A NOTE ON THE THIRD CUBOID
CONJECTURE. PART I.

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ABSTRACT. The problem of finding perfect Euler cuboids or proving their non-existence is an old unsolved problem in mathematics. The third cuboid conjecture is the last of the three propositions suggested as intermediate stages in proving the non-existence of perfect Euler cuboids. It is associated with a certain Diophantine equation of the order 12. In this paper a structural theorem for the solutions of this Diophantine equation is proved.

1. INTRODUCTION.

Let's denote through $P_{abu}(t)$ the following polynomial of the order 12 depending on three integer parameters a, b, and u:

$$
\begin{aligned}
P_{abu}(t) = {} & t^{12} + (6\,u^2 - 2\,a^2 - 2\,b^2)\,t^{10} + (a^4 + b^4 + u^4 + 4\,a^2\,u^2 + \\
& + 4\,b^2\,u^2 - 12\,b^2\,a^2)\,t^8 + (6\,a^4\,u^2 + 6\,u^2\,b^4 - 8\,a^2\,b^2\,u^2 - \\
& - 2\,u^4\,a^2 - 2\,u^4\,b^2 - 2\,a^4\,b^2 - 2\,b^4\,a^2)\,t^6 + (4\,u^2\,b^4\,a^2 + \\
& + 4\,u^2\,a^4\,b^2 - 12\,u^4\,a^2\,b^2 + u^4\,a^4 + u^4\,b^4 + a^4\,b^4)\,t^4 + \\
& + (6\,a^4\,u^2\,b^4 - 2\,u^4\,a^4\,b^2 - 2\,u^4\,a^2\,b^4)\,t^2 + u^4\,a^4\,b^4.
\end{aligned}
\tag{1.1}
$$

There are some special cases where the polynomial $P_{abu}(t)$ is reducible and explicitly splits into lower order factors. Here are these cases:

$$
\begin{array}{llll}
1) & a = b; & 3) & b\,u = a^2; & 5) & a = u; \\
2) & a = b = u; & 4) & a\,u = b^2; & 6) & b = u.
\end{array}
\tag{1.2}
$$

The special cases (1.2) were studied in [1], [2], and [3]. In a general case other than those listed in (1.2) the polynomial (1.1) is described by the following conjecture.

Conjecture 1.1 (third cuboid conjecture). *For any three positive coprime integer numbers a, b, and u such that none of the conditions (1.2) is satisfied the polynomial (1.1) is irreducible in the ring $\mathbb{Z}[t]$.*

The subcases 2, 5, and 6 in (1.2) are trivial. The subcase 1 leads to the first cuboid conjecture. The subcases 3 and 4 lead to the second cuboid conjecture. The first, the second, and the third cuboid conjectures were introduced [1]. They

are associated with the problem of constructing a perfect Euler cuboid (see [4] and [5–39] for more details). As for the polynomial (1.1), it was derived in [40].

Let's write the following equation using the polynomial (1.1):

$$P_{abu}(t) = 0. \tag{1.3}$$

The equation (1.3) can be understood as a Diophantine equation of the order 12 with three integer parameters a, b, and u. The third cuboid conjecture 1.1 implies the following theorem.

Theorem 1.1. *For any three positive coprime integer numbers a, b, and u such that none of the conditions (1.2) is satisfied the polynomial Diophantine equation (1.3) has no integer solutions.*

A similar theorem associated with the first cuboid conjecture was formulated and proved in [2]. A similar theorem associated with the second cuboid conjecture was formulated in [3]. However, it is not yet proved.

Being a weaker proposition than the conjecture 1.1, the theorem 1.1 in our present case is also rather difficult. Probably it is equally difficult as the third cuboid conjecture itself. Below in section 6 we formulate and prove a structural theorem for the solutions of the Diophantine equation (1.3). This structural theorem is the main result of the present paper.

