# ON THE STERN SEQUENCE AND ITS TWISTED VERSION

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**Abstract**

In a recent preprint on ArXiv, Bacher introduced a twisted version of the Stern sequence. His paper contains in particular three conjectures relating the generating series for the Stern sequence and for the twisted Stern sequence. Soon afterwards Coons published two papers in *Integers*: first he proved these conjectures, second he used his result to obtain a correlation-type identity for the Stern sequence. We recall here a simple result of Reznick and we state a similar result for the twisted Stern sequence. We deduce an easy proof of Coons' identity, and a simple proof of Bacher's conjectures. Furthermore we prove identities similar to Coons' for variations on the Stern sequence that include Bacher's sequence.

## 1. Introduction

The Stern sequence is a sequence of integers $\mathbf{s} = (s(n))_{n\geq 0}$ that can be defined inductively by $s(0) = 0$, $s(1) = 1$, and for all $n \geq 1$, $s(2n) = s(n)$ and $s(2n+1) = s(n) + s(n+1)$. (Note that these two equalities are actually true for all $n \geq 0$.) This is sequence A002487 in [10]. Its first few terms are

$$0, 1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1 \ldots$$

Several authors studied that sequence, see, e.g., [12, 9] and the references therein. (Note that some authors call Stern sequence the shifted sequence $(s(n+1))_{n\geq 0}$.)

Bacher introduced recently in [3] a twisted version of the Stern sequence $\mathbf{t} = (t_n)_{n\geq 0}$ defined inductively by $t(0) = 0$, $t(1) = 1$, and for all $n \geq 1$, $t(2n) = -t(n)$, $t(2n+1) = -t(n) - t(n+1)$. He gave several interesting properties of the sequences $\mathbf{s}$ and $\mathbf{t}$ and formulated conjectural relations between the generating series $\sum s(n)X^n$, $\sum t(3.2^e + n)X^n$, $\sum (s(2+n) - s(1+n))X^n$, and $\sum (t(2+n) + t(1+n))X^n$.

In the recent paper [5] Coons proved Bacher's conjectures. He then used in [6] his results to prove the following identity for the Stern sequence: if $e$ and $r$ are integers with $e \geq 0$, then for every integer $n \geq 0$

$$s(r)s(2n+5) + s(2^e - r)s(2n+3) = s(2^e(n+2)+r) + s(2^e(n+1)+r).$$

We recall here (see Section 3) a result of Reznick in [11], and we deduce an easy proof of Coons' identity. We also prove a result similar to Reznick's result for the Bacher-Stern sequence which yields a short proof of Bacher's conjectures Furthermore we prove identities analogous to Reznick's and Coons' identities for sequences satisfying recurrence relations similar to Stern's which include Bacher's sequence.

## 2. Three auxiliary results

We start with three propositions. The first one is [11, Corollary 4] for which Reznick gives a short proof.

**Proposition 1.** [11] *Let $e$ and $r$ be integers with $e \geq 0$ and $0 \leq r \leq 2^e$. Then, for every integer $n \geq 0$, we have*

$$s(2^e n + r) = s(r)s(n+1) + s(2^e - r)s(n).$$

The next Proposition is similar to Proposition 1

**Proposition 2.** *Let $e$ and $r$ be integers with $e \geq 0$ and $0 \leq r \leq 2^e$. Then, for every integer $n \geq 1$, we have*

$$t(2^e n + r) = (-1)^e (s(r)t(n+1) + s(2^e - r)t(n)).$$

*Proof.* We prove by induction on $e \geq 0$ that, for every $r \in [0, 2^e]$, the identity in the proposition holds. This is immediate for $e = 0$ (thus $r \in \{0, 1\}$). If the result is true for some $e$, then, using the definition of $\mathbf{t}$, the induction hypothesis, and the definition of $\mathbf{s}$,
if $2r \in [0, 2^{e+1}]$, then

$$\begin{aligned} t(2^{e+1}n + 2r) &= -t(2^e n + r) = (-1)^{e+1}(s(r)t(n+1) + s(2^e - r)t(n)) \\ &= (-1)^{e+1}(s(2r)t(n+1) + s(2^{e+1} - 2r)t(n)) \end{aligned}$$

