

# Serial Symmetrical Relocation Algorithm for the Equal Sphere

## Packing Problem<sup>†</sup>

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### Abstract

For dealing with the equal sphere packing problem, we propose a serial symmetrical relocation algorithm, which is effective in terms of the quality of the numerical results. We have densely packed up to 200 equal spheres in spherical container and up to 150 equal spheres in cube container. All results are rigorous because of a fake sphere trick. It was conjectured impossible to pack 68 equal spheres of radius 1 into a sphere of radius 5. The serial symmetrical relocation algorithm has proven wrong this conjecture by finding one such packing.

**Key words:** Sphere packing problem; heuristic; serial symmetrical relocation strategy

### Introduction

How to densely pack equal spheres inside a bounded container is a problem of both theoretical and practical value. Although the problem of packing equal spheres in infinite space has been solved by (Hales, 2000), the problem of packing equal spheres inside a bounded container remains unsolved. Besides its theoretical value, it is also of practical value, because in practice all containers are bounded.

The problem of packing equal spheres in spherical containers can be used to model the treat plan of the Gamma Knife (Wang, 1999). The Gamma Knife consists of 201 cobalt-60 radioactive sources and provides an advanced approach to the

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<sup>†</sup>This work is supported by the National Natural Science Foundation of China under Grant No.61070235; the National Natural Science Foundation of China under Grant No.61173180.

treatment of tumor and vascular malformations within the head. Each radioactive source emits a gamma ray. A certain number of gamma rays of the same diameter are designedly pointed at one point in space. At this point, a spherical region (called a shot) of high radiation dose is formed. The tumor can be viewed as an approximate spherical container. All shots can be viewed as ideal equal spheres which shall be packed inside the container. No shot is allowed to extend outside the container so that no normal tissue shall be affected. No two shots should overlap with each other so that no overlapped part with too high dose shall occur in the tumor. Any good method dealing with the problem of packing equal spheres inside a spherical container is conducive to improving the treat plan of the Gamma Knife.

As to packing equal spheres in a cube, it has a usual application in packaging industry.

By means of a max-min optimization approach, (Maranas et al, 1995) dealt with the problem of packing  $n$  equal circles in a square and obtained the quasi optimal solutions for  $n \leq 30$ . (Graham et al, 1998) used the billiard simulation to find out the quasi optimal packings of up to 65 equal circles in a circle. Because the equal sphere packing problem can be viewed as a 3D variant of the equal circle packing problem, we also paid due attention to the methods for packing equal circles in a bounded container.

(Gensane, 2004) has presented his best packings of up to 32 equal spheres in a cube by means of an adaptation of the billiard algorithm proposed by (Lubachevsky, 1991). (Hifi and M'Hallah, 2009) made an extensive review on the circle and sphere packing problems.

(Huang and Xu, 1999), (Huang and Kang, 2004) and (Huang et al, 2005) used the quasi physical (shortly QP) method as the local solver for packing equal or unequal circles inside a large circle. The global search strategy of (Huang and Kang, 2004) is to randomly redistribute all items when the calculation is trapped in a local optimum. The global search strategy of (Huang and Xu, 1999) is to pick the item suffering the largest total overlap and randomly relocate it when trapped.

(Huang et al, 2005) applied greedy algorithms as the global search strategy for

packing unequal circles in a bounded container. Given  $n$  unequal circles, after having placed  $i (i < n)$  circles, they placed next circle by the maximum hole degree rule. The action of placing a circle of appropriate size in some certain vacant region of the container can occupy most area of this region without overlapping other circles in the container. Such placing action can produce larger hole degree than placing too large or too small circles. The greedy algorithms exploit the difference of items sizes and do not work well for equal items.

We design a serial symmetrical relocation strategy as the new global search strategy, and call our algorithm the serial symmetrical relocation algorithm.

Here we clarify two terms: configuration and packing. A configuration of  $n$  equal spheres is a set of locations of  $n$  equal spheres centers. A packing is a configuration which can meet the constraints of the equal sphere packing problem.

This paper is organized as follows. We first introduce the QP model and the QP algorithm as the local solver (denoted by A0). We propose a simple trick called the fake sphere trick in A0 to guarantee the exactness of the results. Then, we propose a serial symmetrical relocation strategy and combine it with A0 to form the serial symmetrical relocation algorithm (denoted by A1). Then, the results of up to 200 equal spheres in spherical container and up to 150 equal spheres in cube container are produced. All our results are rigorous and supported by the spheres centers locations. At last, we have made the conclusions about our work.

### The quasi physical model and method

Let  $(0,0,0)$  be the coordinates of the container center (container radius  $r_0$ ) and  $X$  be the coordinates of  $n$  equal spheres centers (sphere radius  $r$ ), where  $X = \{X_1, \dots, X_n\} = \{x_1, y_1, z_1, \dots, x_n, y_n, z_n\}$ .

We represent  $\sqrt{x_i^2 + y_i^2 + z_i^2}$  by  $|X_i|$ ,  $\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}$  by  $|X_i - X_j|$ . The constraints of the problem of packing  $n$  equal spheres in spherical container are:

$$|X_i| \leq r_0 - r, i = 1 \dots n \quad . \quad (1)$$

$$|X_i - X_j| \geq 2r, i, j = 1 \dots n, j \neq i \quad . \quad (2)$$

A packing is a configuration  $X$  that satisfies constraints (1) and (2).

