A PROBABILISTIC INTERPRETATION OF A SEQUENCE RELATED TO NARAYANA POLYNOMIALS

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Abstract. A sequence of coefficients appearing in a recurrence for the Narayana polynomials is generalized. The coefficients are given a probabilistic interpretation in terms of beta distributed random variables. The recurrence established by M. Lasalle is then obtained from a classical convolution identity. Some arithmetical properties of the generalized coefficients are also established.

1. Introduction

The Narayana polynomials

$$\mathcal{N}_r(z) = \sum_{k=1}^{r} N(r,k) z^{k-1} \tag{1.1}$$

with the Narayana numbers $N(r,k)$ given by

$$N(r,k) = \frac{1}{r}\binom{r}{k-1}\binom{r}{k} \tag{1.2}$$

have a large number of combinatorial properties. In a recent paper, M. Lasalle [19] established the recurrence

$$(z+1)\mathcal{N}_r(z) - \mathcal{N}_{r+1}(z) = \sum_{n \geq 1} (-z)^n \binom{r-1}{2n-1} A_n \mathcal{N}_{r-2n+1}(z). \tag{1.3}$$

The numbers A_n satisfies the recurrence

$$(-1)^{n-1} A_n = C_n + \sum_{j=1}^{n-1} (-1)^j \binom{2n-1}{2j-1} A_j C_{n-j}, \tag{1.4}$$

with $A_1 = 1$ and $C_n = \frac{1}{n+1}\binom{2n}{n}$ the Catalan number. This recurrence is taken here as being the definition of A_n. The first few values are

$$A_1 = 1,\ A_2 = 1,\ A_3 = 5,\ A_4 = 56,\ A_5 = 1092,\ A_6 = 32670. \tag{1.5}$$

Lasalle [19] shows that $\{A_n : n \in \mathbb{N}\}$ is an increasing sequence of positive integers. In the process of establishing the positivity of this sequence, he contacted D. Zeilberger, who suggested the study of the related sequence

$$a_n = \frac{2A_n}{C_n}, \tag{1.6}$$

Date: March 21, 2012.

1991 *Mathematics Subject Classification.* Primary 11B83, Secondary 11B68,60C05.

Key words and phrases. Bessel zeta functions, beta distributions, Catalan numbers, conjugate random variables, cumulants, determinants, Narayana polynomials, random variables, Rayleigh functions.

with first few values

$$a_1 = 2,\ a_2 = 1,\ a_3 = 2,\ a_4 = 8,\ a_5 = 52,\ a_6 = 495,\ a_7 = 6470. \tag{1.7}$$

The recurrence (1.4) yields

$$(-1)^{n-1} a_n = 2 + \sum_{j=1}^{n-1} (-1)^j \binom{n-1}{j-1} \binom{n+1}{j+1} \frac{a_j}{n-j+1}. \tag{1.8}$$

This may be expressed in terms of the numbers

$$\sigma_{n,r} := \frac{2}{n} \binom{n}{r-1} \binom{n+1}{r+1} \tag{1.9}$$

that appear as entry *A*108838 in *OEIS* and count Dyck paths by the number of long interior inclines. The fact that $\sigma_{n,r}$ is an integer also follows from

$$\sigma_{n,r} = \binom{n-1}{r-1}\binom{n+1}{r} - \binom{n-1}{r-2}\binom{n+1}{r+1}. \tag{1.10}$$

The relation (1.8) can also be written as

$$a_n = (-1)^{n-1} \left[2 + \frac{1}{2} \sum_{j=1}^{n-1} (-1)^j \sigma_{n,j} a_j \right]. \tag{1.11}$$

The original approach by M. Lasalle [19] is to establish the relation

$$(z+1)\mathcal{N}_r(z) - \mathcal{N}_{r+1}(z) = \sum_{n \geq 1} (-z)^n \binom{r-1}{2n-1} A_n(r) \mathcal{N}_{r-2n+1}(z) \tag{1.12}$$

for some coefficient $A_n(r)$. The expression

$$\mathcal{N}_r(z) = \sum_{m \geq 0} z^m (z+1)^{r-2m-1} \binom{r-1}{2m} C_m \tag{1.13}$$

given in [12], is then employed to show that $A_n(r)$ is independent of r. This is the definition of A_n given in [19]. Lasalle mentions in passing that "J. Novak observed, as empirical evidence, that the integers $(-1)^{n-1}A_n$ are precisely the (classical) cumulants of a standard semicircular random variable".

The goal of this paper is to revisit Lasalle's results, provide probabilistic interpretation of the numbers A_n and to consider Zeilberger's suggestion.

The probabilistic interpretation of the numbers A_n starts with the semicircular distribution

$$f_1(x) = \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & \text{if } -1 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases} \tag{1.14}$$

Let X be a random variable with distribution f_1. Then $X_* = 2X$ satisfies

$$\mathbb{E}\left[X_*^r\right] = \begin{cases} 0 & \text{if } r \text{ is odd} \\ C_m & \text{if } r \text{ is even, with } r = 2m, \end{cases} \tag{1.15}$$

where $C_n = \frac{1}{m+1}\binom{2m}{m}$ are the Catalan numbers. The moment generating function

$$\varphi(t) = \sum_{n=0}^{\infty} \mathbb{E}\left[X^n\right] \frac{t^n}{n!} \tag{1.16}$$

is expressed in terms of the modified Bessel function of the first kind $I_\alpha(x)$ and the cumulant generating function

$$\psi(t) = \log \varphi(t) = \sum_{n=1}^{\infty} \kappa_1(n) \frac{t^n}{n!} \tag{1.17}$$

has coefficients $\kappa_1(n)$, known as the cumulants of X. The identity

$$A_n = (-1)^{n+1}\kappa_1(2n)2^{2n}, \tag{1.18}$$

is established here. Lasalle's recurrence (1.4) now follows from the convolution identity

$$\kappa(n) = \mathbb{E}\left[X^n\right] - \sum_{j=1}^{n-1} \binom{n-1}{j-1} \kappa(j) \mathbb{E}\left[X^{n-j}\right] \tag{1.19}$$

that holds for any pair of moments and cumulants sequences [24]. The coefficient a_n suggested by D. Zeilberger now takes the form

$$a_n = \frac{2(-1)^{n+1}\kappa_1(2n)}{\mathbb{E}\left[X_*^{2n}\right]}. \tag{1.20}$$

In this paper, these notions are extended to the case of random variables distributed according to the symmetric beta distribution

$$f_\mu(x) = \frac{1}{B(\mu+\frac{1}{2}, \frac{1}{2})}(1-x^2)^{\mu-1/2}, \quad \text{for } |x| \leq 1,\ \mu > -\tfrac{1}{2} \tag{1.21}$$

and 0 otherwise. The semi-circular distribution is the particular case $\mu = 1$. Here $B(a,b)$ is the classical beta function defined by the integral

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\, dt, \quad \text{for } a,\ b > 0. \tag{1.22}$$

These ideas lead to introduce a generalization of the Narayana polynomials and these are expressed in terms of the classical Gegenbauer polynomials $C_n^{\mu+\frac{1}{2}}$. The coefficients a_n are also generalized to a family of numbers $\{a_n(\mu)\}$ with parameter μ. The special cases $\mu = 0$ and $\mu = \pm\frac{1}{2}$ are discussed in detail.

Section 2 produces a recurrence for $\{a_n\}$ from which the fact that a_n is incresaing and positive are established. The recurrence comes from a relation between $\{a_n\}$ and the Bessel function $I_\alpha(x)$. Section 3 gives an expression for $\{a_n\}$ in terms of a determinant of an upper Hessenberg matrix. The standard procedure to evaluate these determinants gives the original recurrence defining $\{a_n\}$. Section 4 introduces the probabilistic interpretation of the numbers $\{a_n\}$. The cumulants of the associated random variable are expressed in terms of the Bessel zeta function. Section 5 presents the Narayana polynomials as expected values of a simple function of a semicircular random variable. These polynomials are generalized in Section 6 and they are expressed in terms of Gegenbauer polynomials. The corresponding extension of $\{a_n\}$ are presented in Section 7. The paper concludes with some arithmetical properties of $\{a_n\}$ and its generalization corresponding to the parameter $\mu = 0$. These are described in Section 8.

2. The sequence $\{a_n\}$ is positive and increasing

In this section a direct proof of the positivity of the numbers a_n defined in (1.8) is provided. Naturally this implies $A_n \geq 0$. The analysis employs the *modified Bessel function of the first kind*

$$I_\alpha(z) := \sum_{j=0}^{\infty} \frac{1}{j!\,(j+\alpha)!} \left(\frac{z}{2}\right)^{2j+\alpha}. \tag{2.1}$$

Formulas for this function appear in [16].

