Expanding alternating André permutations of the second kind: left-to-right minima, right-to-left minima and min-max paths.

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Abstract

The aim of this work was to investigate a new class of permutations enumerated by *Euler* numbers. This class is obtained *expanding* a constrained subset of André permutations of the second kind. The constraint is given by the *d-alternating* property, i.e. alternating starting with a descent, while expansion is provided by an application of the *restriction* operator which has been already used in the literature to define - for example - *simsun* permutations.

We focus on left-to-right minima and right-to-left minima statistics. Both of them are strongly related to the tree-structure of the considered permutations. The connection is given in terms of min-max paths defined on binary *increasing* trees.

As a corollary to some of our results, we indicate the presence of a link between our permutations and those called *cycle-up-down*, recently introduced by Deutsch and Elizalde.

1 Introduction

The aim of this work was to study a new class of permutations counted by *Euler* numbers. This class is derived from *André* permutations of the second kind .

André permutations of the second kind (André permutations) have been introduced in [8] and extensively studied in the literature especially because of their relations with other combinatorial structures [9, 11, 15]. For instance the *cd*-index of the Boolean algebra may be computed by summing the *cd*-variation monomials of André permutations [15].

The n-th Euler number e_n counts André permutations of size n. The first terms are $e_0 = 1, e_1 = 1, e_2 = 1, e_3 = 2, e_4 = 5, e_5 = 16, \dots$. Euler numbers give the enumeration of several other combinatorial structures. In particular they also count rooted binary unordered *increasing* trees. Indeed in [8] the authors describe a bijection - denoted here by ϕ - which maps André permutations to this class of trees and viceversa.

Here we construct a new class of permutations associated with the sequence $(e_n)_n$. We do this by expanding a constrained subset of André permutations. The constraint is given by considering only those permutatons which are *d-alternating*. This is equivalent to say that they are ϕ-associated with *strictly-binary* increasing trees. The expansion is provided by an application of the *restriction* operator. This last tool has been already used in the literature to define *simsun* permutations (see [4] and related references).

What we consider in this manuscript is indeed the class of restrictions of d-alternating André permutations.

In Section 3.1, we provide the enumeration with rispect to the size which is given by Euler numbers.

In Section 3.2 we find the bivariate generating function which counts the restrictions according to two parameters, i.e. the size and the number of left-to-right minima. To achieve this result we make use of a nice correspondence between the set of left-to-right minima and a particular path, called the *min-path*, of the associated increasing tree. As a

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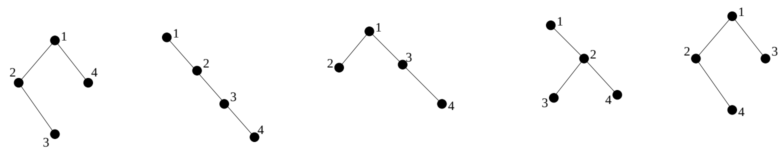

Figure 1: The trees in $\mathcal{B}_4$.

result, fixing the number of left-to-right minima, we provide a nice combinatorial formula which completely describes the desired enumeration in terms of Euler numbers.

In Section 3.3, we study the number of right-to-left minima. We give a functional equation for the associated biivariate generating function. We show how the number of restrictions of size $n+1$ with 2 right-to-left minima is related to the total number of left-to-right minima in restrictions of size n. Finally, we study restrictions with a generic - but fixed - number of right-to-left minima, providing an asymptotic bound for the probability of a random restriction of given size to have a certain number of right-to-left minima.

Interestingly, as a corollary to some of our results, we observe a connection with the class of *cycle-up-down* permutations [5] and this will need to be further investigated.

2 Preliminaries

The set of permutations of size n is denoted by $\mathcal{S}_n$. If $\pi = (\pi_1\pi_2\dots\pi_n) \in \mathcal{S}_n$ the set of its *left-to-right minima* is denoted by $lrm(\pi)$ and its elements are those entries π_i such that if $j < i$, then $\pi_i < \pi_j$. We denote by $rlm(\pi)$ the set of *right-to-left minima* and we recall to the reader that $\pi_i \in rlm(\pi)$ if $j > i$ implies $\pi_i < \pi_j$.

A binary *increasing* tree is a rooted, un-ordered tree with nodes of outdegree $0, 1$ or 2. Nodes of outdegree 0 are also called the *leaves* of the tree. Moreover, for such a tree, we require that each of the n nodes is bijectively labelled by a number in $\{1, 2, ..., n\}$ in a way that going from the root to any leaf we always find an increasing sequence of numbers. If x and y are two nodes, we write $x \prec y$ when the label of x is less than the label of y. The set of binary increasing trees is denoted by $\mathcal{B}$ while we use the symbol $\mathcal{B}_n$ to denote the subset of $\mathcal{B}$ made of those trees with n nodes.

Observe that, according to [8], each tree in $\mathcal{B}$ can be drawn in the plane in a *unique* way respecting the following two conditions c_1) and c_2):

c_1) if a node has only one child, then this child is drawn on the right of its direct ancestor;

c_2) if a node x has two children y and z with $y \prec z$, then y is drawn on the left of x while z on the right.

In Fig. 1 we show those trees belonging to $\mathcal{B}_4$ respecting the previous two conditions.

The set of *André* permutations (of the second kind) $\mathcal{A}$ can be defined in several equivalent ways, see for example Section 2 of [11]. Since $\mathcal{A}_n$ is a subset of $\mathcal{S}_n$ equinumerous to $\mathcal{B}_n$, we choose to characterize the mentioned permutations according to the following injective map $\phi : \mathcal{B}_n \to \mathcal{S}_n$ (see [8]):

1) given $t \in \mathcal{B}_n$, draw t according to c_1) and c_2);
2) each leaf collapses into its direct ancestor whose label is then modified receiving on the left the label of the left child (if any) and on the right the label of its right child. We obtain in this way a new tree whose nodes are labelled with sequences of numbers;
3) starting from the obtained tree go to step 2).

The algorithm ϕ ends when the tree t is reduced to a single node whose label is then a permutation $\phi(t)$ of size n.

The set $\mathcal{A}_n$ can be seen as $\mathcal{A}_n = \{\phi(t) \in \mathcal{S}_n : t \in \mathcal{B}_n\}$. Looking at Fig. 1 the corresponding permutations in $\mathcal{A}_4$ are (from left to right) $(2314), (1234), (2134), (1324)$ and (2413).

André permutations, as binary increasing trees, are enumerated, with respect to the size, by the so called *Euler* numbers $(e_n)_{n\geq 0}$ whose exponential generating function satisfies

$$\int E(z)^2 dz = 2E(z) - z - 2$$

and therefore is equal to

$$E(z) = \sec(z) + \tan(z).$$

The first terms of the sequence are: $1, 1, 1, 2, 5, 16, 61, 272, 1385, ...$ and they correspond to entry $A000111$ in [14].