2. THE INVERSION SYMMETRY.

The polynomial $P_{abu}(t)$ in (1.1) possesses some special properties. They are expressed by the following formulas which can be verified by direct calculations:

$$P_{abu}(t) = P_{bau}(t), \qquad\qquad P_{abu}(-t) = P_{abu}(t). \tag{2.1}$$

The first equality (2.1) means that the polynomial $P_{abu}(t)$ is symmetric with respect to the permutation of the parameters a and b. The second equality is also a symmetry. It is called parity. This symmetry means that that the polynomial $P_{abu}(t)$ is an even function of its argument t.

Apart from the two symmetries (2.1), the polynomial $P_{abu}(t)$ has a third symmetry which is called the inversion symmetry. Having three positive integer numbers a, b, and u, we define the transformation

$$\sigma\colon (a, b, u) \longmapsto (\tilde{a}, \tilde{b}, \tilde{u}) \tag{2.2}$$

by means of the following three formulas:

$$\tilde{a} = \frac{\mathrm{lcm}(a, b, u)}{a}, \qquad \tilde{b} = \frac{\mathrm{lcm}(a, b, u)}{b}, \qquad \tilde{u} = \frac{\mathrm{lcm}(a, b, u)}{u}. \tag{2.3}$$

The numerator $\mathrm{lcm}(a, b, u)$ of the fractions (2.3) is the least common multiple of the integer numbers a, b, and u. The formula

$$P_{\sigma(abu)}(t) = \frac{P_{abu}(\mathrm{lcm}(a, b, u)/t)\, t^{12}}{a^4\, b^4\, u^4} \tag{2.4}$$

is written in terms of the transformation (2.2). This formula is easily verified by means of the direct calculations. The formula (2.4) expresses the inversion symmetry of the polynomial (1.1).

3. Some prerequisites.

Lemma 3.1. *For any three positive integer numbers a, b, and u the numbers $\tilde{a}$, $\tilde{b}$, and $\tilde{u}$ produced by applying the transformation (2.2) are coprime.*

Proof. Let $p_1 \ldots, p_n$ be the prime factors of the numbers a, b, and u. Then we can present a, b, and u in the following way:

$$a = \prod_{i=1}^{n} p_i^{\alpha_i}, \qquad b = \prod_{i=1}^{n} p_i^{\beta_i}, \qquad u = \prod_{i=1}^{n} p_i^{\omega_i}. \qquad (3.1)$$

The multiplicities α_i, β_i, and ω_i in (3.1) obey the inequalities

$$\alpha_i \geqslant 0, \qquad \beta_i \geqslant 0, \qquad \omega_i \geqslant 0. \qquad (3.2)$$

Using the multiplicities (3.2), we define the integer numbers

$$\theta_i = \max(\alpha_i, \beta_i, \omega_i). \qquad (3.3)$$

Then the least common multiple $Z = \operatorname{lcm}(a, b, u)$ in (2.3) is expressed through the above numbers (3.3) in the following way:

$$Z = \operatorname{lcm}(a, b, u) = \prod_{i=1}^{n} p_i^{\theta_i}. \qquad (3.4)$$

Let's substitute the formulas (3.1) and (3.4) into the formulas (2.3). As a result we derive the following expressions for $\tilde{a}$, $\tilde{b}$, and $\tilde{u}$:

$$\tilde{a} = \prod_{i=1}^{n} p_i^{\theta_i - \alpha_i}, \qquad \tilde{b} = \prod_{i=1}^{n} p_i^{\theta_i - \beta_i}, \qquad \tilde{u} = \prod_{i=1}^{n} p_i^{\theta_i - \omega_i}. \qquad (3.5)$$

The greatest common divisor of the numbers $\tilde{a}$, $\tilde{b}$, and $\tilde{u}$ in (3.5) is calculated by a formula very similar to (3.5). Indeed, we have

$$\gcd(\tilde{a}, \tilde{b}, \tilde{u}) = \prod_{i=1}^{n} p_i^{\pi_i}, \qquad (3.6)$$

where the exponents π_i are given by the formula

$$\pi_i = \min(\theta_i - \alpha_i, \theta_i - \beta_i, \theta_i - \omega_i). \qquad (3.7)$$