Viewing all equal spheres as light smooth elastic solids inside a rigid container, we use  $d_{i0}$  to represent the  $i^{th}$  sphere's deformation caused by the container (denoted by the  $0^{th}$  object).  $\vec{d}_{i0}$  represents the elastic repulsion force the  $i^{th}$  sphere suffers for  $d_{i0}$ . Its direction is from the  $i^{th}$  sphere center to the container center.

$$d_{i0} = \left| \vec{d}_{i0} \right| = \begin{cases} |X_i| + r - r0, & \text{if } |X_i| + r > r0 \\ 0, & \text{else} \end{cases} . \quad (3)$$

$(i = 1, \dots, n)$

We use  $d_{ij}$  to represent the  $i^{th}$  sphere's deformation caused by the  $j^{th}$  sphere.  $\vec{d}_{ij}$  represents the elastic repulsion force the  $i^{th}$  sphere suffers for  $d_{ij}$ . Its direction is from the  $j^{th}$  sphere center to the  $i^{th}$  sphere center.

$$d_{ij} = \left| \vec{d}_{ij} \right| = \begin{cases} \frac{1}{2} \times (2r - |X_i - X_j|), & \text{if } |X_i - X_j| < 2r \\ 0, & \text{else} \end{cases} . \quad (4)$$

$(i, j = 1 \dots n, j \neq i)$

We define the potential energy of the  $i^{th}$  sphere as  $u_i = \sum d_{ij}^2$  ( $j = 0, \dots, n, j \neq i$ ).

Thus, the total potential energy of all equal spheres is defined as:

$$U(X, r, r0) = \sum_{i=1}^n u_i = \sum_{i=1}^n d_{i0}^2 + \sum_{i=1}^n \sum_{j=1, j \neq i}^n d_{ij}^2 . \quad (5)$$

We call  $U(X, r, r0)$  as the potential energy function of  $X$ .  $U(X, r, r0)$  can depict the deformations of all  $n$  equal spheres.

Given an initial configuration  $X$ , we use the quasi physical algorithm (Huang and Kang, 2004) as the local solver to locally minimize  $U(X, r, r0)$ . This local solver here is denoted as A0. A0 is a deterministic algorithm. Its basic idea is to simulate the elastic movement of light smooth and elastic spheres jammed in a bounded container. Such moving process is a natural locally minimizing process of the deformations of all spheres.

If  $U(X, r, r0)$ , which is the square sum of all spheres deformations, is reduced to zero, a packing  $X$  of equal spheres of radius  $r$  inside the container of radius  $r0$  is obtained.

Because of the limitation of modern digital computers, we consider  $U(X,r,r0)$  has approximately reached zero when it is locally minimized to less than  $10^{-16}$ . That means the corresponding  $X$  of equal spheres of radius  $r$  still has slight deformations. To get exact packings by the local solver A0, we propose a fake sphere trick.

Without loss of generality, we take standard equal sphere radius as 0.5. In our experiments, we use fake sphere of radius  $0.5+10^{-8}$  instead of standard sphere.

**Theorem**  $U(X,0.5+10^{-8},r0)<10^{-16} \Rightarrow U(X,0.5,r0)=0$ .

**Proof:**

$$U(X,0.5+10^{-8},r0)=\sum_{i=1}^n d_{i0}^2 + \sum_{i=1}^n \sum_{j=1, j \neq i}^n d_{ij}^2 < 10^{-16} \Rightarrow$$

$$d_{i0}^2 < 10^{-16} \text{ and } d_{i0} < 10^{-8} \text{ (} i=1, \dots, n \text{), while } r=0.5+10^{-8} \Rightarrow$$

$$d_{i0} = \begin{cases} \sqrt{x_i^2 + y_i^2 + z_i^2} + 0.5 + 10^{-8} - r0, & \text{if } \sqrt{x_i^2 + y_i^2 + z_i^2} + 0.5 + 10^{-8} > r0 \\ 0, & \text{else} \end{cases} < 10^{-8} \Rightarrow$$

$$(i=1, \dots, n)$$

$$\sqrt{x_i^2 + y_i^2 + z_i^2} + 0.5 + 10^{-8} - r0 < 10^{-8}, i=1, \dots, n \Rightarrow$$

$$\sqrt{x_i^2 + y_i^2 + z_i^2} + 0.5 < r0, i=1, \dots, n \Rightarrow$$

$$d_{i0}=0 \text{ (} i=1, \dots, n \text{), while } r=0.5;$$

$$U(X,0.5+10^{-8},r0)=\sum_{i=1}^n d_{i0}^2 + \sum_{i=1}^n \sum_{j=1, j \neq i}^n d_{ij}^2 < 10^{-16} \Rightarrow$$

$$d_{ij}^2 < 10^{-16} \text{ and } d_{ij} < 10^{-8} \text{ (} i=1, \dots, n; j=1, \dots, i-1, i+1, \dots, n \text{), while } r=0.5+10^{-8} \Rightarrow$$

$$d_{ij} = \begin{cases} \frac{1}{2} \times (2 \times (0.5+10^{-8}) - \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}), & \text{if } \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2} < 2 \times (0.5+10^{-8}) \\ 0, & \text{else} \end{cases} < 10^{-8} \Rightarrow$$

$$(i, j=1..n, j \neq i)$$

$$\frac{1}{2} \times (2 \times (0.5+10^{-8}) - \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}) < 10^{-8} \Rightarrow$$

$$\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2} > 2 \times 0.5 \Rightarrow$$

$$d_{ij}=0 \text{ (} i=1, \dots, n; j=1, \dots, n; j \neq i \text{), while } r=0.5;$$

Because  $d_{i0}=0$  and  $d_{ij}=0$  ( $i=1, \dots, n; j=1, \dots, n; j \neq i$ ) while  $r=0.5$ ,  $U(X,0.5,r0)=0$ .

Thus, a fake sphere configuration  $X$  whose  $U(X,0.5+10^{-8},r0)<10^{-16}$  is an exact

packing  $X$  of standard spheres whose  $U(X, 0.5, r_0) = 0$ .

The above theorem can be illustrated in Fig.1.

