+49x^2+15x^3+2x^4\right))t^6 + \\
&\left(21+124x+174x^2+89x^3+19x^4+2x^5\right)t^7 + \\
&\left(34+273x+546x^2+411x^3+141x^4+23x^5+2x^6\right)t^8 + \\
&\left(55+577x+1557x^2+1635x^3+804x^4+205x^5+27x^6+2x^7\right)t^9 + \cdots.
\end{aligned}$$

We then have the following proposition.

Proposition 1. *1. $Q_{n,132}^{(0,0,2,0)}(0) = F_n$, where F_n is the nth Fibonacci number, and*

2. $Q_{n,132}^{(0,0,2,0)}(x)|_{x^{n-3}} = 3 + 4(n-3)$.

Proof. In this case, we know that $Q_{132}^{(0,0,2,0)}(t,0) = \frac{1}{1-t(C_0+C_1t)} = \frac{1}{1-t-t^2}$, so the sequence $(Q_{n,132}^{(0,0,2,0)}(0))_{n\geq 0}$ is the Fibonacci numbers. This result is known [5, Table 6.1], since the avoidance of $MMP(0,0,2,0)$ is equivalent to the avoidance of the patterns 123 and 213 simultaneously, so in this case we are dealing with the multi-avoidance of the classical patterns 132, 123, and 213.

The fact that $Q_{n,132}^{(0,0,2,0)}(x)|_{x^{n-3}} = 3 + 4(n-3)$ is a special case of Theorem 10. ☐

The sequence $(Q_{n,132}^{(0,0,2,0)}(x)|_x)_{n\geq 3}$, whose initial terms are $2, 7, 21, 53, 124, 273, 577, \ldots$, does not appear in the OEIS.

We have computed that

$$Q_{132}^{(0,0,3,0)}(t,x) = 1 + t + 2t^2 + 5t^3 + (9+5x)t^4 + \left(18 + 19x + 5x^2\right)t^5 + \left(37 + 61x + 29x^2 + 5x^3\right)t^6 + \left(73 + 188x + 124x^2 + 39x^3 + 5x^4\right)t^7 + (146 + 523x + 500x^2 + 207x^3 + 49x^4 + 5x^5)t^8 + (293 + 1387x + 1795x^2 + 1013x^3 + 310x^4 + 59x^5 + 5x^6)t^9 + \cdots.$$

In this case, the sequence $(Q_{n,132}^{(0,0,3,0)}(0))_{n\geq 0}$ whose generating function $Q_{132}^{(0,0,3,0)}(t,0) = \frac{1}{1-t(1+t+2t^2)}$ is A077947 in the OEIS, which also counts the number of sequences of codewords of total length n from the code $C = \{0, 10, 110, 111\}$. For example, for $n = 3$, there are five sequences of length 3 that are in $\{0, 10, 110, 111\}^*$, namely, 000,010,100,110, and 111. The basic idea of a combinatorial explanation of this fact is not that difficult to present. Indeed, a permutation avoiding the patterns 132 and $MMP(0,0,3,0)$ is such that to the left of n, the largest element, one can either have no elements, one element $(n-1)$, two elements in increasing order $(n-2)(n-1)$, or two elements in decreasing order $(n-1)(n-2)$. We can then recursively build the codeword corresponding to the permutation beginning with, say, 0, 10, 110 and 111, respectively, corresponding to the four cases; one then applies the same map to the subpermutation to the right of n.

The sequence $(Q_{n,132}^{(0,0,3,0)}(x)|_x)_{n\geq 4}$, whose initial terms are $5, 19, 61, 188, 532, 1387, \ldots$ does not appear in the OEIS.

We have computed that

$$Q_{132}^{(0,0,4,0)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + (28 + 14x)t^5 + (62 + 56x + 14x^2)t^6 + (143 + 188x + 84x^2 + 14x^3)t^7 + (331 + 603x + 307x^2 + 112x^3 + 14x^4)t^8 + (738 + 1907x + 1455x^2 + 608x^3 + 140x^4 + 14x^5)t^9 + \cdots.$$

Here, neither the sequence $(Q_{n,132}^{(0,0,4,0)}(0))_{n\geq 1}$, whose generating function is $Q_{132}^{(0,4,0,0)}(t,0) = \frac{1}{1-t(1+t+2t^2+5t^3)}$, nor the sequence $(Q_{n,132}^{(0,0,4,0)}(x)|_x)_{n\geq 5}$ appear in the OEIS.

Unlike the situation with the generating functions $Q_{n,132}^{(k,0,0,0)}(t,x)$, there does not seem to be any simple way to extract a simple formula for $Q_{n,132}^{(0,0,k,0)}(t,x)|_x$ from (17).

5 The functions $Q_{132}^{(0,k,0,0)}(t,x) = Q_{132}^{(0,0,0,k)}(t,x)$

In this section, we shall compute the generating functions $Q_{132}^{(0,k,0,0)}(t,x)$ and $Q_{132}^{(0,0,0,k)}(t,x)$ for $k \geq 1$. These two generating functions are equal, since it follows from Lemma 1 that $Q_{n,132}^{(0,k,0,0)}(x) = Q_{n,132}^{(0,0,0,k)}(x)$ for all $k, n \geq 1$. Thus, in this section, we shall only consider the generating functions $Q_{132}^{(0,k,0,0)}(t,x)$.

First let $k = 1$. It is easy to see that $A_i(\sigma)$ will contribute $\mathrm{mmp}^{(0,1,0,0)}(\mathrm{red}(A_i(\sigma)))$ to $\mathrm{mmp}^{(0,1,0,0)}(\sigma)$, since neither n nor any of the elements to the right of n have any effect on whether an element in $A_i(\sigma)$ matches the pattern $MMP(0,1,0,0)$ in σ. Similarly, $B_i(\sigma)$ will contribute $n-i$ to $\mathrm{mmp}^{(0,1,0,0)}(\sigma)$, since the presence of n to the left of these elements guarantees that they all match the pattern $MMP(0,1,0,0)$ in σ. Note that n does not match the pattern $MMP(0,1,0,0)$ in σ. It follows that

$$Q_{n,132}^{(0,1,0,0)}(x) = \sum_{i=1}^{n} Q_{i-1,132}^{(0,1,0,0)}(x) C_{n-i} x^{n-i}. \tag{20}$$

Multiplying both sides of (20) by t^n and summing for $n \geq 1$ will show that

$$-1 + Q_{132}^{(0,1,0,0)}(t,x) = tQ_{132}^{(0,1,0,0)}(t,x)\ C(tx).$$

Thus,

$$Q_{132}^{(0,1,0,0)}(t,x) = \frac{1}{1-tC(tx)},$$

which is the same as the generating function for $Q_{132}^{(1,0,0,0)}(t,x)$.

Next we consider the case $k > 1$. Again, it is easy to see that $A_i(\sigma)$ will contribute $\mathrm{mmp}^{(0,k,0,0)}(\mathrm{red}(A_i(\sigma)))$ to $\mathrm{mmp}^{(0,k,0,0)}(\sigma)$, since neither n nor any of the elements to the right of n have any effect on whether an element in $A_i(\sigma)$ matches the pattern $MMP(0,k,0,0)$ in σ. Now if $i \geq k$, then $B_i(\sigma)$ will contribute $C_{n-i}x^{n-i}$ to $\mathrm{mmp}^{(0,k,0,0)}(\sigma)$, since the presence of n and the elements of $A_i(\sigma)$ guarantee that the elements of $B_i(\sigma)$ all match the pattern $MMP(0,k,0,0)$ in σ. However, if $i < k$, then $B_i(\sigma)$ will contribute $\mathrm{mmp}^{(0,k-i,0,0)}(\mathrm{red}(B_i(\sigma)))$ to $\mathrm{mmp}^{(0,k,0,0)}(\sigma)$, since the presence of n and the elements of $A_i(\sigma)$ to the left of n guarantees that the elements of $B_i(\sigma)$ match the pattern $MMP(0,k,0,0)$ in σ if and only if they match the pattern $MMP(0,k-i,0,0)$ in $B_i(\sigma)$. Note that n does not match the pattern $MMP(0,k,0,0)$ for any $k \geq 1$. It follows that

$$\begin{aligned} Q_{n,132}^{(0,k,0,0)}(x) &= \sum_{i=1}^{k-1} Q_{i-1,132}^{(0,k,0,0)}(x) Q_{n-i,132}^{(0,k-i,0,0)}(x) + \sum_{i=k}^{n} Q_{i-1,132}^{(0,k,0,0)}(x) C_{n-i} x^{n-i} \\ &= \sum_{i=1}^{k-1} C_{i-1} Q_{n-i,132}^{(0,k-i,0,0)}(x) + \sum_{i=k}^{n} Q_{i-1,132}^{(0,k,0,0)}(x) C_{n-i} x^{n-i}. \end{aligned} \tag{21}$$

Here the last equation follows from the fact that $Q_{i-1,132}^{(0,k,0,0)}(x) = C_{i-1}$ if $i \leq k-1$. Multiplying both sides of (21) by t^n and summing for $n \geq 1$ will show that

$$\begin{aligned} &-1 + Q_{132}^{(0,k,0,0)}(t,x) = \\ &\quad t\sum_{i=1}^{k-1} C_{i-1} t^{i-1} Q_{132}^{(0,k-i,0,0)}(t,x) + tC(tx)(Q_{132}^{(0,k,0,0)}(t,x) - (C_0 + C_1 t + \cdots + C_{k-2} t^{k-2})). \end{aligned}$$

Thus, we have the following theorem.