Comparing (3.7) with (3.3), we easily see that $\pi_i = 0$ for all $i = 1, \ldots, n$. Substituting $\pi_i = 0$ into (3.6), we derive $\gcd(\tilde{a}, \tilde{b}, \tilde{u}) = 1$. By definition the equality $\gcd(\tilde{a}, \tilde{b}, \tilde{u}) = 1$ means that the numbers $\tilde{a}$, $\tilde{b}$, and $\tilde{u}$ are coprime. Thus, the lemma 3.1 is proved. $\square$

Lemma 3.2. *If three positive integer numbers a, b, u are coprime and if the num-bers $\tilde{a}$, $\tilde{b}$, $\tilde{u}$ are produced by applying the transformation* (2.2) *to a, b, u, then, applying the transformation* (2.2) *to $\tilde{a}$, $\tilde{b}$, $\tilde{u}$, we get back the numbers a, b, u.*

Proof. In order to prove the lemma 3.2 it is convenient to use the formulas (3.1) for the numbers a, b, u and the formulas (3.5) for the numbers $\tilde{a}$, $\tilde{b}$, and $\tilde{u}$. The coprimality condition for a, b, u is written as

$$\gcd(a, b, u) = \prod_{i=1}^{n} p_i^{\xi_i} = 1. \tag{3.8}$$

For the exponents ξ_i the equality (3.8) yields the formula

$$\xi_i = \min(\alpha_i, \beta_i, \omega_i) = 0. \tag{3.9}$$

Let's denote through $\hat{a}$, $\hat{b}$, $\hat{u}$ the numbers obtained by applying the transforma-tion (2.2) to the numbers $\tilde{a}$, $\tilde{b}$, $\tilde{u}$. Then we have

$$\hat{a} = \frac{\text{lcm}(\tilde{a}, \tilde{b}, \tilde{u})}{\tilde{a}}, \qquad \hat{b} = \frac{\text{lcm}(\tilde{a}, \tilde{b}, \tilde{u})}{\tilde{b}}, \qquad \hat{u} = \frac{\text{lcm}(\tilde{a}, \tilde{b}, \tilde{u})}{\tilde{u}}. \tag{3.10}$$

The numerator of the fractions (3.10) is calculated according to the formula

$$\text{lcm}(\tilde{a}, \tilde{b}, \tilde{u}) = \prod_{i=1}^{n} p_i^{\zeta_i}, \tag{3.11}$$

where the exponents ζ_i are given by the formulas

$$\zeta_i = \max(\theta_i - \alpha_i, \theta_i - \beta_i, \theta_i - \omega_i) = \theta_i - \min(\alpha_i, \beta_i, \omega_i). \tag{3.12}$$

The formula (3.12) is derived from (3.5), while θ_i are given by the formula (3.3).

Now, applying (3.9) to (3.12), we derive $\zeta_i = \theta_i$. The rest is to substitute $\zeta_i = \theta_i$ into (3.11) and then substitute (3.11) into (3.10). And finally, applying the formulas (3.5) to the transformed formulas (3.10), we derive

$$\hat{a} = \prod_{i=1}^{n} p_i^{\alpha_i}, \qquad \hat{b} = \prod_{i=1}^{n} p_i^{\beta_i}, \qquad \hat{u} = \prod_{i=1}^{n} p_i^{\omega_i}. \tag{3.13}$$

Comparing (3.13) with (3.1), we find that the lemma 3.2 is proved. $\square$

The lemma 3.2 means that transformation (2.2) acts as an involution upon co-prime triples of positive integer numbers (a, b, u), i.e. we have the equality

$$\sigma^2 = \sigma \circ \sigma = \text{id}. \tag{3.14}$$

Lemma 3.3. *Let a, b, u be three positive coprime integer numbers and let $\tilde{a}$, $\tilde{b}$, $\tilde{u}$ be the numbers produced from a, b, u by applying the transformation* (2.2). *If one of the conditions* (1.2) *is fulfilled for a, b, u, then the same condition is fulfilled for*

