

# THE DEPTH OF A PERMUTATION

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ABSTRACT. For the elements of a Coxeter group, we present a statistic called *depth*, defined in terms of factorizations of the elements into products of reflections. Depth is bounded above by length and below by reflection length. In this article, we focus on the case of the symmetric group, where we show that depth is equal to  $\sum_i \max\{w(i) - i, 0\}$ . We characterize those permutations for which depth equals length: these are the 321-avoiding permutations (and hence are enumerated by the Catalan numbers). We also characterize those permutations for which depth equals reflection length: these are permutations avoiding both 321 and 3412 (also known as boolean permutations, which we can hence also enumerate). In this case, it also happens that length equals reflection length, leading to a new perspective on a result of Edelman.

## 1. INTRODUCTION

We begin with discussion of a simple recursive sorting algorithm, called “straight selection sort” by Knuth [8]. The algorithm finds the largest element in a list that is not in its proper place, and moves it to its correct position using a single transposition. Having fixed this element, we repeat the procedure, finding the next-largest element that is not in its proper place and moving it to its correct position, and so on. For example, the steps in sorting the permutation  $w = 2431756$  are as follows, where the value being moved is in boldface and each transposition has been labeled by the pair of positions that it swaps.

$$2431756 \xrightarrow{(57)} 2431\mathbf{6}57 \xrightarrow{(56)} 2431567 \xrightarrow{(24)} \mathbf{2}134567 \xrightarrow{(12)} 1234567.$$

From this algorithm we find a minimal length expression for the permutation as a product of transpositions:

$$w = (12)(24)(56)(57),$$

though of course there are usually many other shortest expressions as a product of transpositions. We can measure the “cost” of this algorithm in terms of the distances between the positions changed in each transposition, so in the example above, this measure would be

$$(2 - 1) + (4 - 2) + (6 - 5) + (7 - 5) = 6.$$

In [10], this statistic was called the *sorting index*. The sorting index was shown [10, Corollary 3] (see also [22]) to have the same distribution over all permutations of  $n$  as the inversion number; that is, it is a *Mahonian* statistic.

While straight selection sort optimizes the number of transpositions, one may ask whether we can do better than the sorting index, in terms of cost. Returning to our example of  $w = 2431756$  we see an alternative route from  $w$  to the identity  $e$ :

$$2431756 \xrightarrow{(56)} 24315\mathbf{7}6 \xrightarrow{(67)} 2431567 \xrightarrow{(24)} \mathbf{2}134567 \xrightarrow{(12)} 1234567.$$

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This sorting of the permutation yields

$$w = (12)(24)(67)(56),$$

and

$$(2 - 1) + (4 - 2) + (7 - 6) + (6 - 5) = 5 < 6.$$

In fact, this value of 5 is the best we can hope to do for this  $w$ .

The purpose of this paper is to characterize this sort of minimum sorting measure, which we call the *depth* of a permutation, denoted by  $\text{dp}(w)$ . The term “depth” is chosen because it is natural from the perspective Coxeter groups, as we explain in Section 2.

We will relate depth to the well understood notions of *length* and *reflection length*, denoted  $\ell(w)$  and  $\ell'(w)$ , respectively. For permutations, it is well known that the length is the same as the inversion number, and the reflection length is the size of the permutation minus the number of cycles. We will show that depth is bounded below by reflection length and above by length.

Beyond preliminaries, we study depth primarily in the case of the symmetric group. In particular, we have the following characterization of depth.

**Theorem 1.1.** *Let  $w \in \mathfrak{S}_n$ . Then its depth is given by*

$$\text{dp}(w) = \sum_{w(i) > i} (w(i) - i).$$

We prove Theorem 1.1 by exhibiting an algorithm, based on straight selection sort, that achieves a factorization of  $w$  having both minimum depth and minimum reflection length.

A position  $i$  such that  $w(i) > i$  is called an *excedance* of  $w$ , so this measure is the sum of the sizes of the excedances. It is well known that excedance number and *descent number* are equidistributed over  $\mathfrak{S}_n$ . (Their common generating function is the Eulerian polynomial.) In a similar vein, we find that summing the size of a descent gives a statistic with the same distribution as depth. That is, define the *descent drop* of  $w$  to be

$$\text{drops}(w) = \sum_{w(i) > w(i+1)} (w(i) - w(i+1)).$$

(We note that in [4] Chung, Claesson, Dukes, and Graham study descent numbers in relation to “maximum drop size,” that is,  $\max\{j - w(j)\}$ . Apart from similar terminology, we know of no deeper connection between this work and that one.) It is generally not true that  $\text{dp}(w) = \text{drops}(w)$ . For example,  $\text{dp}(3241) = 2 + 1 = 3$  while  $\text{drops}(w) = 1 + 3 = 4$ . Nonetheless, we will prove the following via a bijection of Steingrímsson [14, Appendix].

**Theorem 1.2.** *For all  $n$ , the pairs of statistics  $(\text{des}, \text{drops})$  and  $(\text{exc}, \text{dp})$  are equidistributed over  $\mathfrak{S}_n$ . That is,*

$$\sum_{w \in \mathfrak{S}_n} q^{\text{drops}(w)} t^{\text{des}(w)} = \sum_{w \in \mathfrak{S}_n} q^{\text{dp}(w)} t^{\text{exc}(w)}.$$

*In particular,*

$$|\{w \in \mathfrak{S}_n : \text{dp}(w) = k\}| = |\{w \in \mathfrak{S}_n : \text{drops}(w) = k\}|.$$

We are also able to characterize, in terms of pattern avoidance, those permutations for which the lower and upper bounds for depth are achieved.

**Theorem 1.3.** *Let  $w \in \mathfrak{S}_n$ . Then,*

- $\text{dp}(w) = \ell(w)$  if and only if  $w$  avoids the pattern 321, and
- $\ell'(w) = \text{dp}(w)$  if and only if  $w$  avoids the patterns 321 and 3412.

*In particular, there are Catalan-many permutations for which  $\text{dp}(w) = \ell(w)$ , and odd-indexed-Fibonacci-many permutations for which  $\ell'(w) = \text{dp}(w)$ .*

Achieving the lower bound here,  $\ell'(w) = \text{dp}(w)$ , also turns out to imply  $\ell'(w) = \ell(w)$ , as discussed in Observation 2.3. The permutations for which  $\ell'(w) = \ell(w)$  were first characterized by Edelman [5, Theorem 3.1]. Here we see Edelman's "unimodal" permutations in a slightly different guise, following work of the second author [20, 19], which provides a dictionary for translating between one-line notation of a permutation and reduced decompositions. The permutations for which  $\text{dp}(w) = \ell(w)$  are characterized directly; we also have a bijection with Dyck paths to help make the characterization more intuitive.

The in-depth discussion of the case of permutations begins in Section 3. In particular, Theorem 1.1 is proved in Section 3.1 and Theorem 1.2 is proved in Section 3.2. Theorem 1.3 is a summary of Theorems 4.1 and 4.7, proved in Sections 4.1 and 4.2, respectively.

As motivation for comparing the three statistics  $\ell'$ ,  $\text{dp}$ , and  $\ell$ , we compare their values for elements of  $\mathfrak{S}_3$  and  $\mathfrak{S}_4$  in Table 1.

## 2. DEPTH IN A COXETER GROUP

We assume some basic knowledge of Coxeter groups in this section. See Humphreys [7] for facts about Coxeter groups that are not explained here. The reader interested only in permutation statistics can safely skip to Section 3.

Let  $W$  be a Coxeter group, with simple reflection set  $S$ . The set of all reflections is the set of conjugates of the simple reflections, denoted  $T = \{wsw^{-1} : w \in W, s \in S\}$ . There are two common measures associated to any element  $w \in W$ . Its *length*,  $\ell(w)$ , is the minimal number of simple reflections needed to express  $w$ , while its *reflection length*,  $\ell'(w)$ , is the minimal number of not-necessarily-simple reflections:

$$\begin{aligned}\ell(w) &= \min\{k : w = s_1 \cdots s_k, s_i \in S\}, \\ \ell'(w) &= \min\{k : w = t_1 \cdots t_k, t_i \in T\}.\end{aligned}$$

Decompositions of the form  $w = s_1 \cdots s_{\ell(w)}$  with  $s_i \in S$  or  $w = t_1 \cdots t_{\ell'(w)}$  with  $t_i \in T$  are called a *reduced decompositions* (with respect to  $S$  or with respect to  $T$ ).