Given a permutation $\pi \in \mathcal{S}_n$, a *restriction* of π is a permutation obtained from π by considering only the entries $1, 2, ..., k$, with $1 \leq k \leq n$. If σ is a restriction of π, then we write $\sigma \lhd \pi$. Restrictions have been used in the literature for example to define *simsun* permutations. Here we use restrictions as an expansion operator. Indeed, if X is a class of permutations, we can define $\mathcal{R}^X = \{\sigma \lhd \pi : \pi \in X\}$. Observe that $X \subseteq \mathcal{R}^X$ and in particular, for André pemutations, we have a strict inclusion given that - for example - $(21) \lhd (213) \in \mathcal{A}$ and $(21) \notin \mathcal{A}$. It follows that, in general, the quantity $|\mathcal{R}^{\mathcal{A}} \cap \mathcal{S}_n|$ can be greater than e_n.

It is then of interest to ask for a constraint on $\mathcal{A}$ such that, expanding the resulting subset $\tilde{\mathcal{A}}$, gives back for $(\mathcal{R}^{\tilde{\mathcal{A}}} \cap \mathcal{S}_n)_n$ the original enumerative sequence $(e_n)_n$. That is what we do with the next definitions.

Given a permutation $\pi = (\pi_1\pi_2 \ldots \pi_n)$, a *descent* (resp. a *ascent*) in π is an entry π_i with $1 \leq i < n$ such that $\pi_i > \pi_{i+1}$ (resp. $\pi_i < \pi_{i+1}$). A permutation is *d-alternating* when it starts with a descent and it has neither two consecutive descents nor two consecutive ascents. The subset of $\mathcal{A}_n$ made of those permutations which are d-alternating is denoted by $\tilde{\mathcal{A}}_n$ and $\tilde{\mathcal{A}} = \bigcup_i \tilde{\mathcal{A}}_i$.

If we define a tree in $\mathcal{B}$ which does *not* have nodes of outdegree 1 to be *strictly-*binary and we let $\tilde{\mathcal{B}}_n$ be the corresponding subset of $\mathcal{B}_n$, we can observe the following correspondence

Proposition 1 *Let $\pi \in \mathcal{A}_n$, then $\pi \in \phi(\tilde{\mathcal{B}}_n)$ if and only if $\pi \in \tilde{\mathcal{A}}_n$.*

Proof. It is a well known fact that André permutations do not have two consecutive descents [11]. It is then sufficient to show that, given $\pi \in \mathcal{A}_n$, $\pi \in \phi(\tilde{B}_n)$ is equivalent to say that π starts with a descent and that it does not contain two consecutive ascents.

($\Rightarrow$) To start observe that if π starts with a ascent, then it cannot be ϕ-associated with a strictly-binary tree. Furthermore, if π has two consecutive ascents, say $\pi = (\ldots \pi_i\, \pi_{i+1}\, \pi_{i+2} \ldots)$ with $\pi_i < \pi_{i+1} < \pi_{i+2}$, let $t \in \mathcal{B}_n$ such that $\phi(t) = \pi$. Looking at π_i, π_{i+1} and π_{i+2} as nodes of t we must have that π_{i+1} belongs to the right subtree of π_i and, similarly, π_{i+2} is a node in the right subtree of π_{i+1}. Furthermore note that π_{i+1} cannot have a left subtree, otherwise π_{i+1} would not be next to π_i in $\phi(t)$. It follows that π_{i+1} is a node of outdegree 1 in t, which means $t \notin \tilde{\mathcal{B}}_n$.

($\Leftarrow$) Let us now take $\pi = \phi(t)$ with t having a node x of outdegree 1. Let w be the first node we encounter in the path from x to the root of t such that x is in the right subtree of w; such a w must exist otherwise π would start with a rise. Let y be the (right) child of x. If t' is the (possibly empty) left subtree of y, then, applying ϕ, we obtain $\pi = \phi(t) = (\ldots w\, x\, \phi(t')\, y \ldots)$ where, either in $w\, x\, \phi(t')$ or in $w\, x\, y$, we find two conscutive ascents. It follows that $\pi \notin \tilde{\mathcal{A}}$. $\square$

The previous proposition allows us to switch from d-alternating André permutations to strictly-binary increasing trees and viceversa.

Let us finally define the object of our work: we consider

$$\mathcal{R}_n^{\tilde{\mathcal{A}}} = \{\sigma \in \mathcal{S}_n : \exists\, \pi \in \tilde{\mathcal{A}} \text{ with } \sigma \lhd \pi\} \text{ and } \mathcal{R}^{\tilde{\mathcal{A}}} = \bigcup_i \mathcal{R}_i^{\tilde{\mathcal{A}}}.$$

According to the definition the reader can check that $\mathcal{R}_5^{\tilde{\mathcal{A}}}$ is made of the following 16 permutations

$$(32514), (42513), (21435), (32415),$$
$$(53241), (32541), (43251), (32154),$$
$$(42153), (52143), (43215), (53214),$$
$$(54213), (21543), (43521), (54321).$$

Observe that in general $\mathcal{R}_n^{\tilde{\mathcal{A}}} \not\subseteq \mathcal{A}_n$. Indeed we have that $(326514) \lhd (3265714) \in \tilde{\mathcal{A}}$ but $(326514) \notin \mathcal{A}_6$.

In Section 3.2 we will focus on left-to-right minima statistics using the symbol $\mathcal{R}_{n,m}^{\tilde{\mathcal{A}}}$ to denote the subset of $\mathcal{R}_n^{\tilde{\mathcal{A}}}$ made of those permutations π with $|lrm(\pi)| = m$, while, in Section 3.3, we will study right-to-left minima using $\mathcal{R}_{n,m}^{\tilde{\mathcal{A}}}$ to refer to the subset $|rlm(\pi)| = m$.

3 Enumeration of $\mathcal{R}_n^{\tilde{\mathcal{A}}}$ and $\mathcal{R}_{n,m}^{\tilde{\mathcal{A}}}$

In this section we provide the enumeration of $\mathcal{R}_n^{\tilde{\mathcal{A}}}$ and $\mathcal{R}_{n,m}^{\tilde{\mathcal{A}}}$ using the existing correspondance between permutations and binary increasing trees.

3.1 The cardinality of $\mathcal{R}_n^{\tilde{\mathcal{A}}}$

The two criteria c_1) and c_2) given in Section 2, which were used to draw each binary increasing tree in a unique and well defined way, are not the only possible ones. Indeed, one can consider the criterion c_2) plus the dual version of c_1). To be more precise, if in a tree a node has only one child, then this child is drawn on the *left* of its direct ancestor and not on the right as it was in c_1).

Trees drawn according to c_2) and the new condition described above are said to be *left oriented*. Given such a tree, we can still apply the procedure ϕ of Section 2 - without considering step 1) - to define a permutation which is not, in general, an André one.

In what follows we show that a permutation belongs to $\mathcal{R}_n^{\tilde{\mathcal{A}}}$ if and only if it is the permutation coming through ϕ from a left oriented binary increasing tree of size n. As a corollary we also have that $|\mathcal{R}_n^{\tilde{\mathcal{A}}}| = |\mathcal{B}_n| = e_n$.

