Lacunary formal power series and the Stern-Brocot sequence

J.-P. Allouche
CNRS, Institut de Mathématiques de Jussieu
Équipe Combinatoire et Optimisation
Université Pierre et Marie Curie, Case 247, 4 Place Jussieu
F-75252 Paris Cedex 05
France
allouche@math.jussieu.fr

M. Mendès France
Université Bordeaux I
Mathématiques
F-33405 Talence Cedex
France
michel.mendes-france@math.u-bordeaux1.fr

À la mémoire de Philippe Flajolet

Abstract

Let $F(X) = \sum_{n \geq 0} (-1)^{\varepsilon_n} X^{-\lambda_n}$ be a real lacunary formal power series, where $\varepsilon_n = 0, 1$ and $\lambda_{n+1}/\lambda_n > 2$. It is known that the denominators $Q_n(X)$ of the convergents of its continued fraction expansion are polynomials with coefficients $0, \pm 1$, and that the number of nonzero terms in $Q_n(X)$ is the nth term of the Stern-Brocot sequence. We show that replacing the index n by any 2-adic integer ω makes sense. We prove that $Q_\omega(X)$ is a polynomial if and only if $\omega \in \mathbb{Z}$. In all the other cases $Q_\omega(X)$ is an infinite formal power series, the algebraic properties of which we discuss in the special case $\lambda_n = 2^{n+1} - 1$.

Keywords: Stern-Brocot sequence, continued fractions of formal power series, automatic sequences, algebraicity of formal power series.

1 Introduction

1.1 Lacunary power series and continued fraction expansions

Let $\Lambda = (\lambda_n)_{n\geq 0}$ be a sequence of positive integers with $0 < \lambda_0 < \lambda_1 < \ldots$ satisfying $\lambda_{n+1}/\lambda_n > 2$ for all $n \geq 0$. Consider the formal power series $F(X) := \sum_{n\geq 0}(-1)^{\varepsilon_n} X^{-\lambda_n}$, where $\varepsilon_n = 0, 1$. As is well known, power series in X^{-1} can be represented by a continued fraction $[A_0(X), A_1(X), A_2(X), \ldots]$, where the A_j's are polynomials in X, and for all $i > 0$, $A_i(X)$ is a non-constant polynomial. Quite obviously, in the case of the above $F(X)$, one has $A_0(X) = 0$.

Let $P_n(X)/Q_n(X) = [0, A_1(X), A_2(X), \ldots, A_n(X)]$ be the nth convergent of $F(X)$. As was already discovered in [3] and [28], the denominators $Q_n(X)$ are particularly interesting to study: their coefficients are $0, \pm 1$.

1.2 A sequence of polynomials and a sequence of integers

The denominators $Q_n(X)$ introduced above can be quite explicitly expressed (see [28]):

$$Q_n(X) = \sum_{k\geq 0} \sigma(k,\varepsilon) \binom{\frac{n+k}{2}}{k}_2 X^{\mu(k,\Lambda)}.$$

The exponent of X is given by $\mu(k,\Lambda) = \sum_{q\geq 0} e_q(k)(\lambda_q - \lambda_{q-1})$, with $\lambda_{-1} = 0$, where $e_q(k)$ is the qth binary digit of $k = \sum_{q\geq 0} e_q(k) 2^q$. The sign of the monomials is given by $\sigma(k,\varepsilon) = (-1)^{\nu(k)+\bar{\mu}(k,\varepsilon)}$ where $\nu(k)$ is the number of occurrences of the block 10 in the usual left-to-right reading of the binary expansion of k (e.g., $\nu(\text{twelve}) = 1$), and where $\bar{\mu}(k,\varepsilon) = \sum_{q\geq 0} e_q(k)(\varepsilon_{k-1} - \varepsilon_{k-2})$, with $\varepsilon_{-1} = \varepsilon_{-2} = 0$. The symbol $\binom{a}{b}_2$ is the **integer** equal to 0 or 1, according to the value modulo 2 of the binomial coefficient $\binom{a}{b}$, with the following convention: if a is not an integer, or if a is a positive integer and $a < b$, then $\binom{a}{b} := 0$. For example, as soon as n and k have opposite parities, $\binom{\frac{n+k}{2}}{k}_2 = 0$. In [3] it was observed that the number of non-zero monomials in $Q_n(X)$ is u_n, the nth term of the celebrated Stern-Brocot sequence defined by $u_0 = u_1 = 1$, and the recursive relations $u_{2n} = u_n + u_{n-1}$, $u_{2n+1} = u_n$ for all $n \geq 1$. This sequence is also called the Stern diatomic series (see sequence A002487 in [30]). It was studied by several authors, see, e.g., [17] and its list of references (including the historical references [9, 31]), see also [33, 29], or see [23] for a relation between the Stern sequence and the Towers of Hanoi. (Note that some authors have the slightly different definition: $v_0 = 0$, $v_{2n} = v_n$, $v_{2n+1} = v_n + v_{n+1}$; clearly $u_n = v_{n+1}$ for all $n \geq 0$.)

Our purpose here is to pursue our previous discussions on the sequence of polynomials $Q_n(X)$ in relationship with the Stern-Brocot sequence.

Remark 1 The sequence $(\nu(n))_{n\geq 0}$ happens to be related to the paperfolding sequence. Indeed, define $v(n) := (-1)^{\nu(n)}$ and $w(n) := v(n)v(n+1)$. From the definition of ν, we have for every $n \geq 0$ the relations $v(2n+1) = v(n)$, $v(4n) = v(2n)$, and $v(4n+2) =$

$-v(n)$. Equivalently, for every $n \geq 0$, we have $v(2n+1) = v(n)$, and $v(2n) = (-1)^n v(n)$. Hence, for every $n \geq 0$, we have $w(n) = v(2n)v(2n+1) = (-1)^n(v(n)^2) = (-1)^n$, and $w(2n+1) = v(2n+1)v(2n+2) = (-1)^{n+1}v(n)v(n+1) = (-1)^{n+1}w(n)$. It it then clear that, if $z(n) := w(2n+1)$, then $z(2n) = -w(2n) = -(-1)^n$ and $z(2n+1) = z(n)$. In other words the sequence $(z(n))_{n\geq 0}$ is the classical paperfolding sequence, and the sequence $(w(n))_{n\geq 0}$ itself is a paperfolding sequence, see e.g., [26, p. 125] where the sequences are indexed by $n \geq 1$ instead of $n \geq 0$.