Lemma 2.1. The numbers a_n satisfy

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1}a_j}{(j+1)!} \frac{x^{j-1}}{(j-1)!} = \frac{2}{\sqrt{x}} \frac{I_2(2\sqrt{x})}{I_1(2\sqrt{x})}. \tag{2.2}$$

Proof. The statement is equivalent to

$$\sqrt{x} I_1(2\sqrt{x}) \times \sum_{j=1}^{\infty} \frac{(-1)^{j-1}a_j}{(j+1)!} \frac{x^{j-1}}{(j-1)!} = 2I_2(2\sqrt{x}). \tag{2.3}$$

This is established by comparing coefficients of x^n on both sides and using (1.8). □

Now change x to x^2 in Lemma 2.1 to write

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1}a_j}{(j+1)!} \frac{x^{2j-2}}{(j-1)!} = \frac{2}{x} \frac{I_2(2x)}{I_1(2x)}. \tag{2.4}$$

The classical relations

$$\frac{d}{dz}\left(z^{-m}I_m(z)\right) = z^{-m}I_{m+1}(z), \text{ and } \frac{d}{dz}\left(z^{m+1}I_{m+1}(z)\right) = z^{m+1}I_m(z) \tag{2.5}$$

give

$$I_1'(z) = I_2(z) + \frac{1}{z}I_1(z). \tag{2.6}$$

Therefore (2.4) may be written as

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1}a_j}{(j+1)!} \frac{x^{2j-2}}{(j-1)!} = \frac{1}{x}\frac{d}{dx} \log\left(\frac{I_1(2x)}{2x}\right). \tag{2.7}$$

The relations (2.5) also produce

$$\frac{d}{dz}\left(\frac{z^{m+1}I_{m+1}(z)}{z^{-m}I_m(z)}\right) = z^{2m+1}\frac{I_m^2(z) - I_{m+1}^2(z)}{I_m^2(z)}. \tag{2.8}$$

In particular,

$$\frac{d}{dz}\left(\frac{z^2 I_2(z)}{z^{-1}I_1(z)}\right) = z^3 - z^3\frac{I_2^2(z)}{I_1^2(z)}. \tag{2.9}$$

Replacing this relation in (2.7) gives the recurrence stated next.

Proposition 2.2. The numbers a_n satisfy the recurrence

$$2na_n = \sum_{k=1}^{n-1} \binom{n}{k-1}\binom{n}{k+1} a_k a_{n-k}, \qquad \text{for } n \geq 2, \tag{2.10}$$

with initial condition $a_1 = 1$.

Corollary 2.3. The numbers a_n are nonnegative.

Proposition 2.4. The numbers a_n satisfy

$$4a_n = \sum_{k=1}^{n-1} \binom{n-1}{k-1}\binom{n-1}{k} a_k a_{n-k} - \sum_{k=2}^{n-2} \binom{n-1}{k-2}\binom{n-1}{k+1} a_k a_{n-k}. \tag{2.11}$$

Proof. This follows from (2.10) and the identity

$$\binom{n}{k-1}\binom{n}{k+1} = \frac{n}{2}\left[\binom{n-1}{k-1}\binom{n-1}{k} - \binom{n-1}{k-2}\binom{n-1}{k+1}\right].$$

□

Corollary 2.5. The numbers a_n are nonnegative integers. Moreover a_n is even if n is odd.

Proof. Corollary 2.3 shows $a_n > 0$. It remains to show $a_n \in \mathbb{Z}$ and to verify the parity statement. This is achieved by simultaneous induction on n.

Assume first $n = 2m+1$ is odd. Then (1.9) shows that $\frac{1}{2}\sigma_{n,r} \in \mathbb{Z}$ and (1.11), written as

$$a_n = (-1)^{n-1}\left[2 + \sum_{r=1}^{n-1} \frac{\sigma_{n,r}}{2} a_r\right], \tag{2.12}$$

proves that $a_n \in \mathbb{Z}$. Now write (2.10) as

$$2(2m+1)a_{2m+1} = 2\sum_{k=1}^{m} \binom{2m+1}{k-1}\binom{2m+1}{k+1} a_k a_{2m+1-k} \tag{2.13}$$

and observe that either k or $2m+1-k$ is odd. The induction hypothesis shows that either a_k or a_{2m+1-k} is even. This shows a_{2m+1} is even.

Now consider the case $n = 2m$ even. If r is odd, then a_r is even; if r is even then $r-1$ is odd and $\frac{1}{2}\sigma_{n,r} \in \mathbb{Z}$ in view of the identity

$$\sigma_{n,r} = \frac{2}{r-1}\binom{n-1}{r-2}\binom{n+1}{r+1}. \tag{2.14}$$

The result follows again from (2.12). □

Corollary 2.6. The numbers A_n are nonnegative integers.

The recurrence in Proposition 2.2 is now employed to prove that $\{a_n\}$ is an increasing sequence. The first few values are 2, 1, 2, 8, 52.

Theorem 2.7. For $n \geq 3$, the inequality $a_n > a_{n-1}$ holds.

Proof. Take the terms $k=1$ and $k=n-1$ in the sum appearing in the recurrence in Proposition (2.2) and use $a_n > 0$ to obtain

$$a_n \geq \frac{1}{2n}\left[\binom{n}{0}\binom{n}{2} a_1 a_{n-1} + \binom{n}{n-2}\binom{n}{2} a_{n-1} a_1\right]. \tag{2.15}$$

Since $a_1 = 2$ the previous inequality yields

$$a_n \geq (n-1)a_{n-1}. \tag{2.16}$$

Hence, for $n \geq 3$, this gives $a_n - a_{n-1} \geq (n-2)a_{n-1} > 0$. □

3. An expression in forms of determinants

The recursion relation (1.8) expressed in the form

$$\sum_{j=1}^{m} (-1)^{j-1} \binom{m}{j-1} \binom{m+1}{j+1} a_j = 2m \tag{3.1}$$

is now employed to produce a system of equations for the numbers a_n by varying m through $1, 2, 3, \cdots, n$. The coefficient matrix has determinant $(-1)^{\binom{n}{2}} n!$ and Cramér's rule gives

$$a_n = \frac{(-1)^{n-1}}{n!} \det \begin{pmatrix} \binom{1}{1-1}\binom{1+1}{1+1} & 0 & 0 & \cdots & 0 & 2 \\ \binom{2}{1-1}\binom{2+1}{1+1} & \binom{2}{2-1}\binom{2+1}{2+1} & 0 & \cdots & 0 & 4 \\ \binom{3}{1-1}\binom{3+1}{1+1} & \binom{3}{2-1}\binom{3+1}{2+1} & \binom{3}{3-1}\binom{3+1}{3+1} & \cdots & 0 & 6 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \binom{n}{1-1}\binom{n+1}{1+1} & \binom{n}{2-1}\binom{n+1}{2+1} & \binom{n}{3-1}\binom{n+1}{3+1} & \cdots & \binom{n}{n-2}\binom{n+1}{n} & 2n \end{pmatrix} \tag{3.2}$$

The power of -1 is eliminated by permuting the columns to produce the matrix

$$B_n = \begin{pmatrix} 2 & \binom{1}{1-1}\binom{1+1}{1+1} & 0 & 0 & 0 \\ 4 & \binom{2}{1-1}\binom{2+1}{1+1} & \binom{2}{2-1}\binom{2+1}{2+1} & 0 & \cdots \\ 6 & \binom{3}{1-1}\binom{3+1}{1+1} & \binom{3}{2-1}\binom{3+1}{2+1} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 2n & \binom{n}{1-1}\binom{n+1}{1+1} & \binom{n}{2-1}\binom{n+1}{2+1} & \binom{n}{3-1}\binom{n+1}{3+1} \cdots & \binom{n}{n-2}\binom{n+1}{n} \end{pmatrix}. \tag{3.3}$$

The representation of a_n in terms of determinants is given in the next result.

Proposition 3.1. The number a_n is given by

$$a_n = \frac{\det B_n}{n!} \tag{3.4}$$

where B_n is the matrix in (3.3).

Recall that an *upper Hessenberg matrix* is one of the form

$$H_n = \begin{pmatrix} \beta_{1,1} & \beta_{1,2} & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ \beta_{2,1} & \beta_{2,2} & \beta_{2,3} & 0 & 0 & \cdots & \cdots & 0 & 0 \\ \beta_{3,1} & \beta_{3,2} & \beta_{3,3} & \beta_{3,4} & 0 & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \beta_{n,1} & \beta_{n,2} & \beta_{n,3} & \beta_{n,4} & \cdots & \cdots & \cdots & \beta_{n,n-1} & \beta_{n,n} \end{pmatrix} \tag{3.5}$$

The matrix B is of this form with

$$\beta_{i,j} = \begin{cases} 2i & \text{if } 1 \leq i \leq n \quad \text{and } j = 1 \\ \binom{i}{j-2}\binom{i+1}{j} & \text{if } j-1 \leq i \leq n \text{ and } j > 1. \end{cases} \tag{3.6}$$

It turns out that the recurrence (1.8) used to define the numbers a_n can be recovered if one employs (3.4).

Proposition 3.2. Define α_n by

$$\alpha_n = \frac{\det B_n}{n!} \tag{3.7}$$

where B is the matrix (3.3). Then $\{\alpha_n\}$ satisfies the recursion

$$(-1)^{n-1}\alpha_n = 2 + \sum_{j=1}^{n-1} (-1)^j \binom{n-1}{j-1}\binom{n+1}{j+1}\frac{\alpha_j}{n-j+1} \tag{3.8}$$

and the initial condition $\alpha_1 = 1$. Therefore $\alpha_n = a_n$.

Proof. For convenience define $\det H_0 = 1$. The determinant of a Hessenberg matrix satisfies the recurrence

$$\det H_n = \sum_{r=1}^{n} (-1)^{n-r} \beta_{n,r} \det H_{r-1} \prod_{i=r}^{n-1} \beta_{i,i+1}. \tag{3.9}$$

A direct application of (3.9) yields

$$\begin{aligned}
\alpha_n &= \frac{1}{n!}\left\{(-1)^{n-1}(2n)(n-1)! + \sum_{r=2}^{n}(-1)^{n-r}\binom{n}{r-2}\binom{n+1}{r}\det B_{r-1}\prod_{i=r}^{n-1} i\right\} \\
&= 2(-1)^{n-1} + \frac{1}{n!}\sum_{r=2}^{n}(-1)^{n-r}\binom{n}{r-2}\binom{n+1}{r}\alpha_{r-1}\,(n-1)! \\
&= 2(-1)^{n-1} + \sum_{r=2}^{n}(-1)^{n-r}\frac{1}{n}\binom{n}{r-2}\binom{n+1}{r}\alpha_{r-1} \\
&= 2(-1)^{n-1} + \sum_{r=2}^{n}(-1)^{n-r}\binom{n}{r-2}\binom{n+1}{r}\frac{\alpha_{r-1}}{n-r+2} \\
&= 2(-1)^{n-1} + (-1)^{n-1}\sum_{r=1}^{n}(-1)^{j}\binom{n-1}{j-1}\binom{n+1}{j+1}\frac{\alpha_j}{n-j+1}.
\end{aligned}$$

This is (3.8). □

Corollary 3.3. The modified Bessel function of the first kind admits a determinant expression

$$I_1(x) = x\,\exp\left(\sum_{j=1}^{\infty}\frac{(-1)^{j-1}\det B_j}{(j+1)!\,j!^2}\left(\frac{x}{2}\right)^{2j}\right). \tag{3.10}$$

Proof. This follows by integrating the identity

$$\frac{2I_2(2x)}{x\,I_1(2x)} = \frac{1}{x}\frac{d}{dx}\log\frac{I_1(2x)}{2x}. \tag{3.11}$$

□

4. The probabilistic background: conjugate random variables

This section provides the probabilistic tools required for an interpretation of the sequence A_n defined in (1.4). The specific connections are given in Section 5.