$\tilde a$, $\tilde b$, $\tilde u$, i. e. 1) $a = b$ implies $\tilde a = \tilde b$, 2) $a = b = u$ implies $\tilde a = \tilde b = \tilde u$, 3) $b\, u = a^2$ implies $\tilde b\, \tilde u = \tilde a^2$, 4) $a\, u = b^2$ implies $\tilde a\, \tilde u = \tilde b^2$, 5) $a = u$ implies $\tilde a = \tilde u$, and finally 6) $b = u$ implies $\tilde b = \tilde u$.

Lemma 3.4. *Let a, b, u be three positive coprime integer numbers and let $\tilde a$, $\tilde b$, $\tilde u$ be the numbers produced from a, b, u by applying the transformation (2.2). If none of the conditions (1.2) is fulfilled for the numbers a, b, u, then none of them is fulfilled for the numbers $\tilde a$, $\tilde b$, $\tilde u$.*

The lemma 3.3 is proved by means of direct calculations with the use of the formula (2.3). The lemma 3.4 is immediate from the lemma 3.3.

Assume that we have an equation (1.3) with the parameters a, b, u satisfying the assumptions of the theorem 1.1. Then due to the lemma 3.4 the equation

$$P_{\sigma(abu)}(t) = 0 \tag{3.15}$$

is also an equation of the form (1.3) whose parameters satisfy the assumptions of the theorem 1.1. For this reason and due to (3.14) the equations (1.3) and (3.15) ate called σ-*conjugate cuboid equations.*

4. INTEGER SOLUTIONS OF σ-CONJUGATE CUBOID EQUATIONS.

Assume that the polynomial $P_{abu}(t)$ has an integer root $t = A_0$. Since a, b, u are nonzero integers, we have $A_0 \neq 0$. Then due to the inversion symmetry in (2.4) the σ-conjugate polynomial $P_{\sigma(abu)}$ has an integer root $t = B_0$, where

$$B_0 = \frac{\mathrm{lcm}(a, b, u)}{A_0}. \tag{4.1}$$

The integer number B_0 in (4.1) is also nonzero. Applying the parity symmetry from (2.1), we conclude that the polynomial $P_{abu}(t)$ has the other integer root $t = -A_0$, while $P_{\sigma(abu)}(t)$ has the other integer root $t = -B_0$. As a result the polynomials $P_{abu}(t)$ and $P_{\sigma(abu)}(t)$ split into factors

$$P_{abu}(t) = (t^2 - A_0^2)\, C_{10}(t), \qquad P_{\sigma(abu)}(t) = (t^2 - B_0^2)\, D_{10}(t) \tag{4.2}$$

with $A_0 > 0$ and $B_0 > 0$. Here $C_{10}(t)$ and $D_{10}(t)$ are tenth order polynomials complementary to $t^2 - A_0^2$ and $t^2 - B_0^2$. Applying (2.1) to (4.2) we derive

$$C_{10}(t) = C_{10}(-t), \qquad D_{10}(t) = D_{10}(-t). \tag{4.3}$$

Due to (4.3) the polynomials $C_8(t)$ and $D_8(t)$ are given by the formulas

$$\begin{aligned}
C_{10}(t) &= t^{10} + C_8\, t^8 + C_6\, t^6 + C_4\, t^4 + C_2\, t^2 + C_0, \\
D_{10}(t) &= t^{10} + D_8\, t^8 + D_6\, t^6 + D_4\, t^4 + D_2\, t^2 + D_0.
\end{aligned} \tag{4.4}$$