1.3 A partial order on the integers

Let $m = e_0(m)e_1(m)...$ and $k = e_0(k)e_1(k)...$ be two nonnegative integers together with their binary expansion, which of course terminates with a tail of 0's. Lucas [25] observed that

$$\binom{m}{k} \equiv \prod_{i\geq 0} \binom{e_i(m)}{e_i(k)} \bmod 2.$$

This implies the following relation (in $\mathbb{Z}$)

$$\binom{m}{k}_2 = \prod_{i\geq 0} \binom{e_i(m)}{e_i(k)},$$

so that we have $\binom{m}{k}_2 = 1$ if and only if $e_i(k) \leq e_i(m)$ for all $i \geq 0$.
We will say that m *dominates* k and we write $k \ll m$, if $e_i(k) \leq e_i(m)$ for all $i \geq 0$. In other words the sequence $k \to \binom{m}{k}_2$ is the characteristic function of the k's dominated by m. (This order was used in, e.g., [2].)

As a consequence of our remarks, the Stern-Brocot sequence has the following representation

$$u_n = \sum_{k \ll \frac{k+n}{2}} 1.$$

Remark 2 This last relation can be easily deduced from a result of Carlitz [11, 12] (Carlitz calls $\theta_0(n)$ what we call u_n):

$$u_n = \sum_{0\leq 2r\leq n} \binom{n-r}{r}_2.$$

Indeed, we have

$$\begin{aligned}\sum_{k \ll \frac{k+n}{2}} 1 &= \sum_{\substack{0\leq k\leq n\\ k\equiv n \bmod 2}} \binom{\frac{k+n}{2}}{k}_2 = \sum_{\substack{0\leq k'\leq n\\ k'\equiv 0 \bmod 2}} \binom{n-\frac{k'}{2}}{n-k'}_2 \quad \text{(by letting } k' = n-k\text{)}\\ &= \sum_{0\leq 2r\leq n} \binom{n-r}{n-2r}_2 = \sum_{0\leq 2r\leq n} \binom{n-r}{r}_2 \quad \text{(by using } \tbinom{a}{b} = \tbinom{a}{a-b}\text{)}.\end{aligned}$$

Also note that in [12] the range $0 \leq 2r < n$ should be replaced by $0 \leq 2r \leq n$ as in [11] (see also [17, Corollary 6.2] where the index n should be adjusted). Let us finally indicate that this remark is also Corollary 13 in [3].

Remark 3 The relation $u_n = \sum_{0\le 2r\le n}\binom{n-r}{r}_2$ can give the idea (inspired by the classical *binomial transform*) of introducing a map on sequences $(a_n)_{n\ge 0} \to (b_n)_{n\ge 0}$ with $b_n := \sum_{0\le 2r\le n}\binom{n-r}{r}_2 a_r$, so that in particular the image of the constant sequence 1 is the Stern-Brocot sequence. One can also go a step further by defining a map $\mathcal{C}$ which associates with two sequences $\mathbf{a} = (a_n)_{n\ge 0}$ and $\mathbf{b} = (b_n)_{n\ge 0}$ the sequence

$$\mathcal{C}(\mathbf{a},\mathbf{b}) := \left(\sum_{0\le 2r\le n}\binom{n-r}{r}_2 a_r b_{n-r}\right)_{n\ge 0}.$$

It is unexpected that some variations on the Stern-Brocot sequences (different from but in the spirit of the twisted Stern sequence of [8]) are related to the celebrated Thue-Morse sequence (see, e.g., [5]). In fact, recall that the ± 1 Thue-Morse sequence $\mathbf{t} = ((t_n)_{n\ge 0}$ can be defined by $t_0 = 1$ and, for all $n \ge 0$, $t_{2n} = t_n$ and $t_{2n+1} = -t_n$. Now define the sequences $\alpha = (\alpha_n)_{n\ge 0}$, $\beta = (\beta_n)_{n\ge 0}$, $\gamma = (\gamma_n)_{n\ge 0}$ by

$$\alpha := \mathcal{C}(\mathbf{t},\mathbf{1}),\ \beta := \mathcal{C}(\mathbf{1},\mathbf{t}),\ \gamma := \mathcal{C}(\mathbf{t},\mathbf{t}).$$

Then the reader can check that these sequences satisfy respectively

$$\begin{array}{lll}
\alpha(0)=1,\ \alpha(1)=1, & \text{and for all } n\ge 1, & \alpha_{2n}=\alpha_n-\alpha_{n-1},\ \alpha_{2n+1}=\alpha_n \\
\beta(0)=1,\ \beta(1)=-1, & \text{and for all } n\ge 1, & \beta_{2n}=\beta_n-\beta_{n-1},\ \beta_{2n+1}=-\beta_n \\
\gamma(0)=1,\ \gamma(1)=-1, & \text{and for all } n\ge 1, & \gamma_{2n}=\gamma_n+\gamma_{n-1},\ \gamma_{2n+1}=-\gamma_n
\end{array}$$

so that, with the notation of [30],

$$(\alpha_n)_{n\ge 0} = (A005590(n+1))_{n\ge 0},\ (\beta_n)_{n\ge 0} = (A177219(n+1))_{n\ge 0},\ (\gamma_n)_{n\ge 0} = (A049347(n))_{n\ge 0}.$$

The last sequence $(\gamma_n)_{n\ge 0}$ is the 3-periodic sequence with period $(1,-1,0)$ (hint: prove by induction on n that for all $j \le n$ one has $(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2}) = (1,-1,0)$).

2 More on the sequence $Q_n(X)$ and a note on $P_n(X)$ for a special Λ

We now specialize to the case $\lambda_n = 2^{n+1} - 1$. In that case, $\mu(k,\Lambda) = k$. Also note that $\sigma(k,\varepsilon) \equiv 1 \bmod 2$. Let $P_n(X)/Q_n(X)$ denote as previously the nth convergent of the continued fraction of the formal power series $\sum_{i\ge 1}(-1)^{\varepsilon_i}X^{1-2^i}$. We begin with a short section on P_n. The rest of the section will be devoted to the "simpler" polynomials Q_n.