Consider a random variable X with the *symmetric beta distribution* given in (1.21). The moments of the symmetric beta distribution, given by

$$\mathbb{E}\left[X^n\right] = \frac{1}{B(\mu+\frac{1}{2},\frac{1}{2})}\int_{-1}^{1} x^n (1-x^2)^{\mu-1/2}\,dx, \tag{4.1}$$

vanish for n odd and for $n = 2m$ they are

$$\mathbb{E}\left[X^{2m}\right] = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1+m)}\frac{(2m)!}{2^{2m}\,m!}. \tag{4.2}$$

Therefore the moment generating function is

$$\varphi_\mu(t) = \mathbb{E}\left[e^{tX}\right] = \sum_{n=0}^{\infty}\mathbb{E}\left[X^n\right]\frac{t^n}{n!} = \Gamma(\mu+1)\sum_{m=0}^{\infty}\frac{t^{2m}}{2^{2m}\,m!\,\Gamma(\mu+m+1)}. \tag{4.3}$$

The next proposition summarizes properties of $\varphi_\mu(t)$. The first one is to recognize the series in (4.3) from (2.1). The zeros $\{j_{\mu,k}\}$ of the Bessel function of the first kind

$$J_\alpha(x) = \sum_{j=0}^{\infty}\frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)}\left(\frac{x}{2}\right)^{2m+\alpha} \tag{4.4}$$

appear in the factorization of φ_μ in view of the relation $I_\mu(z) = e^{-\pi i\mu/2}J_\mu(iz)$.

Proposition 4.1. The moment generating function $\varphi_\mu(t)$ of a random variable $X \sim f_\mu$ is given by

$$\varphi_\mu(t) = \Gamma(\mu+1)\left(\frac{2}{t}\right)^{\mu} I_\mu(t). \tag{4.5}$$

Note 4.2. The Catalan numbers C_n appear as the even-order moments of f_μ when $\mu = 1$. More precisely, if X is distributed as f_1 (written as $X \sim f_1$), then

$$\mathbb{E}\left[(2X)^{2n}\right] = C_n \text{ and } \mathbb{E}\left[(2X)^{2n+1}\right] = 0. \tag{4.6}$$

Note 4.3. The moment generating function of f_μ admits the Weierstrass product representation

$$\varphi_\mu(t) = \prod_{k=1}^{\infty}\left(1+\frac{t^2}{j_{\mu,k}^2}\right) \tag{4.7}$$

where $\{j_{\mu,k}\}$ are the zeros of the Bessel function of the first kind J_μ.

Definition 4.4. The *cumulant generating function* is

$$\begin{aligned}\psi_\mu(t) &= \log\varphi_\mu(t)\\ &= \log\left(\sum_{n=0}^{\infty}\mathbb{E}\left[X^n\right]\frac{t^n}{n!}\right).\end{aligned}$$

The product representation of $\varphi_\mu(t)$ yields

$$\begin{aligned}\log\varphi_\mu(t) &= \sum_{k=1}^{\infty}\log\left(1+\frac{t^2}{j_{\mu,k}^2}\right)\\ &= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{t}{j_{\mu,k}}\right)^{2n}\\ &:= \sum_{n=1}^{\infty}\kappa_\mu(n)\frac{t^n}{n!}.\end{aligned}$$

The series converges for $|t| < j_{\mu,1}$. The first Bessel zero satisfies $j_{\mu,1} > 0$ for all $\mu \geq 0$. It follows that the series has a non-zero radius of convergence.

Note 4.5. The coefficient $\kappa_\mu(n)$ is the n-th *cumulant* of X. An expression that links the moments to the cumulants of X is provided by V. P. Leonov and A. N. Shiryaev [20]:

$$\kappa_\mu(n) = \sum_{\mathcal{V}}(-1)^{k-1}(k-1)!\prod_{i=1}^{k}\mathbb{E}(2X)^{|V_i|} \tag{4.8}$$

where the sum is over all partitions $\mathcal{V} = \{V_1, \cdots, V_k\}$ of the set $\{1, 2, \ldots, n\}$.

In the case $\mu = 0$ the moments are Catalan numbers or 0, in the case $\mu = 1$ the moments are central binomial coefficients. Therefore, in both cases, the cumulants $\kappa_\mu(n)$ are integers. An expression for the general value of μ involves

$$\zeta_\mu(s) = \sum_{k=1}^{\infty}\frac{1}{j_{\mu,k}^s} \tag{4.9}$$

the *Bessel zeta function*, sometimes referred as the *Rayleigh function*.

The next result gives an expression for the cumulants of a random variable X with a distribution f_μ. The special case $\mu = 1$, described in the next section, provides the desired probabilistic interpretation of the original sequence A_n.

Theorem 4.6. Let $X \sim f_\mu$. Then

$$\kappa(n) = \begin{cases}0 & \text{if } n \text{ is odd,}\\ 2(-1)^{n/2+1}(n-1)!\,\zeta_\mu(n) & \text{if } n \text{ is even.}\end{cases} \tag{4.10}$$

Proof. Rearranging the expansion in Definition 4.4 gives

$$\begin{aligned}\log\varphi_\mu(t) &= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(\frac{t}{j_{\mu,k}}\right)^{2n}\\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}t^{2n}\sum_{k=1}^{\infty}\frac{1}{j_{\mu,k}^{2n}}.\end{aligned}$$

Now compare powers of t in this expansion with the definition

$$\log\varphi_\mu(t) = \sum_{n=1}^{\infty}\kappa_\mu(n)\frac{t^n}{n!} \tag{4.11}$$

to obtain the result. □

The next ingredient in the search for an interpretation of the sequence A_n is the notion of conjugate random variables. The properties described below appear in [25]. A complex-valued random variable Z is called a *regular random variable* (*rrv* for short) if $\mathbb{E}|Z|^n < \infty$ for all $n \in \mathbb{N}$ and

$$\mathbb{E}\left[h(Z)\right] = h\left(\mathbb{E}\left[Z\right]\right) \tag{4.12}$$

for all polynomials h. The class of rrv is closed under compositions with polynomials (if Z is rrv and P is a polynomial, then $P(Z)$ is rrv) and it is also closed under addition of independent rrv. The basic definition is stated next.

Definition 4.7. Let X, Y be real random variables, not necessarily independent. The pair (X, Y) is called *conjugate random variables* if $Z = X + iY$ is an rrv. The random variable X is called *self-conjugate* if Y has the same distribution as X.

The property of rrv may be expressed in terms of the function

$$\Phi(\alpha, \beta) := \mathbb{E}\left[\exp(i\alpha X + i\beta Y)\right]$$

The next theorem gives a condition for $Z = X + iY$ to be an rrv. The random variables X and Y are not necessarily independent.

Theorem 4.8. Let $Z = X+iY$ be a complex valued random variable with $\mathbb{E}\left[Z\right] = 0$ and $\mathbb{E}\left[Z^n\right] < \infty$. Then Z is an rrv if and only if $\Phi(\alpha, i\alpha) = 1$ for all $\alpha \in \mathbb{C}$.

This is now reformulated for real and independent random variables.

Theorem 4.9. Let X, Y be independent real valued random variables with finite moments. Define

$$\Phi_X(\alpha) = \mathbb{E}\left[e^{i\alpha X}\right] = \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!}\mathbb{E}\left[X^n\right] \text{ and } \Phi_Y(\beta) = \mathbb{E}\left[e^{i\alpha Y}\right] = \sum_{n=0}^{\infty} \frac{(i\beta)^n}{n!}\mathbb{E}\left[Y^n\right].$$

Then $Z = X + iY$ is an rrv with mean zero if and only if $\Phi_X(\alpha)\Phi_Y(i\alpha) = 1$.

Example 4.10. Let X and Y be independent Gaussian variables with zero mean and the same variance. Then X and Y are conjugate since

$$\varphi_X(t) = \exp\left(\frac{\sigma^2}{2}t^2\right) \text{ and } \varphi_{iY}(t) = \exp\left(-\frac{\sigma^2}{2}t^2\right).$$

Note 4.11. Suppose $Z = X + iY$ is a rrv with $\mathbb{E}\left[Z\right] = 0$ and $z \in \mathbb{C}$. The condition (4.12) becomes

$$\mathbb{E}\left[h(z + X + iY)\right] = h(z). \tag{4.13}$$

Given a sequence of polynomials $\{Q_n(z)\}$ such that $\deg(Q_n) = n$ and with leading coefficient 1, an elementary argument shows that there is a unique sequence of coefficients $\alpha_{j,n}$ such that the relation

$$Q_{n+1}(z) - zQ_n(z) = \sum_{j=0}^{n} \alpha_{j,n} Q_j(z) \tag{4.14}$$

holds. This section discusses this recurrence for the sequence of polynomials

$$P_n(z) := \mathbb{E}(z + X)^n \tag{4.15}$$

associated to a random variable X. The polynomial P_n is of degree n and has leading coefficient 1. It is shown that if the cumulants of odd order vanish, then the even order cumulants provide the coefficients $\alpha_{j,n}$ for the recurrence (4.14).