Theorem 12.

$$Q_{132}^{(0,1,0,0)}(t,x) = \frac{1}{1-tC(tx)}. \tag{22}$$

For $k > 1$,

$$Q_{132}^{(0,k,0,0)}(t,x) = \frac{1 + t\sum_{j=0}^{k-2} C_j t^j (Q_{132}^{(0,k-1-j,0,0)}(t,x) - C(tx))}{1 - tC(tx)} \tag{23}$$

and

$$Q_{132}^{(0,k,0,0)}(t,0) = \frac{1 + t\sum_{j=0}^{k-2} C_j t^j (Q_{132}^{(0,k-1-j,0,0)}(t,0) - 1)}{1 - t}. \tag{24}$$

5.1 Explicit formulas for $Q_{n,132}^{(0,k,0,0)}(x)|_{x^r}$

Note that Theorem 12 gives us a simple recursion for the generating functions for the constant terms in $Q_{n,132}^{(0,k,0,0)}(x)$. For example, one can compute that

$$\begin{aligned}
Q_{132}^{(0,1,0,0)}(t,0) &= \frac{1}{(1-t)}; \\
Q_{132}^{(0,2,0,0)}(t,0) &= \frac{1-t+t^2}{(1-t)^2}; \\
Q_{132}^{(0,3,0,0)}(t,0) &= \frac{1-2t+2t^2+t^3-t^4}{(1-t)^3}; \\
Q_{132}^{(0,4,0,0)}(t,0) &= \frac{1-3t+4t^2-t^3+3t^4-5t^5+2t^6}{(1-t)^4}, \text{ and} \\
Q_{132}^{(0,5,0,0)}(t,0) &= \frac{1-4t+7t^2-5t^3+4t^4+6t^5-21t^6+18t^7-5t^8}{(1-t)^5}.
\end{aligned}$$

We can explain the highest coefficient of x and the second highest coefficient of x in $Q_{n,132}^{(0,k,0,0)}(x)$ for any $k \geq 1$.

Theorem 13.

1. *For all $k \geq 1$ and $n \geq k$, the highest power of x that occurs in $Q_{n,132}^{(0,k,0,0)}(x)$ is x^{n-k}, with $Q_{n,132}^{(0,k,0,0)}(x)|_{x^{n-k}} = C_k C_{n-k}$.*

2. *For all $k \geq 1$ and $n \geq k+1$, $Q_{n,132}^{(0,k,0,0)}(x)|_{x^{n-k-1}} = a_k C_{n-k}$ where $a_1 = 1$ and for $k \geq 2$, $a_k = C_k + \sum_{i=1}^{k-1} C_{i-1} a_{k-i}$.*

Proof. For (1), it is easy to see that to obtain the largest number of $MMP(0,k,0,0)$-matches for a permutation $\sigma \in S_n(132)$, we need only to arrange the largest k elements $n, n-1, \ldots, n-k+1$ such that they avoid 132, followed by the elements $1, \ldots, n-k$ under the same condition. Thus, the highest power of x that occurs in $Q_{n,132}^{(0,k,0,0)}(x)$ is x^{n-k}, and its coefficient is $C_k C_{n-k}$.

For (2), we know that $Q_{n,132}^{(0,1,0,0)}(x) = Q_{n,132}^{(1,0,0,0)}(x)$ and we have proved that for $n \geq 2$ $Q_{n,132}^{(1,0,0,0)}(x)|_{x^{n-2}} = C_{n-1}$. Thus $a_1 = 1$.

Next assume by induction that for $j = 1, \ldots, k-1$,

$$Q_{n,132}^{(j,0,0,0)}(x)|_{x^{n-j-1}} = a_j C_{n-j} \text{ for } n \geq j+1$$

where a_j is a positive integer. Then by the recursion (21), we know that

$$\begin{aligned} Q_{n,132}^{(0,k,0,0)}(x)|_{x^{n-k-1}} &= \sum_{i=1}^{k-1} C_{i-1} Q_{n-i,132}^{(0,k-i,0,0)}(x)|_{x^{n-k-1}} + \sum_{i=k}^{n} \left(Q_{i-1,132}^{(0,k,0,0)}(x) C_{n-i} x^{n-i} \right)|_{x^{n-k-1}} \\ &= \sum_{i=1}^{k-1} C_{i-1} Q_{n-i,132}^{(0,k-i,0,0)}(x)|_{x^{n-k-1}} + \sum_{i=k}^{n} C_{n-i} Q_{i-1,132}^{(0,k,0,0)}(x)|_{x^{i-k-1}}. \end{aligned}$$

By induction, we know that for $i = 1, \ldots, k-1$,

$$Q_{n-i,132}^{(0,k-i,0,0)}(x)|_{x^{n-k-1}} = a_{k-i} C_{n-k} \text{ for } n-i \geq k-i+1$$

and for $i = k+1, \ldots, n$,

$$Q_{i-1,132}^{(0,k,0,0)}(x)|_{x^{i-k-1}} = C_k C_{i-1-k} \text{ for } i-1 \geq k.$$

Moreover, it is clear that for $i = k$, $Q_{k-1,132}^{(0,k,0,0)}(x)|_{x^{k-k-1}} = 0$. Thus we have that for all $n \geq k+1$,

$$\begin{aligned} Q_{n,132}^{(0,k,0,0)}(x)|_{x^{n-k-1}} &= \sum_{i=1}^{k-1} C_{i-1} a_{k-i} C_{n-k} + \sum_{i=k+1}^{n} C_{n-i} C_k C_{i-1-k} \\ &= C_{n-k} \left(\sum_{i=1}^{k-1} C_{i-1} a_{k-i} \right) + C_k \sum_{i=k+1}^{n} C_{n-i} C_{i-1-k} \\ &= C_{n-k} \left(\sum_{i=1}^{k-1} C_{i-1} a_{k-i} \right) + C_k C_{n-k} = C_{n-k} \left(C_k + \sum_{i=1}^{k-1} C_{i-1} a_{k-i} \right). \end{aligned}$$

Thus for $n \geq k+1$, $Q_{n,132}^{(0,k,0,0)}(x)|_{x^{n-k-1}} = a_k C_{n-k}$ where $a_k = C_k + \sum_{i=1}^{k-1} C_{i-1} a_{k-i}$. $\square$

For example,

$$\begin{aligned} a_2 &= C_2 + C_0 a_1 = 2 + 1 = 3, \\ a_3 &= C_3 + C_0 a_2 + C_1 a_1 = 5 + 3 + 1 = 9, \text{ and} \\ a_4 &= C_4 + C_0 a_3 + C_1 a_2 + C_2 a_1 = 14 + 9 + 3 + 2 = 28 \end{aligned}$$

which agrees with the series for $Q_{132}^{(0,2,0,0)}(t,x)$, $Q_{132}^{(0,3,0,0)}(t,x)$, and $Q_{132}^{(0,4,0,0)}(t,x)$ which we give below.

Again we can use Mathematica to compute the first few terms of the generating function $Q_{132}^{(0,k,0,0)}(t,x)$ for small k. Since $Q_{132}^{(0,1,0,0)}(t,x) = Q_{132}^{(1,0,0,0)}(t,x)$, we will not list that generating function again.

We have computed that

$$
\begin{aligned}
&Q_{132}^{(0,2,0,0)}(t,x) = 1 + t + 2t^2 + (3+2x)t^3 + \left(4+6x+4x^2\right)t^4 + \\
&\left(5+12x+15x^2+10x^3\right)t^5 + \left(6+20x+36x^2+42x^3+28x^4\right)t^6 + \\
&\left(7+30x+70x^2+112x^3+126x^4+84x^5\right)t^7 + \\
&\left(8+42x+120x^2+240x^3+360x^4+396x^5+264x^6\right)t^8 + \\
&\left(9+56x+189x^2+450x^3+825x^4+1188x^5+1287x^6+858x^7\right)t^9 + \cdots .
\end{aligned}
$$

The only permutations $\sigma \in S_n(132)$ such that $\mathrm{mmp}^{(0,2,0,0)}(\sigma) = 0$ are the identity permutation plus all the adjacent transpositions

$$(i, i+1) = 12\cdots(i-1)(i+1)i(i+2)\cdots n,$$

which explains why $Q_{n,132}^{(0,2,0,0)}(0) = n$ for all $n \geq 1$. This is a known result [5, Table 6.1], since avoiding $MMP(0,2,0,0)$ is equivalent to avoiding simultaneously the classical patterns 321 and 231. Hence in this case, we are dealing with the simultaneous avoidance of the patterns 132, 321 and 231.