Proposition 2 *Given a permutation $\pi \in \mathcal{S}_n$, we have that π is ϕ-associated with a left oriented binary increasing tree of size n if and only if $\pi \in \mathcal{R}_n^{\tilde{\mathcal{A}}}$.*

Proof. ($\Rightarrow$) Take the left oriented tree corresponding to π and, to every node with outdegree 1, add a child with label a number greater than n. Let t be the resulting tree. Then $t \in \tilde{\mathcal{B}}$ and $\pi \lhd \phi(t) \in \tilde{\mathcal{A}}$.

($\Leftarrow$) Suppose $\pi \lhd \sigma \in \tilde{\mathcal{A}}_m$, with $n \leq m$. Let $t \in \tilde{\mathcal{B}}_m$ be the tree associated with σ. Remove from t all the nodes labelled by a number $k > n$. The obtained tree is left oriented and σ comes from t applying steps 2) and 3) of ϕ. $\square$

Thus we have

Corollary 1 *The cardinality of $\mathcal{R}_n^{\tilde{\mathcal{A}}}$ is given by the n-th Euler number e_n.*

3.2 Left-to-right minima and the min-path

In the previous section we have shown that permutations in $\mathcal{R}^{\tilde{\mathcal{A}}}$ correspond to those permutations coming from left oriented trees when we apply steps 2) and 3) of ϕ. Now we want to refine the bijection described in the proof of Proposition 2 to consider also the set of left-to-right minima of such permutations. To do this we need the following definition.

Given a binary increasing tree t, consider the following path: we start from the root of t and at each step we move to the child with the least label. In Fig. 2 we show a tree t where such path, which will be called the *min-path* of t, is highlighted.

The next proposition states the relation between left-to-right minima of a permutation in $\mathcal{R}^{\tilde{\mathcal{A}}}$ and the min-path of the corresponding tree in $\mathcal{B}$.

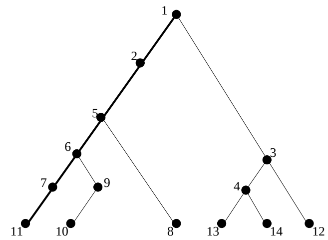

Figure 2: The min-path of a left oriented binary increasing tree.

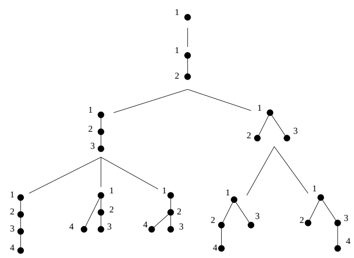

Figure 3: First levels of the generating tree associated with Θ.

Proposition 3 *If $\pi \in \mathcal{R}^{\tilde{\mathcal{A}}}$, then $lrm(\pi)$ consists of the labels belonging to the min-path of $\phi^{-1}(\pi)$.*

Proof. First, observe that when a tree is left oriented then its min-path corresponds to the path going from the root to the leftmost leaf.

If we consider $t = \phi^{-1}(\pi)$, then t is a left oriented binary increasing tree.

If a node x belongs to the min-path of t, then in π the entries which are on the left of x correspond to the nodes belonging to the left subtree of x. Since t is increasing we have that $x \in lrm(\pi)$.

If a node x does not belong to the min-path of t, then, going from x to the root of t, we find a node of outdegree 2 such that x belongs to its right subtree. Let z be the left child of this node and y the right one. Then $z \prec y \preceq x$ which implies $z \prec x$. Moreover, applying the procedure ϕ, the entry z will be placed on the left of x. From this we have that $x \notin lrm(\pi)$. $\square$

As an example consider the binary increasing tree presented in Fig. 2. The associated permutation is $\pi = (11\,7\,6\,10\,9\,5\,8\,2\,1\,13\,4\,14\,3\,12)$ and then the set $lrm(\pi) = \{11, 7, 6, 5, 2, 1\}$ has the same entries of the min-path.

Based on Propositions 2 and 3 we can obtain the enumeration of $\mathcal{R}^{\tilde{\mathcal{A}}}_{n,m}$ by counting the number of binary increasing trees with respect to the size and to the length of the min-path.

3.2.1 A multivariate generating function for $\mathcal{B}$

We now introduce a recursive construction for binary increasing trees. In particular we construct each tree belonging to $\mathcal{B}_{n+1}$ by adding a new node to a tree in $\mathcal{B}_n$. This construction, denoted by Θ, is then translated into a functional equation. Solving the equation yelds a multivariate exponential generating function counting binary increasing trees with respect to size and to the length of the min-path.

Given a tree $t \in \mathcal{B}_n$, Θ simply adds the node labelled '$n+1$' as a child of a node of t having outdegree less than two. More precisely, if $o(t)$ (resp. $p(t)$) is the number of nodes with outdegree 0 (resp. 1) in t, then Θ applied to t produces $o(t) + p(t)$ elements of $\mathcal{B}_{n+1}$ each time adding the new node labelled $n+1$ as a child of the nodes counted in $o(t) + p(t)$. In Fig. 3 the first levels of the generating tree resulting from Θ are shown.

Note that, if $q(t)$ is the number of nodes with outdegree two in a tree $t \in \mathcal{B}_n$, then $o(t) = q(t) + 1$ and $o(t) + p(t) + q(t) = n$. From these relations it follows that $p(t) = n - 2o(t) + 1$. The construction Θ can be translated into the *succession rule* (1) (see [1] [2])

$$(o, l, n) \rightarrow (o, l, n+1)^{o-1}\,(o, l+1, n+1)\,(o+1, l, n+1)^{n-2o+1}. \tag{1}$$

Note that each tree is represented in (1) by the values of its parameters o, l and n, where l represents the length of the min-path (i.e. $l = |lrm(\phi(t))|$) while n corresponds to the size. In particular, given a tree t with parameters $o = o(t)$ and $n = n(t)$, the application of Θ on t produces o new trees having size $n+1$ and the same value for o and $n - 2o + 1$ new trees having both the size and the number of nodes of outdegree zero augmented by one. The starting point of the construction is the unique tree of size one which is associated with the label $(1, 1, 1)$.