2.1 The sequence P_n modulo 2

Theorem 1 *We have $P_n(X) \equiv Q_{n-1}(X) \bmod 2$ for $n \ge 1$.*

Proof. Let $F(X) = \sum_{i\geq 1}(-1)^{\varepsilon_i} X^{1-2^i}$. Define the formal power series $\Phi(X)$ by its continued fraction expansion $\Phi(X) = [0, X, X, \ldots]$. Its nth convergent is given by $\pi_n(X)/\kappa_n(X) = [0, X, \ldots, X]$ (n partial quotients equal to X). An immediate induction shows that $\pi_n(X) = \kappa_{n-1}$ for $n \geq 1$. Reducing $F(X)$ modulo 2, we see that $F^2(X) + XF(X) + 1 \equiv 0 \bmod 2$. On the other hand $\Phi(X) = 1/(X + \Phi(X))$, hence $\Phi^2(X) + X\Phi(X) + 1 \equiv 0 \bmod 2$. This implies that $F(X) \equiv \Phi(X) \bmod 2$. Hence $P_n(X) \equiv \pi_n(X) \bmod 2$ and $Q_n(X) \equiv \kappa_n(X) \bmod 2$: to be sure that the convergents of the reduction modulo 2 of F are equal to the reduction modulo 2 of the convergents of $F(X)$, the reader can look at, e.g., [34]. Thus $P_n(X) \equiv \pi_n(X) = \kappa_{n-1}(X) \equiv Q_{n-1}(X) \bmod 2$.

Corollary 1 *The following congruence is satisfied by $Q_n(X)$ for $n \geq 1$:*

$$Q_n^2(X) - Q_{n+1}(X)Q_{n-1}(X) \equiv 1 \bmod 2.$$

Proof. Use the classical identity $P_{n+1}(X)Q_n(X) - P_n(X)Q_{n+1}(X) = (-1)^n$ for the convergents of a continued fraction.

2.2 The sequence Q_n and the Chebyshev polynomials

We have the formula

$$Q_n(X) \equiv \sum_{k\geq 0} \binom{\frac{n+k}{2}}{k}_2 X^k \equiv \sum_{\substack{0\leq k\leq n \\ k\equiv n \bmod 2}} \binom{\frac{k+n}{2}}{k}_2 X^k \qquad \bmod 2.$$

The Chebyshev polynomials of the second kind (see, e.g., [20, p. 184–185]) are defined by

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}.$$

They have the well-known explicit expansion

$$U_n(X) = \sum_{0\leq k\leq n/2} (-1)^k \binom{n-k}{k} (2X)^{n-2k}.$$

We thus get a relationship between Q_n and U_n (compare with the related but not identical result [17, Proposition 6.1]).

Theorem 2 *The reductions modulo 2 of $Q_n(X)$ and of $U_n(X/2)$ are equal.*

Proof. We can write modulo 2

$$\begin{aligned}
Q_n(X) &\equiv \sum_{\substack{0\leq k'\leq n \\ k'\equiv 0 \bmod 2}} \binom{n-\frac{k'}{2}}{n-k'}_2 X^{n-k'} \quad \text{(by letting } k' = n-k) \\
&\equiv \sum_{0\leq 2r\leq n} \binom{n-r}{n-2r}_2 X^{n-2r} \\
&\equiv \sum_{0\leq 2r\leq n} \binom{n-r}{r}_2 X^{n-2r} \quad \text{(by using } \tbinom{a}{b} = \tbinom{a}{a-b}).
\end{aligned}$$

Hence $Q_n(X) \equiv U_n(X/2) \bmod 2$. □

As an immediate application of Theorem 2 (and of Remark 2) we have the following corollaries.

Corollary 2 *The number of odd coefficients in the (scaled) Chebyshev polynomial of the second kind $U_n(X/2)$ is equal to the Stern-Brocot sequence u_n.*

Remark 4 Corollary 1 above can also be deduced from Theorem 2 using a classical relation for Chebyshev polynomials implied by their expression using sines.

Remark 5 The polynomials $Q_n(X)$ are also related to the Fibonacci polynomials (see, e.g., [19]) and to Morgan-Voyce polynomials which are a variation on the Chebyshev polynomials (on Morgan-Voyce polynomials, introduced by Morgan-Voyce in dealing with electrical networks see, e.g., [32, 7, 22] and the references therein). Indeed, the Fibonacci polynomials satisfy

$$F_{n+1}(X) = \sum_{2j \leq n} \binom{n-j}{j} X^{n-2j}$$

(compare with the proof of Theorem 2), while the Morgan-Voyce polynomials satisfy

$$b_n(X) = \sum_{k \leq n} \binom{n+k}{n-k} X^k \quad \text{and} \quad B_n(X) = \sum_{k \leq n} \binom{n+k+1}{n-k} X^k$$

(note that $\binom{n+k}{n-k} = \binom{n+k}{2k}$, that $\binom{n+k+1}{n-k} = \binom{n+k+1}{2k+1}$, and see Lemmas 1 and 2 below).

Remark 6 The polynomials that we have defined are related to the Stern-Brocot sequence, but they differ from Stern polynomials occurring in the literature, in particular they are not the same as those introduced in [24]. They also differ from the polynomials studied in [17, 18].

2.3 Extension of $Q_n(X)$ to $Q_\omega(X)$ with $\omega \in \mathbb{Z}_2$

Definition 1 Let $\omega = \sum_{i \geq 0} \omega_i 2^i = \omega_0 \omega_1 \omega_2 \ldots \in \mathbb{Z}_2$ be a 2-adic integer, or equivalently an infinite sequence of 0's and 1's. For a nonnegative integer k whose binary expansion is given by $k = \sum_{i \geq 0} k_i 2^i$, we define

$$\binom{\omega}{k}_2 = \prod_{i \geq 0} \binom{\omega_i}{k_i}.$$

The infinite product $\binom{\omega}{k}_2$ is well defined since, for large i, $\binom{\omega_i}{k_i}$ reduces to $\binom{\omega_i}{0} = 1$. It is equal to 0 or 1. The above product extends Lucas' observation to all 2-adic integers ω. In particular, since $-1 = \sum_{i \geq 0} 2^i = 1^\infty$, we see that -1 dominates all $k \in \mathbb{N}$ (where the order introduced in Section 1.3 is generalized in the obvious way). A similar definition (binomials and order) occurs in [27].