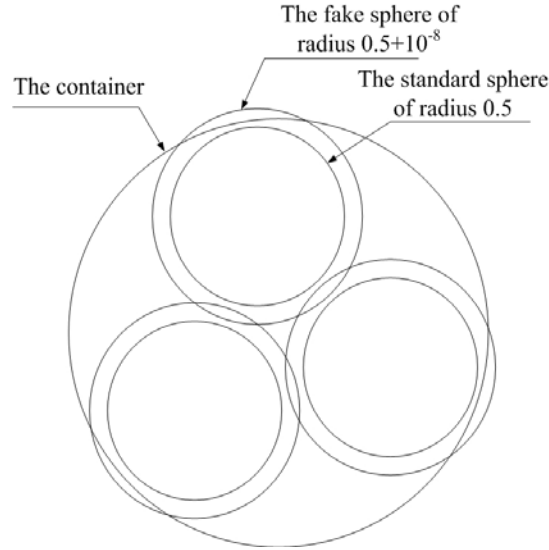


Figure 1: The fake sphere trick.

Henceforth, given a configuration  $X$  of  $n$  equal spheres with radius  $r$ , we denote the process that A0 develops  $X$  into  $X_{halt}$  inside a container of radius  $r_0$ , by  $X_{halt} \leftarrow A0(r, r_0, X)$ .

### The serial symmetrical relocation strategy

Given a local optimal configuration  $X_{local}$  of  $n$  equal spheres, any slight disturbance in it can not change its spatial structure and is unlikely to improve it. But if we move the  $i^{th}$  sphere in this configuration from its current location  $(x_i, y_i, z_i)$  to  $(-x_i, -y_i, -z_i)$  and keep other spheres unmoved, the spatial structure of  $X_{local}$  must be greatly changed.

**Definition (symmetrical relocation)** Given a configuration  $X$ , the symmetrical relocation of the  $i^{th}$  sphere of  $X$  is the action of changing the  $i^{th}$  sphere center from  $x_i, y_i, z_i$  to  $-x_i, -y_i, -z_i$ .

The following denotations are introduced to describe the serial symmetrical relocation strategy.

$X_{local}$  stands for a local optimal configuration which is viewed as a set of the locations of  $n$  equal spheres centers.

$\min U(i,L)$  stands for the  $i$  spheres with the smallest potential energies among all spheres in set  $L$ .  $L$  is a subset of  $X_{local}$ .

$\max U(i,L)$  stands for the  $i$  spheres with the largest potential energies among all spheres in set  $L$ .  $L$  is a subset of  $X_{local}$ .

$\text{invert}(L,X_{local})$  stands for the new configuration obtained by symmetrically relocating all spheres in the subset  $L$  of  $X_{local}$  and keeping all other spheres in  $X_{local}$  unmoved.

The serial symmetrical relocation strategy is to generate  $n(n-1)/2$  configurations from a local optimal configuration  $X_{local}$  of  $n$  spheres by

$$\text{invert}(\max U(j,\min U(i,X_{local})),X_{local}), 1 \leq i \leq n, 1 \leq j < i \quad (6)$$

Since the configuration  $\text{invert}(\max U(n-i,\min U(n,X_{local})),X_{local})$  is identical to the configuration  $\text{invert}(\max U(i,\min U(i,X_{local})),X_{local})$  for symmetry, the variable  $j$  ranges from 1 to  $i-1$ . The number of the new configurations obtained by (6) is  $n(n-1)/2$ .

### The serial symmetrical relocation algorithm

Here, we introduce the serial symmetrical relocation algorithm which is denoted by A1. The inputs of A1 are a roughly estimated value of container radius  $r_0$  and a randomly generated configuration. The outputs are a dense packing and its exact container radius.

The core of A1 is the heuristic strategy: serial symmetrical relocation strategy. In A1, the serial symmetrical relocation strategy each time generates  $n(n-1)/2$  new configurations from a local optimal configuration  $X_{local}$  and calls A0 to examine them one by one. The process of generating  $n(n-1)/2$  new configurations from one  $X_{local}$  and examining them by A0 is called one scan of A1. The first  $X_{local}$  which starts the first scan of A1 is obtained from a randomly generated initial configuration. Each scan of A1, except the first scan, takes the configuration with the smallest potential energy found in its previous scan as its  $X_{local}$ .

The pseudo code of A1 is given as follows:

/\* The beginning of pseudo code of A1\*/

$n \leftarrow$  the number of equal spheres;

```

r0 ← an empirically estimated value of container radius for n sphere;
r ← 0.5+10-8; /* The fake sphere trick. */

Generates one random initial configuration Xrandom of n spheres;

/* From a randomly generated configuration Xrandom, we get the first Xlocal. */
Xlocal ← A0(r,r0,Xrandom);
if (U(Xlocal,r,r0)<10-16) {
    Xfound←Xlocal;
    Goto Binary_Search; } /* A1 finds a packing.*/

l←0; /* Scan counter. */
while (l< 6) /* 6 is a manually predetermined upper limit of scan number. */
{
    for i=1 to n
    {
        for j=1 to i-1
        {
            Xnewlocal(ij) ← A0(r,r0, invert(maxU(j,minU(i,Xlocal)),Xlocal)) ;
            if (U(Xnewlocal(ij),r,r0)<10-16) {
                Xfound←Xnewlocal(ij);
                Goto Binary_Search; } /* A1 finds a packing. */
            }
        }
        Xlocal←the one with the lowest potential energy among all Xnewlocal(ij);
        l←l+1;
    } /* End of while*/
} /* If A1 doesn't directly find any packing, pick the best configuration ever found.*/
Xfound ← the local optimal configuration with the lowest potential energy ever found
during all scans;

Binary_Search:
/* Use a binary search to find the smallest container in which Xfound can just be
developed by A0 into a packing. */