Now consider the exponential generating function

$$F(x, y, z) = \sum_{t \in \mathcal{B}} \frac{x^{o(t)} y^{l(t)} z^{n(t)}}{n(t)\,!},$$

rule (1) can be then translated as follows

$$\begin{aligned}
F(x,y,z) \quad &= \quad xyz \\
&\quad + \sum_{x^o y^l z^n \in \mathcal{B}} \frac{(o-1) x^o y^l z^{n+1}}{n+1\,!} \\
&\quad + \sum_{x^o y^l z^n \in \mathcal{B}} \frac{x^o y^{l+1} z^{n+1}}{n+1\,!} \\
&\quad + \sum_{x^o y^l z^n \in \mathcal{B}} \frac{(n-2o+1)(x^{o+1} y^l z^{n+1})}{n+1\,!} \\
&= \quad xyz \\
&\quad + x(1-2x) \sum_{x^o y^l z^n \in \mathcal{B}} \frac{o x^{o-1} y^l z^{n+1}}{n+1\,!} \\
&\quad + (y-1) \sum_{x^o y^l z^n \in \mathcal{B}} \frac{x^o y^l z^{n+1}}{n+1\,!} \\
&\quad + xz\, F(x,y,z).
\end{aligned}$$

We obtain that

$$\begin{aligned}
F(x,y,z)(1-xz) - xyz \quad &= \quad x(1-2x) \sum_{x^o y^l z^n \in \mathcal{B}} \frac{o x^{o-1} y^l z^{n+1}}{n+1\,!} \\
&\quad + (y-1) \sum_{x^o y^l z^n \in \mathcal{B}} \frac{x^o y^l z^{n+1}}{n+1\,!},
\end{aligned}$$

and differentiating both sides with respect to the variable z we have

$$(1-x-y)\,F(x,y,z) - xy \quad = \quad x(1-2x)\frac{\partial F}{\partial x}(x,y,z) + (xz-1)\frac{\partial F}{\partial z}(x,y,z).$$

The boundary condition to the previous first order partial differential equation can be given as

$$F(x, y, 0) = 0.$$

One can find the desired solution applying the method of *characteristics* (see [12]) or using some computer algebra tools (like Mathematica or Maple). In the latter case, after the substitution $x = 1$ into the solution $F(x, y, z)$ and performing some manipulations, one finds that, when $n \geq 2$ and $2 \leq l \leq n$,

$$|\mathcal{R}^{\tilde{\mathcal{A}}}_{n,l}| = [y^{l-1}] \left[\left(\frac{\partial^{n-1} \tilde{F}}{\partial z^{n-1}} \right)_{z=0} \right],$$

with

$$\tilde{F}(y, z) \quad = \quad \left(\frac{1}{1 - \sin(z)} \right)^y .$$

In other words, we have the following result

Proposition 4 *The (shifted) exponential generating function counting the permutations in $\mathcal{R}^{\tilde{\mathcal{A}}}$ with respect to the size and number of left-to-right minima is given by*

$$\tilde{F}(y, z) = \left(\frac{1}{1 - \sin(z)} \right)^y = \sum_{t \in \mathcal{B}} \frac{y^{l(t)-1} z^{n(t)-1}}{(n(t) - 1)\,!}.$$

Considering $\tilde{F}(1, z)$ provides the (shifted) exponential generating function for Euler numbers.

The first terms of $|\mathcal{R}^{\tilde{\mathcal{A}}}_{n,l}|$ are thus given by the following table.

n/l	2	3	4	5	6	7	8	9	10
2	1	0	0	0	0	0	0	0	0
3	1	1	0	0	0	0	0	0	0
4	1	3	1	0	0	0	0	0	0
5	2	7	6	1	0	0	0	0	0
6	5	20	25	10	1	0	0	0	0
7	16	70	105	65	15	1	0	0	0
8	61	287	490	385	140	21	1	0	0
9	272	1356	2548	2345	1120	266	28	1	0
10	1385	7248	14698	15204	8715	2772	462	36	1

Note that Euler numbers are the entries of the first column. Furthermore observe that, looking at the table column by column, one has

$$\left(\frac{\partial^l \tilde{F}}{\partial y^l} \right)_{y=0} = \left[-\ln\left(1 - \sin(z)\right) \right]^l$$

and then

$$\frac{1}{l!} \left[-\ln\left(1 - \sin(z)\right) \right]^l = \sum_{t \in \mathcal{B},\ l(t) = l+1} \frac{z^{n(t)-1}}{(n(t) - 1)\,!}.$$

Given that $\int E(z) dz = -\ln\left(1 - \sin(z)\right)$, as a corollary we have

Proposition 5 *For every fixed $l \geq 1$*

$$\frac{1}{l!} \left(\sum_{n \geq 1} \frac{e_{n-1}}{n!} z^n \right)^l = \sum_{n \geq l} \frac{|\mathcal{R}^{\tilde{\mathcal{A}}}_{n+1,l+1}|}{n!} z^n, \tag{2}$$

where

$$e_0 = 1, e_1 = 1, e_2 = 1, e_3 = 2, e_4 = 5, e_5 = 16, e_6 = 61, ...$$

are Euler numbers.

We conclude this section recalling that in Chapter 7 of [13] the author studies a family of polynomials corresponding to the rows of the previous table. He also shows a criterion according to which each row defines a partition of the set of *up-down* permutations of a given size. Furthermore, in [5] the authors prove that the rows of the previous table also provide the enumeration of the so called *cycle-up-down* permutations with respect to the size and to the number of cycles. It is then natural to ask for a bijection between the permutations in $\mathcal{R}^{\tilde{\mathcal{A}}}_{n+1}$ and the cycle-up-down ones of size n enlightening the correspondence between left-to-right minima and cycles.

3.3 Right-to-left minima and the max-path

In the previous section we have enumerated the permutations in $\mathcal{R}^{\tilde{A}}$ with respect to the size and to the number of left-to-right minima. We have also seen how the last parameter corresponds to the length of a particuar path in the associated class of trees.

It is interesting to observe that a similar correspondence holds also between the number of *right-to-left minima* and the length of another particular path defined for trees in $\mathcal{B}$. If $t \in \mathcal{B}$, the *max-path* of t is made of those nodes visited according to the following procedure. Start from the root of t and, at each step (until possible), if the current node has *outdegree two* move to the child having the biggest label. It is important to remark that the max-path does *not* represent a dual version of the previously defined min-path. Indeed, according to the procedure for the max-case, we visit a new node only if the current one has outdegree two.

Looking at the tree presented in Fig. 2, its max-path is composed by the nodes in $\{1, 3, 12\}$ which also corresponds to the set of right-to-left minima $rlm(\pi)$, where $\pi = (11\,7\,6\,10\,9\,5\,8\,2\,1\,13\,4\,14\,3\,12)$.

More in general we can state the following

Proposition 6 *If $\pi \in \mathcal{R}^{\tilde{A}}$, then $rlm(\pi)$ consists of the labels belonging to the max-path of $\phi^{-1}(\pi)$.*

Proof. Observe that, when a the tree $t = \phi^{-1}(\pi)$ is drawn according to its left orientation, the max-path of t is made of those nodes we can visit starting from the root and performing only right steps. Then, if π_i belongs to the max-path and $j > i$, π_j must be a descendant of π_i in t and then $\pi_i < \pi_j$. Viceversa, if π_i does not belong to the max-path of t, going from the root of t to π_i we have to perform at least one left step. Corresponding to this left step we find an entry of π which has a smaller value than π_i and it is placed on its right. Then π_i cannot be a right-to-left minima of π. $\square$

As a first step, one can proceed in the enumeration of right-to-left minima using the same generating-tree approach of the previous section. In this case, given a tree t, we denote by $r(t)$ the cardinality of its max-path, by $o(t)$ (resp. $n(t)$) the number of leaves (resp. the size) while $d(t)$ is defined as $d(t) = n(t) - 2o(t) + 1$ (i.e. the number of nodes of outdegree 1). Furthermore we say that a tree is in the class A if the *last* node of its max-path has outdegree 1, while we say that it is in the class B otherwise. The recursive construction Θ defined in Section 3.2.1 gives, in this case, the following set of rules

$$
\begin{array}{lcl}
(o,r,n)_{B,d\geq 0} & \rightarrow & (o,r,n+1)^{o-1}_{B,d+1}\,(o,r,n+1)_{A,d+1}\,(o+1,r,n+1)^{d}_{B,d-1} \\
(o,r,n)_{A,d>0} & \rightarrow & (o,r,n+1)^{o}_{A,d+1}\,(o+1,r+1,n+1)_{B,d-1}\,(o+1,r,n+1)^{d-1}_{A,d-1}
\end{array}
$$

where the construction starts with the tree made of one node whose label is then $(1,1,1)_{B,0}$. Note that we can increase the value of the parameter r only starting from a tree in the class A.