Now let  $\Phi = \Pi \cup -\Pi$  denote the root system associated to  $W$ , and let  $\Delta = \{\alpha_1, \alpha_2, \dots\}$  denote the *simple roots*. Following [3] (see also [16]), the *depth* of a positive root  $\beta \in \Pi$ , denoted  $\text{dp}(\beta)$ , is

$$\text{dp}(\beta) = \min\{k : s_1 \cdots s_k(\beta) \in -\Pi, s_i \in S\}.$$

It is always the case that  $\text{dp}(\beta) \geq 1$ , with  $\text{dp}(\beta) = 1$  if and only if  $\beta \in \Delta$  is a simple root.

There is a one-to-one correspondence between reflections and positive roots: the reflection  $t_\beta$  is the reflection across the hyperplane whose normal vector is  $\beta$ . In particular,  $t_\beta(\beta) = -\beta$ . Now, for any group element  $w \in W$ , we define its depth to be the minimum sum of depths in a factorization of  $w$  into reflections. That is,

$$\text{dp}(w) = \min \left\{ \sum_i \text{dp}(\beta_i) : w = t_{\beta_1} \cdots t_{\beta_k}, t_{\beta_i} \in T \right\}.$$

$w \in \mathfrak{S}_3 = A_2$				$w \in \mathfrak{S}_4$			
$w$	$\ell'(w)$	$\text{dp}(w)$	$\ell(w)$	$w$	$\ell'(w)$	$\text{dp}(w)$	$\ell(w)$
123	0	0	0	1234	0	0	0
213	1	1	1	2134	1	1	1
132	1	1	1	1324	1	1	1
312	2	2	2	1243	1	1	1
231	2	2	2	2314	2	2	2
321	1	2	3	2143	2	2	2
				3124	2	2	2
				1342	2	2	2
				1423	2	2	2
				3214	1	2	3
				1432	1	2	3
				2341	3	3	3
				2413	3	3	3
				3142	3	3	3
				4123	3	3	3
				3241	2	3	4
				2431	2	3	4
				4132	2	3	4
				4213	2	3	4
				3412	2	4	4
				4231	1	3	5
				4312	3	4	5
				3421	3	4	5
				4321	2	4	6

$w \in G_2$			
$w$	$\ell'(w)$	$\text{dp}(w)$	$\ell(w)$
$e$	0	0	0
$s_1$	1	1	1
$s_2$	1	1	1
$s_1s_2$	2	2	2
$s_2s_1$	2	2	2
$s_1s_2s_1$	1	2	3
$s_2s_1s_2$	1	2	3
$s_1s_2s_1s_2$	2	3	4
$s_2s_1s_2s_1$	2	3	4
$s_1s_2s_1s_2s_1$	1	3	5
$s_2s_1s_2s_1s_2$	1	3	5
$s_1s_2s_1s_2s_1s_2$	2	4	6

TABLE 1. Reflection length, depth, and length for the elements of the groups  $\mathfrak{S}_3 = A_2$ ,  $G_2$ , and  $\mathfrak{S}_4 = A_3$ . Note that each element  $w$  satisfies  $\ell'(w) \leq \text{dp}(w) \leq \ell(w)$ .

We can see that, by definition,

$$2 \text{dp}(\beta) = \ell(t_\beta) + 1.$$

It then follows that

$$\text{dp}(t_\beta) = \text{dp}(\beta) = \frac{\ell(t_\beta) + 1}{2},$$

since if

$$t_\beta = t_{\gamma_1} \cdots t_{\gamma_k},$$

is any factorization of  $t_\beta$  as a product of reflections, then

$$\begin{aligned} 2(\text{dp}(\gamma_1) + \cdots + \text{dp}(\gamma_k)) &= \ell(t_{\gamma_1}) + \cdots + \ell(t_{\gamma_k}) + k, \\ &\geq \ell(t_\beta) + k, \\ &\geq 2 \text{dp}(\beta) + k - 1, \\ &\geq 2 \text{dp}(\beta). \end{aligned}$$

In fact, this shows that any expression of  $t_\beta$  using more than one reflection forces a factorization of greater depth.

We can now phrase depth simply in terms of the lengths of the reflections in a reflection factorization of  $w$ .

**Observation 2.1.** *The depth of  $w$  is given by*

$$\begin{aligned} \text{dp}(w) &= \min \left\{ \sum_{i=1}^k \text{dp}(t_i) : w = t_1 \cdots t_k, t_i \in T \right\}, \\ &= \frac{1}{2} \min \left\{ k + \sum_{i=1}^k \ell(t_i) : w = t_1 \cdots t_k, t_i \in T \right\}. \end{aligned}$$

Since  $\ell(t_i) \geq 1$  for  $t_i \in T$ , it is clear from this last observation that  $\text{dp}(w)$  is at least as great as reflection length. Moreover, if  $w = s_1 \cdots s_{\ell(w)}$  is a reduced factorization in  $S$ , then  $\text{dp}(w) \leq (\ell(w) + \sum_{i=1}^{\ell(w)} \ell(s_i))/2 = \ell(w)$ . In summary, we have the following.

**Observation 2.2.** *The depth of an element  $w$  satisfies*

$$\ell'(w) \leq \text{dp}(w) \leq \ell(w).$$

Another easy observation follows from noticing that a factorization of  $w$  can never have fewer than  $\ell'(w)$  terms, and the minimum depth of a term is 1. Thus if depth and reflection length are equal for  $w$ , then the depth of every term is actually equal to 1. In other words, the factorization consists entirely of simple reflections, and thus length equals reflection length. Together with Observation 2.2, we have the following.

**Observation 2.3.** *For  $w \in W$ , we have  $\text{dp}(w) = \ell'(w)$  if and only if  $\ell(w) = \ell'(w)$ .*

For dihedral groups, depth is straightforward. Let  $I_2(m)$  denote the dihedral group of order  $2m$ ,  $m \leq \infty$ , and let  $S = \{s_1, s_2\}$  denote the simple reflections.

**Proposition 2.4.** *For an element  $w \in I_2(m)$ , we have*

$$\text{dp}(w) = \begin{cases} \frac{\ell(w)+1}{2} & \text{if } \ell(w) \text{ odd,} \\ \frac{\ell(w)}{2} + 1 & \text{if } \ell(w) \text{ even.} \end{cases}$$

Hence,

$$\sum_{w \in I_2(m)} q^{\ell(w)} t^{\text{dp}(w)} = \begin{cases} 1 + 2qt + q^m t^{\frac{m}{2}+1} + 2(1+q)t \sum_{i=1}^{\frac{m}{2}-1} q^{2i} t^i & \text{if } m \text{ is even,} \\ 1 + 2qt + q^{m-1} t^{\frac{m+1}{2}} (2+q) + 2(1+q)t \sum_{i=1}^{\frac{m-3}{2}} q^{2i} t^i & \text{if } m \text{ is odd,} \\ 1 + 2qt \cdot \frac{1+qt}{1-q^2t} & \text{if } m = \infty. \end{cases}$$

*Proof.* Indeed, suppose an element  $w = s_1 s_2 s_1 \cdots$ . Then if  $\ell(w) = 2d - 1$  is odd, we know  $w \in T$ , and  $\text{dp}(w) = d$ . Similarly, we can argue that if  $\ell(w) = 2(d-1)$  is even, then  $\text{dp}(w) = d$ . One simply shows that a factorization into two reflections,  $(s_1)(s_2 s_1 \cdots s_2)$  gives this depth, and that a factorization into more than two reflections necessarily increases depth.

The generating function expressions follow, since there are exactly two elements of each length, apart from the identity (length zero) and the long element (length  $m$ ).  $\square$

For example, we see the distributions of length and height for  $I_2(3) = A_2$  and  $I_2(6) = G_2$  in Table 1.

**Remark 2.5** (Depth is additive). Notice that if  $W$  is reducible, say  $W = W_1 \times W_2$ , all the reflections in  $W_1$  commute with the reflections in  $W_2$ . Thus if  $w = uv$  with  $u \in W_1, v \in W_2$ , then  $\text{dp}(w) = \text{dp}(u) + \text{dp}(v)$ . Hence, to characterize depth, it suffices to study irreducible Coxeter groups.

**Remark 2.6** (Depth versus height). Recall that every root is a linear combination of simple roots,  $\sum_i c_i \alpha_i$ , and the *height* of a root is  $\sum_i c_i$ . Thus, one can then assign a minimum “height” to any reflection factorization just as we have assigned a depth.

Whenever  $\Phi$  is a finite crystallographic root system, depth and height agree for short roots. If  $\Phi$  is a root system of type  $A_n$ , all the roots are the same length, and so the depth of a root is the same as its height. Thus, we could just as easily have called the permutation statistic we study in Sections 3 and 4 the “height” of a permutation.