```



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 $r0_{low} \leftarrow 1/2 \times r0;$ 
 $r0_{up} \leftarrow 2 \times r0;$ 
 $\varepsilon \leftarrow 10^{-12};$  /*  $\varepsilon$  is a precision. */
 $X \leftarrow X_{found};$ 
while ( $r0_{up} - r0_{low} > \varepsilon$ )
{
     $r0 \leftarrow (r0_{up} + r0_{low})/2;$ 
     $X \leftarrow A0(r, r0, X);$ 
    if ( $U(X, r, r0) < 10^{-16}$ )
         $r0_{up} \leftarrow r0;$ 
    if ( $U(X, r, r0) \geq 10^{-16}$ )
         $r0_{low} \leftarrow r0;$ 
}
 $r0_{min} \leftarrow r0_{up};$ 
/*  $r0_{min}$  is the smallest container radius for  $X_{found}$  to become a packing through A0.*/
 $X_{dense} \leftarrow A0(r, r0_{min}, X);$ 
/*  $X_{dense}$  is the corresponding dense packing of  $X_{found}$  in the container of  $r0_{min}$ .*/
/* The end of pseudo code of A1*/

```

A1 examines only  $O(n^2)$  configurations, thus it can evaluate the instances of up to 200 spheres within feasible runtime.

For each instance, we performed A1 five times and got five dense packings. We chose the best one of them as the result of the instance.

## Numerical results

We performed A1 on a personal computer with Pentium E6500 2.93GHz and 2GB DDR2 800MHZ RAM and obtained dense packings of equal spheres in spherical and cube containers. All our packings are rigorous because of the fake sphere trick.

Because of the basic nature of the equal sphere packing problem, seemingly “insignificant” improvements on the former records may imply new packings which

are unknown before. The nature of the equal sphere packing problem is analogous to that of the equal circle packing problem. (Specht, 2012) collects the best known records of the equal circle packing problem from different researchers. Among these best known records, (Buddenhagen, 2010) reduced the circumcircle radius of 64 unit circles from the former record 8.96197110850392353216121 to 8.96197110848573830216130. This is a real improvement. Thus, according to the quality of the results found by it, we assume the serial symmetrical relocation algorithm (denoted by A1) lead to substantial improvements.

Especially, the serial symmetrical relocation algorithm packed 68 equal spheres of radius 1 into a large sphere of radius  $1/0.20000222$ , and proved wrong a conjecture (Pfoertner, 2011) which alleges a large sphere of radius 5 can contain at most 67 equal spheres of radius 1.

### The results for packing equal spheres inside spherical containers

We list in Table 1 the containers sizes for our best packings in spherical containers, along with the best known records to our knowledge. The quality of spherical container is represented by the ratio  $r/r_0$ , where  $r$  is 0.5.

Table 1. Records for packing equal spheres in spherical containers.

$n$	A1 $r/r_0$	Average runtime for A1 $r/r_0$ (second)	Hugo(a) $r/r_0$	Hugo(b) $r/r_0$	Dave $r/r_0$	A1 $r/r_0$ minus Hugo(b) $r/r_0$
1	1.00000000	0.001	1.00000000	1.00000000	-	0.00000000
2	0.50000000	0.001	0.50000000	0.50000000	-	0.00000000
3	0.46410160	0.006	0.46410160	0.46410160	-	0.00000000
4	0.44948974	0.007	0.44948970 <sup>(+)</sup>	0.44948970 <sup>(+)</sup>	-	0.00000004
5	0.41421350	0.025	0.41421350	0.41421350	-	0.00000000
6	0.41421350	0.043	0.41421350	0.41421350	-	0.00000000
7	0.38591355	0.079	0.38591350 <sup>(+)</sup>	0.38591360 <sup>(-)</sup>	-	-0.00000005
8	0.37802480	0.134	0.37802480	0.37802480	-	0.00000000
9	0.36602539	1.216	0.36602530 <sup>(+)</sup>	0.36602540 <sup>(-)</sup>	-	-0.00000001
10	0.35304942	0.926	0.35304940 <sup>(+)</sup>	0.35304940 <sup>(+)</sup>	-	0.00000002
11	0.34457650	0.776	0.34457640 <sup>(+)</sup>	0.34457650	-	0.00000000
12	0.34457650	0.675	0.34457640 <sup>(+)</sup>	0.34457650	-	0.00000000
13	0.33333332	1.028	0.33333330	0.33333330	-	0.00000002