Now let's apply the inversion symmetries from (2.4) to (4.2). As a result we get

$$C_{10}(t) = -\frac{D_{10}(\mathrm{lcm}(\tilde a, \tilde b, \tilde u)/t)\, t^{10}}{\tilde a^4\, \tilde b^4\, \tilde u^4/B_0^2}, \quad D_{10}(t) = -\frac{C_{10}(\mathrm{lcm}(a, b, u)/t)\, t^{10}}{a^4\, b^4\, u^4/A_0^2}. \tag{4.5}$$

Applying the symmetries (4.5) to (4.4), we derive a series of relationships for the coefficients of the polynomials $C_{10}(t)$ and $D_{10}(t)$:

$$
\begin{aligned}
&C_0\,Z^2 = -a^4\,b^4\,u^4\,B_0^2, &\qquad &C_2\,Z^4 = -a^4\,b^4\,u^4\,B_0^2\,D_8,\\
&C_4\,Z^6 = -a^4\,b^4\,u^4\,B_0^2\,D_6, &\qquad &C_6\,Z^8 = -a^4\,b^4\,u^4\,B_0^2\,D_4,\\
&C_8\,Z^{10} = -a^4\,b^4\,u^4\,B_0^2\,D_2, &\qquad &Z^{12} = -a^4\,b^4\,u^4\,B_0^2\,D_0,
\end{aligned}
\tag{4.6}
$$

$$
\begin{aligned}
&D_0\,Z^2 = -\tilde{a}^4\,\tilde{b}^4\,\tilde{u}^4\,A_0^2, &\qquad &D_2\,Z^4 = -\tilde{a}^4\,\tilde{b}^4\,\tilde{u}^4\,A_0^2\,C_8,\\
&D_4\,Z^6 = -\tilde{a}^4\,\tilde{b}^4\,\tilde{u}^4\,A_0^2\,C_6, &\qquad &D_6\,Z^8 = -\tilde{a}^4\,\tilde{b}^4\,\tilde{u}^4\,A_0^2\,C_4,\\
&D_8\,Z^{10} = -\tilde{a}^4\,\tilde{b}^4\,\tilde{u}^4\,A_0^2\,C_2, &\qquad &Z^{12} = -\tilde{a}^4\,\tilde{b}^4\,\tilde{u}^4\,A_0^2\,C_0.
\end{aligned}
\tag{4.7}
$$

Here we use the notations (3.4), i. e. the relationships

$$
Z = \operatorname{lcm}(a, b, u) = \operatorname{lcm}(\tilde{a}, \tilde{b}, \tilde{u})
\tag{4.8}
$$

are fulfilled for the parameter Z in (4.6) and (4.7). The equations (4.6) and (4.7) are excessive. Due to (2.3) and (4.8) some of them are equivalent to some others. For this reason we can eliminate excessive variables:

$$
\begin{aligned}
&C_0 = -\frac{a^4\,b^4\,u^4\,B_0^2}{Z^2}, &\qquad &C_2 = -\frac{a^4\,b^4\,u^4\,B_0^2\,D_8}{Z^4},\\[2mm]
&C_4 = -\frac{a^4\,b^4\,u^4\,B_0^2\,D_6}{Z^6}, &\qquad &D_0 = -\frac{\tilde{a}^4\,\tilde{b}^4\,\tilde{u}^4\,A_0^2}{Z^2},\\[2mm]
&D_2 = -\frac{\tilde{a}^4\,\tilde{b}^4\,\tilde{u}^4\,A_0^2\,C_8}{Z^4}, &\qquad &D_4 = -\frac{\tilde{a}^4\,\tilde{b}^4\,\tilde{u}^4\,A_0^2\,C_6}{Z^6}.
\end{aligned}
\tag{4.9}
$$