Definition 2 In the general case for Λ, with $\lambda_{n+1}/\lambda_n > 2$, and $\varepsilon = 0, 1$, the polynomials $Q_n(X)$ above naturally extend to formal power series $Q_\omega(X)$ defined for $\omega = \omega_0\omega_1\omega_2 \ldots \in \mathbb{Z}_2$ by

$$Q_\omega(X) = \sum_{k \geq 0} \sigma(k, \varepsilon) \binom{\frac{\omega+k}{2}}{k}_2 X^{\mu(k,\Lambda)} = \sum_{\substack{k \equiv \omega \bmod 2 \\ k \ll \frac{\omega+k}{2}}} \sigma(k, \varepsilon) X^{\mu(k,\Lambda)}.$$

Remark 7 The reader can check (e.g., by using integer truncations of ω tending to ω) that

$$\binom{\omega}{k} \equiv \binom{\omega}{k}_2 \bmod 2$$

where the binomial coefficient $\binom{\omega}{k}$ is defined by

$$\binom{\omega}{k} = \frac{\omega(\omega-1)\ldots(\omega-k+1)}{k!} \in \mathbb{Z}_2.$$

In particular, we see that for any 2-adic integer ℓ,

$$\binom{-\ell}{k} = (-1)^k \binom{\ell+k-1}{k}, \quad \text{hence} \quad \binom{-\ell}{k}_2 = \binom{\ell+k-1}{k}_2.$$

Now for $n \in \mathbb{N}$ we have

$$Q_{-n}(X) = \sum_{k \geq 0} \sigma(k, \varepsilon) \binom{\frac{-n+k}{2}}{k}_2 X^{\mu(k,\Lambda)} = \sum_{k \geq 0} \sigma(k, \varepsilon) \binom{-\frac{(n-k)}{2}}{k}_2 X^{\mu(k,\Lambda)},$$

thus

$$Q_{-n}(X) = \sum_{k \geq 0} \sigma(k, \varepsilon) \binom{\frac{n-k}{2}+k-1}{k}_2 X^{\mu(k,\Lambda)} = \sum_{k \geq 0} \sigma(k, \varepsilon) \binom{\frac{n-2+k}{2}}{k}_2 X^{\mu(k,\Lambda)} = Q_{n-2}(X).$$

In particular Q_{-n} and Q_{n-2} have same degree. Also note that the definition of Q_{-n} for $n \in \mathbb{N}$ yields

$$Q_{-1}(X) = \sum_{k \geq 0} \sigma(k, \varepsilon) \binom{\frac{k-1}{2}}{k}_2 X^{\mu(k,\Lambda)} = 0.$$

Remark 8 If $\lambda_n = 2^{n+1} - 1$, Corollary 1 can be extended to 2-adic integers: using again truncations of ω tending to ω yields, for any 2-adic integer ω,

$$Q_\omega^2(X) - Q_{\omega+1}(X) Q_{\omega-1}(X) \equiv 1 \bmod 2.$$

2.4 Extension of the sequence $(u_n)_{n\geq 0}$ to negative indices

What precedes suggests two ways of extending the sequence $(u_n)_{n\geq 0}$ to negative integer indices. First, we noted the relation $u_n = \sum_{k \ll \frac{n+k}{2}} 1$, i.e., u_n is the number of monomials with non-zero coefficient in $Q_n(X)$. But from the previous section, we can define $Q_{-n}(X)$ for $n \in \mathbb{N}$, and we have that $Q_{-n}(X) = Q_{n-2}(X)$. This suggests the definition

$$u_{-n} := u_{n-2} \text{ for all } n \geq 2.$$

Strictly speaking, this definition leaves the value u_{-1} indeterminate, but, since u_n is the number of monomials with nonzero coefficients in Q_n, the remark above that $Q_{-1} = 0$ implies $u_{-1} = 0$.

Another way of generalizing u_n to negative indices would be to use the recursion

$$u_{2n} = u_n + u_{n-1}, \quad u_{2n+1} = u_n, \quad \text{for all } n \geq 1,$$

allowing non-positive values for n. Allowing first $n = 0$ leads to $u_0 = u_0 + u_{-1}$, hence $u_{-1} = 0$. On the other hand we claim that the relation $u_{-n} := u_{n-2}$ for all $n \geq 2$ leads to the same recursion formulas with u_{2n} and u_{2n+1} for non-positive n. Indeed, let $m = -n$, with $n \geq 2$. Then

$$u_{2m} = u_{-2n} = u_{2n-2} = u_{2(n-1)} = u_{n-1} + u_{n-2} = u_{-n-1} + u_{-n} = u_{m-1} + u_m$$

and

$$u_{2m+1} = u_{-2n+1} = u_{2n-3} = u_{2(n-2)+1} = u_{n-2} = u_{-n} = u_m.$$

We thus finally have a generalization compatible with both approaches, yielding

$$\ldots\ u_{-4} = 2,\ u_{-3} = 1,\ u_{-2} = 1,\ u_{-1} = 0,\ u_0 = 1,\ u_1 = 1,\ u_2 = 2,\ u_3 = 1,\ u_4 = 3\ \ldots$$

and the definition

Definition 3 The Stern-Brocot sequence $(u_n)_{n\geq 0}$ can be extended to a sequence $(u_n)_{n\in\mathbb{Z}}$ by letting $u_{-n} = u_{n-2}$ for $n \geq 2$, and $u_{-1} = 0$. This sequence satisfies the same recursive relations as the initial sequence $(u_n)_{n\geq 0}$, namely $u_{2n} = u_n + u_{n-1}$ and $u_{2n+1} = u_n$ for all $n \in \mathbb{Z}$.