It seems natural at this point to consider

$$G = G(x,w,v,z) = \sum_{t \in \mathcal{B}} x^{o(t)} w^{r(t)} v^{d(t)} z^{n(t)}$$

as the sum

$$G = \sum_{k \geq 0} (G_{A,k}(x,w,v,z) + G_{B,k}(x,w,v,z)),$$

where $G_{A,k}$ (resp. $G_{B,k}$) is the ordinary generating function counting those trees in the class A (resp. B) having $d = k$.

The functional equations look then as follows

$$
\begin{aligned}
G_{A,0} &= 0,\\
G_{A,k} &= \frac{-xz}{v}G_{A,k+1} + xz\frac{\partial G_{A,k+1}}{\partial v} + xvz\frac{\partial G_{A,k-1}}{\partial x} + vzG_{B,k-1},\\
G_{B,0} &= xwz + \frac{xwz}{v}G_{A,1} + xz\frac{\partial G_{B,1}}{\partial v},\\
G_{B,k} &= \frac{xwz}{v}G_{A,k+1} + xz\frac{\partial G_{B,k+1}}{\partial v} + xvz\frac{\partial G_{B,k-1}}{\partial x} - vzG_{B,k-1} \text{ (with } k > 0).
\end{aligned}
$$

Then, considering that $G_{B,0}$ does not depend on v, we can write

$$
\begin{aligned}
G_A &= \sum_{k\geq 0} G_{A,k}\\
&= \frac{-xz}{v}G_A + xz\frac{\partial G_A}{\partial v} + xvz\frac{\partial G_A}{\partial x} + vzG_B \quad \text{and}\\
G_B &= \sum_{k\geq 0} G_{B,k}\\
&= xwz + \frac{xwz}{v}G_A + xz\frac{\partial G_B}{\partial v} + xvz\frac{\partial G_B}{\partial x} - vzG_B.
\end{aligned}
$$

We are not able to solve the system of equations for G_A and G_B but we can still use it, as a recursive algorithm, to find the number of trees belonging to $\mathcal{B}_n$ having the cardinality of the max-path equal to r. Based on Proposition 6 this value is the number of permutations in $\mathcal{R}_n^{\tilde{A}}$ having a fixed number r of right-to-left minima.

Recursion: Let us define $G_A^{(n)}$ as the polynomial

$$G_A^{(n)} = \sum_{t\in\mathcal{B}_n\cap A} x^{o(t)}w^{r(t)}v^{d(t)}z^n$$

and analogously consider

$$G_B^{(n)} = \sum_{t\in\mathcal{B}_n\cap B} x^{o(t)}w^{r(t)}v^{d(t)}z^n.$$

We are interested in the polynomial $G^{(n)} = G_A^{(n)} + G_B^{(n)}$.

The starting point of the procedure is given by $G_A^{(1)} = 0$ and $G_B^{(1)} = xwz$. By induction, given $G_A^{(n)}$ and $G_B^{(n)}$, we can compute

$$
\begin{aligned}
G_A^{(n+1)} &= \frac{-xz}{v}G_A^{(n)} + xz\frac{\partial G_A^{(n)}}{\partial v} + xvz\frac{\partial G_A^{(n)}}{\partial x} + vzG_B^{(n)} \quad \text{and}\\
G_B^{(n+1)} &= \frac{xwz}{v}G_A^{(n)} + xz\frac{\partial G_B^{(n)}}{\partial v} + xvz\frac{\partial G_B^{(n)}}{\partial x} - vzG_B^{(n)}.
\end{aligned}
$$

Using the previously defined procedure we have computed the entries of the following table showing, for all $(n,r) \in \{1,...,10\}^2$, the number of permutations in $\mathcal{R}_n^{\tilde{A}}$ having r right-to-left minima.

n/r	1	2	3	4	5	6	7	8	9	10
1	1	0	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0	0	0
3	1	1	0	0	0	0	0	0	0	0
4	2	3	0	0	0	0	0	0	0	0
5	5	10	1	0	0	0	0	0	0	0
6	16	38	7	0	0	0	0	0	0	0
7	61	165	45	1	0	0	0	0	0	0
8	272	812	288	13	0	0	0	0	0	0
9	1385	4478	1936	136	1	0	0	0	0	0
10	7936	27408	13836	1320	21	0	0	0	0	0

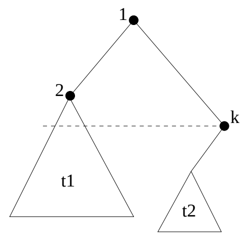

Figure 4: Decomposition of a tree with max-path of length two.

In the first column we find (shifted) Euler numbers. It is also interesting to observe that the entries in the second column

$$1, 3, 10, 38, 165, 812, 4478, 27408, 184529, 1356256, 10809786, 92892928...$$

belong to sequence A186367 of [14]. This sequence counts the number of cycles in all *cycle-up-down* permutations of size n (see also [5]) and, furthermore, it is strongly related to the total number of left-to-right minima in the permutations of $\mathcal{R}^{\tilde{A}}$ having fixed size. Indeed, we will prove that the exponential generating function associated with the non-zero entries of the column $r = 2$ of the table above is given by

$$\left(\frac{\partial \tilde{F}}{\partial y}\right)_{y=1} = \frac{-\ln(1-\sin(z))}{1-\sin(z)},$$

where $\tilde{F}$ is the same of Proposition 4.

In order to prove the correspondence, we observe that each tree in $\mathcal{B}$ having max-path of length exactly two can be decomposed as shown in Fig. 4.