14	0.32350466	1.148	0.32350460 <sup>(+)</sup>	0.32350460 <sup>(+)</sup>	0.32331300 <sup>(+)</sup>	0.00000006
15	0.31830481	1.345	0.31830470 <sup>(+)</sup>	0.31830480 <sup>(+)</sup>	0.31830500 <sup>(-)</sup>	0.00000001
16	0.31097591	2.330	0.31097580 <sup>(+)</sup>	0.31097590 <sup>(+)</sup>	0.31097600 <sup>(-)</sup>	0.00000001
17	0.30569395	2.914	0.30569390 <sup>(+)</sup>	0.30569400 <sup>(-)</sup>	0.30569400 <sup>(-)</sup>	-0.00000005
18	0.30129658	3.016	0.30129650 <sup>(+)</sup>	0.30129650 <sup>(+)</sup>	0.30129600 <sup>(-)</sup>	0.00000008
19	0.29533232	4.635	0.29533230 <sup>(+)</sup>	0.29533230 <sup>(+)</sup>	0.29533200 <sup>(+)</sup>	0.00000002
20	0.28789082	5.678	0.28789070 <sup>(+)</sup>	0.28789070 <sup>(+)</sup>	0.28785100 <sup>(+)</sup>	0.00000012
21	0.28683281	8.321	0.28683270 <sup>(+)</sup>	0.28683270 <sup>(+)</sup>	0.28683300 <sup>(-)</sup>	0.00000011
22	0.27934262	6.916	0.27934250 <sup>(+)</sup>	0.27934250 <sup>(+)</sup>	0.27933400 <sup>(+)</sup>	0.00000012
23	0.27567069	9.199	0.27567070 <sup>(-)</sup>	0.27567070 <sup>(-)</sup>	0.27508100 <sup>(+)</sup>	-0.00000001
24	0.27134130	10.783	0.27134130	0.27134130	0.27133600 <sup>(+)</sup>	0.00000000
25	0.27119182	13.595	0.27119170 <sup>(+)</sup>	0.27119180 <sup>(+)</sup>	0.27112000 <sup>(+)</sup>	0.00000002
26	0.26685126	14.057	0.26684960 <sup>(+)</sup>	0.26685130 <sup>(-)</sup>	0.26667920 <sup>(+)</sup>	-0.00000004
27	0.26223207	17.722	0.26223200 <sup>(+)</sup>	0.26223210 <sup>(-)</sup>	0.26212000 <sup>(+)</sup>	-0.00000003
28	0.26030547	21.977	0.26030080 <sup>(+)</sup>	0.26030550 <sup>(-)</sup>	0.26009600 <sup>(+)</sup>	-0.00000003
29	0.25792545	35.987	0.25792510 <sup>(+)</sup>	0.25792550 <sup>(-)</sup>	0.25781900 <sup>(+)</sup>	-0.00000005
30	0.25533055	26.826	0.25533000 <sup>(+)</sup>	0.25533060 <sup>(-)</sup>	0.25478000 <sup>(+)</sup>	-0.00000005
31	0.25311620	79.014	0.25311410 <sup>(+)</sup>	0.25311620	0.25311500 <sup>(+)</sup>	0.00000000
32	0.25078744	38.078	0.25078740 <sup>(+)</sup>	0.25078740 <sup>(+)</sup>	0.25071200 <sup>(+)</sup>	0.00000004
33	0.24876230	39.542	0.24871040 <sup>(+)</sup>	0.24876240 <sup>(+)</sup>	0.24870300 <sup>(+)</sup>	-0.00000010
34	0.24705265	43.430	0.24704220 <sup>(+)</sup>	0.24705270 <sup>(-)</sup>	0.24700600 <sup>(+)</sup>	-0.00000005
35	0.24483365	47.870	0.24482840 <sup>(+)</sup>	0.24483370 <sup>(-)</sup>	0.24477300 <sup>(+)</sup>	-0.00000005
36	0.24313216	56.587	0.24313210 <sup>(+)</sup>	0.24313220 <sup>(-)</sup>	0.24178700 <sup>(+)</sup>	-0.00000004
37	0.24068655	61.810	0.24062170 <sup>(+)</sup>	0.24068660 <sup>(-)</sup>	0.24044100 <sup>(+)</sup>	-0.00000005
38	0.24051936	98.767	0.24051470 <sup>(+)</sup>	0.24051560 <sup>(+)</sup>	0.24036000 <sup>(+)</sup>	0.00000376
39	0.23674523	77.186	0.23655210 <sup>(+)</sup>	0.23674520 <sup>(+)</sup>	0.23670300 <sup>(+)</sup>	0.00000003
40	0.23499923	85.288	0.23497560 <sup>(+)</sup>	0.23499920 <sup>(+)</sup>	0.23487100 <sup>(+)</sup>	0.00000003
41	0.23275597	92.837	0.23266430 <sup>(+)</sup>	0.23275590 <sup>(+)</sup>	0.23270200 <sup>(+)</sup>	0.00000007
42	0.23211860	109.130	0.23184150 <sup>(+)</sup>	0.23211900 <sup>(-)</sup>	0.23211900 <sup>(-)</sup>	-0.00000040
43	0.22973295	113.118	0.22934050 <sup>(+)</sup>	0.22973300 <sup>(-)</sup>	0.22966700 <sup>(+)</sup>	-0.00000005
44	0.22816302	128.813	0.22815860 <sup>(+)</sup>	0.22816290 <sup>(+)</sup>	0.22808300 <sup>(+)</sup>	0.00000012
45	0.22691155	233.270	0.22674120 <sup>(+)</sup>	0.22691160 <sup>(-)</sup>	0.22659400 <sup>(+)</sup>	-0.00000005
46	0.22516818	245.517	0.22511610 <sup>(+)</sup>	0.22516810 <sup>(+)</sup>	0.22516400 <sup>(+)</sup>	0.00000008
47	0.22350704	148.588	0.22348200 <sup>(+)</sup>	0.22350700 <sup>(+)</sup>	0.22334200 <sup>(+)</sup>	0.00000004
48	0.22240593	178.593	0.22235400 <sup>(+)</sup>	0.22240590 <sup>(+)</sup>	0.22236300 <sup>(+)</sup>	0.00000003