if $2r+1 \in [0, 2^{e+1}]$, then

$$\begin{array}{ll} t(2^{e+1}n+2r+1) & = t(2(2^e n+r)+1) = -t(2^e n+r) - t(2^e n+r+1) \\ & = \left\{ \begin{array}{l} (-1)^{e+1}(s(r)t(n+1)+s(2^e-r)t(n)) \\ \quad +(-1)^{e+1}(s(r+1)t(n+1)+s(2^e-r-1)t(n)) \end{array} \right. \\ & = \left\{ \begin{array}{l} (-1)^{e+1}(s(r)+s(r+1))t(n+1) \\ \quad +(-1)^{e+1}(s(2^e-r)+s(2^e-r-1))t(n) \end{array} \right. \\ & = (-1)^{e+1}(s(2r+1)t(n+1)+s(2(2^e-r-1)+1) \\ & = (-1)^{e+1}(s(2r+1)t(n+1)+s(2^{e+1}-2r-1)t(n). \end{array}$$

The last result we need is a consequence of Proposition 1.

**Proposition 3.** *Let* $S(X) = \sum_{n \geq 0} s(N)X^n$*. Then*

$$S(X) = S(X^{2^e}) \sum_{0 \leq r \leq 2^e-1} (s(2^e-r)X^r + s(r)X^{r-2^e}).$$

*Proof.* This is an easy consequence of Proposition 1 (also recall that $s(0)=0$): we write

$$\begin{array}{ll} S(X) & = \displaystyle\sum_{n \geq 0} s(n)X^n = \sum_{0 \leq r \leq 2^e-1} \sum_{k \geq 0} s(k.2^e+r)X^{k.2^e+r} \\ & = \displaystyle\sum_{0 \leq r \leq 2^e-1} X^r \sum_{k \geq 0} (s(r)s(k+1)+s(2^e-r)s(k))X^{k.2^e} \\ & = \displaystyle\sum_{0 \leq r \leq 2^e-1} (s(r)X^{r-2^e}+s(2^e-r)X^r) \sum_{k \geq 0} s(k)X^{k.2^e} \\ & = S(X^{2^e}) \displaystyle\sum_{0 \leq r \leq 2^e-1} (s(r)X^{r-2^e}+s(2^e-r)X^r). \end{array}$$

## 3. A direct proof of Coons' identity

Theorem 1 of [6] is a straightforward corollary of Reznick's result (Proposition 1 above).

**Corollary 4.** *Let* $e$ *and* $r$ *be integers with* $e \geq 0$ *and* $0 \leq r \leq 2^e$*. Then, for every integer* $n \geq 0$*, we have*

$$s(r)s(2n+5) + s(2^e-r)s(2n+3) = s(2^e(n+2)+r) + s(2^e(n+1)+r).$$

*Proof.* Let $S(e,r,n) = s(2^e(n+2)+r) + s(2^e(n+1)+r)$. Applying Proposition 1 with $n$ replaced by $n+2$ and $n+1$, and the definition of the sequence $\mathbf{s}$ yields

$$\begin{array}{lll} S(e,r,n) & = & s(r)s(n+3)+s(2^e-r)s(n+2)+s(r)s(n+2)+s(2^e-r)s(n+1) \\ & = & s(r)(s(n+3)+s(n+2))+s(2^e-r)(s(n+2)+s(n+1)) \\ & = & s(r)s(2n+5)+s(2^e-r)s(2n+3). \end{array}$$

## 4. A simple proof of Bacher's conjectures

We can now prove the three conjectures that Bacher proposed in [3] (Conjectures 1.3, 3.2 (i), and 3.2 (ii)) as Theorems 5, 6, and 7 below.

**Theorem 5.** *Let* $S(X) = \sum_{n \geq 0} s(n)X^n$ *and* $T(X) = \sum_{n \geq 0} t(n)X^n$ *be the generating series of* $\mathbf{s}$ *and* $\mathbf{t}$. *Then, there exists a series* $U(X) = \sum_{n \geq 0} u(n)X^n$ *with integral coefficients, such that*

$$\forall e \geq 0, \ \sum_{n \geq 0} t(3.2^e + n)X^n = (-1)^e U(X^{2^e})S(X).$$

*Proof.* The series $U(X)$ must satisfy in particular $\sum_{n \geq 0} t(3+n)X^n = U(X)S(X)$. This relation *defines* a series $U(X)$ that clearly has integer coefficients ($s(1) = 1$, and $s(0) = 0$). Now, using Proposition 2 above, the definition of $U(X)$, and Proposition 3, we have