Theorem 4.12. Let X be a random variable with cumulants $\kappa(m)$. Assume the odd-order cumulants vanish and that X has a conjugate random variable Y. Define the polynomials

$$P_n(z) = \mathbb{E}\left[(z+X)^n\right]. \tag{4.16}$$

Then P_n satisfies the recurrence

$$P_{n+1}(z) - zP_n(z) = \sum_{m\geq 1} \binom{n}{2m-1} \kappa(2m) P_{n-2m+1}(z). \tag{4.17}$$

Proof. Let X_1, X_2 independent copies of X. Then

$$\mathbb{E}\left[X_1\left((X_1+iY_1+z+X_2)^n-(z+X_2)^n\right)\right] = \\ = \sum_{j=0}^{n} \binom{n}{j} \mathbb{E}\left[X_1(X_1+iY_1)^j(z+X_2)^{n-j}\right] - \mathbb{E}\left[X_1(z+X_2)^n\right].$$

This last expression becomes

$$\sum_{j=1}^{n} \binom{n}{j} \mathbb{E}\left[X_1(X_1+iY_1)^j(z+X_2)^{n-j}\right] = \sum_{j=1}^{n} \binom{n}{j} \mathbb{E}\left[X_1(X_1+iY_1)^j\right] \mathbb{E}\left[(z+X_2)^{n-j}\right].$$

On the other hand

$$\mathbb{E}\left[X_1\left((X_1+z+X_2+iY_1)^n-(z+X_2)^n\right)\right] = \\ \sum_{r=0}^{n} \binom{n}{r} \mathbb{E}\left[X_1(X_1+z)^{n-r}\right] \mathbb{E}\left[(X_2+iY_1)^r\right] - \mathbb{E}\left[X_1(z+X_2)^n\right].$$

The cancellation property (4.28) shows that the only surviving term in the sum is $r=0$, therefore

$$\mathbb{E}\left[X_1\left((X_1+z+X_2+iY_1)^n-(z+X_2)^n\right)\right] = \\ \mathbb{E}\left[X_1(X_1+z)^n\right] - \mathbb{E}\left[X_1\right]\mathbb{E}\left[(z+X_2)^n\right]$$

and $\mathbb{E}\left[X_1\right]=0$ since $\kappa(1)=0$. This shows the identity

$$\sum_{j=1}^{n} \binom{n}{j} \mathbb{E}\left[X_1(X_1+iY_1)^j\right] \mathbb{E}\left[(z+X_2)^{n-j}\right] = \mathbb{E}\left[X_1(X_1+z)^n\right]. \tag{4.18}$$

The cumulants of X satisfy

$$\kappa(m) = \mathbb{E}X(X+iY)^{m-1}, \quad \text{for } m \geq 1, \tag{4.19}$$

(see Theorem 3.3 in [13]), therefore in the current situation

$$\mathbb{E}\left[X_1(X_1+iY_1)^j\right] = \begin{cases} 0 & \text{if } j \text{ is even} \\ \kappa(2m) & \text{if } j=2m+1 \text{ is odd.} \end{cases} \tag{4.20}$$

On the other hand

$$\begin{aligned} \mathbb{E}\left[X_1(X_1+z)^n\right] &= \mathbb{E}\left[(X_1+z)^{n+1} - z(X_1+z)^n\right] \\ &= P_{n+1}(z) - zP_n(z). \end{aligned}$$

Replacing in (4.18) yields the result. ∎

Recall that a random variable has a Laplace distribution if its distribution function is

$$f_L(x) = \tfrac{1}{2}e^{-|x|}. \tag{4.21}$$

Assume X_μ has a distribution f_μ defined in (1.21) and moment generating function given by (4.7). The next lemma constructs a random variable Y_μ conjugate to X_μ.

Lemma 4.13. Let $Y_{\mu,n}$ be a random variable defined by

$$Y_{\mu,n} = \sum_{k=1}^{n} \frac{L_k}{j_{\mu,k}} \tag{4.22}$$

where $\{L_k : k \in \mathbb{N}\}$ is a sequence of independent, identically distributed Laplace random variables. Then $\lim\limits_{n\to\infty} Y_{\mu,n} = Y_\mu$ exists and is a random variable with continuous probability density. Moreover, the moment generating function of iY_μ is

$$\mathbb{E}\left[e^{itY_\mu}\right] = \prod_{k=1}^{\infty}\left(1+\frac{t^2}{j_{\mu,k}^2}\right)^{-1}. \tag{4.23}$$

the reciprocal of the moment generating function of f_μ given in (4.7).

Proof. The characteristic function of a Laplace random variable $iL_k/j_{\mu,k}$ is

$$\varphi_{iL_k}(t) = \frac{1}{1+\frac{t^2}{j_{\mu,k}^2}}. \tag{4.24}$$

The values

$$\mathbb{E}\left[\frac{L_k}{j_{\mu,k}}\right] = 0, \text{ and } \mathbb{E}\left[\frac{L_k^2}{j_{\mu,k}^2}\right] = \frac{2}{j_{\mu,k}^2}, \tag{4.25}$$

guarantee the convergence of the series

$$\sum_{k=1}^{\infty}\mathbb{E}\left[\frac{L_k}{j_{\mu,k}}\right] \text{ and } \sum_{k=1}^{\infty}\mathbb{E}\left[\frac{L_k^2}{j_{\mu,k}^2}\right]. \tag{4.26}$$

(The last series evaluates to $1/(2\mu+2)$). This ensures the existence of the limit defining Y_μ (see [17] for details). The continuity of the limiting probability density Y_μ is ensured by the fact that at least one term (in fact all) in the defining sum has a continuous probability density that is of bounded variation. □

Note 4.14. In the case $X_\mu \sim f_\mu$ is independent of Y_μ, then the conjugacy property states that if h is an analytic function in a neighborhood $\mathcal{O}$ of the origin, then

$$\mathbb{E}\left[h(z+X_\mu+iY_\mu)\right] = h(z), \quad \text{for } z \in \mathcal{O}. \tag{4.27}$$

In particular

$$\mathbb{E}\left[(X_\mu+iY_\mu)^n\right] = \begin{cases} 1 & \text{if } n=0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.28}$$

Note 4.15. In the special case $\mu = n/2-1$ for $n \in \mathbb{N}$, $n \geq 3$, the function (4.23) has been characterized in [11] as the moment generating function of the total time T_n spent in the sphere S^{n-1} by an n-dimensional Brownian motion starting at the origin.

5. The Narayana polynomials and the sequence A_n

The result of Theorem 4.12 is now applied to a random variable $X \sim f_1$. In this case the polynomials P_n correspondi, up to a change of variable, to the Narayama polynomials $\mathcal{N}_n$. The recurrence established by M. Lasalle comes from the results in Section 4. In particular, this provides an interpretation of the sequence $\{A_n\}$ in terms of cumulants and the Bessel zeta function.

Recall the distribution function f_1

$$f_1(x) = \begin{cases} 2\sqrt{1-x^2}/\pi, & \text{for } |x| \leq 1 \\ 0, & \text{otherwise.} \end{cases} \tag{5.1}$$

Lemma 5.1. Let $X \sim f_1$. The Narayana polynomials appear as the moments

$$\mathcal{N}_r(z) = \mathbb{E}\left[\left(1+z+2\sqrt{z}X\right)^{r-1}\right], \tag{5.2}$$

for $r \geq 1$.

Proof. The binomial theorem gives

$$\mathbb{E}\left[(1+z+2\sqrt{z}X)^{r-1}\right] = \sum_{j=0}^{r-1} \binom{r-1}{j} (z+1)^{r-1-j} z^{j/2} \mathbb{E}\left[(2X)^j\right].$$

The result now follows from (4.6) and (1.13). □

In order to apply Theorem 4.12 consider the identities

$$\begin{aligned} \mathcal{N}_r(z) &= \mathbb{E}\left[\left(1+z+2\sqrt{z}X\right)^{r-1}\right] \\ &= (2\sqrt{z})^{r-1}\mathbb{E}\left[\left(X+z_*\right)^{r-1}\right] \\ &= (2\sqrt{z})^{r-1} P_{r-1}(z_*), \end{aligned} \tag{5.3}$$

with

$$z_* = \frac{1+z}{2\sqrt{z}}. \tag{5.4}$$

The recurrence (4.17) applied to the polynomial $P_n(z_*)$ yields

$$\frac{\mathcal{N}_{n+2}(z)}{(2\sqrt{z})^{n+1}} - \frac{(1+z)}{2\sqrt{z}}\frac{\mathcal{N}_{n+1}(z)}{(2\sqrt{z})^n} = \sum_{m\geq 1} \binom{n}{2m-1} \kappa(2m) \frac{\mathcal{N}_{n-2m+2}(z)}{(2\sqrt{z})^{n-2m+1}} \tag{5.5}$$

that reduces to

$$(1+z)\mathcal{N}_r(z) - \mathcal{N}_{r+1}(z) = -\sum_{m\geq 1} \binom{r-1}{2m-1} \kappa(2m) 2^{2m} z^m \mathcal{N}_{r+1-2m}(z), \tag{5.6}$$

by using $r = n+1$. This recurrence has the form of (1.12).

Theorem 5.2. Let $X \sim f_1$. Then the coefficients A_n in Definition 1.3 are given by

$$A_n = (-1)^{n+1}\kappa(2n)2^{2n}. \tag{5.7}$$

The expression in (4.10) gives the next result.