We claim that $(Q_{n,132}^{(0,2,0,0)}(x)|_x = (n-1)(n-2)$ for all $n \geq 3$. This can be easily proved by induction. That is, $Q_{3,132}^{(0,2,0,0)}(x)|_x = 2$, so our formula holds for $n = 3$. Now suppose that $n \geq 4$ and $\sigma = \sigma_1 \cdots \sigma_n \in S_n(132)$ is such that $\mathrm{mmp}^{(0,2,0,0)}(\sigma) = 1$. We claim there are only three possibilities for the position of n in σ. That is, it cannot be that $\sigma_i = n$ for $2 \leq i \leq n-2$, since then both σ_n and σ_{n-1} would match $MMP(0,2,0,0)$ in σ. Thus, it must be the case that $\sigma_n = n$, $\sigma_{n-1} = n$, or $\sigma_1 = n$. Clearly, if $\sigma_n = n$, then we must have that $\mathrm{mmp}^{(0,2,0,0)}(\sigma_1 \cdots \sigma_{n-1}) = 1$, so there are $(n-2)(n-3)$ choices of $\sigma_1 \cdots \sigma_{n-1}$ by induction. If $\sigma_{n-1} = n$, then $\sigma_n = 1$, so σ_n will match $MMP(0,2,0,0)$ in σ. Thus, it must be the case that $\mathrm{mmp}^{(0,2,0,0)}(\mathrm{red}(\sigma_1 \cdots \sigma_{n-2})) = 0$, which means that we have $n-2$ choices for $\sigma_1 \cdots \sigma_{n-2}$ in this case. Finally, if $\sigma_1 = n$, then we must have that $\mathrm{mmp}^{(0,1,0,0)}(\mathrm{red}(\sigma_2 \cdots \sigma_n)) = 1$. Using the fact that $Q_{132}^{(0,1,0,0)}(t,x) = Q_{132}^{(1,0,0,0)}(t,x)$ and that $Q_{n,132}^{(1,0,0,0)}(x)|_x = n-1$, it follows that there are $n-2$ choices for $\sigma_2 \cdots \sigma_n$ in this case. Thus, it follows that

$$Q_{n,132}^{(0,2,0,0)}(x)|_x = (n-2)(n-3) + 2(n-2) = (n-1)(n-2).$$

The sequence $(Q_{n,132}^{(0,2,0,0)}(x)|_{x^{n-3}})_{n \geq 3}$ is sequence A120589 in the OEIS which has no listed combinatorial interpretation so that we have give a combinatorial interpretation to this sequence.

We have computed that

$$
\begin{aligned}
&Q_{132}^{(0,3,0,0)}(t,x) = 1 + t + 2t^2 + 5t^3 + (9+5x)t^4 + \left(14+18x+10x^2)\right)t^5 + \\
&\left(20+42x+45x^2+25x^3\right)t^6 + \left(27+80x+126x^2+126x^3+70x^4\right)t^7 + \\
&\left(35+135x+280x^2+392x^3+378x^4+210x^5\right)t^8 + \\
&\left(44+210x+540x^2+960x^3+1260x^4+1088x^5+660x^6\right)t^9 + \cdots
\end{aligned}
$$

and

$$Q_{132}^{(0,4,0,0)}(t,x) = 1+t+2t^2+5t^3+14t^4+(28+14x)t^5+\left(48+56x+28x^2\right)t^6+ \left(75+144x+140x^2+70x^3\right)t^7+\left(110+300x+432x^2+392x^3+196x^4\right)t^8+ \left(154+550x+1050x^2+1344x^3+1176x^4+588x^5\right)t^9+\cdots.$$

The sequences $(Q_{n,132}^{(0,3,0,0)}(0))_{n\geq 1}$, $(Q_{n,132}^{(0,3,0,0)}(x)|_x)_{n\geq 4}$, $(Q_{n,132}^{(0,4,0,0)}(0))_{n\geq 1}$, and $(Q_{n,132}^{(0,4,0,0)}(x)|_x)_{n\geq 5}$ do not appear in the OEIS.

We have now considered all the possibilities for $Q_{132}^{(a,b,c,d)}(t,x)$ for $a,b,c,d\in\mathbb{N}$ where all but one of the parameters a,b,c,d are zero. There are several alternatives for further study. One is to consider $Q_{132}^{(a,b,c,d)}(t,x)$ for $a,b,c,d\in\mathbb{N}$ where at least two of the parameters a,b,c,d are non-zero. This will be the subject of [7]. A second alternative is to allow some of the parameters to be equal to $\emptyset$. In the next two sections, we shall give two simple examples of this type of alternative.

6 The function $Q_{132}^{(k,0,\emptyset,0)}(t,x)$

In this section, we shall consider the generating function $Q_{132}^{(k,0,\emptyset,0)}(t,x)$, where $k\in\mathbb{N}\cup\{\emptyset\}$. Given a permutation $\sigma=\sigma_1\cdots\sigma_n\in S_n$, we say that σ_j is a *right-to-left maximum* (*left-to-right minimum*) of σ if $\sigma_j>\sigma_i$ for all $i>j$ ($\sigma_j<\sigma_i$ for all $i<j$). We let $\mathrm{RLmax}(\sigma)$ denote the number of right-to-left maxima of σ and $\mathrm{LRmin}(\sigma)$ denote the number of left-to-right minima of σ. One can view the pattern $MMP(k,0,\emptyset,0)$ as a generalization of the number of left-to-right minima statistic (which corresponds to the case $k=0$).

First we compute the generating function for $Q_{n,132}^{(\emptyset,0,\emptyset,0)}(x)$, which corresponds to the elements that are both left-to-right minima and right-to-left maxima. Consider the permutations $\sigma\in S_n(132)$ where $\sigma_1=n$. Clearly such permutations contribute $xQ_{n-1,132}^{(\emptyset,0,\emptyset,0)}(x)$ to $Q_{n,132}^{(\emptyset,0,\emptyset,0)}(x)$. For $i>1$, it is easy to see that $A_i(\sigma)$ will contribute nothing to $\mathrm{mmp}^{(\emptyset,0,\emptyset,0)}(\sigma)$, since the presence of n to the right of these elements ensures that no point in $A_i(\sigma)$ matches the pattern $MMP(\emptyset,0,\emptyset,0)$. Similarly, $B_i(\sigma)$ will contribute $\mathrm{mmp}^{(\emptyset,0,\emptyset,0)}(\mathrm{red}(B_i(\sigma)))$ to $\mathrm{mmp}^{(\emptyset,0,\emptyset,0)}(\sigma)$, since neither n nor any of the elements to the left of n have any effect on whether an element in $B_i(\sigma)$ matches the pattern $MMP(\emptyset,0,\emptyset,0)$ in σ. Thus,

$$Q_{n,132}^{(\emptyset,0,\emptyset,0)}(x) = xQ_{n-1,132}^{(\emptyset,0,\emptyset,0)}(x)+\sum_{i=2}^{n} C_{i-1}Q_{n-i,132}^{(\emptyset,0,\emptyset,0)}(x). \tag{25}$$

Multiplying both sides of (25) by t^n and summing over all $n\geq 1$, we obtain that

$$-1+Q_{132}^{(\emptyset,0,\emptyset,0)}(t,x) = txQ_{132}^{(\emptyset,0,\emptyset,0)}(t,x)+tQ_{132}^{(\emptyset,0,\emptyset,0)}(t,x)\ (C(t)-1).$$

Thus, we have the following theorem.