$$\begin{aligned}
\sum_{n \geq 0} t(3.2^e + n)X^n &= \sum_{0 \leq r \leq 2^e - 1} \sum_{k \geq 0} t(3.2^e + k.2^e + r)X^{k.2^e + r} \\
&= \sum_{0 \leq r \leq 2^e - 1} X^r \sum_{k \geq 0} t(2^e(3+k) + r)X^{k.2^e} \\
&= \sum_{0 \leq r \leq 2^e - 1} X^r \sum_{k \geq 0} (-1)^e (s(r)t(k+4) + s(2^e - r)t(k+3))X^{k.2^e} \\
&= (-1)^e \sum_{0 \leq r \leq 2^e - 1} (s(2^e - r)X^r + s(r)X^{r - 2^e}) \sum_{k \geq 0} t(3+k)X^{k.2^e} \\
&= (-1)^e \sum_{0 \leq r \leq 2^e - 1} (s(2^e - r)X^r + s(r)X^{r - 2^e}) U(X^{2^e})S(X^{2^e}) \\
&= (-1)^e S(X)U(X^{2^e}).
\end{aligned}$$

**Theorem 6.** *Let* $A(X) = \dfrac{1}{S(X)} \sum_{n \geq 0} (s(2+n) - s(1+n))X^n$. *Then*

$$\sum_{n \geq 0} (s(2^{e+1} + n) - s(2^e + n))X^n = A(X^{2^e})S(X).$$

*Proof.* Let $A_e(X) = \sum_{n \geq 0} (s(2^{e+1} + n) - s(2^e + n))X^n$. We write, using Proposition 1 and Proposition 3 (recall that $s(2) - s(1) = 0$),

$$\begin{aligned}
A_e(X) &= \sum_{0 \leq r \leq 2^e - 1} \sum_{k \geq 0} (s(2^{e+1} + k.2^e + r) - s(2^e + k.2^e + r))X^{k.2^e} \\
&= \sum_{0 \leq r \leq 2^e - 1} X^r \sum_{k \geq 0} (s(2^e(k+2) + r) - s(2^e(k+1) + r))X^{k.2^e}.
\end{aligned}$$

Thus

$$\begin{aligned}
A_e(X) &= \begin{cases} \displaystyle\sum_{0\le r\le 2^e-1} X^r \sum_{k\ge 0}((s(r)s(k+3)+s(2^e-r)s(k+2))X^{k.2^e} \\ \displaystyle - \sum_{0\le r\le 2^e-1} X^r(s(r)s(k+2)+s(2^e-r)s(k+1))X^{k.2^e} \end{cases} \\
&= \begin{cases} \displaystyle\sum_{0\le r\le 2^e-1} s(r)X^r \sum_{k\ge 0}(s(k+3)-s(k+2))X^{k.2^e} \\ \displaystyle + \sum_{0\le r\le 2^e-1} s(2^e-r)X^r \sum_{k\ge 0}(s(k+2)-s(k+1))X^{k.2^e} \end{cases} \\
&= \sum_{0\le r\le 2^e-1} (s(r)X^{r-2^e}+s(2^e-r))X^r \sum_{k\ge 0}(s(k+2)-s(k+1))X^{k.2^e} \\
&= \frac{S(X)}{S(X^{2^e})}\sum_{k\ge 0}(s(k+2)-s(k+1))X^{k.2^e} = S(X)A(X^{2^e}).
\end{aligned}$$

**Theorem 7.** *Let* $B(X) = \dfrac{1}{S(X)}\displaystyle\sum_{n\ge 0}(t(2+n)+t(1+n))X^n$. *Then*

$$(-1)^{e+1}\sum_{n\ge 0}(t(2^{e+1}+n)+t(2^e+n))X^n = B(X^{2^e})S(X).$$

*Proof.* The proof is the same as the proof of Theorem 6, except that we use Propositions 2 and 3 instead of Propositions 1 and 3.