Corollary 5.3. Let

$$\zeta_\mu(s) = \sum_{k=1}^{\infty} \frac{1}{j_{\mu,k}^s} \tag{5.8}$$

be the Bessel zeta function. Then the coefficients A_n are given by

$$A_n = 2^{2n+1}(2n-1)!\,\zeta_1(2n).$$

The scaled coefficients a_n are now expressed in terms of the Bessel zeta function.

Corollary 5.4. The coefficients a_n are given by

$$a_n = 2^{2n+1}(n+1)!(n-1)!\,\zeta_1(2n). \tag{5.9}$$

Note 5.5. This expression for the coefficients and the recurrence

$$(n+\mu)\zeta_\mu(2n) = \sum_{r=1}^{n-1} \zeta_\mu(2r)\zeta_\mu(2n-2r). \tag{5.10}$$

given in [15], provides a new proof of the recurrence in Proposition (2.2).

6. The generalized Narayana polynomials

The Narayama polynomials $\mathcal{N}_r(z)$, defined in (1.1), have been expressed as the moments

$$\mathcal{N}_r(z) = \mathbb{E}\left[\left(1+z+2\sqrt{z}X\right)^{r-1}\right], \tag{6.1}$$

for $r \geq 1$. Here X is a random variable with distribution function f_1. This suggests the extension

$$\mathcal{N}_n^\mu(z) = \mathbb{E}\left[\left(1+z+2\sqrt{z}X\right)^{n-1}\right], \tag{6.2}$$

with $X \sim f_\mu$. Therefore, $\mathcal{N}_n = \mathcal{N}_n^1$.

Note 6.1. The same argument given in (5.6) gives the recurrence

$$(1+z)\mathcal{N}_r^\mu(z) - \mathcal{N}_{r+1}^\mu(z) = -\sum_{m\geq 1} \binom{r-1}{2m-1} \kappa(2m) 2^{2m} z^m \mathcal{N}_{r+1-2m}^\mu(z), \tag{6.3}$$

where $\kappa(2n)$ are the cumulants of $X \sim f_\mu$. Theorem 5.2 gives an expression for the generalization of the Lasalle numbers:

$$A_n^\mu := (-1)^{n+1}\kappa(2n)2^{2n} \tag{6.4}$$

and the corresponding expression in terms of the Bessel zeta function:

$$A_n^\mu := 2^{2n+1}(2n-1)!\zeta_\mu(2n). \tag{6.5}$$

The generalized Narayana polynomials are now expressed in terms of the Gegenbauer polynomials $C_n^\mu(x)$ defined by the generating function

$$\sum_{n=0}^{\infty} C_n^\mu(x) t^n = (1-2xt+t^2)^{-\mu}. \tag{6.6}$$

These polynomial admit several hypergeometric representations:

$$(6.7)\quad C_n^\mu(x) = \frac{(2\mu)_n}{n!}\,{}_2F_1\left(-n, n+2\mu; \mu+\tfrac{1}{2}; \frac{1-x}{2}\right)$$
$$= \frac{2^n(\mu)_n}{n!}(x-1)^n\,{}_2F_1\left(-n, -n-\mu+\tfrac{1}{2}; -2n-2\mu+1; \frac{2}{1-x}\right)$$
$$= \frac{(2\mu)_n}{n!}\left(\frac{x+1}{2}\right)^n\,{}_2F_1\left(-n, -n-\mu+\tfrac{1}{2}; \mu+\tfrac{1}{2}; \frac{x-1}{x+1}\right).$$

The connection between Narayana and Gegenbauer polynomials comes from the expression for $C_n^\mu(z)$ given in the next proposition.

Proposition 6.2. The Gegenbauer polynomials are given by

$$(6.8)\qquad C_n^\mu(z) = \frac{(2\mu)_n}{n!}\mathbb{E}\left[\left(z+\sqrt{z^2-1}X_{\mu-1/2}\right)^n\right].$$

Proof. The Laplace integral representation

$$(6.9)\qquad C_n^\mu(\cos\theta) = \frac{\Gamma(n+2\mu)}{2^{2\mu-1}n!\Gamma^2(\mu)}\int_0^\pi (\cos\theta + i\sin\theta\cos\phi)^n \sin^{2\mu-1}\phi\, d\phi$$

appears as Theorem 6.7.4 in [3]. The change of variables $z = \cos\theta$ and $X = \cos\phi$ gives

$$C_n^\mu(z) = \frac{\Gamma(n+2\mu)}{2^{2\mu}\, n!\Gamma^2(\mu)}\int_{-1}^1 \left(z+\sqrt{z^2-1}X\right)^n (1-X^2)^{\mu-1}\, dX$$
$$= \frac{(2\mu)_n}{n!}\mathbb{E}\left[\left(z+\sqrt{z^2-1}X_{\mu-1/2}\right)^n\right],$$

as claimed. Since this is a polynomial identity in z, it can be extended to all $z \in \mathbb{C}$. □

Theorem 6.3. The Gegenbauer polynomial C_n^μ and the generalized polynomial $\mathcal{N}_n^\mu$ satisfy the relation

$$(6.10)\qquad \mathcal{N}_{n+1}^\mu(z) = \frac{n!}{(2\mu+1)_n}(1-z)^n C_n^{\mu+\frac{1}{2}}\left(\frac{1+z}{1-z}\right).$$

Proof. Introduce the variable

$$(6.11)\qquad Z = \frac{1+z}{1-z}$$

so that

$$(6.12)\qquad z = \frac{Z-1}{Z+1} \text{ and } \frac{Z}{\sqrt{Z^2-1}} = \frac{1+z}{2\sqrt{z}}.$$

Then

$$C_n^{\mu+\frac{1}{2}}\left(\frac{1+z}{1-z}\right) = \frac{(2\mu+1)_n}{n!}\left(\frac{2\sqrt{z}}{1-z}\right)^n \mathbb{E}\left[\left(\frac{1+z}{2\sqrt{z}}+X_\mu\right)^n\right]$$
$$= \frac{(2\mu+1)_n}{n!\,(1-z)^n}\mathbb{E}\left[\left(1+z+2\sqrt{z}X_\mu\right)^n\right]$$
$$= \frac{(2\mu+1)_n}{n!\,(1-z)^n}\mathcal{N}_{n+1}^\mu(z),$$

using $Z^2-1 = 4z/(1-z)^2$. □

The expression (6.7) now provides hypergeometric expressions for the original Narayana polynomials

$$\mathcal{N}_{n+1}(z) = \frac{2(1-z)^n}{(n+2)(n+1)} C_n^{3/2}\left(\frac{1+z}{1-z}\right). \tag{6.13}$$

Corollary 6.4. The Narayana polynomials are given by

$$\begin{aligned} \mathcal{N}_{n+1}(z) &= (1-z)^n {}_2F_1\left(-n, n+3; 2; \frac{z}{z-1}\right) \\ &= \frac{(2n+2)!}{(n+2)!\,(n+1)!} z^n {}_2F_1\left(-n, -n-1; -2n-2; \frac{z-1}{z}\right) \\ &= {}_2F_1(-n, -n-1; 2; z). \end{aligned} \tag{6.14}$$

This yields the representation as finite sums

$$\begin{aligned} \mathcal{N}_{n+1}(z) &= \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}\binom{n+k+2}{k} z^k (1-z)^{n-k} \\ &= \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}\binom{2n+2-k}{n-k} z^{n-k}(1-z)^k \\ &= \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k+1}\binom{n+1}{k} z^k. \end{aligned} \tag{6.15}$$

Note that the first two expressions coincide up to the change of summation variable $k \to n-k$ while the third identity is nothing but (1.1).

Note 6.5. The representation

$$C_n^{\mu}(z) = \frac{(\mu)_n}{n!}(2x)^n {}_2F_1\left(-\frac{n}{2}, \frac{1-n}{2}; 1-n-\mu; \frac{1}{x^2}\right) \tag{6.16}$$

that appears in as 6.4.12 in [3], gives the expression

$$\mathcal{N}_{n+1}(z) = \frac{(2n+2)!}{(n+1)!\,(n+2)!}\left(\frac{1+z}{2}\right)^n {}_2F_1\left(-\frac{n}{2}, \frac{1-n}{2}; -n-\frac{1}{2}; \left(\frac{1-z}{1+z}\right)^2\right) \tag{6.17}$$

equal to the finite sum representation

$$\mathcal{N}_{n+1}(z) = \frac{1}{2^{n-1}(n+2)}\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n}{k}\binom{2n+1-2k}{n-2k}(1-z)^{2k}(1+z)^{n-2k}. \tag{6.18}$$

Note 6.6. The polynomials $S_n(z) = z\mathcal{N}_n^1(z)$ satisfy the symmetry identity

$$S_n(z) = z^{n+1} S_n(z^{-1}). \tag{6.19}$$

These polynomials were expressed in [21] as

$$S_n(z) = (z-1)^{n+1}\int_0^{z/(z-1)} P_n(2x-1)\,dx \tag{6.20}$$

where $P_n(x) = C_n^{1/2}(x)$ are the Legendre polynomials. An equivalent formulation is provided next.