Theorem 14.

$$Q_{132}^{(\emptyset,0,\emptyset,0)}(t,x) = \frac{1}{1-tx+t-tC(t)} \tag{26}$$

and

$$Q_{132}^{(\emptyset,0,\emptyset,0)}(t,0) = \frac{1}{1+t-tC(t)}. \tag{27}$$

Next we compute the generating function for $Q_{n,132}^{(0,0,\emptyset,0)}(x)$. First consider the permutations $\sigma \in S_n^{(1)}(132)$. Clearly such permutations contribute $xQ_{n-1,132}^{(0,0,\emptyset,0)}(x)$ to $Q_{n,132}^{(0,0,\emptyset,0)}(x)$. For $i > 1$, it is easy to see that $A_i(\sigma)$ will contribute $\mathrm{mmp}^{(0,0,\emptyset,0)}(\mathrm{red}(A_i(\sigma)))$ to $\mathrm{mmp}^{(0,0,\emptyset,0)}(\sigma)$, since neither n nor any of the elements to the right of n have any effect on whether an element in $A_i(\sigma)$ matches the pattern $MMP(0,0,\emptyset,0)$ in σ. Similarly, $B_i(\sigma)$ will contribute $\mathrm{mmp}^{(0,0,\emptyset,0)}(\mathrm{red}(B_i(\sigma)))$ to $\mathrm{mmp}^{(0,0,\emptyset,0)}(\sigma)$, since neither n nor any of the elements to the left of n have any effect on whether an element in $B_i(\sigma)$ matches the pattern $MMP(0,0,\emptyset,0)$ in σ. Thus,

$$Q_{n,132}^{(0,0,\emptyset,0)}(x) = xQ_{n-1,132}^{(0,0,\emptyset,0)}(x) + \sum_{i=2}^{n} Q_{i-1,132}^{(0,0,\emptyset,0)}(x)Q_{n-i,132}^{(0,0,\emptyset,0)}(x). \tag{28}$$

Multiplying both sides of (28) by t^n and summing over all $n \geq 1$, we obtain that

$$-1 + Q_{132}^{(0,0,\emptyset,0)}(t,x) = txQ_{132}^{(0,0,\emptyset,0)}(t,x) + tQ_{132}^{(0,0,\emptyset,0)}(t,x)\ (Q_{132}^{(0,0,\emptyset,0)}(t,x) - 1),$$

so

$$0 = 1 + Q_{132}^{(0,0,\emptyset,0)}(t,x)(tx-t-1) + t(Q_{132}^{(0,0,\emptyset,0)}(t,x))^2.$$

Thus,

$$Q_{132}^{(0,0,\emptyset,0)}(t,x) = \frac{(1+t-tx) - \sqrt{(1+t-tx)^2 - 4t}}{2t}.$$

Next we compute a recursion for $Q_{n,132}^{(k,0,\emptyset,0)}(x)$, where $k \geq 1$. It is clear that n can never match the pattern $MMP(k,0,\emptyset,0)$ for $k \geq 1$ in any $\sigma \in S_n(132)$. For $i \geq 1$, it is easy to see that $A_i(\sigma)$ will contribute $\mathrm{mmp}^{(k-1,0,\emptyset,0)}(\mathrm{red}(A_i(\sigma)))$ to $\mathrm{mmp}^{(k,0,\emptyset,0)}(\sigma)$, since none of the elements to the right of n have any effect on whether an element in $A_i(\sigma)$ matches the pattern $MMP(k,0,\emptyset,0)$ in σ and the presence of n ensures that an element in $A_i(\sigma)$ matches $MMP(k,0,\emptyset,0)$ in σ if and only if it matches $MMP(k-1,0,\emptyset,0)$ in $A_i(\sigma)$. Similarly, $B_i(\sigma)$ will contribute $\mathrm{mmp}^{(k,0,\emptyset,0)}(\mathrm{red}(B_i(\sigma)))$ to $\mathrm{mmp}^{(k,0,\emptyset,0)}(\sigma)$, since neither n nor any of the elements to the left of n have any effect on whether an element in $B_i(\sigma)$ matches the pattern $MMP(k,0,\emptyset,0)$ in σ. Thus,

$$Q_{n,132}^{(k,0,\emptyset,0)}(x) = \sum_{i=1}^{n} Q_{i-1,132}^{(k-1,0,\emptyset,0)}(x)Q_{n-i,132}^{(k,0,\emptyset,0)}(x). \tag{29}$$

Multiplying both sides of (29) by t^n and summing over all $n \geq 1$, we obtain that

$$-1 + Q_{132}^{(0,0,\emptyset,0)}(t,x) = tQ_{132}^{(k-1,0,\emptyset,0)}(t,x)\ Q_{132}^{(k,0,\emptyset,0)}(t,x).$$

Thus, we have the following theorem.

Theorem 15.

$$Q_{132}^{(0,0,\emptyset,0)}(t,x) = \frac{(1+t-tx)-\sqrt{(1+t-tx)^2-4t}}{2t}. \tag{30}$$

For $k \geq 1$,

$$Q_{132}^{(k,0,\emptyset,0)}(t,x) = \frac{1}{1-tQ_{132}^{(k-1,0,\emptyset,0)}(t,x)}. \tag{31}$$

Thus,

$$Q_{132}^{(0,0,\emptyset,0)}(t,0) = 1$$

and for $k \geq 1$,

$$Q_{132}^{(k,0,\emptyset,0)}(t,0) = \frac{1}{1-tQ_{132}^{(k-1,0,\emptyset,0)}(t,0)}. \tag{32}$$

6.1 Explicit formulas for $Q_{n,132}^{(k,0,\emptyset,0)}(x)|_{x^r}$

We have computed that

$$\begin{aligned}
&Q_{132}^{(\emptyset,0,\emptyset,0)}(t,x) = 1 + xt + \left(1+x^2\right)t^2 + \left(2+2x+x^3\right)t^3 + \left(6+4x+3x^2+x^4\right)t^4 + \\
&\left(18+13x+6x^2+4x^3+x^5\right)t^5 + \left(57+40x+21x^2+8x^3+5x^4+x^6\right)t^6 + \\
&\left(186+130x+66x^2+30x^3+10x^4+6x^5+x^7\right)t^7 + \\
&\left(622+432x+220x^2+96x^3+40x^4+12x^5+7x^6+x^8\right)t^8 + \\
&\left(2120+1466x+744x^2+328x^3+130x^4+51x^5+14x^6+8x^7+x^9\right)t^9 + \cdots .
\end{aligned}$$

Clearly the highest degree term in $Q_{n,132}^{(\emptyset,0,\emptyset,0)}(x)$ is x^n, which comes from the permutation $n(n-1)\cdots 21$. It is easy to see that $Q_{n,132}^{(\emptyset,0,\emptyset,0)}(x)|_{x^{n-1}} = 0$. That is, suppose $\sigma = \sigma_1 \cdots \sigma_n \in S_n(132)$ and $\mathrm{mmp}^{(\emptyset,0,\emptyset,0)}(\sigma) = n-1$. It can not be the case that $\sigma_i = n$, where $i \geq 2$, since in such a situation, none of $\sigma_1, \ldots, \sigma_i$ would match $MMP(\emptyset,0,\emptyset,0)$ in σ. Thus, it must be the case that $\sigma_1 = n$. But then it must be that case that $\mathrm{mmp}^{(\emptyset,0,\emptyset,0)}(\sigma_2\cdots\sigma_n) = n-1$, which would mean that $\sigma_2\cdots\sigma_n = (n-1)(n-2)\cdots 21$. But then $\sigma = n(n-1)\cdots 21$ and $\mathrm{mmp}^{(\emptyset,0,\emptyset,0)}(\sigma) = n$, which contracts our choice of σ. Thus, there can be no such σ. Similarly, the coefficient of x^{n-2} in $Q_{n,132}^{(\emptyset,0,\emptyset,0)}(x)$ is $n-1$, which comes from the permutations $n(n-1)\cdots(i+2)i(i+1)(i-1)\cdots 21$ for $i = 1, \ldots, n-1$.

The sequence $(Q_{n,132}^{(\emptyset,0,\emptyset,0)}(0))_{n\geq 1}$ is the Fine numbers (A000957 in the OEIS). The Fine numbers $(\mathbb{F}_n)_{n\geq 0}$ can be defined by the generating function

$$\mathbb{F}(t) = \sum_{n\geq 0} \mathbb{F}_n t^n = \frac{1-\sqrt{1-4t}}{3t-\sqrt{1-4t}}.$$

It is straightforward to verify that

$$\frac{1-\sqrt{1-4t}}{3t-\sqrt{1-4t}} \cdot \frac{1+\sqrt{1-4t}}{1+\sqrt{1-4t}} = \frac{1}{1+t-tC(t)}.$$

$\mathbb{F}_n$ counts the number of 2-Motzkin paths with no level steps at height 0; see [2, 3]. Here, a Motzkin path is a lattice path starting at $(0,0)$ and ending at $(n,0)$ that is formed by three types of steps, up-steps $(1,1)$, level steps $(1,0)$, and down steps $(1,-1)$, and never goes below the x-axis. A c-Motzkin path is a Motzkin path where the level steps can be colored with any of c colors. $\mathbb{F}_n$ also counts the number of ordered rooted trees with n edges that have root of even degree.

Problem 3. Find simple bijective proofs of the facts that the number of $\sigma \in S_n(132)$ such that $\mathrm{mmp}^{(\emptyset,0,\emptyset,0)}(\sigma) = 0$ equals the number of 2-Motzkin paths with no level steps at height 0 and that the number of $\sigma \in S_n(132)$ such that $\mathrm{mmp}^{(\emptyset,0,\emptyset,0)}(\sigma) = 0$ equals the number of ordered rooted trees with n-edges that have root of even degree.