49	0.22127817	188.396	0.22091760 <sup>(+)</sup>	0.22127510 <sup>(+)</sup>	0.22098000 <sup>(+)</sup>	0.00000307
50	0.21975827	233.125	0.21934840 <sup>(+)</sup>	0.21975290 <sup>(+)</sup>	-	0.00000537
51	0.21855027	229.869	0.21788800 <sup>(+)</sup>	0.21788880 <sup>(+)</sup>	-	0.00066147
52	0.21693040	287.637	-	0.21676650 <sup>(+)</sup>	-	0.00016390
53	0.21628805	259.448	-	0.21577720 <sup>(+)</sup>	-	0.00051085
54	0.21492066	296.349	-	0.21468460 <sup>(+)</sup>	-	0.00023606
55	0.21344168	297.398	-	0.21326800 <sup>(+)</sup>	-	0.00017368
56	0.21306786	379.628	-	0.21261360 <sup>(+)</sup>	-	0.00045426

57	0.21130935	340.078	-	0.21086610 <sup>(+)</sup>	-	0.00044325
58	0.21048009	397.783	-	0.21030220 <sup>(+)</sup>	-	0.00017789
59	0.20975944	391.824	-	0.20962520 <sup>(+)</sup>	-	0.00013424
60	0.20942699	483.227	-	0.20942690 <sup>(+)</sup>	-	0.00000009
61	0.20910865	412.989	-	0.20891230 <sup>(+)</sup>	-	0.00019635
62	0.20673915	452.234	-	0.20581700 <sup>(+)</sup>	-	0.00092215
63	0.20595491	519.972	-	0.20471400 <sup>(+)</sup>	-	0.00124091
64	0.20408715	685.259	-	0.20366460 <sup>(+)</sup>	-	0.00042255
65	0.20307603	589.248	-	0.20274750 <sup>(+)</sup>	-	0.00032853
66	0.20210875	636.278	-	0.20191580 <sup>(+)</sup>	-	0.00019295
67	0.20121662	854.971	-	0.20044290 <sup>(+)</sup>	-	0.00077372
68	0.20000222	694.894	-	0.19972200 <sup>(+)</sup>	-	0.00028022
69	0.19928152	706.689	-	0.19879220 <sup>(+)</sup>	-	0.00048932
70	0.19868872	786.326	-	0.19833100 <sup>(+)</sup>	-	0.00035772
71	0.19722212	805.879	-	0.19697870 <sup>(+)</sup>	-	0.00024342
72	0.19628591	795.408	-	0.19580010 <sup>(+)</sup>	-	0.00048581
73	0.19562734	992.946	-	-	-	-
74	0.19514678	1074.563	-	-	-	-
75	0.19396442	1068.320	-	-	-	-
76	0.19294742	1183.277	-	-	-	-
77	0.19225411	1243.473	-	-	-	-
78	0.19145892	1208.989	-	-	-	-
79	0.19065973	1229.879	-	-	-	-
80	0.18972728	1336.218	-	-	-	-
81	0.18896712	1493.084	-	-	-	-
82	0.18829304	1452.398	-	-	-	-
83	0.18764017	1592.395	-	-	-	-
84	0.18695602	1608.082	-	-	-	-
85	0.18641122	1791.520	-	-	-	-
86	0.18561006	1776.278	-	-	-	-
87	0.18495292	1809.917	-	-	-	-
88	0.18427588	1849.809	-	-	-	-
89	0.18348126	2239.128	-	-	-	-
90	0.18293732	2102.385	-	-	-	-
91	0.18235327	2848.673	-	-	-	-
92	0.18177412	2307.977	-	-	-	-
93	0.18108200	2331.690	-	-	-	-
94	0.18047196	2537.434	-	-	-	-
95	0.18011439	2847.453	-	-	-	-
96	0.17956685	2660.676	-	-	-	-
97	0.17893218	2807.216	-	-	-	-
98	0.17830007	2999.697	-	-	-	-
99	0.17787965	3021.932	-	-	-	-

100	0.17743921	3378.980	-	-	-	-
105	0.17459442	3902.393	-	-	-	-
110	0.17212934	4958.305	-	-	-	-
115	0.16970936	5782.270	-	-	-	-
120	0.16749907	6789.297	-	-	-	-
125	0.16543987	5985.572	-	-	-	-
130	0.16373513	9786.393	-	-	-	-
135	0.16210499	12794.897	-	-	-	-
140	0.16029921	15986.278	-	-	-	-
145	0.15859944	12363.627	-	-	-	-
150	0.15665832	16893.134	-	-	-	-
155	0.15470631	19769.023	-	-	-	-
160	0.15335971	20689.204	-	-	-	-
165	0.15178309	25019.026	-	-	-	-
170	0.15034579	22134.377	-	-	-	-
175	0.14889938	31281.456	-	-	-	-
180	0.14741802	34897.343	-	-	-	-
185	0.14604949	38968.805	-	-	-	-
190	0.14445529	40354.659	-	-	-	-
195	0.14337725	39691.728	-	-	-	-
200	0.14224761	50982.358	-	-	-	-

In Table 1, a superscript (+) of one record indicates that the corresponding record of A1  $r/r0$  is larger (better) than this record. For example, the record for 72 equal spheres in Hugo(b)  $r/r0$  has a superscript (+), because the corresponding record of A1  $r/r0$  is larger than it by  $4.8581 \times 10^{-4}$ . The superscript (-) has the opposite meaning.

In Table 1, A1  $r/r0$  is obtained by the serial symmetrical relocation algorithm. Hugo(a)  $r/r0$  is a set of verifiable records provided by Hugo Pfoertner (Pfoertner, 2008a), which are supported by their coordinates of spheres centers. Hugo(b)  $r/r0$  is a set of best known records collected by Hugo Pfoertner (Pfoertner, 2008b), which is produced by Hugo Pfoertner, Thierry Gensane and Dave Boll. Each record of this collection is the best one of their respective records. Dave  $r/r0$  is provided by Dave Boll (Boll, 2005).