Substituting (4.9) into the formulas (4.4) for $C_{10}(t)$ and $D_{10}(t)$, we get

$$
\begin{aligned}
C_{10}(t) &= t^{10} + C_8\,t^8 + C_6\,t^6 - a^4\,b^4\,u^4\,B_0^2\,D_6\,Z^{-6}\,t^4 - \\
&\quad - a^4\,b^4\,u^4\,B_0^2\,D_8\,Z^{-4}\,t^2 - a^4\,b^4\,u^4\,B_0^2\,Z^{-2},\\[3mm]
D_{10}(t) &= t^{10} + D_8\,t^8 + D_6\,t^6 - \tilde{a}^4\,\tilde{b}^4\,\tilde{u}^4\,A_0^2\,C_6\,Z^{-6}\,t^4 - \\
&\quad - \tilde{a}^4\,\tilde{b}^4\,\tilde{u}^4\,A_0^2\,C_8\,Z^{-4}\,t^2 - \tilde{a}^4\,\tilde{b}^4\,\tilde{u}^4\,A_0^2\,Z^{-2}.
\end{aligned}
\tag{4.10}
$$

Having derived the formulas (4.10), we substitute them back into the relationships (4.2). As a result we derive the following formulas:

$$
\begin{aligned}
P_{abu}(t) &= t^{12} + (C_8 - A_0^2)\,t^{10} + (C_6 - A_0^2\,C_8)\,t^8 - \\[2mm]
&- \frac{A_0^2\,C_6\,Z^6 + B_0^2\,a^4\,b^4\,u^4\,D_6}{Z^6}\,t^6 - \frac{B_0^2\,a^4\,b^4\,u^4\,(D_8\,Z^2 - A_0^2\,D_6)}{Z^6}\,t^4 - \\[2mm]
&- \frac{B_0^2\,a^4\,b^4\,u^4\,(Z^2 - A_0^2\,D_8)}{Z^4}\,t^2 + \frac{A_0^2\,B_0^2\,a^4\,b^4\,u^4}{Z^2},
\end{aligned}
\tag{4.11}
$$

$$P_{\sigma(abu)}(t) = t^{12} + (D_8 - B_0^2)\, t^{10} + (D_6 - B_0^2\, D_8)\, t^8 -$$

$$-\,\frac{B_0^2\, D_6\, Z^6 + A_0^2\, \tilde{a}^4\, \tilde{b}^4\, \tilde{u}^4\, C_6}{Z^6}\, t^6 - \frac{A_0^2\, \tilde{a}^4\, \tilde{b}^4\, \tilde{u}^4\, (C_8\, Z^2 - B_0^2\, C_6)}{Z^6}\, t^4 - \qquad (4.12)$$

$$-\,\frac{A_0^2\, \tilde{a}^4\, \tilde{b}^4\, \tilde{u}^4\, (Z^2 - B_0^2\, C_8)}{Z^4}\, t^2 + \frac{B_0^2\, A_0^2\, \tilde{a}^4\, \tilde{b}^4\, \tilde{u}^4}{Z^2}.$$

The polynomial $P_{abu}(t)$ in (4.11) is initially given by the formula (1.1). As for the polynomial $P_{\sigma(abu)}(t)$ it is produced from the polynomial (1.1) by substituting $\tilde{a}$, $\tilde{b}$, and $\tilde{u}$ for the parameters a, b, and u respectively:

$$\begin{aligned}
P_{\sigma(abu)}(t) = {}& t^{12} + (6\,\tilde{u}^2 - 2\,\tilde{a}^2 - 2\,\tilde{b}^2)\, t^{10} + (\tilde{a}^4 + \tilde{b}^4 + \tilde{u}^4 + 4\,\tilde{a}^2\,\tilde{u}^2 + \\
& +\, 4\,\tilde{b}^2\,\tilde{u}^2 - 12\,\tilde{b}^2\,\tilde{a}^2)\, t^8 + (6\,\tilde{a}^4\,\tilde{u}^2 + 6\,\tilde{u}^2\,\tilde{b}^4 - 8\,\tilde{a}^2\,\tilde{b}^2\,\tilde{u}^2 - \\
& -\, 2\,\tilde{u}^4\,\tilde{a}^2 - 2\,\tilde{u}^4\,\tilde{b}^2 - 2\,\tilde{a}^4\,\tilde{b}^2 - 2\,\tilde{b}^4\,\tilde{a}^2)\, t^6 + (4\,\tilde{u}^2\,\tilde{b}^4\,\tilde{a}^2 + \\
& +\, 4\,\tilde{u}^2\,\tilde{a}^4\,\tilde{b}^2 - 12\,\tilde{u}^4\,\tilde{a}^2\,\tilde{b}^2 + \tilde{u}^4\,\tilde{a}^4 + \tilde{u}^4\,\tilde{b}^4 + \tilde{a}^4\,\tilde{b}^4)\, t^4 + \\
& +\, (6\,\tilde{a}^4\,\tilde{u}^2\,\tilde{b}^4 - 2\,\tilde{u}^4\,\tilde{a}^4\,\tilde{b}^2 - 2\,\tilde{u}^4\,\tilde{b}^4\,\tilde{a}^2)\, t^2 + \tilde{u}^4\,\tilde{a}^4\,\tilde{b}^4.
\end{aligned} \qquad (4.13)$$