## 5. Similar sequences

Proposition 2 gives an expression of $t(2^e n + r)$ in terms of $t(n)$ and $t(n+1)$ with coefficients in terms of $\mathbf{s}$. One might want to find relations of the same kind but involving $\mathbf{t}$ only. In this section we give such a relation. More generally we prove such relations for sequences satisfying recurrence relations similar to the recurrences defining the Stern sequence.

**Theorem 8.** *Let* $\mathbf{v} = (v(n))_{n\ge 0}$ *be a sequence of real numbers satisfying*

$\exists n_0 \ge 0,\ \exists (a,b,c)\in\mathbb{R}^3,\ \forall n\ge n_0\ v(2n) = av(n)$ and $v(2n+1) = bv(n)+cv(n+1)$.

*Then, for all integers* $(e,r)$ *with* $e\ge 0$ *and* $r\in[0,2^e]$, *there exist* $A = A(e,r)$ *and* $B = B(e,r)$ *such that for all* $n\ge n_0$

$$v(2^e n + r) = A(e,r)v(n) + B(e,r)v(n+1).$$

*Proof.* We prove by induction on $e$ that for all $r\in[0,2^e]$, there exist $A(e,r)$ and $B(e,r)$ satisfying the conditions in the theorem. For $e=0$, hence $r\in\{0,1\}$ one gets from the definition of $\mathbf{v}$ that $A(0,0) = 1$, $B(0,0) = 0$, $A(0,1) = 0$, and $B(0,1) = 1$. Going from $e$ to $e+1$ yields $A(e+1, 2r) = aA(e,r)$, $B(e+1, 2r) = aB(e,r)$, if $0 \leq 2r \leq 2^{e+1}$, and $A(e+1, 2r+1) = bA(e,r) + cA(e, r+1)$, $B(e+1, 2r+1) = bB(e,r) + cB(e, r+1)$, if $0 \leq 2r+1 \leq 2^{e+1}$.

**Corollary 9.** *Let $\mathbf{v} = (v(n))_{n \geq 0}$ be a sequence of real numbers satisfying*

$$\exists n_0 \geq 0,\ \exists (a,b,c) \in \mathbb{R}^3,\ \forall n \geq n_0\ v(2n) = av(n) \text{ and } v(2n+1) = bv(n) + cv(n+1).$$

*Then, for all integers $(e,r)$ with $e \geq 0$ and $r \in [0, 2^e]$, there exist $A = A(e,r)$ and $B = B(e,r)$ such that for all $n \geq n_0$*

$$A(e,r)v(2n+3) + B(e,r)v(2n+5) = cv(2^e(n+2)+r) + bv(2^e(n+1)+r).$$

*Proof.* Apply Theorem 8 with $n$ replaced by $n+2$ and $n+1$ to the left side of the identity to be proven.

**Remark 10.** The quantities $A(e,r)$ and $B(e,r)$ can of course be computed in terms of $e, r$ and of certain values of $\mathbf{v}$. For example if the sequence $\mathbf{v}$ is not trivial, there exist two integers $x_0$ and $y_0$ with $x_0, y_0 \geq n_0$ such that $\begin{vmatrix} v(x_0) & v(x_0+1) \\ v(y_0) & v(y_0+1) \end{vmatrix} \neq 0$. Then

$$\begin{aligned} v(2^e x_0 + r) &= A(e,r)v(x_0) + B(e,r)v(x_0+1) \\ v(2^e y_0 + r) &= A(e,r)v(y_0) + B(e,r)v(y_0+1) \end{aligned}$$

yields

$$A(e,r) = (v(y_0)v(x_0+1) - v(x_0)v(y_0+1))^{-1}(v(x_0+1)v(2^e y_0 + r) - v(y_0+1)v(2^e x_0 + r))$$

and

$$B(e,r) = (v(x_0)v(y_0+1) - v(y_0)v(x_0+1))^{-1}(v(x_0)v(2^e y_0 + r) - v(y_0)v(2^e x_0 + r)).$$

## 6. Examples

### 6.1. The Stern sequence again

One can apply Theorem 8 to the Stern sequence, for which $n_0 = 0$, $a = b = c = 1$. The values of $A$ and $B$ can be obtained by taking $n = 0$ and $n = 1$ in the relation $s(2^e n + r) = A(e,r)s(n) + B(e,r)s(n+1)$, yielding $B(e,r) = s(r)$ and $A(e,r) = s(2^e n + r) - s(r)$. To obtain the result of Proposition 1 and Corollary 4 this way, it remains to prove that for all $e \geq 0$ and $r \in [0, 2^e]$ one has $s(2^e + r) - s(r) = s(2^e - r)$. This last equality can be proven by induction on $e$, but this is also Corollary 3.1 in [7] (see also [3, Theorem 1.2] where the author adds that this identity "is probably well-known to the experts")