Theorem 6.7. The polynomials $S_n(z)$ are given by

$$S_n(z) = \frac{1}{2^{n+1}} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{n+1-k} \binom{2n-2k}{n-k} \binom{n+1-k}{k} (z-1)^{2k}(z+1)^{n+1-2k}.$$

Proof. The integration rule

$$\int C_n^{\mu}(x)\, dx = \frac{1}{2(\mu-1)} C_{n+1}^{\mu-1}(x) \tag{6.21}$$

implies

$$\int_0^{z/(z-1)} C_n^{1/2}(2x-1)\, dx = -\frac{1}{2} C_{n+1}^{-1/2}\left(\frac{z+1}{z-1} \right), \tag{6.22}$$

since the generating function

$$\sum_{n=0}^{\infty} t^n C_n^{-1/2}(z) = (1-2zt+t^2)^{1/2} \tag{6.23}$$

gives $C_{n+1}^{-1/2}(-1) = 0$ for $n > 1$. Then (6.20) yields

$$S_n(z) = -\frac{1}{2}(z-1)^{n+1} C_{n+1}^{-1/2}\left(\frac{z+1}{z-1} \right). \tag{6.24}$$

A classical formula for the Gegenbauer polynomials states

$$C_n^{\mu}(z) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{k!} \frac{(\mu)_{n-k}}{(n-2k)!} (2z)^{n-2k} \tag{6.25}$$

and the identity

$$\left(-\frac{1}{2} \right)_k = -\frac{1}{2^{2k-1}} \frac{(2k-2)!}{(k-1)!}$$

produce

$$C_n^{-1/2}(z) = \frac{1}{2^{n-1}} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^{k+1}}{n-k} \binom{2n-2k-2}{n-k-1} \binom{n-k}{k} z^{n-2k}. \tag{6.26}$$

The result now follows from (6.24). ∎

7. The generalization of the numbers a_n

The terms forming the original suggestion of Zeilberger

$$a_n = \frac{2A_n}{C_n} \tag{7.1}$$

have been given a probabilistic interpretation: let X be a random variable with a symmetric beta distribution function with parameter $\mu = 1$ given explicitly in (5.1). The numerator A_n is

$$A_n = (-1)^{n+1} \kappa(2n) 2^{2n} \tag{7.2}$$

where $\kappa(2n)$ is the even-order cumulant of the scaled random variable $X_* = 2X$. The denominator C_n is interpreted as the even-order moment of X_*:

$$C_n = \mathbb{E}\left[X_*^{2n} \right]. \tag{7.3}$$

These notions are used now to define an extension of the coefficients a_n.

Definition 7.1. Let X be a random variable with vanishing odd cumulants. The numbers $a_n(\mu)$ are defined by

$$a_n(\mu) = \frac{2(-1)^{n+1}\kappa(2n)}{\mathbb{E}\left[X_*^{2n}\right]} \tag{7.4}$$

In the special case $X_* = 2X$ with $X \sim f_\mu$, these numbers are computed using the cumulants

$$\kappa_\mu(2n) = (-1)^{n+1}2^{2n+1}(2n-1)!\zeta_\mu(2n) \tag{7.5}$$

and the even order moments

$$\mathbb{E}\left[X_*^{2n}\right] = \frac{(2n)!}{n!}\frac{1}{(\mu+1)_n} \tag{7.6}$$

to produce

$$a_n(\mu) = 2^{2n+1}\,(n-1)!\,(\mu+1)_n\,\zeta_\mu(2n). \tag{7.7}$$

The value

$$\zeta_\mu(2) = \frac{1}{4(\mu+1)} \tag{7.8}$$

yields the initial condition $a_1(\mu) = 2$.

The recurrence (5.10) now provides the next result. Recall that when x is not necessarily a positive integer, the binomial coefficient is given by

$$\binom{x}{k} = \frac{\Gamma(x+1)}{\Gamma(x-k+1)\,k!}. \tag{7.9}$$

Proposition 7.2. The coefficients $a_n(\mu)$ satisfy the recurrence

$$a_n(\mu) = \frac{1}{2\binom{n+\mu-1}{n-1}}\sum_{k=1}^{n-1}\binom{n+\mu-1}{n-k-1}\binom{n+\mu-1}{k-1}a_k(\mu)a_{n-k}(\mu), \tag{7.10}$$

with initial condition $a_1(\mu) = 2$.

Proof. Start with the convolution identity for Bessel zeta functions (5.10) and replace each zeta function by its expression in terms of $a_n(\mu)$ from (7.7), which gives

$$(n+\mu)\frac{a_n(\mu)}{2^{2n+1}(n-1)!(\mu+1)_n} = \sum_{k=1}^{n-1}\frac{a_k(\mu)}{2^{2k+1}(k-1)!(\mu+1)_k}\,\frac{a_{n-k}(\mu)}{2^{2n-2k+1}(n-k-1)!(\mu+1)_{n-k}}$$

and after simplification

$$a_n(\mu) = \frac{1}{2(n+\mu)}\sum_{k=1}^{n-1}\frac{(n-1)!}{(k-1)!(n-k-1)!}\frac{(\mu+1)_n}{(\mu+1)_k(\mu+1)_{n-k}}a_k(\mu)a_{n-k}(\mu).$$

The resut now follows by elementary algebra. □

Note 7.3. In the case $\mu = 1$, the recurrence (7.10) becomes (2.10) and the coefficients $a_n(1)$ are the original numbers a_n.

Note 7.4. The recurrence (7.10) can be written as

$$a_n(\mu) = \frac{1}{2}\sum_{k=1}^{n-1} \frac{\Gamma(n)\Gamma(\mu+1)\Gamma(n+\mu)}{\Gamma(\mu+k+1)\Gamma(n+\mu-k+1)\Gamma(n-k)\Gamma(k)} a_k(\mu)a_{n-k}(\mu).$$

Theorem 7.5. The coefficients $a_n(\mu)$ are positive and increasing for $n \geq \lfloor \frac{\mu+3}{2} \rfloor$.

Proof. The positivity is clear from (7.7). Now take the terms corresponding to $k=1$ and $k=n-1$ in (7.10) to obtain

$$a_n(\mu) \geq \frac{n-1}{\mu+1} a_1(\mu)a_{n-1}(\mu) = \frac{2(n-1)}{\mu+1} a_{n-1}(\mu). \tag{7.11}$$

This yields

$$a_n(\mu) - a_{n-1}(\mu) \geq \frac{2n-3-\mu}{\mu+1} a_{n-1}(\mu) \tag{7.12}$$

and the result follows. □

Some other special cases are considered next.

The case $\mu = 0$. In this situation the distribution is the arcsine distribution given by

$$f_0(x) = \begin{cases} \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}}, & \text{for } |x| \leq 1 \\ 0, & \text{otherwise.} \end{cases} \tag{7.13}$$

By the recurrence on the ζ_0 function, the coefficients

$$a_n(0) = 2^{2n}(n-1)!\, n!\, \zeta_0(2n) \tag{7.14}$$

satisfy the recurrence

$$a_n(0) = \frac{1}{2}\sum_{k=1}^{n-1} \binom{n-1}{k}\binom{n-1}{k-1} a_k(0)a_{n-k}(0) \tag{7.15}$$

with $a_1(0) = 2$. Now define as Lasalle $b_n = \frac{1}{2}a_n(0)$ and then (7.15) becomes

$$\begin{aligned} b_n &= \sum_{k=1}^{n-1} \binom{n-1}{k}\binom{n-1}{k-1} b_k b_{n-k}, \\ b_1 &= 1. \end{aligned} \tag{7.16}$$

In particular b_n is a positive integer.

The following comments are obtained by an analysis similar to that for a_n.

Note 7.6. The recurrence

$$\sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j}\binom{n-1}{j-1} b_j = 1$$

gives the generating function

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1} b_j}{j!} \frac{x^{2j-2}}{(j-1)!} = \frac{I_1(2x)}{x\, I_0(2x)} = \frac{1}{2x}\frac{d}{dx}\log I_0(2x).$$

Note 7.7. The sequence b_n admits a determinant representation $b_n = \det(M_n)$, where

$$M_n = \begin{pmatrix} 1 & \binom{1}{1}\binom{1-1}{1-1} & 0 & 0 & \cdots & 0 \\ 1 & \binom{2}{1}\binom{2-1}{1-1} & \binom{2}{2}\binom{2-1}{2-1} & 0 & \cdots & 0 \\ 1 & \binom{3}{1}\binom{3-1}{1-1} & \binom{3}{2}\binom{3-1}{2-1} & \binom{3}{3}\binom{3-1}{3-1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \binom{n}{1}\binom{n-1}{1-1} & \binom{n}{2}\binom{n-1}{2-1} & \binom{n}{3}\binom{n-1}{3-1} & \cdots & \binom{n}{n-1}\binom{n-1}{n-2} \end{pmatrix} \tag{7.17}$$

Note 7.8. The identity $I_2(x) = I_0(x) - \frac{2}{x} I_1(x)$ is expressed as

$$\frac{I_1(2x)}{xI_0(2x)} \left[1 + \frac{1}{2} x^2 \frac{2 I_2(2x)}{x I_1(2x)} \right] = 1 \tag{7.18}$$

provides the relation

$$b_n = \frac{1}{2} \sum_{j=1}^{n-1} \binom{n-1}{j} \binom{n}{j-1} b_j a_{n-j}. \tag{7.19}$$

The case $\mu = \frac{1}{2}$. In this situation the distribution is the uniform distribution on $[-1,1]$ with even moments

$$\mathbb{E} X_*^{2n} = \frac{2^{2n}}{2n+1} \tag{7.20}$$

and vanishing odd moments. The sequence of cumulants is

$$\kappa_{1/2}(2n) = 2(-1)^{n+1}(2n-1)!\, \zeta_{1/2}(2n) \tag{7.21}$$

where the Bessel zeta function is

$$\zeta_{1/2}(2n) = \sum_{k=1}^{\infty} \frac{1}{\pi^{2n} \, k^{2n}} = \frac{1}{\pi^{2n}} \zeta(2n) = \frac{2^{2n-1}}{(2n)!} |B_{2n}|, \tag{7.22}$$

where B_n are the Bernoulli numbers. This follows from the identity

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x. \tag{7.23}$$

This yields

$$\kappa_{1/2}(2n) = 2^{2n} \frac{B_{2n}}{2n} \text{ and } \kappa_{1/2}(2n+1) = 0, \tag{7.24}$$

with $\kappa_{1/2}(0) = 0$. These are the coefficients of $u^n/n!$ in the cumulant moment generating function

$$\log \varphi_{1/2}(u) = \log \frac{\sinh u}{u} = \frac{1}{6} u^2 - \frac{1}{180} u^4 + \frac{1}{2835} u^6 + \cdots . \tag{7.25}$$