3 The arithmetical nature of the power series $Q_\omega(X)$

Recall that the formal series $Q_\omega(X)$, where $\omega = \omega_0\omega_1 \ldots$ belongs to $\mathbb{Z}_2$, is given by

$$Q_\omega(X) = \sum_{k\geq 0} \sigma(k,\varepsilon) \binom{\frac{\omega+k}{2}}{k}_2 X^{\mu(k,\Lambda)} = \sum_{\substack{k\equiv\omega \bmod 2 \\ k \ll \frac{\omega+k}{2}}} \sigma(k,\varepsilon) X^{\mu(k,\Lambda)}.$$

We have seen that $Q_\omega(X)$ reduces to a polynomial if ω belongs to $\mathbb{Z}$. We will prove that this is a necessary and sufficient condition for this series being a polynomial. Then we will address the question of the algebraicity of $Q_\omega(X)$, on $\mathbb{Q}(X)$ and on $\mathbb{Z}/2\mathbb{Z}(X)$, in the special case $\lambda_n = 2^{n+1} - 1$. We begin with a lemma.

Lemma 1 *Let $\omega = \omega_0\omega_1\ldots$ belong to $\mathbb{Z}_2$. Then the following properties hold.*

(i) *For every $j \geq 0$,*
$$\binom{\omega + 2^j}{2^{j+1}}_2 \equiv \omega_j + \omega_{j+1} \bmod 2.$$

(ii) *The sequence $(\binom{\omega+2^j}{2^{j+1}}_2)_{j\geq 0}$ is ultimately periodic if and only if ω is rational.*

(iii) *The sequence $(\binom{\omega+2^j}{2^{j+1}}_2)_{j\geq 0}$ is ultimately equal to 0 if and only if ω is an integer.*

(iv) *For every $k \geq 0$*
$$\binom{\frac{\omega+k}{2}}{k}_2 = \binom{\omega + k + 1}{2k+1}_2.$$

(v) *If $\omega \neq -1$, there exist an integer $\ell \geq 0$ and a 2-adic integer ω' such that $\omega = 2^\ell - 1 + 2^{\ell+1}\omega'$. Let $f_\omega(k) := \binom{\frac{\omega+k}{2}}{k}_2 = \binom{\omega+k+1}{2k+1}_2$. Then for any integer k' we have $f_\omega(2^\ell - 1 + 2^{\ell+1}k') = \binom{\omega'+k'}{2k'}_2$.*

(vi) *Suppose that there exist $\ell \geq 0$ and $j \geq 0$ such that $\omega = 2^\ell - 1 + 2^{\ell+1}(2^j(2\omega'+1))$. Then for any integer k' we have $f_\omega(2^\ell - 1 + 2^{\ell+1}(2^j(2k'+1))) = \binom{\omega'+k'+1}{2k'+1}_2$.*

Proof. In order to prove (i) we write

$$\begin{array}{rcccccccc} \omega + 2^j & = & & \omega_0 & \omega_1 & \ldots & \omega_j & \omega_{j+1} & \ldots \\ & & + & 0 & 0 & \ldots & 1 & 0 & \ldots \\ & = & & \omega_0 & \omega_1 & \ldots & \alpha_j & \alpha_{j+1} & \ldots \end{array}$$

where α_j and α_{j+1} are given by

$$\begin{array}{ll} \text{if } \omega_j = 0 \text{ and } \omega_{j+1} = 0, & \text{then } \alpha_j = 1 \text{ and } \alpha_{j+1} = 0 \\ \text{if } \omega_j = 0 \text{ and } \omega_{j+1} = 1, & \text{then } \alpha_j = 1 \text{ and } \alpha_{j+1} = 1 \\ \text{if } \omega_j = 1 \text{ and } \omega_{j+1} = 0, & \text{then } \alpha_j = 0 \text{ and } \alpha_{j+1} = 1 \\ \text{if } \omega_j = 1 \text{ and } \omega_{j+1} = 1, & \text{then } \alpha_j = 0 \text{ and } \alpha_{j+1} = 0. \end{array}$$

By inspection we see that $\alpha_{j+1} \equiv \omega_j + \omega_{j+1} \bmod 2$. Now we write

$$\binom{\omega + 2^j}{2^{j+1}}_2 = \left(\prod_{0\leq k\leq j-1} \binom{\omega_k}{0}_2\right) \binom{\alpha_j}{0}_2 \binom{\alpha_{j+1}}{1}_2 \left(\prod_{k\geq j+2} \binom{\alpha_k}{0}_2\right) = \alpha_{j+1} \equiv \omega_j + \omega_{j+1} \bmod 2.$$

Let us prove (ii). We note that the sequence $((\omega_j + \omega_{j+1}) \bmod 2)_{j\geq 0}$ is ultimately periodic if and only if the sequence $(\omega_j \bmod 2)_{j\geq 0}$ is ultimately periodic (hence if and only if the sequence $(\omega_j)_{j\geq 0}$ itself is ultimately periodic): indeed, $((\omega_j + \omega_{j+1}) \bmod 2)_{j\geq 0}$ is ultimately periodic if and only if the formal power series $G(X) := \sum_{j\geq 0}(\omega_j + \omega_{j+1})X^j$ is rational (as an element of $\mathbb{Z}/2\mathbb{Z}[[X]]$). But, if we let $H(X)$ denote the formal power series $H(X) := \sum_{j\geq 0} \omega_j X^j \in \mathbb{Z}/2\mathbb{Z}[[X]]$, then $XG(X) + \omega_0 = (1 + X)H(X)$. So $G(X)$ is rational if and only if H is, if and only if $(\omega_j \bmod 2)_{j\geq 0}$ is ultimately periodic, i.e., if the 2-adic integer ω is rational.

To prove (iii), we note that $\binom{\omega+2^j}{2^{j+1}}_2 = 0$ for j large enough implies from Lemma 1 (i) that $\omega_j + \omega_{j+1} \equiv 0 \bmod 2$ for j large enough. This means that $\omega_j \equiv \omega_{j+1} \bmod 2$ for j large enough, or equivalently $\omega_j = \omega_{j+1}$ for j large enough. But then either $\omega_j = \omega_{j+1} = 0$ for large j, hence ω is a nonnegative integer, or $\omega_j = \omega_{j+1} = 1$ for large j, hence ω is a negative integer. We thus finally get that ω belongs to $\mathbb{Z}$. The converse is straightforward.