However, if  $\Phi$  has both long and short roots, depth and height need not agree, and it is our feeling that depth is the better behaved of the two statistics. In fact, it need not even be the case that the height of a reflection  $t$  be the same as the height of its corresponding positive root! (The reflection  $s_1 s_2 s_1$  in  $G_2$  corresponds to a root of height 4, for example.) Moreover, depth makes sense for all root systems, not only crystallographic ones. Observation 2.1 shows depth is independent of the choice of root system; that is, it depends only on the group.

We now turn our focus to the case of the symmetric group  $\mathfrak{S}_n$ ; that is, the Coxeter group of type  $A_{n-1}$ .

### 3. THE DEPTH STATISTIC

For a positive integer  $n$ , let  $[n] = \{1, 2, \dots, n\}$ , and let  $\mathfrak{S}_n$  be the symmetric group on  $[n]$ ; that is, the set  $\{w : [n] \rightarrow [n]\}$  of all bijections from  $[n]$  to itself. This group is a Coxeter group of type  $A_{n-1}$ . The set of simple reflections in  $\mathfrak{S}_n$  is denoted by  $S_n = \{s_1, \dots, s_{n-1}\}$ , where  $s_i = (i \ i+1) \in \mathfrak{S}_n$  is the adjacent transposition interchanging  $i$  and  $i+1$ , and fixing all other elements.

The reflections in  $\mathfrak{S}_n$  are the transpositions. We denote the set of all transpositions by

$$T_n = \{t_{ij} : 1 \leq i < j \leq n\},$$

where  $t_{ij} = (i \ j) \in \mathfrak{S}_n$  is the permutation interchanging  $i$  and  $j$ , and fixing all other elements. To avoid confusion with the subscripts, we will occasionally write  $t_{i,j}$  for  $t_{ij}$ . Notably,  $s_i = t_{i,i+1}$ . It is easily checked that the depth of a transposition is

$$\text{dp}(t_{ij}) = |i - j|.$$

We will freely write elements of  $w \in \mathfrak{S}_n$  as permutations in one-line notation, permutations in cycle notation, words on the set  $S_n$ , or words on the set  $T_n$ . For example,

$$2431756 = (124)(3)(576) = s_1 s_2 s_3 s_2 s_6 s_5 = t_{12} t_{24} t_{67} t_{56}.$$

It is well known that length is equal to the number of *inversions* of  $w$ . Let  $\text{inv}(w)$  denote the number of pairs  $i < j$  such that  $w(i) > w(j)$ . For example, we see that  $\text{inv}(2431756) = 6$ .

Also well known is that reflection length is equal to  $n - c(w)$ , where  $c(w)$  denotes the number of cycles of  $w$ . With  $w = (124)(3)(576)$  as above, we see that  $7 - c(w) = 4$ . It is interesting that length is easily computed from the one-line notation of a permutation, whereas reflection length is easily computed from cycle notation. It would be very interesting if there was a notation for permutations in which both length and reflection length are easily seen.

**3.1. An algorithm and a characterization of depth.** We begin our discussion of depth of a permutation by reconsidering straight selection sort applied to  $w = 3715246$ . We have:

$$\begin{array}{rcc}
 & 3715246 & \\
 (67) \cdot & \downarrow & \cdot (27) \\
 & 3615247 & \\
 (46) \cdot & \downarrow & \cdot (26) \\
 & 3415267 & \\
 (25) \cdot & \downarrow & \cdot (45) \\
 & 3412567 & \\
 (24) \cdot & \downarrow & \cdot (24) \\
 & 3214567 & \\
 (13) \cdot & \downarrow & \cdot (13) \\
 & 1234567 & 
 \end{array}$$

In each line, we have written on the left the transposition  $(w(i) i)$  for the highlighted letter  $i$  (the largest number not yet in its proper place), while on the right we have  $(w^{-1}(i) i)$ . The difference is only whether we consider the action of left multiplication, which swaps the specified digits, or right multiplication, which swaps the digits in the specified positions.

We see that we could associate many different factorizations of  $w$  to the straight selection sort algorithm. For the sorting index of [10], we use right multiplication at each step to obtain a reduced reflection decomposition for  $w$ . This gives  $w = t_{13}t_{24}t_{45}t_{26}t_{27}$ , and

$$\text{dp}(w) \leq (3 - 1) + (4 - 2) + (5 - 4) + (6 - 2) + (7 - 2) = 14.$$

On the other hand, we can see equally well that  $t_{13}t_{24}t_{46}t_{67} \cdot w \cdot t_{45} = e$ , and so  $w = t_{67}t_{46}t_{24}t_{13}t_{45}$ . This factorization shows

$$\text{dp}(w) \leq (7 - 6) + (6 - 4) + (4 - 2) + (3 - 1) + (5 - 4) = 8.$$

In fact, by choosing the transposition of least depth in each step of the straight selection sort, we achieve a minimal depth factorization. We make the procedure precise with the algorithm SHALLOW-DECOMP below.

**Algorithm SHALLOW-DECOMP.**

The input of the algorithm is a permutation  $w \in \mathfrak{S}_n$ . The output is a pair of reduced decompositions  $u = u(w)$ , and  $v = v(w)$ , such that  $w = uv$ .

- (1) If  $n = 1$ , then  $u := e$ ,  $v := e$ .
- (2) If  $w(n) = n$ , then let  $w' := w(1) \cdots w(n-1)$  and  $u := u(w')$ ,  $v := v(w')$ .
- (3) If  $w(n) < n$  then  $w' := t_{w(n),n} \cdot w$  and:
  - (a) if  $w(n) \leq w^{-1}(n)$  then  $u := u(w')$ ,  $v := v(w') \cdot t_{w^{-1}(n),n}$ ,
  - (b) if  $w(n) > w^{-1}(n)$  then  $u := t_{w(n),n} \cdot u(w')$ ,  $v := v(w')$ .

With the example of  $w = 3715246$  as above, we can trace through the algorithm to find

$$u = t_{67}t_{46}t_{24}t_{13} \quad \text{and} \quad v = t_{45}.$$

We now prove the following, which establishes Theorem 1.1.

**Theorem 3.1.** *For  $w \in \mathfrak{S}_n$ , the output of SHALLOW-DECOMP gives a reduced expression  $w = uv$  of minimal reflection length and minimal depth. Moreover, this depth is equal to*

$$\text{dp}(w) = \sum_{w(i) > i} (w(i) - i).$$

*Proof.* The fact that  $uv = w$  is obvious by construction. That the factorization has  $\ell'(w)$  terms follows from the fact that straight selection sort achieves minimal reflection length: a transposition can only change the number of cycles by  $\pm 1$ , and the algorithm uses steps that each increase the number of fixed points, and hence, cycles.

To see that the algorithm achieves the minimal depth, we proceed by induction on  $n$ . When  $n = 1$ , this is obviously true, and the given formula for depth works.

Now suppose  $n > 1$  and let

$$d(w) = \sum_{w(i) > i} (w(i) - i).$$

Suppose  $\text{dp}(w') = d(w')$  for all  $w' \in \mathfrak{S}_{n-1}$ . Let  $w \in \mathfrak{S}_n$ . If  $w(n) = n$ , then we can view  $w$  as lying in  $\mathfrak{S}_{n-1}$ , and so  $\text{dp}(w) = d(w)$ .

Now let  $j = w^{-1}(n) < n$ . We will first show  $\text{dp}(w) \leq d(w)$ . As in step (2) of SHALLOW-DECOMP, we let  $w' = t_{w(n),n} \cdot w$ . There are two cases to consider, given in steps (3a) and (3b). In the first case, from (3a), we suppose that  $w(n) \leq j$ , in which case  $w'(j) = w(n)$  is not an excedance of  $w'$ , so

$$\begin{aligned} d(w) &= \sum_{w(i) > i} (w(i) - i), \\ &= (n - j) + \sum_{i \neq j: w(i) > i} (w(i) - i), \\ &= (n - j) + \sum_{w'(i) > i} (w'(i) - i), \\ &= (n - j) + d(w'), \\ &= (n - j) + \text{dp}(w'). \end{aligned}$$

We get that  $d(w)$  is the sum of the depths in the factorization

$$w = u(w') \cdot v(w') \cdot t_{jn},$$

and so  $\text{dp}(w) \leq d(w)$ .