In particular, note that tree t_1 must contain at least one node (labelled with 2) while tree t_2 could be empty. The class of trees consisting of a (possibly empty) tree appended to a node - denoted by k in Fig. 4 - is counted by the exponential generating function

$$f_{t_2}(z) = \sum_{m>0} \frac{\tilde{e}_m z^m}{m!} = \int E(z) = -\ln(1-\sin(z)) \ \ (\text{with } \tilde{e}_m = e_{m-1})$$

while

$$f_{t_1}(z) = \sum_{n>0} \frac{e_n z^n}{n!} = E(z) - 1$$

counts those trees having at least one node. Appending t_1 of size n and t_2 of size $m-1$ as shown in Fig. 4 we can build exactly

$$\binom{n+m-1}{m} = \frac{(n+m-1)!}{m!(n-1)!}$$

different trees. Indeed we have to merge the total order given by k and the nodes of t_2 into the order of t_1 placing k after 2. It follows that, in the previous table, the entries $n \geq 1$ of the column $r = 2$ correspond to the coefficients of the following exponential generating function

$$\begin{aligned} g_2(z) &= \sum_{n>0}\sum_{m>0} \frac{e_n \tilde{e}_m z^{n+m+1}}{(n+m+1)!} \frac{(n+m-1)!}{(m!)(n-1)!} \\ &= \sum_{n>0}\sum_{m>0} \frac{e_n \tilde{e}_m z^{n+m+1}}{(n+m)(n+m+1)(m!)(n-1)!}. \end{aligned}$$

Finally observe that

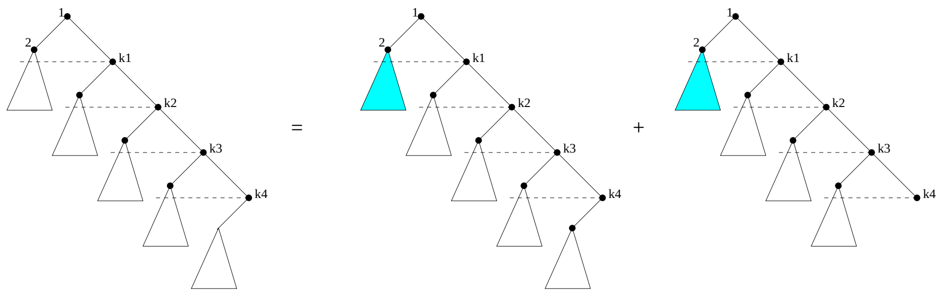

Figure 5: Tree-decomposition for a fixed number of right-to-left minima. The max-path is made of the nodes $1, k_1, k_2, k_3, k_4$.

$$g_2'' = f_{t_2} \cdot f_{t_1}' = \ln\left(\frac{1}{1-\sin(z)}\right) \cdot \left(\frac{1}{1-\sin(z)}\right) = \left(\frac{\partial \tilde{F}}{\partial y}\right)_{y=1}.$$

Given the above calculations, we obtain the following result

Proposition 7 *The following equality holds for all $n \geq 2$:*

$$\begin{aligned} |\mathcal{R}_{n+1,2}^{\tilde{\mathcal{A}}}| &= \sum_{l\geq 2}(l-1)\cdot|\{\pi \in \mathcal{R}_n^{\tilde{\mathcal{A}}} : |lrm(\pi)| = l\}| \\ &= (n-1)! \cdot [z^{n-1}]\left(\frac{-\ln(1-\sin(z))}{1-\sin(z)}\right) \end{aligned}$$

from which we have the next corollary

Corollary 2 *For $n \geq 2$:*

$$|\mathcal{R}_{n+1,2}^{\tilde{\mathcal{A}}}| + |\mathcal{R}_n^{\tilde{\mathcal{A}}}| = \sum_{l\geq 2} l \cdot |\{\pi \in \mathcal{R}_n^{\tilde{\mathcal{A}}} : |lrm(\pi)| = l\}| \tag{3}$$

Observe that, starting from (3) and dividing by $|\mathcal{R}_n^{\tilde{\mathcal{A}}}|$, one obtains that the expected number of left-to-right minima in a random permutation of $\mathcal{R}_n^{\tilde{\mathcal{A}}}$ is given by $1+|\mathcal{R}_{n+1,2}^{\tilde{\mathcal{A}}}|/|\mathcal{R}_n^{\tilde{\mathcal{A}}}|$.

3.3.1 Fixing the number of right-to-left minima

Given what we have shown above, it is interesting to investigate more in details what happens when we fix the number of right-to-left minima in $\mathcal{R}_n^{\tilde{\mathcal{A}}}$. Let

$$g_r(z) = \sum_n \frac{|\mathcal{R}_{n,r}^{\tilde{\mathcal{A}}}|}{n!} \cdot z^n.$$

Looking at the associated tree-decomposition, the situation looks as shown in Fig. 5. A tree with max-path of length r is obtained by appending to the same root a tree of max-path $r-1$ and a generic tree of size greater than zero. We must put the root of the last tree above the root of the first one. Furthermore, recall that we can distinguish among two possible trees of max-path size r depending on the out-degree of the last node belonging to the max-path. If this has out-degree one, then we say that the tree belongs to the class A_r otherwise it is in B_r. We denote by $g_r^A(z)$ and $g_r^B(z)$ the associated generating function and obviously $g_r(z) = g_r^A(z) + g_r^B(z)$.

If $r > 0$, one has

$$g_r^A = \sum_{t_1\in\mathcal{B}}\sum_{t_2\in A_r} \frac{z^{n_1+n_2+1}}{(n_1+n_2+1)!} \cdot \binom{n_1+n_2-1}{n_1-1}$$

and similarly for g_r^B.

It follows that we can write two independent recursions for $g_r^A(z)$ and $g_r^B(z)$, namely

$$(g_r^A)'' = g_{r-1}^A \cdot (E(z)-1)', \text{ with } g_0^A = 1$$

and

$$(g_r^B)'' = g_{r-1}^B \cdot (E(z)-1)', \text{ with } g_1^B = z.$$

Applying the first one to compute g_1^A gives $g_1^A = \int E(z) - 1\ dz = -\ln(1-\sin(z)) - z$ and then we obtain a recursion for $(g_r)'' = (g_r^A + g_r^B)''$ when $r > 1$. Indeed we have the following result which shows the link between two contiguous generating functions in the family $(g_r)_r$.

Proposition 8 *For $r > 1$, the family of generating functions $(g_r)_r$ satisfies*

$$g_r(z) = \int\int g_{r-1}(z) \cdot E'(z) dz \tag{4}$$

being $E'(z) = \left(\frac{1}{1-\sin(z)}\right)$ and $g_1(z) = \int E(z)dz = -\ln(1-\sin(z))$.

Observe that as a corollary, if we define $g = g(w,z) = \sum_{r\geq 1} w^r g_r(z)$, one has the following equation

$$\begin{aligned} g =& wg_1 + \sum_{r\geq 2} w^r g_r = wg_1 + \sum_{r\geq 2} w^r \left(\int\int E' g_{r-1}\right) = wg_1 + \left(\int\int wE' \sum_{r\geq 2} w^{r-1} g_{r-1}\right) \\ =& wg_1 + w\left(\int\int E' g\right) = w\left(\int E\right) + w\left(\int\int E' g\right) \end{aligned}$$

which gives

$$\frac{\partial^2 g}{\partial z^2} = wE' + wE' g.$$

Considering now $\tilde{g} = g + 1$ we have - as expected - the following result

Corollary 3 *The bivariate generating function $\tilde{g}(w,z) = 1 + \sum_{r\geq 1} w^r g_r(z)$ satisfies the differential equation*

$$\frac{\partial^2 \tilde{g}}{\partial z^2} = wE'(z) \cdot \tilde{g}, \tag{5}$$

where $E'(z) = \left(\frac{1}{1-\sin(z)}\right)$.