We prove (iv) by considering the parities of ω and k. First note that if ω and k have opposite parities, then $\binom{\frac{\omega+k}{2}}{k}_2 = 0$ while $\binom{\omega+k+1}{2k+1}_2 = 0$ (use Definition 1 and look at the last digit of $\omega+k+1$ and of $2k+1$). Now if $\omega = 2\omega'$ and $k = 2k'$, we have $\binom{\frac{\omega+k}{2}}{k}_2 = \binom{\omega'+k'}{2k'}_2$ while $\binom{\omega+k+1}{2k+1}_2 = \binom{2(\omega'+k')+1}{4k'+1}_2 = \binom{\omega'+k'}{2k'}_2$ (use Definition 1 again). Finally if $\omega = 2\omega'+1$ and $k = 2k'+1$, we have $\binom{\frac{\omega+k}{2}}{k}_2 = \binom{\omega'+k'+1}{2k'+1}_2$ while $\binom{\omega+k+1}{2k+1}_2 = \binom{2(\omega'+k'+1)+1}{4k'+3}_2 = \binom{\omega'+k'+1}{2k'+1}_2$ (use Definition 1 again).

Let us prove (v). Since $\omega \neq -1$, its 2-adic expansion contains at least one zero. Write $\omega = 11\ldots10\omega_{\ell+1}\omega_{\ell+2}\ldots$, so that the 2-adic expansion of ω begins with exactly $\ell \geq 0$ ones. Defining $\omega' := \omega_{\ell+1}\omega_{\ell+2}\ldots$, we thus have $\omega = 2^\ell - 1 + 2^{\ell+1}\omega'$. Now for any integer k' we have from Definition 1

$$f_\omega(2^\ell - 1 + 2^{\ell+1}k') = \binom{\omega + 2^\ell + 2^{\ell+1}k'}{2^{\ell+1} - 1 + 2^{\ell+1}(2k')}_2 = \binom{2^{\ell+1} - 1 + 2^{\ell+1}(\omega'+k')}{2^{\ell+1} - 1 + 2^{\ell+1}(2k')}_2 = \binom{\omega'+k'}{2k'}_2.$$

We finally prove (vi). Using (v) we see that

$$f_\omega(2^\ell - 1 + 2^{\ell+1}(2^j(2k'+1))) = \binom{2^j(2\omega'+1+2k'+1)+1}{2^{j+1}(2k'+1)+1}_2 = \binom{\omega'+k'+1}{2k'+1}_2.$$

Now we can prove the following result.

Theorem 3 *Let ω be a 2-adic integer. The formal power series $Q_\omega(X)$ is a polynomial if and only if ω belongs to $\mathbb{Z}$.*

Proof. If n is a nonnegative integer, then $Q_n(X)$ is a polynomial. So is $Q_{-n}(X)$ for $n \neq 1$ because $Q_{-n} = Q_{n-2}$ as we have seen in Remark 7. On the other hand $Q_{-1}(X)$ is also a polynomial since $Q_{-1}(X) = 0$. Conversely suppose that $Q_\omega(X)$ is a polynomial, for some $\omega = \omega_0\omega_1\ldots$ in $\mathbb{Z}_2$. The coefficients of the monomials $X^{\mu(k,\Lambda)}$ in $Q_\omega(X)$, i.e., $\sigma(k,\varepsilon)\binom{\frac{\omega+k}{2}}{k}_2$, are equal to zero for k large enough. Thus $f_\omega(k) = \binom{\frac{\omega+k}{2}}{k}_2$ is equal to zero for k large enough. We may suppose that $\omega \neq -1$; thus, using the notation in Lemma 1 (v), we certainly have that $f_\omega(2^\ell - 1 + 2^{\ell+1}k') = 0$ for k' large enough. Using Lemma 1 (v), we thus have $\binom{\omega'+k'}{2k'}_2 = 0$ for k' large enough. This implies $\binom{\omega'+2^j}{2^{j+1}}_2 = 0$ for j large enough. Lemma 1 (iii) yields that ω', hence ω, belongs to $\mathbb{Z}$.

Before proving our Theorem 4 characterizing the algebraicity of the series $Q_\omega(X)$ for a special Λ, we need a lemma.

Lemma 2 *Let $\omega = \omega_0\omega_1\dots$ be a 2-adic integer. We let $(f_\omega(k))_{k\geq 0}$, $(g_\omega(k))_{k\geq 0}$, $(h_\omega(k))_{k\geq 0}$ denote the sequences*

$$f_\omega(k) := \binom{\omega+k+1}{2k+1}_2, \quad g_\omega(k) := \binom{\omega+k}{2k}_2, \quad h_\omega(k) := \binom{\omega+k}{2k+1}_2.$$

Then we have the following relations.

$$\begin{array}{llll} f_{2\omega}(2k) = g_\omega(k), & f_{2\omega+1}(2k) = 0, & f_{2\omega}(2k+1) = 0, & f_{2\omega+1}(2k+1) = f_\omega(k) \\ g_{2\omega}(2k) = g_\omega(k), & g_{2\omega+1}(2k) = g_\omega(k), & g_{2\omega}(2k+1) = h_\omega(k), & g_{2\omega+1}(2k+1) = f_\omega(k) \\ h_{2\omega}(2k) = 0, & h_{2\omega+1}(2k) = g_\omega(k), & h_{2\omega}(2k+1) = h_\omega(k), & h_{2\omega+1}(2k+1) = 0 \end{array}$$

Proof. The proof is easy: it uses the definition of $\binom{\omega}{\ell}_2$, which in particular shows for any 2-adic integer ω and any integer ℓ that

$$\begin{array}{ll} \binom{2\omega}{2\ell}_2 = \binom{\omega}{\ell}_2\binom{0}{0}_2 = \binom{\omega}{\ell}_2 & \binom{2\omega+1}{2\ell}_2 = \binom{\omega}{\ell}_2\binom{1}{0}_2 = \binom{\omega}{\ell}_2 \\ \binom{2\omega}{2\ell+1}_2 = \binom{\omega}{\ell}_2\binom{0}{1}_2 = 0 & \binom{2\omega+1}{2\ell+1}_2 = \binom{\omega}{\ell}_2\binom{1}{1}_2 = \binom{\omega}{\ell}_2. \end{array}$$

Remark 9 The sequences above occur in the OEIS [30] when $\omega = n$ is an integer. In particular, $(\binom{\frac{n+k}{2}}{k})_{n,k} = (\binom{n+k+1}{2k+1})_{n,k}$ is equal to A168561; also $(\binom{n+k}{2k})_{n,k}$ is equal to A085478; finally, up to shifting k, we have that $(\binom{n+k}{2k+1})_{n,k}$ is equal to A078812.