For the next case, from (3b) of the algorithm, we suppose that  $w(n) = w'(j) > j$ . This time,

$$\begin{aligned}
 d(w) &= \sum_{w(i) > i} (w(i) - i), \\
 &= (n - j) + \sum_{i \neq j: w(i) > i} (w(i) - i), \\
 &= (n - j) + \sum_{i \neq j: w'(i) > i} (w'(i) - i), \\
 &= (n - j) + d(w') - (w(n) - j), \\
 &= (n - w(n)) + d(w'), \\
 &= (n - w(n)) + \text{dp}(w').
 \end{aligned}$$

Again we conclude that  $d(w)$  is the sum of the depths in the factorization

$$w = t_{w(n),n} \cdot u(w') \cdot v(w'),$$

yielding  $\text{dp}(w) \leq d(w)$ .

Thus, we see that  $d(w)$  is an upper bound for depth in all cases. On the other hand, we know  $d(w) \leq \text{dp}(w)$ , because, at the very least, all of the excedances  $w(i)$  in a permutation are  $w(i) - i$  places away from their initial positions. Combining the inequalities we have  $\text{dp}(w) = d(w)$ , as desired.  $\square$

We include here for reference Table 2, which contains the distribution of depths for  $n \leq 8$ . This array can be found as entry A062869 of [13].

$n$	$k=0$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1																
2	1	1															
3	1	2	3														
4	1	3	7	9	4												
5	1	4	12	24	35	24	20										
6	1	5	18	46	93	137	148	136	100	36							
7	1	6	25	76	187	366	591	744	884	832	716	360	252				
8	1	7	33	115	327	765	1523	2553	3696	4852	5708	5892	5452	4212	2844	1764	576

TABLE 2. The number of permutations  $w \in \mathfrak{S}_n$  with depth  $\text{dp}(w) = k$ .

Given the formula for depth in Theorem 3.1, we can prove the following.

**Proposition 3.2.** *For all  $w \in \mathfrak{S}_n$ , we have  $\text{dp}(w) \leq \lfloor n^2/4 \rfloor$ , and this bound is sharp.*

Further, we can say precisely how many permutations achieve this upper bound.

**Proposition 3.3.** *The number of permutations in  $\mathfrak{S}_n$  achieving maximal depth is*

$$|\{w \in \mathfrak{S}_n : \text{dp}(w) = \lfloor n^2/4 \rfloor\}| = \begin{cases} (k!)^2 & \text{if } n = 2k, \\ n(k!)^2 & \text{if } n = 2k + 1. \end{cases}$$

We remark that both these propositions can be found in the remarks (and links therein) for entry A062870 of [13]. In particular, Alekseyev has a webpage [1] with a proof of Proposition 3.3.

To prove the propositions, it will be helpful to have the following lemma.

**Lemma 3.4.** *Let  $w \in \mathfrak{S}_n$ , and suppose  $i < j$  and  $w(i) < w(j)$ . Then,*

$$\mathrm{dp}(w \cdot t_{ij}) = \begin{cases} \mathrm{dp}(w) & \text{if } j \leq w(i) \text{ or } w(j) < i, \\ \mathrm{dp}(w) + \min\{w(j), j\} - \max\{w(i), i\} & \text{otherwise.} \end{cases}$$

*In particular,  $\mathrm{dp}(w \cdot t_{ij}) \geq \mathrm{dp}(w)$  for all such  $w, i$ , and  $j$ .*

*Proof.* Each of these cases is straightforward to verify. We will do the first few. The remaining cases follow similar lines of reasoning.

Let  $w'$  denote  $wt_{ij}$ .

**(Case:  $w(i) < w(j) < i < j$ .)** If  $w(i) < w(j) < i < j$ , then, in both  $w$  and  $w'$ , neither  $w(i)$  nor  $w(j)$  are excedances. Since these are the only positions in which  $w$  and  $w'$  differ, they have exactly the same excedance set, and so  $\mathrm{dp}(w) = \mathrm{dp}(w')$  by the characterization of depth given in Theorem 3.1.

**(Case:  $i < j \leq w(i) < w(j)$ .)** Now suppose that  $i < j \leq w(i) < w(j)$ . Then

$$\begin{aligned} \mathrm{dp}(w) &= \sum_{w(k) > k} (w(k) - k) \\ &= \cdots + (w(i) - i) + \cdots + (w(j) - j) + \cdots \\ &= \cdots + (w(j) - i) + \cdots + (w(i) - j) + \cdots \\ &= \cdots + (w'(i) - i) + \cdots + (w'(j) - j) + \cdots \\ &= \sum_{w'(k) > k} (w'(k) - k) = \mathrm{dp}(w'), \end{aligned}$$

as desired.

**(Case:  $i \leq w(i) < w(j) < j$ .)** If  $i \leq w(i) < w(j) < j$ , then we get

$$\begin{aligned} \mathrm{dp}(w) &= \sum_{w(k) > k} (w(k) - k) \\ &= (w(i) - i) + \sum_{k \neq i: w(k) > k} (w(k) - k) \\ &= w(i) - w(j) + (w(j) - i) + \sum_{k \neq i: w(k) > k} (w(k) - k) \\ &= w(i) - w(j) + (w'(i) - i) + \sum_{k \neq i: w'(k) > k} (w'(k) - k) \\ &= w(i) - w(j) + \sum_{w'(k) > k} (w'(k) - k) = w(i) - w(j) + \mathrm{dp}(w'), \end{aligned}$$

as desired.

The remaining cases are similar. □

With the lemma in hand, we now prove Proposition 3.2.

*Proof of Proposition 3.2.* Observe that the longest permutation,  $w_0 = n(n-1)\cdots 21$ , achieves the claimed upper bound of  $\lfloor n^2/4 \rfloor$ . Indeed, if  $n$  is even, then

$$\begin{aligned} \text{dp}(w_0) &= (n-1) + ((n-1)-2) + \cdots = (n-1) + (n-3) + \cdots + 5 + 3 + 1 \\ &= \left(\frac{n}{2}\right)^2. \end{aligned}$$

If  $n$  is odd, then

$$\begin{aligned} \text{dp}(w_0) &= (n-1) + ((n-1)-2) + \cdots = (n-1) + (n-3) + \cdots + 4 + 2 \\ &= \left(\frac{n+1}{2}\right) \left(\frac{n+1}{2} - 1\right) = \frac{n^2-1}{4}. \end{aligned}$$

Now, we claim that  $\text{dp}(w_0) \geq \text{dp}(w)$  for any  $w \in \mathfrak{S}_n$ .

First suppose  $w_0 = wt_{ij}$ . Then since  $w_0$  is totally decreasing, it must be that  $i < j$  and  $w(i) < w(j)$ . Thus  $\text{dp}(w_0) \geq \text{dp}(w)$  by Lemma 3.4.

In general, we can proceed from any permutation  $w$  to  $w_0$  by transpositions of this type. To see this, define  $A(w) = \{t_{ij} : i < j, w(i) < w(j)\}$ . Then  $|A(w_0)| = 0$ , and it is the only permutation with this property. Generally, if  $t \in A(w)$ , then

$$A(wt) \subseteq (A(w) \setminus \{t\}).$$

In particular,  $|A(wt)| < |A(w)|$ . So by some choice of transpositions  $t_{ij}$  with  $i < j$  and  $w(i) < w(j)$ , we can move from any  $w$  to  $w_0$ . Lemma 3.4 shows that this cannot decrease depth. So there exists a sequence of transpositions  $t_1, \dots, t_r$  such that

$$|A(w)| > |A(wt_1)| > \cdots > |A(wt_1 \cdots t_r)| = |A(w_0)| = 0,$$

and

$$\text{dp}(w_0) = \text{dp}(wt_1 \cdots t_r) \geq \cdots \geq \text{dp}(wt_1) \geq \text{dp}(w).$$

This completes the proof.  $\square$

Now we will prove Proposition 3.3, characterizing the permutations with maximal depth.

*Proof of Proposition 3.3.* To begin, suppose  $n = 2k$  is even. We will show that the permutations for which  $\text{dp}(w) = k^2$  are precisely those permutations for which

$$\{w(1), w(2), \dots, w(k)\} = \{k+1, k+2, \dots, n\},$$

or, equivalently,

$$\{w(k+1), w(k+2), \dots, w(n)\} = \{1, 2, \dots, k\}.$$

Denote this set by  $\mathfrak{S}_{k,k}$ . Obviously there are  $(k!)^2$  permutations in this set, since the first  $k$  elements can be permuted independently of the final  $k$  elements.