The sequence $(Q_{n,132}^{(\emptyset,0,\emptyset,0)}(x)|_x)_{n\geq 1}$ is sequence A065601 in the OEIS, which counts the number of Dyck paths of length $2n$ with exactly one hill. A hill in a Dyck path is an up-step that starts on the x-axis and that is immediately followed by a down-step.

Next we consider the constant term and the coefficient of x in $Q_{n,132}^{(k,0,\emptyset,0)}(x)$ for $k \geq 1$.

Proposition 2. *For all $k \geq 1$,*

$$Q_{132}^{(k,0,\emptyset,0)}(t,0) = Q_{132}^{(k,0,0,0)}(t,0).$$

Proof. Note that $Q_{132}^{(1,0,\emptyset,0)}(t,0) = \frac{1}{1-t} = Q_{132}^{(1,0,0,0)}(t,0)$. If we compare the recursions (32) and (8), we see that we have that $Q_{132}^{(k,0,\emptyset,0)}(t,0) = Q_{132}^{(k,0,0,0)}(t,0)$ for all $k \geq 1$. This fact is easy to see directly. That is, suppose that $\sigma \in S_n(132)$ has a $MMP(k,0,0,0)$-match. Then it is easy to see that if i is the smallest t such that σ_t matches $MMP(k,0,0,0)$ in σ, then there can be no $j < i$ with $\sigma_j < \sigma_i$ because otherwise, σ_j would match $MMP(k,0,0,0)$. That is, σ_i is also a $MMP(k,0,\emptyset,0)$-match. Thus, if σ has a $MMP(k,0,0,0)$-match, then it also has a $MMP(k,0,\emptyset,0)$-match. The converse of this statement is trivial. Hence the number of $\sigma \in S_n(132)$ with no $MMP(k,0,0,0)$-matches equals the number of $\sigma \in S_n(132)$ with no $MMP(k,0,\emptyset,0)$-matches. ◻

The recursion (31) has the same form as the recursion (6). Thus, we can use the same method of proof that we did to establish the recursion (11) to prove that

$$Q_{132}^{(k,0,\emptyset,0)}(t,x)|_x = Q_{132}^{(k-1,0,\emptyset,0)}(t,x)|_x \frac{t\frac{d}{dt}Q_{132}^{(k,0,\emptyset,0)}(t,0)}{\frac{d}{dt}tQ_{132}^{(k-1,0,\emptyset,0)}(t,0)}. \tag{33}$$

For example, we know that

$$Q_{132}^{(1,0,\emptyset,0)}(t,x)|_x = Q_{132}^{(0,0,1,0)}(t,x)|_x = \sum_{n\geq 2}\binom{n}{2}t^n = \frac{t^2}{(1-t)^3}. \tag{34}$$

Since $Q_{132}^{(k,0,\emptyset,0)}(t,0) = Q_{132}^{(k,0,0,0)}(t,0)$ for all $k \geq 1$, one can use (33) and Mathematica to show that

$$\begin{aligned}
Q_{132}^{(2,0,\emptyset,0)}(t,x)|_x &= \frac{t^3}{(1-t)(1-2t)^2}, \\
Q_{132}^{(3,0,\emptyset,0)}(t,x)|_x &= \frac{t^4}{(1-t)(1-3t+t^2)^2}, \\
Q_{132}^{(4,0,\emptyset,0)}(t,x)|_x &= \frac{t^5}{(1-t)^3(1-3t)^2}, \text{ and} \\
Q_{132}^{(5,0,\emptyset,0)}(t,x)|_x &= \frac{t^6}{(1-t)(1-5t+6t^2-t^3)^2}.
\end{aligned}$$

We also have the following proposition concerning the coefficient of the highest power of x in $Q_{n,132}^{(k,0,\emptyset,0)}(x)$.

Proposition 3. *For all $k \geq 1$, the highest power of x appearing in $Q_{n,132}^{(k,0,\emptyset,0)}(x)$ is x^{n-k}, and for all $n \geq k$, $Q_{n,132}^{(k,0,\emptyset,0)}(x)|_{x^{n-k}} = 1$.*

Proof. It is easy to see that for any $k \geq 1$, the permutation $\sigma \in S_n(132)$ with the maximal number of $MMP(k,0,\emptyset,0)$-matches for $n \geq k+1$, will be of the form $(n-k)(n-k-1)\cdots 21(n-k+1)(n-k+2)\cdots n$. Thus, the highest power of x that occurs in $Q_{n,132}^{(k,0,\emptyset,0)}(x)$is x^{n-k} which appears with coefficient 1. $\square$

Using Theorem 15, one can compute that

$$\begin{aligned}
&Q_{132}^{(0,0,\emptyset,0)}(t,x) = 1 + xt + x(1+x)t^2 + x\left(1+3x+x^2\right)t^3 + x\left(1+6x+6x^2+x^3\right)t^4 + \\
&\left(1+10x+20x^2+10x^3+x^4\right)t^5 + x\left(1+15x+50x^2+50x^3+15x^4+x^5\right)t^6 + \\
&x\left(1+21x+105x^2+175x^3+105x^4+21x^5+x^6\right)t^7 + \\
&x\left(1+28x+196x^2+490x^3+490x^4+196x^5+28x^6+x^7\right)t^8 + \\
&x\left(1+36x+336x^2+1176x^3+1764x^4+1176x^5+336x^6+36x^7+x^8\right)t^9 + \cdots.
\end{aligned}$$

If we compare $Q_{132}^{(0,0,\emptyset,0)}(t,x)$ to $Q_{132}^{(0,0,1,0)}(t,x)$, we see that for $n \geq 1$,

$$Q_{n,132}^{(0,0,\emptyset,0)}(x) = xQ_{n,132}^{(0,0,1,0)}(x). \tag{35}$$

Note the $Q_{n,132}^{(0,0,\emptyset,0)}(x)$ has an obvious symmetry property. That is, the following holds.

Theorem 16. *For all $n \geq 1$,*

$$x^{n+1}Q_{n,132}^{(0,0,\emptyset,0)}\left(\frac{1}{x}\right) = Q_{n,132}^{(0,0,\emptyset,0)}(x).$$

Proof. For $\sigma \in S_n$, define the statistic non-LRmin$(\sigma) = n - \text{LRmin}(\sigma)$. Since the statistic $\text{mmp}^{(0,0,\emptyset,0)}$ is the same as the LRmin statistic and the statistic $\text{mmp}^{(0,0,1,0)}$ is the same as the non-LRmin statistic, Theorem 16 shows that the statistics LRmin and $1+\text{non-LRmin}$ are equidistributed on 132-avoiding permutations. In fact, it proves a more general claim, namely that on $S_n(132)$, the joint distribution of the pair $(\text{mmp}^{(0,0,\emptyset,0)} - 1, \text{mmp}^{(0,0,1,0)})$ is the same as the distribution of $(\text{mmp}^{(0,0,1,0)}, \text{mmp}^{(0,0,\emptyset,0)} - 1)$, which often is not the case but is here because the sum $\text{mmp}^{(0,0,\emptyset,0)}(\sigma) + \text{mmp}^{(0,0,1,0)}(\sigma)$ equals the length of the permutation σ. That is, if we let

$$R_n(x,y) = \sum_{\sigma \in S_n(132)} x^{\text{mmp}^{(0,0,\emptyset,0)}(\sigma)} y^{\text{mmp}^{(0,0,1,0)}(\sigma)}, \tag{36}$$

then the theorem shows that $yR_n(x,y)$ is symmetric in x and y for all n.

We shall sketch a combinatorial proof of this fact. First we construct a bijection T from $S_n(132)$ onto $S_n(123)$ that will make the fact that $yR_n(x,y)$ is symmetric apparent. If $\sigma = \sigma_1 \cdots \sigma_k \in S_k$ and $\tau = \tau_1 \cdots \tau_\ell \in S_\ell$, then we let

$$\sigma \oplus \tau = \sigma_1 \cdots \sigma_k (k+\tau_1) \cdots (k+\tau_\ell)$$

and

$$\sigma \ominus \tau = (\ell+\sigma_1) \cdots (\ell+\sigma_k) \tau_1 \cdots \tau_\ell.$$

Then $\bigcup_n S_n(132)$ is recursively generated by starting with the permutation 1 and closing under the operations of $\sigma \ominus \tau$ and $\sigma \oplus 1$. Then we can define a recursive bijection $T : \bigcup_n S_n(132) \to \bigcup_n S_n(123)$ by letting $T(1) = 1$, $T(\sigma \ominus \tau) = T(\sigma) \ominus T(\tau)$, and $T(\sigma \oplus 1) = X(T(\sigma))$, where $X(\sigma)$ is constructed from σ as follows.