Finally, the corresponding sequence

$$a_n\left(\tfrac{1}{2}\right) = \frac{2(-1)^{n+1} \kappa(2n)}{\mathbb{E}\left[X_*^{2n}\right]} \tag{7.26}$$

is given by

$$a_n\left(\tfrac{1}{2}\right) = 2^{2n} \frac{2n+1}{n} |B_{2n}|. \tag{7.27}$$

The first few terms are

$$a_1\left(\tfrac{1}{2}\right)=2,\ a_2\left(\tfrac{1}{2}\right)=\frac{4}{3},\ a_3\left(\tfrac{1}{2}\right)=\frac{32}{9},\ a_4\left(\tfrac{1}{2}\right)=\frac{96}{5},\ a_5\left(\tfrac{1}{2}\right)=\frac{512}{3}, \tag{7.28}$$

as expected, this is an increasing sequence for $n \geq 3$. The convolution identity (5.10) for Bessel zeta functions gives the well-known quadratic relation for the Bernoulli numbers

$$\sum_{k=1}^{n-1}\binom{2n}{2k}B_{2k}B_{2n-2k}=-(2n+1)B_{2n}, \quad \text{for } n>1. \tag{7.29}$$

Moreover, the moment-cumulants relation (1.19) gives, replacing n by $2n$ and after simplification, the other well-known identity

$$\sum_{j=1}^{n}\binom{2n+1}{2j}2^{2j}B_{2j}=2n, \text{ for } n \geq 1. \tag{7.30}$$

Note 7.9. The generating function of the sequence $a_n\left(\frac{1}{2}\right)$ is given by

$$\frac{I_{3/2}(x)}{xI_{1/2}(x)}=\frac{x\tanh x-1}{x^2}=\sum_{j=1}^{\infty}\frac{(-1)^{j-1}2a_j\left(\frac{1}{2}\right)}{(2j+1)(2j-1)!}x^{2j-2}.$$

The limiting case $\mu=-\frac{1}{2}$ has the probability distribution

$$f_{-1/2}(x)=\frac{1}{2}\delta(x-1)+\frac{1}{2}\delta(x+1) \tag{7.31}$$

(the discrete Rademacher distribution). For a Rademacher random variable X, the odd moments of $X_*=2X$ vanish while the even order moments are

$$\mathbb{E}\left[X_*^{2n}\right]=2^{2n}. \tag{7.32}$$

Therefore

$$\kappa_{-1/2}(2n)=(-1)^{n+1}2^{2n+1}(2n-1)!\,\zeta_{-1/2}(2n). \tag{7.33}$$

The identity

$$J_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\cos x \tag{7.34}$$

shows that $j_{k,-1/2}=(2k-1)\pi/2$ and therefore

$$\zeta_{-1/2}(2n)=\sum_{k=1}^{\infty}\frac{2^{2n}}{\pi^{2n}(2k-1)^{2n}}=\frac{2^{2n}-1}{\pi^{2n}}\zeta(2n). \tag{7.35}$$

The expression for $\kappa_{-1/2}(2n)$ may be simplified by the relation

$$E_n=-\frac{2}{n+1}(2^{n+1}-1)B_{n+1} \tag{7.36}$$

between the Euler numbers E_n and the Bernoulli numbers. It follows that

$$\kappa_{-1/2}(2n)=-2^{4n-1}E_{2n-1}. \tag{7.37}$$

The corresponding sequence $a_n\left(-\frac{1}{2}\right)$ is now given by

$$a_n\left(-\tfrac{1}{2}\right)=(-1)^n2^{2n}E_{2n-1} \tag{7.38}$$

and its first few values are

$$a_1\left(-\tfrac{1}{2}\right)=2,\ a_2\left(-\tfrac{1}{2}\right)=4,\ a_3\left(-\tfrac{1}{2}\right)=32,\ a_4\left(-\tfrac{1}{2}\right)=544,\ a_5\left(-\tfrac{1}{2}\right)=15872,$$

Note 7.10. The generating function of the sequence $a_n\left(-\frac{1}{2}\right)$ is given by

$$\frac{I_{1/2}(x)}{xI_{-1/2}(x)} = \frac{\tanh x}{x} = \sum_{j=1}^{\infty} \frac{(-1)^{j-1} 2a_j\left(-\frac{1}{2}\right)}{(2j-1)!} x^{2j-2}.$$

Note 7.11. The convolution identity (5.10) yields the well-known quadratic recurrence relation

$$\sum_{k=1}^{n-1} \binom{2n-2}{2k-1} E_{2k-1} E_{2n-2k-1} = 2E_{2n-1}, \text{ for } n > 1, \tag{7.39}$$

and the moment-cumulant relation (1.19) gives the other well-known identity

$$\sum_{k=1}^{n} \binom{2n-1}{2k-1} 2^{2k-1} E_{2k-1} = 1, \text{ for } n \geq 1. \tag{7.40}$$

8. Some arithmetic properties of the sequences a_n and b_n

Given a sequence of integers $\{x_n\}$ it is often interesting to examine its arithmetic properties. For instance, given a prime p, this is measured by the p-adic valuation $\nu_p(x_n)$, defined as the largest power of p that divides x_n. Examples of this process appear in [2] for the Stirling numbers and in [1, 22] for a sequence of coefficients arising from a definite integral.

The statements described below give information about $\nu_p(a_n)$. These results will be presented in a future publication. M. Lasalle [19] established the next theorem by showing that A_n and C_n have the same parity. The fact that the Catalan numbers are odd if and only if $n = 2^r - 1$ for some $r \geq 2$ provides the proof. This result appears in [14, 18].

Theorem 8.1. The integer a_n is odd if and only if $n = 2(2^m - 1)$.

The previous statement may be expressed in terms of the sequence of binary digits of n.

Experimental Fact 8.2. Let $B(n)$ be the binary digits of n and denote $\bar{x}$ a sequence of a arbitrary length consisting of the repetitions of the symbol x. The following statements hold (experimentally)
1) $\nu_2(a_n) = 0$ if and only if $B(n) = \{\bar{1}, 0\}$.
2) $\nu_2(a_n) = 1$ if and only if $B(n) = \{\bar{1}\}$ or $\{1, \bar{0}\}$.
3) $\nu_2(a_n) = 2$ if and only if $B(n) = \{1, 0, \bar{1}, 0\}$.

The experimental findings for the prime $p = 3$ are described next.

Experimental Fact 8.3. Suppose n is not of the form $3^m - 1$. Then

$$\nu_3(a_{3n-2}) = \nu_3(a_{3n-1}) = \nu_3(a_{3n}). \tag{8.1}$$

Define $w_j = 3^j - 1$. Suppose n lies in the interval $w_j + 1 \leq n \leq w_{j+1} - 1$. Then

$$\nu_3(a_{3n+2}) = j - \nu_3(n+1). \tag{8.2}$$

If $n = w_j$, then $\nu_3(a_{3n}) = 0$.

Now assume that $n = 3^m - 1$. Then

$$\nu_3(a_{3n}) = \nu_3(a_{3n-1}) - 1 = \nu_3(a_{3n-2}) - 1 = m. \tag{8.3}$$

Experimental Fact 8.4. The last observation deals with the sequence $\{a_n(\mu)\}$. Consider it now as defined by the recurrence (7.10). The initial condition $a_1(\mu) = 2$, motivated by the origin of the sequence, in general does not provide integer entries. For example, if $\mu = 2$, the sequence is

$$\left\{2, \frac{2}{3}, \frac{8}{9}, \frac{7}{3}, \frac{88}{9}, \frac{1594}{27}, \frac{1448}{3}\right\},$$

and for $\mu = 3$

$$\left\{2, \frac{1}{2}, \frac{1}{2}, \frac{39}{40}, 3, \frac{263}{20}, \frac{309}{4}\right\}.$$

Observe that the denominators of the sequence for $\mu = 2$ are always powers of 3, but for $\mu = 3$ the arithmetic nature of the denominators is harder to predict. On the other hand if in the case $\mu = 3$ the initial condition is replaced by $a_1(3) = 4$, then the resulting sequence has denominators that are powers of 5. This motivates the next definition.

Definition 8.5. Let x_n be a sequence of rational numbers and p be a prime. The sequence is called p-integral if the denominator of x_n is a power of p.

Therefore if $a_1(3) = 4$, then the sequence $a_n(3)$ is 5-integral. The same phenomena appeas for other values of μ, the data is summarized in the next table.

μ	2	3	4	5	6	7	8
$a_1(\mu)$	2	4	10	12	84	264	990
p	3	5	7	7	11	11	13

Note 8.6. The sequence $\{2, 4, 10, 12, 84, 264, 990\}$ does not appear in Sloane's sequences list OEIS.

This suggests the next conjecture.

Conjecture 8.7. Let $\mu \in \mathbb{N}$. Then there exists an initial condition $a_1(\mu)$ and a prime p such that the sequence $a_n(\mu)$ is p-integral.

Some elementary arithmetical properties of a_n are discussed next. A classical result of E. Lucas states that a prime p divides the binomial coefficient $\binom{a}{b}$ if and only if at least one of the base p digits of b is greater than the corresponding digit of a.

Proposition 8.8. Assume n is odd. Then a_n is even.