### 6.2. The case of Bacher's twisted Stern sequence

The definition of Bacher's twisted Stern sequence $\mathbf{t} = (t(n))_{n\geq 0}$ recalled in the Introduction shows that $\mathbf{t}$ satisfies the hypotheses of Theorem 8 with $a = b = c = -1$, and $n_0 = 1$. Note that the first few terms of $\mathbf{t}$ are:

$$0, 1, -1, 0, 1, 1, 0, -1, -1, -2, -1, -1, 0, 1, 1, 2, \ldots$$

Applying Theorem 8 and Corollary 9 we get the following results.

**Theorem 11.** *Let $e$ and $r$ be integers with $e \geq 0$ and $0 \leq r \leq 2^e$. Then, for every integer $n \geq 1$, we have*

$$t(2^e n + r) = -t(2^{e+1} + r)t(n) - t(3.2^e - r)t(n + 1).$$

*Proof.* From Theorem 8 we have the existence of $A'$ and $B'$ such that $t(2^e n + r) = A'(e, r)t(n) + B'(e, r)t(n + 1)$ for $n \geq 1$. Taking $n = 2$ and using that $t(2) = -1$ and $t(3) = 0$, we get $A'(e, r) = -t(2^{e+1} + r)$. Now taking $n = 1$ yields $t(2^e + r) = A'(e, r) - B'(e, r)$. Hence $B'(e, r) = A'(e, r) - t(2^e + r)$, i.e., $B'(e, r) = -t(2^{e+1} + r) - t(2^e + r)$. An immediate induction on $e$ shows that for $r \in [0, 2^e]$ one has $t(2^{e+1} + r) + t(2^e + r) = t(3.2^e - r)$. Hence the result

**Corollary 12.** *Let $e$ and $r$ be integers with $e \geq 0$ and $0 \leq r \leq 2^e$. Then, for every integer $n \geq 0$, we have*

$$t(2^{e+1} + r)t(2n + 3) + t(3.2^e - r)t(2n + 5) = t(2^e(n + 2) + r) + t(2^e(n + 1) + r).$$

### 6.3. Other variations on Stern's sequence

Let the three sequences $(z_1(n))_{n\geq 0}$, $(z_2(n))_{n\geq 0}$, and $(z_3(n))_{n\geq 0}$ defined by (using the notation of [10]): for all $n \geq 0$, $z_1(n) = A005590(n)$, and for all $n \geq 1$, $z_2(n) = A177219(n)$, and $z_3(n) = A049347(n)$ with $z_2(0) = z_3(0) = 0$. These sequences satisfy respectively

$$(z_1(0), z_1(1)) = (0, 1), \text{ and } \forall n \geq 1, z_1(2n) = z_1(n),\ z_1(2n+1) = -z_1(n) + z_1(n+1),$$

$$(z_2(0), z_1(1)) = (0, 1), \text{ and } \forall n \geq 1, z_2(2n) = -z_2(n),\ z_2(2n+1) = -z_2(n) + z_2(n+1),$$

$$(z_3(0), z_3(1)) = (0, 1), \text{ and } \forall n \geq 1, z_3(2n) = -z_3(n),\ z_3(2n+1) = z_3(n) + z_3(n+1).$$

Note that he last sequence $(z_3(n))_{n\geq 0}$ is the 3-periodic sequence with period $(0, 1, -1)$ (hint: prove by induction on $n$ that for all $j \leq n$ one has $(z_3(3j), z_3(3j + 1), z_(3j + 2)) = (0, 1, -1)$). Also note that all relations $z_i(2n) = \pm z_i(n)$ and $z_i(2n + 1) = \pm z_i(n) + z_i(n + 1)$, $i = 1, 2, 3$, are actually valid for $n \geq 0$.