Suppose  $w \in \mathfrak{S}_{k,k}$ . If  $i, j \in \{1, 2, \dots, k\}$  or if  $i, j \in \{k+1, k+2, \dots, n\}$ , then  $wt_{ij} \in \mathfrak{S}_{k,k}$  as well. Moreover, from Lemma 3.4, we see that

$$\text{dp}(wt_{ij}) = \text{dp}(w).$$

To see that these depths are equal to  $k^2 = \lfloor n^2/4 \rfloor$ , we note that, just as in the proof of Proposition 3.2, a succession of such transpositions can be used to obtain the decreasing permutation  $w_0 \in \mathfrak{S}_{k,k}$ , whose depth was already established as  $\text{dp}(w_0) = n^2/4$ .

Finally, we observe that if  $w \notin \mathfrak{S}_{k,k}$ , then there are indices  $i$  and  $j$ , with  $i \leq k < j$ , such that  $w(i) \leq k < w(j)$ . Using Lemma 3.4 and Proposition 3.2 we have  $\text{dp}(w) < \text{dp}(wt_{ij}) \leq \text{dp}(w_0)$ .

The case when  $n$  is odd proceeds analogously, except that we will define  $n = 2k + 1$  different sets  $\mathfrak{S}_{i;k,k}$ , one for each choice  $i \in [n]$  of the the middle value,  $w(k + 1)$ , of the permutation.

If  $i \leq k + 1$ , then  $\mathfrak{S}_{i;k,k}$  is the set of permutations for which

$$\{w(1), \dots, w(k)\} = \{k + 2, \dots, n\},$$

or, equivalently,

$$\{w(k + 2), \dots, w(n)\} = \{1, \dots, k + 1\} - \{i\}.$$

If  $i > k + 1$ , then  $\mathfrak{S}_{i;k,k}$  is the set of permutations for which

$$\{w(1), \dots, w(k)\} = \{k + 1, \dots, n\} - \{i\},$$

or, equivalently,

$$\{w(k + 2), \dots, w(n)\} = \{1, \dots, k\}.$$

Although  $w_0 \notin \mathfrak{S}_{i;k,k}$  if  $i \neq k + 1$ , it is not hard to see that each  $\mathfrak{S}_{i;k,k}$  is closed under depth-preserving transpositions that do not involve  $i$ , and each  $\mathfrak{S}_{i;k,k}$  contains a “nearly decreasing” element  $w$  whose non-middle entries are mapped as

$$w : (1, 2, \dots, \widehat{k + 1}, \dots, n) \mapsto (n, n - 1, \dots, \widehat{i}, \dots, 1).$$

Moreover, this  $w$  has depth

$$\frac{n^2 - 1}{4}$$

if  $i \leq k + 1$  and depth

$$\frac{n^2 - 1}{4} - 1(i - (k + 1)) + (i - (k + 1)) = \frac{n^2 - 1}{4}$$

if  $i > k + 1$ . There are  $(k!)^2$  permutations in each  $\mathfrak{S}_{i;k,k}$ , and so a total of  $n(k!)^2$  permutations  $w \in \mathfrak{S}_{2k+1}$  with  $\text{dp}(w) = \lfloor n^2/4 \rfloor$ .

If  $w \notin \mathfrak{S}_{i;k,k}$  for any  $i$ , it follows that there are  $i < k + 1 < j$  such that  $w(i) < k + 1 < w(j)$ , and so again using Lemma 3.4 and Proposition 3.2 we have  $\text{dp}(w) < \text{dp}(wt_{ij}) \leq \text{dp}(w_0)$ .

This completes the proof.  $\square$

**3.2. Depth is equidistributed with descent drop.** We will now prove Theorem 1.2. Recall that  $\text{des}(w) = |\{i : w(i) > w(i + 1)\}|$  and  $\text{exc}(w) = |\{i : w(i) > i\}|$ . In the introduction we defined the *descent drop* of  $w$  to be

$$\text{drops}(w) = \sum_{w(i) > w(i+1)} (w(i) - w(i + 1)),$$

and Theorem 1.1 shows that

$$\text{dp}(w) = \sum_{w(i) > i} (w(i) - i).$$

Theorem 1.2 claims the following:

$$(1) \quad \sum_{w \in \mathfrak{S}_n} q^{\text{drops}(w)} t^{\text{des}(w)} = \sum_{w \in \mathfrak{S}_n} q^{\text{dp}(w)} t^{\text{exc}(w)}.$$

The proof of equation (1) follows from a bijection that carries descents to excedances, due to Steingrímsson [14, Appendix]. (Note that this is related to the “transformation fondamentale” of Foata and Schützenberger [6].) We follow the description of the bijection given in Theorem 51 of [14].

Given  $w \in \mathfrak{S}_n$ , we form  $v \in \mathfrak{S}_n$  with the following process. For each  $j \in [n]$ :

- if  $w(j) > w(k)$  for some  $k > j$ , then

$$v(w(j+1)) = w(j),$$

- otherwise, let  $i < j$  be maximal such that  $w(i) < w(j)$  (with  $w(0) = 0$ ), and declare

$$v(w(i+1)) = w(j).$$

We now let  $\phi : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$  be given by  $\phi(w) = v$ .

For example, let  $w = 7213645$ . In working from left to right in  $w$ : we put  $w(1) = 7$  in position  $w(2) = 2$  of  $v$  because it 7 greater than something to its right in  $w$ ; we put  $w(2) = 2$  in position  $w(3) = 1$  of  $v$  because 2 is greater than something to its right in  $w$ ; we put  $w(3) = 1$  in position  $w(1) = 7$  of  $v$  since it is *not* greater than something to its right in  $w$  and  $w(0) = 0$  is the rightmost number less than  $w(3)$ , and so on. Continuing, we ultimately achieve  $v = 2736541$ .

$w$	7	2	1	3	6	4	5
$v$	$v(1)$	$v(2)$	$v(3)$	$v(4)$	$v(5)$	$v(6)$	$v(7)$
		7					
	2	7					
	2	7					1
	2	7	3				1
	2	7	3	6			1
	2	7	3	6		4	1
	2	7	3	6	5	4	1

That the map  $\phi$  is well-defined and injective (hence bijective) follows from close inspection of its definition, and we will omit proof of this fact. See [15, Section 4.1].

More important from our perspective is the following lemma that follows by construction of  $\phi$ .

**Lemma 3.5** ([14], Section 4.1). *Let  $w \in \mathfrak{S}_n$  and  $v = \phi(w)$ . Then  $w(i) > w(i+1)$  is a descent of  $w$  if and only if  $w(i) = v(w(i+1))$  is an excedance of  $v$ . In particular,  $\text{des}(w) = \text{exc}(v)$ .*

Notice that in the example above,  $w = \mathbf{7213645}$  has descent pairs  $7 > 2$ ,  $2 > 1$ , and  $6 > 5$ , so that  $\text{drops}(w) = 5 + 1 + 1 = 7$ . On the other hand,  $v = \mathbf{2736541}$  has excedances 2 (in position 1), 7 (in position 2), and 6 (in position 5), so  $\text{dp}(v) = 1 + 5 + 1 = 7$ .

From Lemma 3.5 we can now prove Theorem 1.2.

*Proof of Theorem 1.2.* It suffices to show that  $\phi$  carries descents to excedances, and descent drops to depth. Lemma 3.5 states that  $\text{des}(w) = \text{exc}(\phi(w))$ , and moreover, with  $v = \phi(w)$ ,

$$\begin{aligned} \text{drops}(w) &= \sum_{w(i) > w(i+1)} (w(i) - w(i+1)) \\ &= \sum_{w(i) > w(i+1)} (v(w(i+1)) - w(i+1)) \\ &= \sum_{v(k) > k} (v(k) - k) \\ &= \text{dp}(v) = \text{dp}(\phi(w)), \end{aligned}$$

as desired. □

#### 4. COINCIDENCES OF DEPTH, LENGTH, AND REFLECTION LENGTH

From Observation 2.2, we have that reflection length gives a lower bound for depth, while length gives an upper bound for depth. In this section and the next, we will establish Theorem 1.3, in which we characterize, in terms of pattern avoidance, those permutations for which  $\text{dp}(w) = \ell(w)$  and those for which  $\text{dp}(w) = \ell'(w)$ .

First, recall the notion of pattern avoidance. Let  $w \in \mathfrak{S}_n$  and  $p \in \mathfrak{S}_k$ , where  $n \geq k$ . We say that  $w$  *contains* a  $p$ -pattern if there exist indices  $\{i_1 < \dots < i_k\}$  such that  $w(i_1) \dots w(i_k)$  is in the same relative order as  $p(1) \dots p(k)$ . Otherwise, we say that  $w$  *avoids*  $p$ , or is  *$p$ -avoiding*. For example, the permutation 3241576 contains the pattern 1234 (in positions  $\{1, 3, 5, 6\}$ , for example), and avoids the pattern 4321.