Some packings found by the serial symmetrical relocation algorithm inside spherical containers are illustrated in Fig.2.

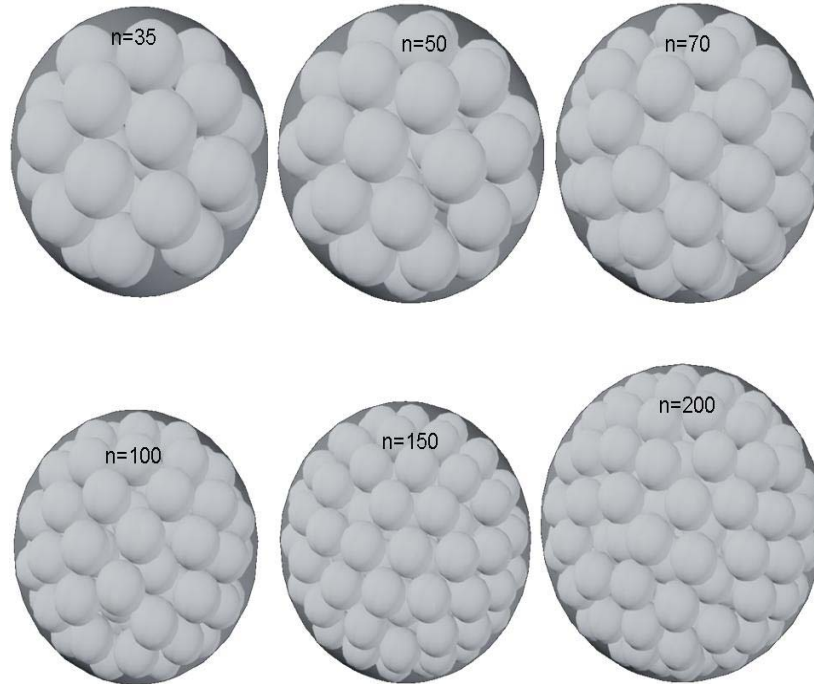


Figure 2: Six packings of equal spheres inside sphere

### The results for packing equal spheres inside cubes

To test the generality of A1, we have also packed equal spheres inside cubes. The cubes sizes of our best packings are listed in table 2, along with the best known records to our knowledge. Here, we take one half of cube edge as cube radius  $r_0$ . The quality of cube is represented by the ratio  $r/r_0$ , where  $r$  is 0.5.

Table 2. Records for packing equal spheres in cubes.

$n$	A1 $r/r_0$	Average runtime for A1 $r/r_0$ (second)	Hugo $r/r_0$	A1 $r/r_0$ minus Hugo $r/r_0$
1	1.00000000	0.001	1.00000000	0.00000000
2	0.63397458	0.001	0.63397460 <sup>(-)</sup>	-0.00000002
3	0.58578640	0.008	0.58578640	0.00000000
4	0.58578640	0.006	0.58578640	0.00000000
5	0.52786400	0.032	0.52786400	0.00000000
6	0.51471861	0.033	0.51471860 <sup>(+)</sup>	0.00000001
7	0.50027231	0.058	0.50027230 <sup>(+)</sup>	0.00000001
8	0.50000000	0.182	0.50000000	0.00000000

9	0.46410161	1.016	0.46410160 <sup>(+)</sup>	0.00000001
10	0.42857142	1.126	0.42857140 <sup>(+)</sup>	0.00000002
11	0.41524447	0.976	0.41524450 <sup>(-)</sup>	-0.00000003
12	0.41421355	0.682	0.41421350 <sup>(+)</sup>	0.00000005
13	0.41421355	1.223	0.41421350 <sup>(+)</sup>	0.00000005
14	0.41421355	1.198	0.41421350 <sup>(+)</sup>	0.00000005
15	0.38461537	1.648	0.38461540 <sup>(-)</sup>	-0.00000003
16	0.37759353	2.832	0.37759350 <sup>(+)</sup>	0.00000003
17	0.37737047	2.936	0.37737050 <sup>(-)</sup>	-0.00000003
18	0.37536119	3.523	0.37536110 <sup>(+)</sup>	0.00000009
19	0.36637060	4.358	0.36637040 <sup>(+)</sup>	0.00000020
20	0.35681439	5.925	0.35681440 <sup>(-)</sup>	-0.00000001
21	0.35443808	7.368	0.35443810 <sup>(-)</sup>	-0.00000002
22	0.34654620	9.932	0.34654620	0.00000000
23	0.34363356	8.996	0.34363330 <sup>(+)</sup>	0.00000026
24	0.34108137	11.653	0.34108110 <sup>(+)</sup>	0.00000027
25	0.33560809	12.893	0.33560750 <sup>(+)</sup>	0.00000059
26	0.33381032	15.959	0.33381020 <sup>(+)</sup>	0.00000012
27	0.33333332	20.732	0.33333330 <sup>(+)</sup>	0.00000002
28	0.32038200	22.803	0.32038200	0.00000000
29	0.32037723	23.682	0.32037720 <sup>(+)</sup>	0.00000003
30	0.32037723	27.376	0.32037720 <sup>(+)</sup>	0.00000003
31	0.32037723	33.518	0.32037720 <sup>(+)</sup>	0.00000003
32	0.32037723	35.142	0.32037720 <sup>(+)</sup>	0.00000003
33	0.30921071	38.252	0.30921070 <sup>(+)</sup>	0.00000001
34	0.30423418	33.426	0.30423380 <sup>(+)</sup>	0.00000038
35	0.30333704	46.853	0.30333690 <sup>(+)</sup>	0.00000014
36	0.29861826	57.087	0.29861740 <sup>(+)</sup>	0.00000086
37	0.29812325	66.823	0.29812320 <sup>(+)</sup>	0.00000005
38	0.29807753	68.367	0.29807750 <sup>(+)</sup>	0.00000003
39	0.29523475	78.236	0.29523470 <sup>(+)</sup>	0.00000005
40	0.29411764	95.218	0.29411760 <sup>(+)</sup>	0.00000004
41	0.28967575	91.825	0.28967510 <sup>(+)</sup>	0.00000065
42	0.28608925	107.653	0.28608530 <sup>(+)</sup>	0.00000395
43	0.28330422	115.127	0.28330320 <sup>(+)</sup>	0.00000102
44	0.28172121	118.423	0.28172060 <sup>(+)</sup>	0.00000061
45	0.28126386	133.370	0.28126390 <sup>(-)</sup>	-0.00000004
46	0.28049373	143.527	0.28049340 <sup>(+)</sup>	0.00000033
47	0.27991768	146.982	0.27991770 <sup>(-)</sup>	-0.00000002
48	0.27991768	288.125	0.27991770 <sup>(-)</sup>	-0.00000002
49	0.27285349	198.592	0.27285340 <sup>(+)</sup>	0.00000009
50	0.27190872	225.216	0.27190740 <sup>(+)</sup>	0.00000132
51	0.27004592	236.169	0.27001290 <sup>(+)</sup>	0.00003302

52	0.26764001	272.237	0.26763980 <sup>(+)</sup>	0.00000021