We know from Theorem 8 that, for all $e \geq 0$ and $r \in [0, 2^e]$, there exist $A_i(e, r)$ and $B_i(e, r)$ such that for all $n \geq 0$ we have

$$z_i(2^e n + r) = A_i(e, r)z_i(n) + B_i(e, r)z_i(n + 1).$$

Taking $n = 0$ yields $B_i(e,r) = z_i(r)$ (for $i = 1,2,3$). Taking $n = 2$, and using that $z_1(2) = 1$, $z_3(2) = -1$, and $z_1(3) = z_3(3) = 0$, we get $A_1(e,r) = z_1(2^{e+1}+r)$ and $A_3(e,r) = -z_3(2^{e+1}+r)$. Now taking $n = 1$ yields $A_2(e,r) = z_2(2^e+r) - z_2(r)$. An immediate induction on $e$ proves that, for $r \in [0, 2^e]$, one has $z_2(2^e + r) - z_2(r) = -z_2(5.2^e + r)$. Hence we can state the following theorem.

**Theorem 13.** *Let $(z_1(n))_{n\geq 0}$, $(z_2(n))_{n\geq 0}$, $(z_3(n))_{n\geq 0}$ be the sequences defined above. Let $e \geq 0$ and $r \in [0, 2^e]$. Then, for all $n \geq 0$ we have*

$$\begin{array}{rcl} z_1(2^e n + r) & = & z_1(2^{e+1}+r)z_1(n) + z_1(r)z_1(n+1) \\ z_2(2^e n + r) & = & -z_2(5.2^e n + r)z_2(n) + z_2(r)z_2(n+1) \\ z_3(2^e n + r) & = & -z_3(2^{e+1}+r)z_1(n) + z_3(r)z_3(n+1) \end{array}$$

*and*

$$z_1(2^{e+1}+r)z_1(2n+5) + z_1(r)z_1(2n+3) = -z_1(2^e(n+2)+r) + z_1(2^e(n+1)+r)$$

$$-z_2(5.2^e+r)z_2(2n+5) + z_2(r)z_2(2n+3) = -z_2(2^e(n+2)+r) + z_2(2^e(n+1)+r)$$

$$-z_3(2^{e+1}+r)z_3(2n+5) + z_3(r)z_3(2n+3) = z_3(2^e(n+2)+r) + z_3(2^e(n+1)+r).$$

### 6.4. Block-complexity of the Thue-Morse sequence

Other sequences satisfy the hypotheses of Theorem 8, e.g., sequence A145865 in [10]. An example that we would like to mention is the sequence $(y(n))_{n\geq 0} = (A005942(n+1))_{n\geq 0}$ with the notation of [10]. The sequence $(A005942(n))_{n\geq 0}$ is the (block-)complexity of the Thue-Morse sequence (the Thue-Morse sequence is the fixed point beginning with 0 of the morphism $0 \to 01$, $1 \to 10$, see, e.g., [2]; its block-complexity is the number of distinct factors (blocks) of each length occurring in that sequence). It satisfies $A005942(2n) = A005942(n) + A005942(n+1)$, and $A005942(2n+1) = 2A005942(n+1)$ if $n \geq 2$ (see [4, 8]). Hence the sequence $(y(n))_{n\geq 0}$ satisfies the hypotheses of Theorem 8 with $n_0 = 2$, $a = 2$, $b = c = 1$. Note that $y(0) = 2$ and $y(1) = 4$.

**Remark 14.** The sequence $(A006165(n))_{n\geq 0}$ satisfies the same recurrence properties as the sequence $(y(n))_{n\geq 0}$ above, but is equal to 1 for $n = 1$ and $n = 2$. As indicated in [10] this sequence is related to the Josephus problem.

## 7. Final remarks

For sequences $(z(n))_{n\geq 0}$ satisfying the hypotheses of Theorem 8, any subsequence of the form $(z(2^e n + r))_{n\geq 0}$ with $e \geq 0$ and $r \in [0, 2^e]$ is a linear expression in $(z(n))_{n\geq 0}$ and $(z(n+1))_{n\geq 0}$ for $n \geq n_0$ with coefficients depending on $r$ and $e$ only: this proves the 2-regularity of these sequences (see [1]).

Also note that, as visible in the proof of Theorem 8 above, several other relations can be found between the terms of sequences satisfying the hypotheses of that theorem.

## 8. Acknowledgments

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