Unfortunately equation (5) does not give an explicit solution for $\tilde{g}$. Still, as we will see later, it can be used to explore the structure of the solution in a neighbourhood of the singularity $z = \pi/2$.

Let us now focus on the exact computation of g_r. To do that one can apply the result of Proposition 8 together with the fact that $E = E(z)$ satisfies $\int E^2 = 2E - z - 2$. Here we compute explicitly the generating function g_r for the first values of r, say $r = 1, 2, 3$. Indeed, if we define

$$\int^{(i)} H = \overbrace{\int\int \ldots \int}^{i-\text{times}} H(z) dz,$$

for $r = 1, 2$ we have

$$\begin{aligned} g_1 =& \int E \\ g_2 =& \left(\int^{(2)} \left(\int E\right) E'\right) = \left(\int \left(E \int E\right)\right) - \int^{(2)} E^2 \\ =& \frac{1}{2} \cdot \left(\int E\right)^2 - \int (2E - z - 2) \\ =& \frac{1}{2} \cdot \left(\int E\right)^2 - 2\left(\int E\right) + \frac{z^2}{2} + 2z, \end{aligned}$$

while, for $r = 3$, we obtain:

$$
\begin{aligned}
g_3 =& \left(\int^{(2)} \frac{(\int E)^2}{2} E'\right) - 2\left(\int^{(2)} \left(\int E\right) E'\right) + \left(\int^{(2)} \left(\frac{z^2}{2} + 2z\right) E'\right) \\
=& \left(\int E \frac{(\int E)^2}{2}\right) - \left(\int^{(2)} E^2 \left(\int E\right)\right) - 2\left[\left(\int E \left(\int E\right)\right) - \int^{(2)} E^2\right] \\
&+ \left(\frac{z^2}{2} + 2z\right)\left(\int E\right) + (-2z-4)\left(\int^{(2)} E\right) + 3\left(\int^{(3)} E\right) \\
=& \frac{1}{6}\cdot\left(\int E\right)^3 - \left[\left(\int (2E - z - 2)\left(\int E\right)\right) - \left(\int^{(2)} (2E - z - 2)E\right)\right] \\
&- 2\left[\frac{1}{2}\cdot\left(\int E\right)^2 - \int (2E - z - 2)\right] + \left(\frac{z^2}{2} + 2z\right)\left(\int E\right) \\
&+ (-2z-4)\left(\int^{(2)} E\right) + 3\left(\int^{(3)} E\right) \\
=& \frac{1}{6}\left(\int E\right)^3 - 2\left(\int E\right)^2 + \left(8 + 2z + \frac{z^2}{2}\right)\left(\int E\right) - 2z^2 - 8z \\
&+ (-2z-4)\left(\int^{(2)} E\right) + 4\left(\int^{(3)} E\right).
\end{aligned}
$$

For values of r greater than 3 the exact computation of g_r becomes more difficult. In this cases we can still use the results of Proposition 8 to obtain asymptotic estimates of the coefficients $[z^n]g_r(z)$. Using standard methods of analytic combinatorics (see [7]) it is sufficient to know an approximation of the function g_r near its dominant singularity to describe the behaviour of $([z^n]g_r(z))_n$. In this case, the main idea is to iteratively recover an approximation for g_{r+1} by integration of an approximation for $(g_r \cdot E')$. By using this approach, we provide, for a given value of r, an estimate of the probability for a random permutation $\pi \in \mathcal{R}_n^{\tilde{A}}$ to have $|rlm(\pi)| = r$. To illustrate the idea we start, as a firs step, considering just rough estimates.

Near the dominant singularity $z = \pi/2$ we have

$$
E'(z) = \frac{1}{1 - \sin(z)} = \frac{2}{\left(\frac{\pi}{2} - z\right)^2} + \mathcal{O}(1) \tag{6}
$$

and, for every $A > 0$,

$$
g_1 = \int E(z)dz = \ln\left(\frac{1}{1 - \sin(z)}\right) = -2\ln\left(\frac{\pi}{2} - z\right) + \mathcal{O}(1) = \mathcal{O}\left(\left(\frac{\pi}{2} - z\right)^{-A}\right). \tag{7}
$$

Then, as a first approximation, one has

$$
(g_1 \cdot E')(z) = \mathcal{O}\left(\left(\frac{\pi}{2} - z\right)^{-2-A}\right),
$$

which gives by Proposition 8 and Th. VI.9 of [7]

$$
g_2(z) = \mathcal{O}\left(\left(\frac{\pi}{2} - z\right)^{-A}\right).
$$

We remark that, by the mentioned theorem, we can obtain a singular approximation of g_2 by integrating, according to classical rules, the singular expansion of $(g_1 \cdot E')$.

Iterating the procedure one has that, independently on r, for every $A > 0$

$$
g_r(z) = \mathcal{O}\left(\left(\frac{\pi}{2} - z\right)^{-A}\right). \tag{8}
$$

Applying Th. VI.3 of [7] to (8) gives that, when n is large, for every $A > 0$

$$\frac{|\mathcal{R}_{n,r}^{\tilde{A}}|}{n!} = [z^n]g_r(z) = \mathcal{O}\left(\left(\frac{2}{\pi}\right)^n \cdot n^{A-1}\right). \tag{9}$$

A more precise analysis can be now carried out, using exactly the same procedure, to provide a refinement of (9) showing the dependance of $|\mathcal{R}_{n,r}^{\tilde{A}}|$ on the parameter r. One can indeed iteratively consider more terms in the estimates for g_r and $(g_r \cdot E')$ and then use a more general version of Th. VI.9 of [7]. Indeed, according to Th. 7 of [6] and to the related references, in order to obtain a singular approximation of g_{r+1} we are allowed to classically integrate the singular expansions of $(g_r \cdot E')$ even when this is of the form $(\pi/2 - z)^{-\alpha} \cdot (-\ln(\pi/2 - z))^{\beta}$, where $\alpha \neq 1$ and β is a positive integer. Furthermore, this applies also to error terms with the same structure, i.e., the corresponding $\mathcal{O}$-transfer holds true. This property is in our case very useful because, for example, one can exactly compute integrals of the form

$$\int^{(2)} \frac{\left[\ln\left(\frac{\pi}{2} - z\right)\right]^r}{\left(\frac{\pi}{2} - z\right)^2} = -\frac{\left[\ln\left(\frac{\pi}{2} - z\right)\right]^{r+1}}{r+1} + \cdots,$$

where the remaining term is $\mathcal{O}\left(-\left[\ln\left(\frac{\pi}{2} - z\right)\right]^r\right)$ for z near $\pi/2$.