53	0.26646158	236.426	0.26646040 <sup>(+)</sup>	0.00000118
54	0.26330123	299.028	0.26326990 <sup>(+)</sup>	0.00003133
55	0.26214603	293.256	0.26176990 <sup>(+)</sup>	0.00037613
56	0.26130505	279.338	0.26121580 <sup>(+)</sup>	0.00008925
57	0.26120393	306.120	0.26120380 <sup>(+)</sup>	0.00000013
58	0.26120387	396.085	0.26120380 <sup>(+)</sup>	0.00000007
59	0.26120387	386.516	0.26120380 <sup>(+)</sup>	0.00000007
60	0.26120387	459.205	0.26120380 <sup>(+)</sup>	0.00000007
61	0.26120387	412.989	0.26120390 <sup>(-)</sup>	-0.00000003
62	0.26120387	483.122	0.26120380 <sup>(+)</sup>	0.00000007
63	0.26120387	536.268	0.26120380 <sup>(+)</sup>	0.00000007
64	0.25348654	583.427	0.25348040 <sup>(+)</sup>	0.00000614
65	0.25236517	589.248	0.25235460 <sup>(+)</sup>	0.00001057
66	0.25147929	616.163	0.25146920 <sup>(+)</sup>	0.00001009
67	0.24896216	653.073	0.24896000 <sup>(+)</sup>	0.00000216
68	0.24686760	687.253	0.24675700 <sup>(+)</sup>	0.00011060
69	0.24627801	702.585	0.24593500 <sup>(+)</sup>	0.00034301
70	0.24607500	883.166	0.24588280 <sup>(+)</sup>	0.00019220
71	0.24605515	805.879	0.24569490 <sup>(+)</sup>	0.00036025
72	0.24605515	795.408	0.24566920 <sup>(+)</sup>	0.00038595
73	0.24357499	998.258	-	-
74	0.24267720	1182.591	-	-
75	0.24264288	1008.150	-	-
76	0.24036710	1283.368	-	-
77	0.23810069	1353.386	-	-
78	0.23709060	1198.385	-	-
79	0.23602798	1329.125	-	-
80	0.23483879	1306.856	-	-
81	0.23414746	1503.165	-	-
82	0.23366493	1497.893	-	-
83	0.23256887	1539.263	-	-
84	0.23256724	1678.925	-	-
85	0.23256724	2098.535	-	-
86	0.23256724	1769.892	-	-
87	0.23256724	1822.735	-	-
88	0.23104286	1856.203	-	-
89	0.22753889	2369.213	-	-
90	0.22651347	2086.563	-	-
91	0.22558935	3003.356	-	-
92	0.22500279	2687.076	-	-
93	0.22453635	2030.548	-	-
94	0.22398206	2612.231	-	-



95	0.22347071	2896.546	-	-
96	0.22329903	2698.323	-	-
97	0.22309749	2936.285	-	-
98	0.22303798	2597.805	-	-
99	0.22277482	3136.025	-	-
100	0.22276469	3295.895	-	-
105	0.22048120	3979.295	-	-
110	0.21470034	5085.573	-	-
115	0.21027862	5968.735	-	-
120	0.20959841	7937.739	-	-
125	0.20710709	8283.382	-	-
130	0.20297413	10063.240	-	-
135	0.20005504	11373.453	-	-
140	0.19972284	16901.682	-	-
145	0.19642072	18313.383	-	-
150	0.19339963	15293.625	-	-

In Table 2, the superscripts (+) and (-) have the same meanings as before. A1  $r/r0$  is obtained by the serial symmetrical relocation algorithm. Hugo  $r/r0$  is a collection of records gathered by Hugo Pfoertner (Pfoertner, 2005), which is produced by J. Schaer, M. Goldberg, Hugo Pfoertner, Thierry Gensane and Dave Boll.

Some packings found by the serial symmetrical relocation algorithm inside cubes are illustrated in Fig.3.

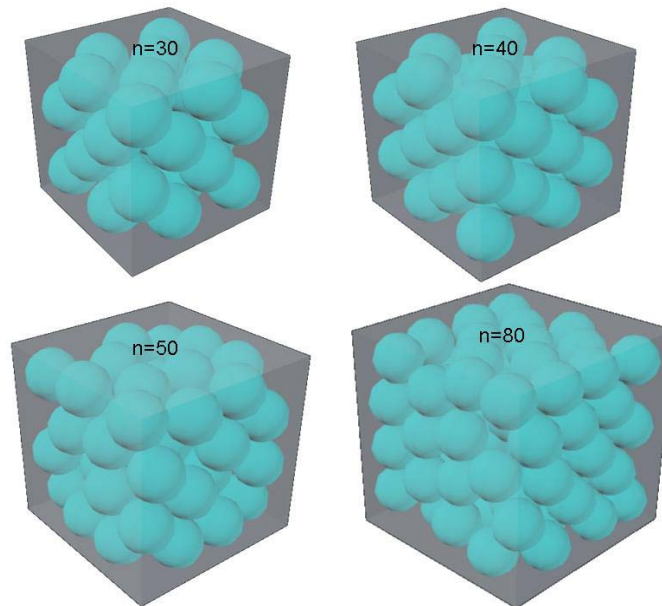


Figure 3: Four packings of equal spheres inside cube

## Conclusions

The serial symmetrical relocation strategy as the global search strategy is the core of our work. Experiments show that it led to some improvements over the current best known records about the equal sphere packing problem.

The fake sphere trick guarantees that all results found by the serial symmetrical relocation algorithm are rigorous. We stress the exactness of results because any approximate result including overlaps may have the exaggerated quality and, thus may cover any possible subtle improvements.

By intuition, the serial symmetrical relocation algorithm could be applied for packing circles or spheres in any centrosymmetric container. We will further develop the ideas that presented in this paper and try to find out a generic and highly efficient method for packing circles or spheres in arbitrary containers.

## Acknowledgements

The authors thank Hugo et al for their records. The authors also thank anonymous referees for their helpful comments and suggestions.

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