Take the permutation $\sigma \in S_n(123)$ and fix the positions and values of the left-to-right minima. Append one position to the end of σ, and renumber the non-left-to-right minima in decreasing order. For example, if $\sigma = 4762531$, then 4, 2, and 1 are the left-to-right minima. After fixing those positions and values and appending one position, the permutation looks like $4xx2xx1x$. Then we fill in the xs with 8, 7, 6, 5, 3, in that order, to obtain 48726513. The map X is essentially based on the *Simion-Schmidt bijection* described in [5, Chapter 4].

It is straightforward to prove by induction that if $T(\sigma) = \tau$, then σ_j matches the pattern $MMP(0,0,\emptyset,0)$ in σ if and only if τ_j matches the pattern $MMP(0,0,\emptyset,0)$ in τ. That is, the map T preserves left-to-right minima. Note that if σ_j does not match the pattern $MMP(0,0,\emptyset,0)$ in σ, then it must match the pattern $MMP(0,0,1,0)$ in σ. Thus, it follows that

$$\begin{aligned} R_n(x,y) &= \sum_{\sigma \in S_n(132)} x^{\text{mmp}^{(0,0,\emptyset,0)}(\sigma)} y^{\text{mmp}^{(0,0,1,0)}(\sigma)} \\ &= \sum_{\sigma \in S_n(123)} x^{\text{LRmin}(\sigma)} y^{\text{non-LRmin}(\sigma)}. \end{aligned}$$

Next observe that specifying the left-to-right minima of a permutation $\sigma \in S_n(123)$ completely determines σ. That is, if $\sigma_{i_1} > \sigma_{i_2} > \cdots > \sigma_{i_k}$ are the left-to-right minima of σ, where $1 = i_1 < i_2 < \cdots < i_k \leq n$, then the remaining elements must be placed in decreasing order, as in the map X, since any pair that are not decreasing will form a 123-pattern with a previous left-to-right minimum. This means that $X : S_n(123) \to S_{n+1}(123)$ is one-to-one, and since $\text{LRmin}(X(\sigma)) = \text{LRmin}(\sigma)$ and $\text{non-LRmin}(X(\sigma)) = 1 + \text{non-LRmin}(\sigma)$, it follows that

$$yR_n(x,y) = \sum_{\sigma \in S_n(123)} x^{\text{LRmin}(X(\sigma))} y^{\text{non-LRmin}(X(\sigma))}.$$

But it is easy to see that for any permutation $X(\sigma)$, reversing and then complementing $X(\sigma)$, which rotates the graph of $X(\sigma)$ by 180° around its center, produces a permutation of the form $X(\tau)$ for some $\tau \in S_n(123)$ such that $\text{LRmin}(X(\sigma)) = \text{non-LRmin}(X(\tau))$ and $\text{non-LRmin}(X(\sigma)) = \text{LRmin}(X(\tau))$. Thus,

$$\sum_{\sigma \in S_n(123)} x^{\text{LRmin}(X(\sigma))} y^{\text{non-LRmin}(X(\sigma))}$$

is symmetric in x and y. Hence, $yR_n(x,y)$ is symmetric in x and y. Thus, if r and c are the reverse and complement maps, respectively, then $Y : S_n(132) \to S_n(132)$ given by $Y(\sigma) = T^{-1}X^{-1}rcXT(\sigma)$ is a bijection that swaps the statistics $\text{mmp}(0,0,\emptyset,0) - 1$ and $\text{mmp}(0,0,1,0)$. $\square$

We have computed that

$$\begin{aligned}
&Q_{132}^{(1,0,\emptyset,0)}(t,x) = 1 + t + (1+x)t^2 + \left(1 + 3x + x^2\right)t^3 + \\
&\left(1 + 6x + 6x^2 + x^3\right)t^4 + \left(1 + 10x + 20x^2 + 10x^3 + x^4\right)t^5 + \\
&\left(1 + 15x + 50x^2 + 50x^3 + 15x^4 + x^5\right)t^6 + \\
&\left(1 + 21x + 105x^2 + 175x^3 + 105x^4 + 21x^5 + x^6\right)t^7 + \\
&\left(1 + 28x + 196x^2 + 490x^3 + 490x^4 + 196x^5 + 28x^6 + x^7\right)t^8 + \\
&\left(1 + 36x + 336x^2 + 1176x^3 + 1764x^4 + 1176x^5 + 336x^6 + 36x^7 + x^8\right)t^9 + \cdots.
\end{aligned}$$

One can observe that $Q_{132}^{(1,0,\emptyset,0)}(t,x) = Q_{132}^{(0,0,1,0)}(t,x)$. We provide here a combinatorial proof of this fact. Actually, we will prove a stronger statement that we record as the following theorem.

Theorem 17. *The two pairs of statistics* $(MMP(1,0,\emptyset,0), MMP(0,0,1,0))$ *and* $(MMP(0,0,1,0), MMP(1,0,\emptyset,0))$ *have the same joint distributions on* $S_n(132)$.

Proof. We will construct a map φ on $\cup_n S_n(132)$, recursively interchanging occurrences of the involved patterns. The base case, $n = 1$, obviously holds: $\varphi(1) := 1$ and neither of the patterns occur in 1.

Assume that the claim holds for 132-avoiding permutations of length less than n, and consider a permutation $\sigma \in S_n^{(i)}$ for some i. Consider two cases.

Case 1. $i = 1$. In this case, we can define $\varphi(\pi) := n\varphi(B_i(\sigma))$. Since n is neither an occurrence of $MMP(1, 0, \emptyset, 0)$ nor an occurrence of $MMP(0, 0, 1, 0)$, we get the desired property by the induction hypothesis.

Case 2. $i > 1$. Note that n is an occurrence of the pattern $MMP(0, 0, 1, 0)$, and because of n, each left-to-right minimum in $A_i(\sigma)$ is actually an occurrence of the pattern $MMP(1, 0, \emptyset, 0)$. Further, each non-left-to-right minimum in $A_i(\sigma)$ is obviously an occurrence of the pattern $MMP(0, 0, 1, 0)$. If $i = n$, we let $\varphi(\sigma) := Y(\text{red}(A_i(\sigma))) \oplus 1$, where Y is as defined in the proof of Theorem 16, and for $1 < i < n$, we let $\varphi(\sigma) := (Y(\text{red}(A_i(\sigma))) \oplus 1) \ominus \varphi(B_i(\sigma))$. Indeed, $\varphi(B_i(\sigma))$ will interchange the occurrences of the patterns by the induction hypothesis. Also, as in the proof of Theorem 16, $Y(\text{red}(A_i(\sigma)))\oplus 1$ will exchange the number of occurrences of the patterns in $A_i(\sigma)n$. $\square$

We have computed that

$$\begin{aligned}
&Q_{132}^{(2,0,\emptyset,0)}(t,x) = 1 + t + 2t^2 + (4+x)t^3 + \left(8 + 5x + x^2\right)t^4 + \\
&\left(16 + 17x + 8x^2 + x^3\right)t^5 + \left(32 + 49x + 38x^2 + 12x^3 + x^4\right)t^6 + \\
&\left(64 + 129x + 141x^2 + 77x^3 + 17x^4 + x^5\right)t^7 + \\
&\left(128 + 321x + 453x^2 + 361x^3 + 143x^4 + 23x^5 + x^6\right)t^8 + \\
&\left(256 + 769x + 1326x^2 + 1399x^3 + 834x^4 + 247x^5 + 30x^6 + x^7\right)t^9 + \cdots .
\end{aligned}$$

The sequence $(Q_{n,132}^{(2,0,\emptyset,0)}(x)|_x)_{n\geq 2}$ is sequence A000337 in the OEIS, whose nth term is $(n-1)2^n + 1$. Thus, $Q_{n,132}^{(2,0,\emptyset,0)}(x)|_x = (n-3)2^{n-2} + 1$ for $n \geq 2$.

We have computed that

$$\begin{aligned}
&Q_{132}^{(3,0,\emptyset,0)}(t,x) = 1 + t + 2t^2 + 5t^3 + (13+x)t^4 + \left(34 + 7x + x^2\right)t^5 + \\
&\left(89 + 32x + 10x^2 + x^3\right)t^6 + \left(233 + 122x + 59x^2 + 14x^3 + x^4\right)t^7 + \\
&\left(610 + 422x + 272x^2 + 106x^3 + 19x^4 + x^5\right)t^8 + \\
&\left(1597 + 1376x + 1090x^2 + 591x^3 + 182x^4 + 25x^5 + x^6\right)t^9 + \cdots ,
\end{aligned}$$

$$\begin{aligned}
&Q_{132}^{(4,0,\emptyset,0)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + (41+x)t^5 + \left(122 + 9x + x^2\right)t^6 + \\
&\left(365 + 51x + 12x^2 + x^3\right)t^7 + \left(1094 + 235x + 84x^2 + 16x^3 + x^4\right)t^8 + \\
&\left(3281 + 966x + 454x^2 + 139x^3 + 21x^4 + x^5\right)t^9 + \cdots , \text{ and}
\end{aligned}$$

$$\begin{aligned}
&Q_{132}^{(5,0,\emptyset,0)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + (131+x)t^6 + \left(417 + 11x + x^2\right)t^7 + \\
&\left(1341 + 74x + 14x^2 + x^3\right)t^8 + \left(4334 + 396x + 113x^2 + 18x^3 + x^4\right)t^9 + \cdots .
\end{aligned}$$

The second highest power of x that occurs in $Q_{n,132}^{(k,0,\emptyset,0)}(x)$ is x^{n-k-1}. Our next result will show that $Q_{n,132}^{(k,0,\emptyset,0)}(x)|_{x^{n-k-1}}$ has a regular behavior for large enough n. That is, we have the following theorem.