**4.1. When depth equals length.** We now present a characterization of those permutations for which  $\text{dp}(w) = \ell(w)$ .

**Theorem 4.1.** *For any  $w$ ,  $\text{dp}(w) = \ell(w)$  if and only if  $w$  avoids 321.*

**Corollary 4.2.** *The number of  $w \in \mathfrak{S}_n$  for which  $\text{dp}(w) = \ell(w)$  is given by the Catalan number  $\text{Cat}_n = \frac{1}{n+1} \binom{2n}{n}$ .*

Recall that a permutation  $w$  is called *fully commutative* if every reduced expression  $w = s_1 \dots s_{\ell(w)}$  for  $w$  (with simple reflections) can be obtained from any other by swapping two letters:  $s_i s_j = s_j s_i$ . In [2, Theorem 2.1] it was shown that  $w$  is fully commutative if and only if  $w$  avoids 321. (See [17] for generalizations.) Thus, we obtain another characterization of when length equals depth.

**Corollary 4.3.** *A permutation  $w \in \mathfrak{S}_n$  has  $\text{dp}(w) = \ell(w)$  if and only if  $w$  is fully commutative.*

We will prove Theorem 4.1 by exhibiting a bijection between  $\{w : \text{dp}(w) = \ell(w)\}$  and Dyck paths. More specifically, we will demonstrate that the fibers of a certain map from  $\mathfrak{S}_n$  to Dyck paths of length  $2n$  have unique minimal length representatives, and that these representatives are the 321-avoiding permutations.

Let  $\text{Dyck}_n$  denote the set of Dyck paths of length  $2n$ ; that is, those lattice paths from  $(0, 0)$  to  $(n, n)$  that take steps North and East, and never pass below the line  $y = x$ .

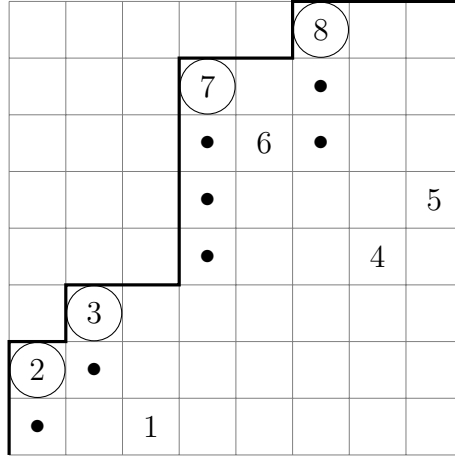


FIGURE 1. An example of the map from  $\mathfrak{S}_n$  to  $\text{Dyck}_n$ .

Let  $\text{lr-max}(w)$  denote the set of *left-right maxima* of  $w$ , written as pairs  $(i, w(i))$ ; that is,

$$\text{lr-max}(w) = \{(i, w(i)) : w(j) < w(i) \text{ for all } j < i\}.$$

From  $w$  we can then form a Dyck path by putting the outer corners of the path at coordinates  $(i - 1, w(i))$  for each left-right maximum. This is easiest to understand with an example. Let  $w = 23176845$ . We will draw  $w$  as a collection of non-attacking rooks on a chessboard, where the value  $w(i)$  is placed in the box appearing in the  $i$ th column from the left and the  $w(i)$ th row from the bottom. We circle the left-right maxima, then draw the Dyck path with these positions as outer corners. Let  $P : \mathfrak{S}_n \rightarrow \text{Dyck}_n$  denote this function. We see  $P(23176845)$  in Figure 1.

It will be useful to have the following lemma.

**Lemma 4.4.** *Fix  $w \in \mathfrak{S}_n$ . If  $w(i) - i > 0$ , then*

$$|\{j : i < j \text{ and } w(i) > w(j)\}| \geq w(i) - i,$$

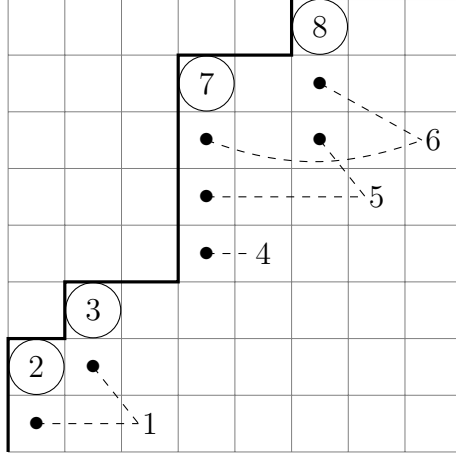
*with equality if and only if  $(i, w(i))$  is a left-right maximum. In particular,*

$$\begin{aligned} |\{(i, j) : i < j, w(i) > w(j), \text{ and } (i, w(i)) \in \text{lr-max}(w)\}| &= \sum_{(i, w(i)) \in \text{lr-max}(w)} (w(i) - i) \\ &\leq \text{dp}(w) \\ &\leq \ell(w). \end{aligned}$$

*Proof.* Every left-right maximum  $(i, w(i))$  is the greater value in  $w(i) - i$  inversions, since there are only  $n - w(i)$  numbers greater than  $w(i)$  and each of these must be placed among the  $n - i$  positions  $\{i + 1, i + 2, \dots, n\}$ . So there are  $n - i - (n - w(i)) = w(i) - i$  numbers  $j$  such that  $i < j$  and  $w(i) > w(j)$ .

If  $(i, w(i))$  is not a left-right maximum, then there are strictly fewer than  $n - w(i)$  numbers greater than  $w(i)$ , to be placed among the  $n - i$  positions to its right. Thus, the number of positions  $j$  such that  $i < j$  and  $w(i) > w(j)$  is:

$$n - i - |\{\text{numbers larger than } w(i) \text{ to the right of } w(i)\}| > n - i - (n - w(i)) = w(i) - i.$$

FIGURE 2. The minimal length element  $w'$ .

□

We have highlighted with bullet points the spaces below left-right maxima and on or above the main diagonal. Note that each bullet point can be identified with an inversion pair in which the left-right maximum above the bullet point is the greater value in the pair, and the value connected to the bullet point by a dashed line is the smaller value in the pair.

Let  $p \in \text{Dyck}_n$  be a Dyck path, and let  $P^{-1}(p)$  denote its preimage under  $P$ , that is,

$$P^{-1}(p) = \{w \in S_n : P(w) = p\}.$$

**Theorem 4.5.** *For any Dyck path  $p$ , the following statements are true of  $P^{-1}(p)$ .*

- (a) *There is a unique element  $w' \in P^{-1}(p)$  of minimal length such that  $\text{dp}(w') = \ell(w')$ .*
- (b) *For any other element  $w' \neq w \in P^{-1}(p)$ ,  $\text{dp}(w) < \ell(w)$ .*

*Proof.* For part (a), we simply observe that if the elements that are not left-right maxima are arranged in increasing order, then the *only* inversions have the left-right maxima as the larger elements in the inversion pairs. Letting  $w'$  denote this permutation, we have

$$\{(i, j) : i < j, w'(i) > w'(j)\} = \{(i, j) : i < j, w'(i) > w'(j), \text{ and } (i, w'(i)) \in \text{lr-max}(w')\}.$$

That is, the inequalities in Lemma 4.4 are equalities. For example the path in Figure 1 has  $w' = 23174856$ , where  $\{1, 4, 5, 6\}$  have been inserted into  $23\_7\_8\_$  in increasing order. See Figure 2.

For part (b), there are two cases to consider for  $w \in P^{-1}(p)$ .

If  $w$  and  $w'$  have the same excedance set, then  $\text{dp}(w) = \text{dp}(w')$ . If  $w \neq w'$ , the elements that are not left-right maxima cannot be increasing in  $w$ , and thus there must be an inversion among those numbers, giving  $\ell(w) > \ell(w') = \text{dp}(w') = \text{dp}(w)$ .

If  $w$  has excedances apart from the left-right maxima, then we rely on Lemma 4.4 again, which says that if  $w(i) - i = k > 0$  and  $(i, w(i))$  is *not* a left-right maximum, then  $w(i)$  is



the greater value in more than  $k$  inversions, and so

$$\begin{aligned} \text{dp}(w) &= \sum_{w(i)>i} (w(i) - i) \\ &< \sum_{w(i)>i} |\{j : i < j, w(i) > w(j)\}| \\ &\leq \text{inv}(w) = \ell(w), \end{aligned}$$

as desired.  $\square$

**Proposition 4.6.** *The minimal length fiber representatives,  $w'$ , are precisely the 321-avoiding permutations.*

*Proof.* Let  $w'$  be the minimal length representative of  $P^{-1}(p)$ . By definition, any left-right maximum can only play the role of ‘3’ in the pattern 321. Thus, the other two elements in the pattern must be elements that are not left-right maxima. But by construction, the subword of these elements is strictly increasing, so it avoids the pattern 21.