Based on this one finds that for $z \to \pi/2$

$$g_r(z) = \frac{2^r}{r!}\left[-\ln\left(\frac{\pi}{2} - z\right)\right]^r + \mathcal{O}\left(\left[-\ln\left(\frac{\pi}{2} - z\right)\right]^{r-1}\right) \tag{10}$$

which gives, by Th. VI.4 of [7], that for $n \to \infty$

$$[z^n]g_r(z) = \frac{2^r}{r!} \cdot [z^n]\left[\left[-\ln\left(\frac{\pi}{2} - z\right)\right]^r\right] + [z^n]\left[\mathcal{O}\left(\left[-\ln\left(\frac{\pi}{2} - z\right)\right]^{r-1}\right)\right].$$

By Th. VI.2 of [7] (see special cases) one has that there is a constant $c_{1,r} > 0$ such that

$$[z^n]\left[\left[-\ln\left(\frac{\pi}{2} - z\right)\right]^r\right] \sim c_{1,r}\left(\frac{2}{\pi}\right)^n \cdot n^{-1}\left[\ln(n)\right]^{r-1} \quad (n \to \infty)$$

while, using Th. VI.3 of [7], one finds that there is $c_{2,r} > 0$ such that

$$[z^n]\left[\mathcal{O}\left(\left[-\ln\left(\frac{\pi}{2} - z\right)\right]^{r-1}\right)\right] \leq c_{2,r}\left(\frac{2}{\pi}\right)^n \cdot n^{-1}\left[\ln(n)\right]^{r-1} \quad (n \to \infty).$$

All together this gives

$$\frac{2^r}{r!} \cdot c_{1,r} \leq \frac{[z^n]g_r(z)}{\left(\frac{2}{\pi}\right)^n \cdot n^{-1}\left[\ln(n)\right]^{r-1}} \leq c_{1,r} + c_{2,r}. \tag{11}$$

which says that the ratio

$$\frac{|\mathcal{R}_{n,r}^{\tilde{A}}|}{n! \cdot \left(\frac{2}{\pi}\right)^n \cdot n^{-1}\left[\ln(n)\right]^{r-1}}$$

is bounded from both sides.

In conclusion we remark that, even if the recursion of Proposition 8 requires a double integration, we can still efficiently use it to compute the singular expansions at $z = \pi/2$ of the functions g_r. Indeed, near the mentioned singularity, we are allowed to use standard integration and the integrals involved are explicitely solvable.

Comparing the value of $|\mathcal{R}_{n,r}^{\tilde{A}}|/n!$ as described in (11) with the probability for a random permutation of size n to be in $\mathcal{R}_n^{\tilde{A}}$, i.e. $\frac{4}{\pi} \cdot \left(\frac{2}{\pi}\right)^n$, we obtain the following result

Proposition 9 *For n large, the probability for a random permutation in $\mathcal{R}_n^{\tilde{A}}$ to have r right-to-left minima satisfies*

$$c_r \leq \frac{P\left(|rlm(\pi)| = r \,\middle|\, \pi \in \mathcal{R}_n^{\tilde{A}}\right)}{n^{-1}\left[\ln(n)\right]^{r-1}} \leq C_r,$$

where c_r and C_r are two positive constants.

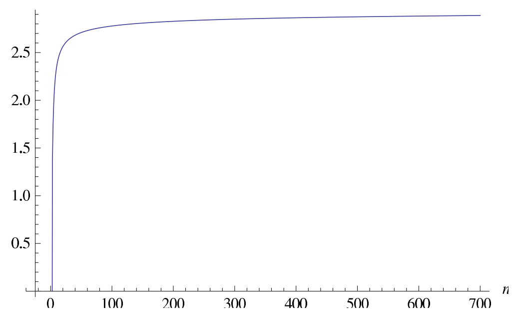

Figure 6: Plot of $P\left(|rlm(\pi)| = 2 \middle| \pi \in \mathcal{R}_n^{\tilde{\mathcal{A}}}\right) / (n^{-1}\ln(n))$.

In Fig. 6 we show for n large the behaviour of $|\mathcal{R}_{n,2}^{\tilde{\mathcal{A}}}|/|\mathcal{R}_n^{\tilde{\mathcal{A}}}|$ divided by $n^{-1}\ln(n)$.

Structural properties of $\tilde{g}$ near the singularity. To conclude our asymptotic analysis we go back to equation (5) to describe a structural property of the solution $\tilde{g}$. Indeed, treating w as a constant, we can apply Th. VII.9 of [7] finding that near the *regular* singular point $z = \pi/2$ the desired solution $\tilde{g}$ can be expressed as

$$\tilde{g} = a_w \cdot \left(\frac{\pi}{2} - z\right)^{\frac{1+\sqrt{1+8w}}{2}} A_w\left(z - \frac{\pi}{2}\right) + b_w \cdot \left(\frac{\pi}{2} - z\right)^{\frac{1-\sqrt{1+8w}}{2}} B_w\left(z - \frac{\pi}{2}\right),$$

where w could appear in $a_w, A_w(z), b_w, B_w(z)$ and the functions $A_w(z), B_w(z)$ are analytic at $z = 0$.

It is interesting to note that, taking $a_w = 0$ and $b_w = B_w = 1$, one obtains

$$\tilde{g}_\alpha = \left(\frac{\pi}{2} - z\right)^{\frac{1-\sqrt{1+8w}}{2}}$$

whose expansion at $w = 0$ looks as

$$\begin{aligned}\tilde{g}_\alpha =& 1 - 2w\ln(\pi/2 - z) + w^2\left(4\ln(\pi/2 - z) + 2\left[\ln(\pi/2 - z)\right]^2\right) \\ &+ w^3\left(-16\ln(\pi/2 - z) - 8\left[\ln(\pi/2 - z)\right]^2 - \frac{4}{3}\left[\ln(\pi/2 - z)\right]^3\right) + \cdots.\end{aligned}$$

Based on (7), this reflects well enough the asymptotic behaviour of the expressions for g_1, g_2 and g_3 which have been previously computed. More in general, we can consider $([w^r]\tilde{g}_\alpha)(z)$ as an approximation of the desired $g_r(z)$ near the singularity $z = \pi/2$. This can be justified recalling what is shown in (10) and observing that $\tilde{g}_\alpha$ satisfies

$$\frac{\partial^2 \tilde{g}_\alpha}{\partial z^2} = w \cdot \frac{2}{(\pi/2 - z)^2} \cdot \tilde{g}_\alpha$$

which is obtained by substituting in (5), i.e. the defining equation for $\tilde{g}$, the term $E'(z)$ by $2/(\pi/2 - z)^2$, the latter being the main part of the singular approximation (6). Therefore, as for $g_r(z)$, we have that in a neighbourhood of $z = \pi/2$

$$([w^r]\tilde{g}_\alpha)(z) = \frac{2^r}{r!}\left[-\ln\left(\frac{\pi}{2} - z\right)\right]^r + \mathcal{O}\left(\left[-\ln\left(\frac{\pi}{2} - z\right)\right]^{r-1}\right).$$

Acknowledgements

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