Comparing the formula (4.11) with (1.1) and comparing the formula (4.12) with (4.13), we derive twelve equations for the coefficients of the polynomials (4.10). Two of them are equivalent to the equation (4.1) written as

$$A_0\, B_0 = Z. \qquad (4.14)$$

The other ten of these equations are written as follows:

$$\begin{aligned}
C_8 - A_0^2 &= 6\, u^2 - 2\, a^2 - 2\, b^2, \\
D_8 - B_0^2 &= 6\, \tilde{u}^2 - 2\, \tilde{a}^2 - 2\, \tilde{b}^2,
\end{aligned} \qquad (4.15)$$

$$\begin{aligned}
C_6 - A_0^2\, C_8 &= a^4 + b^4 + u^4 + 4\, a^2\, u^2 + 4\, b^2\, u^2 - 12\, b^2\, a^2, \\
D_6 - B_0^2\, D_8 &= \tilde{a}^4 + \tilde{b}^4 + \tilde{u}^4 + 4\, \tilde{a}^2\, \tilde{u}^2 + 4\, \tilde{b}^2\, \tilde{u}^2 - 12\, \tilde{b}^2\, \tilde{a}^2,
\end{aligned} \qquad (4.16)$$

$$\begin{aligned}
-(A_0^2\, C_6\, Z^6 + B_0^2\, a^4\, b^4\, u^4\, D_6)\, Z^{-6} &= 6\, a^4\, u^2 + 6\, u^2\, b^4 - \\
&- 8\, a^2\, b^2\, u^2 - 2\, u^4\, a^2 - 2\, u^4\, b^2 - 2\, a^4\, b^2 - 2\, b^4\, a^2, \\
-(B_0^2\, D_6\, Z^6 + A_0^2\, \tilde{a}^4\, \tilde{b}^4\, \tilde{u}^4\, C_6)\, Z^{-6} &= 6\, \tilde{a}^4\, \tilde{u}^2 + 6\, \tilde{u}^2\, \tilde{b}^4 - \\
&- 8\, \tilde{a}^2\, \tilde{b}^2\, \tilde{u}^2 - 2\, \tilde{u}^4\, \tilde{a}^2 - 2\, \tilde{u}^4\, \tilde{b}^2 - 2\, \tilde{a}^4\, \tilde{b}^2 - 2\, \tilde{b}^4\, \tilde{a}^2,
\end{aligned} \qquad (4.17)$$