Theorem 18. *For $n \geq 3$ and $k \geq 1$,*

$$Q_{n+k-1,132}^{(k,0,\emptyset,0)}|_{x^{n-2}} = 2(k-1) + \binom{n}{2}. \tag{37}$$

Proof. Note that $Q_{132}^{(1,0,\emptyset,0)}(t,x) = Q_{132}^{(0,0,1,0)}(t,x)$ and by Theorem 11, we have that $Q_{n,132}^{(0,0,1,0)}(x)|_{x^{n-2}} = \binom{n}{2}$ for $n \geq 3$. Thus, the theorem holds for $k = 1$.

By induction, assume that $Q_{n+k-1,132}^{(k,0,\emptyset,0)}|_{x^{n-2}} = 2(k-1) + \binom{n}{2}$. We know by (29) that

$$Q_{n+k,132}^{(k+1,0,\emptyset,0)}(x) = \sum_{i=1}^{n+k} Q_{i-1,132}^{(k,0,\emptyset,0)}(x) Q_{n+k-i,132}^{(k+1,0,\emptyset,0)}(x). \tag{38}$$

Note that for $2 \leq i \leq n-k-2$, the highest coefficient of x that appears in $Q_{n+k-i,132}^{(k+1,0,\emptyset,0)}(x)$ is $x^{n+k-i-(k+1)} = x^{n-i-1}$. However the highest coefficient of x in $Q_{i-1,132}^{(k,0,\emptyset,0)}(x)$ is x^{i-2} so that the only terms on the RHS of (38) that can contribute to the coefficient of x^{n-2} are $i = 1$, $i = n+k-1$, and $i = n+k$. By Proposition 3, we know that

$$Q_{n+k-1,132}^{(k+1,0,\emptyset,0)}(x)|_{x^{n-2}} = 1 = Q_{n+k-2,132}^{(k,0,\emptyset,0)}(x)|_{x^{n-2}},$$

so the $i = 1$ and $i = n+k-1$ terms in (38) contribute 2 to $Q_{n+k,132}^{(k+1,0,\emptyset,0)}(x)|_{x^{n-2}}$. Now the $i = n+k$ term in (38) contributes

$$Q_{n+k-1,132}^{(k,0,\emptyset,0)}(x)|_{x^{n-2}} = 2(k-1) + \binom{n}{2}$$

to $Q_{n+k,132}^{(k+1,0,\emptyset,0)}(x)|_{x^{n-2}}$. Thus,

$$Q_{n+k,132}^{(k+1,0,\emptyset,0)}(x)|_{x^{n-2}} = 2k + \binom{n}{2}.$$

□

The sequences $(Q_{n,132}^{(3,0,\emptyset,0)}(x)|_x)_{n\geq 4}$, $(Q_{n,132}^{(4,0,\emptyset,0)}(x)|_x)_{n\geq 5}$, and $(Q_{n,132}^{(5,0,\emptyset,0)}(x)|_x)_{n\geq 5}$ do not appear in the OEIS.

7 The function $Q_{132}^{(\emptyset,0,k,0)}(t,x)$

In this section, we shall compute $Q_{132}^{(\emptyset,0,k,0)}(t,x)$ for $k \geq 0$. First we compute the generating function for $Q_{n,132}^{(\emptyset,0,0,0)}(x)$. Observe that n will always match the pattern $MMP(\emptyset,0,0,0)$ in any $\sigma \in S_n$. For $i \geq 1$, it is easy to see that $A_i(\sigma)$ will contribute nothing to $\text{mmp}^{(\emptyset,0,0,0)}(\sigma)$, since the presence of n to the right of an element in $A_i(\sigma)$ ensures that it does not match the pattern $MMP(\emptyset,0,0,0)$ in σ. Similarly, $B_i(\sigma)$ will contribute $\text{mmp}^{(\emptyset,0,0,0)}(\text{red}(B_i(\sigma)))$

to $\mathrm{mmp}^{(\emptyset,0,0,0)}(\sigma)$, since neither n nor any of the elements to the left of n have any effect on whether an element in $B_i(\sigma)$ matches the pattern $MMP(\emptyset,0,0,0)$ in σ. Thus,

$$Q_{n,132}^{(\emptyset,0,0,0)}(x) = x\sum_{i=1}^{n} C_{i-1} Q_{n-i,132}^{(\emptyset,0,0,0)}(x). \tag{39}$$

Multiplying both sides of (39) by t^n and summing over all $n \geq 1$, we obtain that

$$-1 + Q_{132}^{(\emptyset,0,0,0)}(t,x) = txC(t)\ Q_{132}^{(\emptyset,0,0,0)}(t,x),$$

so

$$Q_{132}^{(\emptyset,0,0,0)}(t,x) = \frac{1}{1-txC(t)}.$$

Next suppose that $k \geq 1$. In this case n in $\sigma \in S_n^{(i)}(132)$ will match the pattern $MMP(\emptyset,0,k,0)$ in σ if and only if $i > k$. For $i \geq 1$, it is easy to see that $A_i(\sigma)$ will contribute nothing to $\mathrm{mmp}^{(\emptyset,0,k,0)}(\sigma)$, since the presence of n to the right ensures that none of these elements will match the pattern $MMP(\emptyset,0,k,0)$ in σ. Similarly, $B_i(\sigma)$ will contribute $\mathrm{mmp}^{(\emptyset,0,k,0)}(\mathrm{red}(B_i(\sigma)))$ to $\mathrm{mmp}^{(\emptyset,0,k,0)}(\sigma)$, since neither n nor any of the elements to the left of n have any effect on whether an element in $B_i(\sigma)$ matches the pattern $MMP(\emptyset,0,k,0)$ in σ. Thus,

$$Q_{n,132}^{(\emptyset,0,k,0)}(x) = \sum_{i=1}^{k} C_{i-1} Q_{n-i,132}^{(\emptyset,0,k,0)}(x) + x\sum_{i=k+1}^{n} C_{i-1} Q_{n-i,132}^{(\emptyset,0,k,0)}(x). \tag{40}$$

Multiplying both sides of (40) by t^n and summing over all $n \geq 1$, we obtain that

$$-1 + Q_{132}^{(\emptyset,0,k,0)}(t,x) = t(\sum_{j=0}^{k-1} C_j t^j) Q_{132}^{(\emptyset,0,k,0)}(t,x) + xtQ_{132}^{(\emptyset,0,k,0)}\ (C(t) - \sum_{j=0}^{k-1} C_j t^j).$$

Thus, we have the following theorem.

Theorem 19.

$$Q_{132}^{(\emptyset,0,0,0)}(t,x) = \frac{1}{1-txC(t)}. \tag{41}$$

For $k \geq 1$,

$$Q_{132}^{(\emptyset,0,k,0)}(t,x) = \frac{1}{1-txC(t)-t(1-x)(\sum_{j=0}^{k-1} C_j t^j)} \tag{42}$$

and

$$Q_{132}^{(\emptyset,0,k,0)}(t,0) = \frac{1}{1-t(\sum_{j=0}^{k-1} C_j t^j)}. \tag{43}$$

7.1 Explicit formulas for $Q_{n,132}^{(\emptyset,0,k,0)}(x)|_{x^r}$

We have seen the constant terms $Q_{n132}^{(\emptyset,0,k,0)}(0)$ previously. That is, we have the following proposition.