Proof. Let $n = 2m + 1$. The recurrence (2.10) gives

$$\begin{aligned} 2(2m+1)a_{2m+1} &= \sum_{k=1}^{2m} \binom{2m+1}{k-1}\binom{2m+1}{k+1} a_k a_{2m+1-k} \\ &= 2\sum_{k=1}^{m} \binom{2m+1}{k-1}\binom{2m+1}{k+1} a_k a_{2m+1-k}. \end{aligned}$$

For k in the range $1 \leq k \leq m$, one of the indices k or $2m + 1 - k$ is odd. The induction argument shows that for each such k, either a_k or a_{2m+1-k} is an even integer. This completes the argument. □

Lemma 8.9. Assume $n = 2^m - 1$. Then $\frac{1}{2}a_n$ is an odd integer.

Proof. Proposition 8.8 shows that $\frac{1}{2}a_n$ is an integer. The relation (1.8) may be written as

$$(-1)^{n-1}a_n = 2 + \frac{1}{n}\sum_{j=1}^{n-1}(-1)^j\binom{n}{j-1}\binom{n+1}{j+1}a_j. \tag{8.4}$$

This implies

$$n\left[(-1)^{n-1}\tfrac{1}{2}a_n - 1\right] = \frac{1}{2}\sum_{j=1}^{n-1}(-1)^j\binom{n}{j-1}\binom{n+1}{j+1}a_j. \tag{8.5}$$

Observe that if j is odd, then a_j is even and $\binom{n+1}{j+1}$ is also even. Therefore the corresponding term in the sum is divisible by 4. If j is even, then Lucas's theorem shows that 4 divides $\binom{n+1}{j+1}$. It follows that the right hand side is an even number. This implies that $\frac{1}{2}a_n$ is odd, as claimed. □

The next statement, which provides the easier part of Theorem 8.1, describes the indices that produce odd values of a_n.

Theorem 8.10. If $n = 2(2^m - 1)$, then a_n is odd.

Proof. Isolate the term $j = n/2$ in the identity (8.4) to produce

$$\begin{aligned}\left[(-1)^n a_n + 2\right](2^m - 1) &= \binom{2^{m+1}-2}{2^m-2}\binom{2^{m+1}-1}{2^m}\tfrac{1}{2}a_{n/2} \\ &+ \frac{1}{2}\sum_{j\neq n/2}(-1)^j\binom{n}{j-1}\binom{n+1}{j+1}a_j.\end{aligned}$$

Lemma 8.9 shows that $\frac{1}{2}a_{n/2}$ is odd and the binomial coefficients on the first term of the right-hand side are also odd by Lucas' theorem. Each term of the sum is even because a_j is even if j is odd and for j even $\binom{n}{j-1}$ is even. Therefore the entire right-hand side is even which forces a_n to be odd. □

The final result discussed here deals with the parity of the sequence b_n. The main tool is the recurrence

$$b_n = \sum_{k=1}^{n-1}\binom{n-1}{k}\binom{n-1}{k-1}b_k b_{n-k} \tag{8.6}$$

with $b_1 = 1$. Observe that the binomial coefficients appearing in this recurrence are related to the Narayana numbers $N(n,k)$ (1.2) by

$$\binom{n-1}{k}\binom{n-1}{k-1} = (n-1)N(n-1,k-1). \tag{8.7}$$

Arithmetic properties of the Narayana numbers have been discussed by M. Bona and B. Sagan [5]. It is established that if $n = 2^m - 1$ then $N(n,k)$ is odd for $0 \leq k \leq n-1$; while if $n = 2^m$ then $N(n,k)$ is even for $1 \leq k \leq n-2$.

The next theorem is the analog of M. Lasalle's result for the sequence b_n.

Theorem 8.11. The coefficient b_n is an odd integer if and only if $n = 2^m$, for some $m \geq 0$.

Proof. The first few terms $b_1 = 1$, $b_2 = 1$, $b_3 = 4$ support the base case of an inductive proof.

If n is odd, then

$$b_n = (n-1)\sum_{k=1}^{n-1} N(n-1,k-1) b_k b_{n-k} \tag{8.8}$$

shows that b_n is even.

Consider now the case $n = 2^m$. Then Lucas' theorem shows that $\binom{2^m-1}{k}\binom{2^m-1}{k-1}$ is odd for all k. The inductive step states that b_k is even if $k \neq 2^r$. In the case $k = 2^r$, then b_{n-k} is odd if and only if $k = 2^{m-1}$, in which case all the terms in (8.8) are even with the single expection $\binom{2^m-1}{2^{m-1}}\binom{2^m-1}{2^{m-1}-1}b_{2^{m-1}}^2$. This shows that b_n is odd.

Finally, if $n = 2j$ is even with $j \neq 2^r$, then

$$b_n = \binom{2j-1}{j}\binom{2j-1}{j-1}b_j^2 + 2\sum_{k=1}^{j-1}\binom{n-1}{k}\binom{n-1}{k-1}b_k b_{n-k}. \tag{8.9}$$

Now simply observe that $j \neq 2^r$, therefore b_j is even by induction. It follows that b_n itself is even.

This completes the proof. □

9. One final question

Sequences of combinatorial origin often turn out to be unimodal or logconcave. Recall that a sequence $\{x_j : 1 \leq j \leq n\}$ is called *unimodal* if there is an index m_* such that $x_1 \leq x_2 \leq \cdots \leq x_{m_*}$ and $x_{m_*+1} \geq x_{m_*+2} \geq \cdots \geq x_n$. The sequence is called *logconcave* if $x_{n+1}x_{n-1} \geq x_n^2$. An elementary argument shows that a logconcave sequence is always unimodal. The reader will find in [4, 6, 7, 8, 9, 10, 23, 26] a variety of examples of these type of sequences.

Conjecture 9.1. The sequences $\{a_n\}$ and $\{b_n\}$ are logconcave.

Acknowledgements. The work of the second author was partially supported by nsf-dms 0070567.

References

[1] T. Amdeberhan, D. Manna, and V. Moll. The 2-adic valuation of a sequence arising from a rational integral. *Jour. Comb. A*, 115:1474–1486, 2008.

[2] T. Amdeberhan, D. Manna, and V. Moll. The 2-adic valuation of Stirling numbers. *Experimental Mathematics*, 17:69–82, 2008.

[3] G. E. Andrews, R. Askey, and R. Roy. *Special Functions*, volume 71 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, New York, 1999.

[4] M. Bona. A combinatorial proof of the logconcavity of a famous sequence counting permutations. *Elec. Jour. Comb.*, 11:1–4, 2004.

[5] M. Bona and B. Sagan. On divisibility of Narayana numbers by primes. *Jour. Int. Seq.*, 8:#05.2.4, 2005.

[6] G. Boros and V. Moll. A criterion for unimodality. *Elec. Jour. Comb.*, 6:1–6, 1999.

[7] F. Brenti. Log-concave and unimodal sequences in Algebra, Combinatorics and Geometry: an update. *Contemporary Mathematics*, 178:71–89, 1994.

[8] L. M. Butler. A unimodality result in the enumeration of subgroups of a finite abelian group. *Proc. Amer. Math. Soc.*, 101:771–775, 1987.

[9] L. M. Butler. The q-logconcavity of q-binomial coefficients. *J. Comb. Theory, Ser. A*, 54:54–63, 1990.

[10] J. Y. Choi and J. D. H. Smith. On the unimodality and combinatorics of Bessel numbers. *Disc. Math.*, 264:45–53, 2003.

[11] D. S. Ciesielski and S. J. Taylor. First passage times and sojourn times for Brownian motion in space and the exact Hausdorff measure of the sample path. *Trans. Amer. Math. Soc.*, 103:434–450, 1962.

[12] C. Coker. Enumerating a class of lattice paths. *Disc. Math.*, 271:13–28, 2003.

[13] E. Di Nardo, P. Petrullo, and D. Senato. Cumulants and convolutions via Abel polynomials. *Europ. Journal of Comb.*, 31:1792–1804, 2010.

[14] O. Egecioglu. The parity of the Catalan numbers via lattice paths. *Fib. Quart.*, 21:65–66, 1983.

[15] E. Elizalde, S. Leseduarte, and A. Romeo. Sum rules for zeros of Bessel functions and an application to spherical Aharonov-Bohm quantum bags. *J. Phys. A: Math. Gen.*, 26:2409–2419, 1993.

[16] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series, and Products*. Edited by A. Jeffrey and D. Zwillinger. Academic Press, New York, 7th edition, 2007.

[17] B. Jessen and Wintner. A. Distribution functions and the Riemann zeta function. *Trans. Amer. Math. Soc.*, 38:48–88, 1935.

[18] T. Koshy and M. Salmassi. Parity and primality of Catalan numbers. *Coll. Math. J.*, 37:52–53, 2006.

[19] M. Lasalle. Two integer sequences related to Catalan numbers. *J. Comb. Theory Ser. A*, 119:923–935, 2012.

[20] V. P. Leonov and A. N. Shiryaev. On a method of calculation of semi-invariants. *Theory Prob. Appl.*, IV-3:312–328, 1959.

[21] T. Mansour and Y. Sun. Identities involving Narayana polynomials and Catalan numbers. *Disc. Math.*, 309:4079–4088, 2009.

[22] V. Moll and X. Sun. A binary tree representation for the 2-adic valuation of a sequence arising from a rational integral. *INTEGERS*, 10:211–222, 2010.

[23] B. Sagan. Inductive proofs of q-logconcavity. *Disc. Math.*, 99:289–306, 1992.

[24] P. J. Smith. A recursive formulation of the old problem of obtaining moments from cumulants and viceversa. *The American Statistician*, 49:217–218, 1995.

[25] F. Spitzer. On a class of random variables. *Proc. Amer. Math. Soc.*, 6:494–505, 1955.

[26] R. Stanley. Log-concave and unimodal sequences in Algebra, Combinatorics and Geometry. graph theory and its applications: East and West (Jinan, 1986). *Ann. New York Acad. Sci.*, 576:500–535, 1989.

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