On the other hand, it is straightforward to check that if  $u$  and  $v$  are two distinct permutations that avoid 321, then  $\text{lr-max}(u) \neq \text{lr-max}(v)$ , and so they correspond to distinct Dyck paths:  $P(u) \neq P(v)$ .  $\square$

We have now proved Theorem 4.1. Note that the class of permutations defined in Theorem 4.1 is entry P0002 of [18], enumerated by sequence A000108 of [13].

**4.2. When reflection length, depth, and length coincide.** If we are given a permutation  $w$  such that length equals reflection length, then Observation 2.2 clearly implies that the depth of  $w$  has the same value. On the other hand, if we know only that depth equals reflection length, then Observation 2.3 claims that length equals reflection length and again all three are equal:

$$\ell'(w) = \text{dp}(w) = \ell(w).$$

It is this family of permutations that we study in this section.

The question of when  $\ell'(w) = \ell(w)$  has been studied before. Edelman [5] studied the question of the joint distribution of the number of cycles,  $n - \ell'(w)$ , together with the inversion number,  $\ell(w)$ . In particular, [5, Theorem 3.1] characterizes those  $w$  for which  $\ell'(w) = \ell(w)$ . These are what Edelman calls “unimodal” permutations; that is, permutations in which the cycles are disjoint intervals and each cycle is unimodal. Here, we give a different characterization, in the language of pattern avoidance.

**Theorem 4.7.** *The length and the reflection length of a permutation  $w$  agree if and only if  $w$  is 321- and 3412-avoiding.*

In the context of [19], and the subsequent papers [11, 12] by Ragnarsson and the second author, such permutations were called *boolean*.

The proof of Theorem 4.7 relies heavily on the main result of [20], where the second author provided a dictionary for translating between the one-line notation of a permutation and its reduced decompositions. The second author extended this work in [19].

Before proceeding, we recall that simple reflections obey the following *Coxeter relations*.

$$\begin{aligned} s_i^2 &= 1 && \text{for all } i \\ s_i s_j &= s_j s_i && \text{for all } i, j \text{ with } |i - j| > 1 \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} && \text{for all } i \end{aligned}$$

It was shown, independently, by Matsumoto [9] and Tits [21] that any reduced decomposition of  $w$  can be obtained from any other by applying a sequence of these Coxeter relations. Thus, the set of letters appearing in any reduced decomposition of a permutation is fixed by the permutation, regardless of which reduced decomposition is examined. For a permutation  $w \in \mathfrak{S}_n$ , let  $\text{supp}(w) \subseteq S_n$  be the set of letters appearing in reduced decompositions of  $w$ . We call this set the *support* of  $w$ .

We will need the following lemmas.

**Lemma 4.8** (see [20]). (a) *If  $w$  contains a 321-pattern, then there exists a reduced decomposition of  $w$  of the form*

$$\alpha s_k s_{k+1} s_k \beta,$$

where  $\alpha$  and  $\beta$  are products of simple reflections.

(b) *If  $w$  contains a 3412-pattern, then there exists a reduced decomposition of  $w$  of the form*

$$\alpha s_k s_{k-1} s_{k+1} s_k \beta,$$

where  $\alpha$  and  $\beta$  are products of simple reflections.

**Lemma 4.9** ([19], Theorem 4.3). *A permutation  $w$  avoids both 321 and 3412 if and only if each of its reduced decompositions contains no repeated letters; that is, if and only if  $\ell(w) = |\text{supp}(w)|$ .*

*Proof of Theorem 4.7.* Suppose that  $w$  contains a 321-pattern. Then, by Lemma 4.8(a), there is a decomposition of  $w$  having the form

$$\alpha t_{k,k+2} \beta.$$

This decomposition uses  $\ell(w) - 2$  reflections, and thus  $\ell'(w) \leq \ell(w) - 2 < \ell(w)$ .

Suppose that  $w$  contains a 3412-pattern. Then, by Lemma 4.8(b), there is a decomposition of  $w$  having the form

$$\alpha t_{k-1,k+1} t_{k,k+2} \beta.$$

This decomposition uses  $\ell(w) - 2$  reflections, and thus  $\ell'(w) \leq \ell(w) - 2 < \ell(w)$ .

The result is easy to check for permutations of length 0 and 1. We assume, inductively, that the result holds for all permutations of length less than  $\ell$ . Now suppose that  $\ell(w) = \ell$ , and that  $w$  is a minimal-length counterexample to the theorem; that is,  $w$  avoids 321 and 3412, and  $\ell'(w) = r < \ell$ . To be concrete, let

$$w = t_1 \cdots t_r$$

be a representation of  $w$  as a product of reflections. Consider  $w' = wt_r$ , and say that  $t_r$  is the reflection  $t_{ij}$ . We necessarily have that  $\ell(w') < \ell(w)$ . It follows, then, that  $w' < w$  in the Bruhat order. This means that some reduced decomposition of  $w'$  is a substring of a reduced decomposition of  $w$ . Lemma 4.9 implies that the reduced decompositions of  $w$  had no repeated letters. Thus it follows from the same lemma that the reduced decompositions of  $w'$

have no repeated letters, and so  $w'$  is both 321- and 3412-avoiding. Therefore  $\ell(w') = \ell'(w')$ , by the inductive hypothesis. We know, furthermore, that  $\ell'(w') \leq \ell'(w) - 1$ , because  $t_1 \cdots t_{r-1}$  is a way to write  $w'$  as the product of  $r - 1$  transpositions.

Consider the two values  $\ell(w)$  and  $\ell(w')$ . To understand their relationship, we must ask how the collection of inversions in  $w$  have been changed. Because both  $w$  and  $w'$  are 321-avoiding, there are no inversions of the form  $\{i, x\}$  or  $\{x, j\}$  with  $x \in (i, j)$  in either of the permutations. Thus we can make a bijective correspondence between the inversions of  $w$  and the inversions of  $w'$ , with a single exception, as given in the following table.

Inversion in $w$	Inversion in $w'$	conditions
$\{a, b\}$	$\{a, b\}$	$\{a, b\} \cap \{i, j\} = \emptyset$
$\{h, i\}$	$\{h, j\}$	$h < i$
$\{h, j\}$	$\{h, i\}$	$h < i$
$\{i, h\}$	$\{j, h\}$	$h > j$
$\{j, h\}$	$\{i, h\}$	$h > j$
$\{i, j\}$	—	—

Thus, since length is the inversion number of a permutation, we have that  $\ell(w') = \ell(w) - 1$ , which yields the following string of inequalities:

$$\ell(w') = \ell(w) - 1 \geq \ell'(w) > \ell'(w') = \ell(w').$$

This is a contradiction. Thus the hypothesis that  $\ell(w) > \ell'(w)$  must be false, and so  $\ell(w) = \ell'(w)$ .  $\square$

It is worth noting that we can also give an alternative proof to Theorem 4.7. This alternative proof uses the following lemma.

**Lemma 4.10.** *If a permutation is both 321- and 3412-avoiding, then each cycle in its standard cycle notation consists of consecutive integers.*

*Proof.* The product of two cycles in which one consists of the integers  $[a, b]$  and the other consists of the integers  $[b, c]$  is a cycle consisting of the integers  $[a, c]$ . By Lemma 4.9, a permutation avoiding 321 and 3412 has reduced decompositions with no repeated letters. If each  $s_k$  in such a reduced decomposition is written as the cycle  $(k \ k + 1)$ , then we see that

- every cycle in the resulting product consists of consecutive integers, and
- each letter appears in at most two cycles.

The first sentence of this proof guarantees that these two properties are maintained as we start to combine cycles by multiplication when they are not disjoint. The process halts when no such multiplication is possible, meaning that no letter appears in more than one cycle.  $\square$

With Lemma 4.10 in hand, the proof of Theorem 4.7 only requires showing that the boolean Coxeter elements in  $\mathfrak{S}_n$  (321- and 3412-avoiding permutations consisting of a single  $n$ -cycle) have length  $n - 1$ . This is easily done by induction on  $n$ .

The class of permutations defined in Theorem 4.7 is entry P0006 of [18], enumerated by sequence A001519 of [13], and enumerated by length in sequence A105306 of the same database. These enumerations were calculated by the second author in [19], in the context of boolean permutations.