We can also note that $f_\omega(k) \equiv g_\omega(k) + h_\omega(k) \bmod 2$, for any integer $k \geq 0$.

Theorem 4 *Suppose that $\lambda_n = 2^{n+1} - 1$. The following results then hold.*

– The formal power series $Q_\omega(X)$ is either a polynomial if $\omega \in \mathbb{Z}$ or a transcendental series over $\mathbb{Q}(X)$ if $\omega \in \mathbb{Z}_2 \setminus \mathbb{Z}$.

– The formal power series $Q_\omega(X)$ is algebraic over $\mathbb{Z}/2\mathbb{Z}(X)$ if and only if ω is rational. It is rational if and only if it is a polynomial, which happens if and only if ω is a rational integer.

Proof. The first assertion is a consequence of a classical theorem of Fatou [21] which states that *a power series $\sum_{n\geq 0} a_n z^n$ with integer coefficients that converges inside the unit disk is either rational or transcendental over $\mathbb{Q}(z)$.* This implies that the formal power series $Q_\omega(X)$ is either rational or transcendental over $\mathbb{Q}(X)$. We then have to prove that if Q_ω is a rational function, then it is a polynomial, or equivalently that ω is a rational integer (use Theorem 3). Now to say that Q_ω is rational is to say that the sequence of its coefficients is ultimately periodic, which implies that the sequence of their absolute values $(f_\omega(k))_{k\geq 0} = (\binom{\omega+k+1}{2k+1}_2)_{k\geq 0}$ is ultimately periodic. Let θ be its period. We have, for large k, that $\binom{\omega+k+1}{2k+1}_2 = \binom{\omega+k+\theta+1}{2(k+\theta)+1}_2$. If θ is odd, the left side is zero for $\omega + k$ odd while the right side is zero for $\omega + k$ even. Thus $\binom{\omega+k+1}{2k+1}_2 = 0$ for large k, and Q_ω is a polynomial. So suppose that θ is even. Let us suppose that ω does not belong to $\mathbb{Z}$, then its 2-adic expansion contains infinitely many blocks 01. Consider the first such block: there exist $\ell \geq 0$ and $j \geq 0$ such that $\omega = 2^\ell - 1 + 2^{\ell+1}(2^j(2\omega' + 1))$. Then for any integer k' we have $f_\omega(2^\ell - 1 + 2^{\ell+1}(2^j(2k'+1))) = \binom{\omega'+k'+1}{2k'+1}_2$. The sequence $(f_\omega(2^\ell - 1 + 2^{\ell+1}(2^j(2k'+1))))_{k'\geq 0}$ is ultimately periodic and $\theta/2$ is a period. But from Lemma 1 (vi) this sequence is equal to $(\binom{\omega'+k'+1}{2k'+1}_2)_{k'\geq 0}$. As previously, either $\theta/2$ is odd and this sequence is ultimately equal to zero or $\theta/2$ is even. In the first case, as above, ω' belongs to $\mathbb{Z}$, so does ω, which is impossible. In the second case, we iterate the reasoning that used Lemma 1 (vi), with ω replaced by ω' and k by k', where the first block 01 occurring in ω is replaced by the first such block occurring in ω'. The fact that θ cannot be divisible by arbitrarily large powers of 2 gives the desired contradiction.

In order to prove the second assertion, we first suppose that $Q_\omega(X)$ is algebraic over $\mathbb{Z}/2\mathbb{Z}(X)$. If $\omega = -1$, $Q_\omega(X) = 0$. Otherwise write $\omega = 2^\ell - 1 + 2^{\ell+1}\omega'$ as in Lemma 1 (v). The algebraicity of $Q_\omega(X)$ over $\mathbb{Z}/2\mathbb{Z}(X)$ implies that the sequence $(\binom{\frac{\omega+k}{2}}{k}_2 \bmod 2)_{n\geq 0}$ is 2-automatic (from a theorem of Christol, see [15, 16] or [6]). Using Lemma 2 (i) we thus have that the sequence $(\binom{\omega+k+1}{2k+1}_2)_{k\geq 0}$ is 2-automatic. Thus its subsequence obtained for $k = 2^\ell - 1 + 2^{\ell+1}k'$, namely $(\binom{\omega+2^\ell+2^{\ell+1}k'}{2^{\ell+1}-1+2^{\ell+1}(2k')}_2)_{k'\geq 0}$ is also 2-automatic (see, e.g., [6, Theorem 6.8.1, page 189]). But this last sequence is equal to $(\binom{2^{\ell+1}-1+2^{\ell+1}(\omega'+k')}{2^{\ell+1}-1+2^{\ell+1}(2k')}_2)_{k'\geq 0}$, i.e., to $(\binom{\omega'+k'}{2k'}_2)_{k'\geq 0}$ (look at the 2-adic expansions and use Definition 1). But this in turns implies (see, e.g., [6, Corollary 5.5.3, page 167]) that the subsequence $(\binom{\omega'+2^j}{2^{j+1}}_2)_{j\geq 0}$ is ultimately periodic. Using Lemma 1 (ii) this means that ω is rational.