Proposition 4. *$Q_{132}^{(\emptyset,0,k,0)}(t,0) = Q^{(0,0,k,0)}(t,0)$ for all $k \geq 1$.*

Proof. The proposition follows immediately from Theorems 9 and 19. That is, we have

$$Q_{132}^{(0,0,k,0)}(t,0) = \frac{1}{1-t(\sum_{j=0}^{k-1} C_j t^j)} = Q_{132}^{(\emptyset,0,k,0)}(t,0).$$

This fact is easy to see directly. That is, suppose that $\sigma = \sigma_1 \cdots \sigma_n \in S_n(132)$ and σ contains a $MMP(0,0,k,0)$-match. Then it is easy to see that if i is the largest such that σ_i matches $MMP(0,0,k,0)$, then there can be no $j > i$ with $\sigma_j > \sigma_i$ because otherwise, σ_j would match $MMP(0,0,k,0)$. Thus, if σ has a $MMP(0,0,k,0)$-match, then it also has a $MMP(\emptyset,0,k,0)$-match. Again, the converse is trivial. Hence the number of $\sigma \in S_n(132)$ with no $MMP(0,0,k,0)$-matches equals the number of $\sigma \in S_n(132)$ with no $MMP(\emptyset,0,k,0)$-matches. $\square$

We have computed that

$$Q_{132}^{(\emptyset,0,0,0)}(t,x) = 1 + xt + \left(x+x^2\right)t^2 + \left(2x+2x^2+x^3\right)t^3 + \left(5x+5x^2+3x^3+x^4\right)t^4 + \left(14x+14x^2+9x^3+4x^4+x^5\right)t^5 + \left(42x+42x^2+28x^3+14x^4+5x^5+x^6\right)t^6 + \left(132x+132x^2+90x^3+48x^4+20x^5+6x^6+x^7\right)t^7 + \left(429x+429x^2+297x^3+165x^4+75x^5+27x^6+7x^7+x^8\right)t^8 + \left(1430x+1430x^2+1001x^3+572x^4+275x^5+110x^6+35x^7+8x^8+x^9\right)t^9 + \cdots.$$

Recall that $Q_{132}^{(1,0,0,0)}(t,x) = \frac{1}{1-tC(tx)}$, so $Q_{132}^{(1,0,0,0)}(tx, \frac{1}{x}) = Q_{132}^{(\emptyset,0,0,0)}(t,x)$. This can easily be explained by the fact that every σ_i, $1 \leq i \leq n$, matches either $MMP(1,0,0,0)$ or $MMP(\emptyset,0,0,0)$.

We have computed that

$$Q_{132}^{(\emptyset,0,1,0)}(t,x) = 1 + t + (1+x)t^2 + (1+4x)t^3 + \left(1+12x+x^2\right)t^4 + \left(1+34x+7x^2\right)t^5 + \left(1+98x+32x^2+x^3\right)t^6 + \left(1+294x+124x^2+10x^3\right)t^7 + \left(1+919x+448x^2+61x^3+x^4\right)t^8 + \left(1+2974x+1576x^2+298x^3+13x^4\right)t^9 + \cdots.$$

In this case, it is easy to see that the only $\sigma \in S_n(132)$ that avoids the pattern $MMP(\emptyset,0,1,0)$ is the strictly decreasing permutation. Thus, $Q_{n,132}^{(\emptyset,0,1,0)}(0) = 1$ for all $n \geq 1$.

It is also easy to see that the permutation that maximizes the number of matches of $MMP(\emptyset,0,1,0)$ in $S_{2n}(132)$ is $(2n-1)(2n)(2n-3)(2n-2)\cdots 12$, which explains why the highest power of x in $Q_{2n,132}^{(\emptyset,0,1,0)}(x)$ is x^n, which has coefficient 1.

More generally, we have the following proposition.

Proposition 5. *For all $k \geq 1$, the highest power of x occurring in $Q_{kn,132}^{(\emptyset,0,k-1,0)}(x)$ is x^n, with coefficient $(C_{k-1})^n$.*

Proof. It is easy to see that the permutations that maximize the number of matches of $MMP(\emptyset, 0, k-1, 0)$ in $S_{kn}(132)$ are the permutations that have blocks consisting of

$$\tau^{(n)}(kn)\tau^{(n-1)}(k(n-1))\tau^{(n-2)}(k(n-2))\cdots\tau^{(1)}k,$$

where for each $i = 1, \ldots, n$, $\tau^{(i)}$ is a permutation of $(i-1)k+1, \ldots, (i-1)k+k-1$ that avoids 132. Since there are C_{k-1} choices for each $\tau^{(i)}$, the result follows. $\square$

It is also not difficult to see that the highest power of x in $Q_{2n+1,132}^{(\emptyset,0,1,0)}(x)$ is x^n, which has the coefficient $3n+1$. That is, if $\sigma \in S_n(132)$ and $\mathrm{mmp}^{(\emptyset,0,1,0)}(\sigma) = n$, then σ must be equal to either

$$\begin{aligned}&(2n+1)(2n-1)(2n)(2n-3)(2n-2)\cdots 12,\\&(2n-1)(2n)(2n+1)(2n-3)(2n-2)\cdots 12, \text{ or}\\&(2n)(2n-1)(2n+1)(2n-3)(2n-2)\cdots 12,\end{aligned}$$

or be of the form $(2n)(2n+1)\tau$, where $\tau \in S_{2n-1}(132)$, which has $n-1$ occurrences of $MMP(\emptyset, 0, 1, 0)$. Thus, for $n \geq 2$,

$$Q_{2n+1,132}^{(\emptyset,0,1,0)}(x)|_{x^n} = 3 + Q_{2n-1,132}^{(\emptyset,0,1,0)}(x)|_{x^{n-1}}.$$

The result now follows by induction, since $Q_{3,132}^{(\emptyset,0,1,0)}(x)|_x = 4$.

The sequence $(Q_{n,132}^{(\emptyset,0,1,0)}(x)|_x)_{n\geq 2}$ is A014143 in the OEIS, which has the generating function $\frac{1-2t\sqrt{1-4t}}{2t^2(1-t)^2}$. That is, one can easily compute that

$$\begin{aligned}Q_{132}^{(\emptyset,0,1,0)}(t,x)|_x &= \frac{1}{1+t(x-1)-xtC(t)}|_x = \frac{1}{1-(tx(C(t)-1)+t)}|_x\\&= \sum_{n\geq 1}(tx(C(t)-1)+t)^n|_x = \sum_{n\geq 1}\binom{n}{1}t(C(t)-1)t^{n-1}\\&= (C(t)-1)\sum_{n\geq 1}nt^n = \left(\frac{1-\sqrt{1-4t}}{2t}-1\right)\frac{t}{(1-t)^2}\\&= \frac{1-2t-\sqrt{1-4t}}{2(1-t)^2}.\end{aligned}$$

We have computed that

$$\begin{aligned}&Q_{132}^{(\emptyset,0,2,0)}(t,x) = 1+t+2t^2+(3+2x)t^3+(5+9x)t^4+\\&(8+34x)t^5+\left(13+115x+4x^2\right)t^6+\left(21+376x+32x^2\right)t^7+\\&\left(34+1219x+177x^2\right)t^8+\left(55+3980x+819x^2+8x^3\right)t^9+\cdots.\end{aligned}$$

The sequence $(Q_{n,132}^{(\emptyset,0,2,0)}(0))_{n\geq 2}$ is the Fibonacci numbers. We can give a combinatorial explanation for this fact as well. That is, the permutations in $S_n(132)$ that avoid the pattern $MMP(\emptyset,0,2,0)$ are of the form $n\alpha$, where α is a permutation in $S_{n-1}(132)$ that avoids $MMP(\emptyset,0,2,0)$, or of the form $(n-1)n\beta$, where β is a permutation in $S_{n-2}(132)$ that avoids $MMP(\emptyset,0,2,0)$. It follows that

$$Q_{n,132}^{(\emptyset,0,2,0)}(0) = Q_{n-1,132}^{(\emptyset,0,2,0)}(0) + Q_{n-2,132}^{(\emptyset,0,2,0)}(0).$$

The sequence $(Q_{n,132}^{(\emptyset,0,2,0)}(x)|_x)_{n\geq 3}$ does not appear in the OEIS.

We have computed that

$$Q_{132}^{(\emptyset,0,3,0)}(t,x) = 1 + t + 2t^2 + 5t^3 + (9+5x)t^4 + (18+24x)t^5 + (37+95x)t^6 + (73+356x)t^7 + \left(146+1259x+25x^2\right)t^8 + \left(293+4354x+215x^2\right)t^9 + \cdots .$$

The sequence $(Q_{n,132}^{(\emptyset,0,3,0)}(0))_{n\geq 0}$ is sequence A077947 in the OEIS, which has the generating function $\frac{1}{1-x-x^2-2x^3}$. However, the sequence $(Q_{n,132}^{(\emptyset,0,3,0)}(x)|_x)_{n\geq 4}$ does not appear in the OEIS.

We have computed that

$$Q_{132}^{(\emptyset,0,4,0)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + (28+14x)t^5 + (62+70x)t^6 + (143+286x)t^7 + (331+1099x)t^8 + (738+4124x)t^9 + \cdots .$$

The sequence $(Q_{n,132}^{(\emptyset,0,4,0)}(0))_{n\geq 0}$ does not appear in the OEIS.

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