**Corollary 4.11** ([19]). *Fix a positive integer  $n$ . Then we have the following enumerative results, where  $\{F_0, F_1, \dots\}$  are the Fibonacci numbers.*

$$|\{w \in \mathfrak{S}_n : \ell'(w) = \text{dp}(w) = \ell(w)\}| = F_{2n-1}$$

$$|\{w \in \mathfrak{S}_n : \ell'(w) = \text{dp}(w) = \ell(w) = k\}| = \sum_{i=1}^k \binom{n-i}{k+1-i} \binom{k-1}{i-1}$$

As mentioned earlier, Theorem 4.7 recovers a result of Edelman, although his result was phrased slightly differently. It follows that the class of permutations defined in [5] coincides with the set of permutations avoiding the patterns 321 and 3412.

**Corollary 4.12.** *Unimodal permutations are exactly those permutations which avoid the patterns 321 and 3412.*

*Proof.* In [5, Theorem 3.1], it was shown that a permutation  $w$  satisfies  $\ell(w) = \ell'(w)$  if and only if  $w$  is unimodal. It follows from Theorem 4.7, then, that unimodal permutations are precisely those that avoid 321 and 3412.  $\square$

It is also possible to see the equivalence between unimodal permutations and permutations that avoid 321 and 3412 using Lemma 4.10.

Consider a permutation  $w \in \mathfrak{S}_n$ . We now have the following equivalent statements.

- $\ell(w) = \ell'(w)$
- all paths from  $e$  to  $w$  in the Bruhat graph have  $\ell(w) = \ell'(w)$  edges
- $w$  avoids 321 and 3412
- $|\text{supp}(w)| = \ell(w) = \ell(w')$
- the reduced decompositions of  $w$  contain no repeated letters

We can now explore when, provided that  $\ell(w) = \ell'(w)$ , all of the edges along all of these paths from  $e$  to  $w$  in the Bruhat graph are weighted by 1. (And thus produce a path whose edge weights add up to depth.) That is, we can consider when all of these edges represent simple reflections  $s \in S_n$ , not elements of  $T_n \setminus S_n$ .

**Definition 4.13.** Suppose that a permutation  $w$  has the property that if  $s_k \in \text{supp}(w)$  then  $s_{k\pm 1} \notin \text{supp}(w)$ . Then we will say that  $w$  is *free*.

Note that all of the letters in a reduced decomposition of a free permutation  $w$  are distinct, and they all commute with each other. Thus  $w$  has  $2^{\ell(w)}$  reduced decompositions. The following lemma follows from the main result in [20].

**Lemma 4.14** (See [20]). *A permutation  $w$  is free if and only if it avoids 231, 312, and 321.*

**Corollary 4.15.** *The number of free permutations in  $\mathfrak{S}_n$  is the Fibonacci number  $F_{n+1}$ .*

*Proof.* The free permutations in  $\mathfrak{S}_n$  can be partitioned into two sets, determined by whether or not  $s_{n-1}$  appears in their supports. Those for which  $s_{n-1}$  does not appear are in bijective correspondence with free permutations in  $\mathfrak{S}_{n-1}$ . Those  $w$  for which  $s_{n-1} \in \text{supp}(w)$  must have  $s_{n-2} \notin \text{supp}(w)$ , meaning that they are in bijective correspondence with free permutations in  $\mathfrak{S}_{n-2}$ . Thus

$$|\{w \in \mathfrak{S}_n : w \text{ is free}\}| = |\{w \in \mathfrak{S}_{n-1} : w \text{ is free}\}| + |\{w \in \mathfrak{S}_{n-2} : w \text{ is free}\}|.$$

Observing that  $|\{w \in \mathfrak{S}_1 : w \text{ is free}\}| = 1$  and  $|\{w \in \mathfrak{S}_2 : w \text{ is free}\}| = 2$  completes the proof.  $\square$

It is also possible to prove Corollary 4.15 by noting, using Lemma 4.14, that the only allowable positions for  $n$  in a free permutation  $w$  are  $w(n-1)$  and  $w(n)$ .

**Corollary 4.16.** *Suppose that  $w \in \mathfrak{S}_n$  satisfies the property  $\ell(w) = \ell'(w)$ . Then  $w$  is free (that is,  $w$  avoids 231, 312, and 321) if and only if every decomposition  $w = t_1 \cdots t_{\ell(w)}$  with  $t_i \in T_n$  actually satisfies  $t_i \in S_n$ .*

*Proof.* It follows from Theorem 4.7 and Lemma 4.9 that any reduced decomposition of  $w$  consists of all distinct letters.

Suppose that  $w$  is not free. Without loss of generality, we can assume that  $w$  has a reduced decomposition of the form

$$\alpha s_k \beta s_{k+1} \gamma,$$

where  $\alpha$ ,  $\beta$ , and  $\gamma$  are products of simple reflections. It is possible to use Coxeter relations to move the letters of  $\beta$  in order to yield a reduced decomposition of  $w$  having the form  $\alpha' s_k s_{k+1} \gamma'$ : any letter  $s_j \in \beta$  with  $j < k$  moves past  $s_{k+1}$  to the right, and any letter  $s_j$  with  $j > k+1$  moves past  $s_k$  to the left. But then we see the following equivalence

$$\alpha' s_k s_{k+1} \gamma' = \alpha' t_{k,k+2} s_k \gamma',$$

giving an undesirable decomposition of  $w$  into  $\ell$  reflections.

Now suppose that  $w$  is free. Then, for  $k > 1$  the only  $k$ -cycles in the standard cycle form of  $w$  must be 2-cycles of the form  $(i, i+1)$ . In a decomposition of  $w$  into reflections  $t_1 t_2 \cdots t_\ell$ , where  $\ell = \ell'(w)$ , each successive permutation  $w, wt_\ell, wt_\ell t_{\ell-1}, \dots$  must have smaller length than the preceding permutation. However, the only inversions in the permutation are of the form  $\{i, i+1\}$ , and so each  $t_j$  must be a simple reflection.  $\square$

**Example 4.17.** Consider the permutation  $231 \in \mathfrak{S}_3$ . By Lemma 4.14, we know that 231 is not free. It is easy to compute  $\ell(231) = \ell'(231) = 2$ , and we see that

$$231 = s_1 s_2 = t_{13} s_1.$$

The class of permutations described in Corollary 4.16 is entry P0026 of [18], enumerated by sequence A000045 of [13].

## 5. OPEN QUESTIONS AND FURTHER REMARKS

There are many possible directions for the future study of depth. For permutations, we have characterized depth combinatorially (Theorem 1.1), and shown that another statistic, descent drop, has the same distribution. We know the uniform upper bound for these statistics is  $\lfloor n^2/4 \rfloor$ , and we know how many permutations achieve this bound (Proposition 3.3). There are obviously  $n$  elements of depth 1 (the simple reflections) and it is also not difficult to see that (for  $n \geq 3$ ):

$$|\{w \in \mathfrak{S}_n : \text{dp}(w) = 2\}| = \binom{n-1}{2} + 2(n-2) = \frac{(n+3)(n-2)}{2}.$$

In terms of reduced decompositions, these are all elements of the form:

- $s_i s_j$ , with  $1 \leq i < j \leq n-1$ ,

$B_n$	$k = 0$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$n = 1$	1	1														
2	1	2	3	2												
3	1	3	7	12	16	8	1									
4	1	4	12	28	53	70	89	54	60	12	1					
5	1	5	18	51	118	215	347	456	594	558	505	466	325	164	16	1

TABLE 3. Depth distribution for  $B_n$ , that is,  $|\{w \in B_n : \text{dp}(w) = k\}|$ .

- $s_{i+1}s_i$ , with  $1 \leq i \leq n - 2$ ,
- $s_i s_{i+1} s_i$ , with  $1 \leq i \leq n - 2$ .

But what of the other values for depth?

**Question 5.1.** *How many permutations in  $\mathfrak{S}_n$  have depth (or descent drop)  $k$ ? What is  $|\{w \in \mathfrak{S}_n : \text{dp}(w) = k\}|$ ? (or  $|\{w \in \mathfrak{S}_n : \text{drops}(w) = k\}|$ ?)*

Another direction for generalization is to study depth more closely in other Coxeter groups. It would be nice to find useful characterizations of depth analogous to the “sum of sizes of excedances” characterization we have given for permutations. Surely such a model exists in type  $B_n$ .

**Question 5.2.** *What is the analogue of Theorem 1.1 for signed permutations in type  $B_n$ ?*

We have the distributions of depth in  $B_n$  for  $n \leq 5$  in Table 3. It certainly appears that  $\text{dp}(w)$  is bounded above by  $\binom{n+1}{2}$ , although we have no proof of this.

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