Now suppose that ω is rational. Denote by $T\omega$ the 2-adic integer defined by $T\omega = (\omega - \omega_0)/2$ (i.e., $T\omega$ is the 2-adic integer obtained by shifting the sequence of digits of ω). Also note T^j the j-th iteration of T. Define (with the notation of Lemma 2) the set $\mathcal{K}$ by

$$\mathcal{K} =: \{(f_{T^j\omega}(k))_{k\geq 0},\ j \in \mathbb{N}\} \cup \{(g_{T^j\omega}(k))_{k\geq 0},\ j \in \mathbb{N}\} \cup \{(h_{T^j\omega}(k))_{k\geq 0},\ j \in \mathbb{N}\}.$$

As a consequence of Lemma 2, we see that $\mathcal{K}$ is stable by the maps defined on $\mathcal{K}$ by $(v_k)_{k\geq 0} \to (v_{2k})_{k\geq 0}$ and $(v_k)_{k\geq 0} \to (v_{2k+1})_{k\geq 0}$ (use that for any 2-adic integer $\omega = \omega_0\omega_1\ldots$ one has $\omega = 2T\omega + \omega_0$). On the other hand we have from Lemma 1 (iv) that $\binom{\frac{\omega+k}{2}}{k}_2 = f_\omega(k)$. Hence the 2-kernel of the sequence $(\binom{\frac{\omega+k}{2}}{k}_2)_{k\geq 0}$, i.e., the smallest set of sequences containing that sequence and stable under the maps $(v_k)_{k\geq 0} \to (v_{2k})_{k\geq 0}$ and $(v_k)_{k\geq 0} \to (v_{2k+1})_{k\geq 0}$, is a subset of $\mathcal{K}$. Now, since ω is rational, the set of 2-adic integers $\{T^j\omega,\ j \in \mathbb{N}\}$ is finite. Hence the 2-kernel of $(\binom{\frac{\omega+k}{2}}{k}_2)_{k\geq 0}$ is finite and this sequence is 2-automatic (see, e.g., [6]). This implies that the formal power series $Q_\omega(X)$ is algebraic over $\mathbb{Z}/2\mathbb{Z}(X)$ (using again Christol's theorem, see [15, 16] or [6]).

Finally, $Q_\omega(X)$ reduced modulo 2 is rational if and only if the sequence of its coefficients $(f_\omega(k))_{k\geq 0} = (\binom{\omega+k+1}{2k+1}_2)_{k\geq 0}$ modulo 2 is ultimately periodic, which is the same as saying that the sequence $(f_\omega(k))_{k\geq 0} = (\binom{\omega+k+1}{2k+1}_2)_{k\geq 0}$ itself is ultimately periodic. But from the first part of the proof this implies that $Q_\omega(X)$ (not reduced modulo 2) is a polynomial, hence that $Q_\omega(X)$ modulo 2 is a polynomial. Conversely, if $Q_\omega(X)$ modulo 2 is a polynomial, then the sequence of its coefficients $(f_\omega(k))_{k\geq 0} = (\binom{\omega+k+1}{2k+1}_2)_{k\geq 0}$ modulo 2 is ultimately equal to 0, and so is $(f_\omega(k))_{k\geq 0}$ not reduced modulo 2. Thus $Q_\omega(X)$ not reduced modulo 2 is a polynomial, thus ω is a rational integer by using Theorem 3.

Remark 10

- The authors of [4] prove that the formal power series $(1+X)^\omega = \sum_{k\geq 0}\binom{\omega}{k}_2 X^k$ is algebraic over $\mathbb{Z}/2\mathbb{Z}(X)$ if and only if ω is rational. They do not ask when that series is rational, i.e., belongs to $\mathbb{Z}/2\mathbb{Z}(X)$, but this is clear since for rational $\omega = a/b$, with integers $a, b > 0$, we have $((1+X)^\omega)^b \equiv (1+X)^a$ mod 2. Hence if $(1+X)^\omega$ is a rational function A/B with A and B coprime polynomials, then $A^b \equiv (1+X)^a B^b$ hence B is constant, i.e., $(1+X)^\omega$ is a polynomial. Now if $a < 0$ and $b > 0$, we have that $(1+X)^{-\omega}$ is a polynomial, hence $(1+X)^\omega$ is the inverse of a polynomial. Finally $(1+X)^\omega$ is a rational function if and only if $\omega \in \mathbb{Z}$.

- In the same vein, the authors of [4] prove that, if ω_1, ω_2, ..., ω_d are 2-adic integers, then the formal power series $(1+X)^{\omega_1}$, $(1+X)^{\omega_2}$, ..., $(1+X)^{\omega_d}$ are algebraically independent over $\mathbb{Z}/2\mathbb{Z}(X)$ if and only if 1, ω_1, ω_2,..., ω_d are linearly independent over $\mathbb{Z}$. Is a similar statement true for the Q_ω?

- Another question is to ask whether a similar study can be done in the p-adic case (here $p = 2$). The two papers [13, 14] might prove useful.

- Results of transcendence, hypertranscendence, and algebraic independence of values for the generating function of the Stern-Brocot sequence have been obtained very recently by Bundschuh (see [10], see also the references therein).

- A last question is the arithmetic nature of the real numbers $A(\varepsilon,\omega,g)$ defined by $A(\varepsilon,\omega,g) = \sum_{k\ll\frac{k+\omega}{2}} \sigma(k,\varepsilon) g^{-k}$ where $g \geq 2$ is an integer, the sequence $(\varepsilon_n)_n$ is ultimately periodic, and $\omega \in \mathbb{Z}_2 \setminus \mathbb{Z}$. Take in particular $\varepsilon = 0$ (thus $\sigma(k,\varepsilon) = (-1)^{\nu(k)}$). We already know that the number $A(0,\omega,g)$ is transcendental for $\omega \in (\mathbb{Q}\cap\mathbb{Z}_2)\setminus\mathbb{Z}$ by using [1], the fact that the sequence $((-1)^{\nu(k)})_{k\geq 0}$ is 2-automatic as recalled above, and the fact that the sequence $(\binom{\frac{k+\omega}{2}}{k}_2)_{k\geq 0}$ is 2-automatic for ω rational as seen in the course of the proof of Theorem 4 (the fact that $A(0,\omega,g)$ is not rational is a consequence of the non-ultimate periodicity of the sequence $((-1)^{\nu(k)}\binom{\frac{k+\omega}{2}}{k}_2)_{k\geq 0}$ for ω rational but not a rational integer, which has also been seen in the course of the proof of Theorem 4).

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