$$\begin{aligned}
B_0^2\, a^4\, b^4\, u^4\, (A_0^2\, D_6 - D_8\, Z^2)\, Z^{-6} &= 4\, u^2\, b^4\, a^2 + \\
+ 4\, u^2\, a^4\, b^2 - 12\, u^4\, a^2\, b^2 &+ u^4\, a^4 + u^4\, b^4 + a^4\, b^4, \\
A_0^2\, \tilde{a}^4\, \tilde{b}^4\, \tilde{u}^4\, (B_0^2\, C_6 - C_8\, Z^2)\, Z^{-6} &= 4\, \tilde{u}^2\, \tilde{b}^4\, \tilde{a}^2 + \\
+ 4\, \tilde{u}^2\, \tilde{a}^4\, \tilde{b}^2 - 12\, \tilde{u}^4\, \tilde{a}^2\, \tilde{b}^2 &+ \tilde{u}^4\, \tilde{a}^4 + \tilde{u}^4\, \tilde{b}^4 + \tilde{a}^4\, \tilde{b}^4,
\end{aligned} \qquad (4.18)$$

$$\begin{aligned}
B_0^2\, a^4\, b^4\, u^4\, (A_0^2\, D_8 - Z^2)\, Z^{-4} &= 6\, a^4\, u^2\, b^4 - 2\, u^4\, a^4\, b^2 - 2\, u^4\, b^4\, a^2, \\
A_0^2\, \tilde{a}^4\, \tilde{b}^4\, \tilde{u}^4\, (B_0^2\, C_8 - Z^2)\, Z^{-4} &= 6\, \tilde{a}^4\, \tilde{u}^2\, \tilde{b}^4 - 2\, \tilde{u}^4\, \tilde{a}^4\, \tilde{b}^2 - 2\, \tilde{u}^4\, \tilde{b}^4\, \tilde{a}^2.
\end{aligned} \qquad (4.19)$$

Note that the parameters a, b, u and $\tilde{a}$, $\tilde{b}$, $\tilde{u}$ are related with each other by means of the formulas (2.3). Now we write these formulas as follows:

$$a\,\tilde{a} = Z, \qquad\qquad a\,\tilde{b} = Z, \qquad\qquad u\,\tilde{u} = Z. \qquad (4.20)$$

The equations (4.15), (4.16), (4.17), (4.18), (4.19) are excessive. Indeed, the equations (4.18) can be derived from the equations (4.16) by applying the equalities (4.14) and (4.20). Similarly, the equations (4.19) can be derived from the equations (4.15) by applying the equalities (4.14) and (4.20). As for the equations (4.15), (4.16), (4.17), when complemented with the equation (4.14), they constitute a system of Diophantine equations with respect to the integer variables C_8, D_8, C_6, D_6, A_0 and B_0. The results of the above calculations are summarized as a lemma.

Lemma 4.1. *For any three positive coprime integer numbers a, b, and u the polynomial $P_{abu}(t)$ has integer roots if and only if the system of Diophantine equations (4.14), (4.15), (4.16), and (4.17) is solvable with respect to the integer variables C_8, D_8, C_6, D_6, $A_0 > 0$, and $B_0 > 0$.*

Now let's consider the equations (4.17). They are not independent. The second equation (4.17) can be derived from the first one. Indeed, it is sufficient to multiply the first equation (4.17) by $tu^4\,ta^4\,tb^4\,Z^{-6}$ and then apply the relationships (4.20). Due to this observation we can omit the second equation (4.17) preserving the first equation (4.17) only. We write this equation as follows:

$$\tilde{a}^2\,\tilde{b}^2\,\tilde{u}^2\,A_0^2\,C_6 + a^2\,b^2\,u^2\,B_0^2\,D_6 = Z^4\,(8\,Z^2 - 6\,\tilde{b}^2\,a^2 -$$
$$- 6\,b^2\,\tilde{a}^2 + 2\,\tilde{a}^2\,u^2 + 2\,\tilde{b}^2\,u^2 + 2\,\tilde{u}^2\,a^2 + 2\,\tilde{u}^2\,b^2). \qquad (4.21)$$

The equation (4.21) is produced from the first equation (4.17) by multiplying it by $\tilde{a}^2\,\tilde{b}^2\,\tilde{u}^2$ and then applying the relationships (4.20). In terms of the equation (4.21) the above lemma 4.1 is reformulated as follows.

Lemma 4.2. *For any three positive coprime integer numbers a, b, and u the polynomial $P_{abu}(t)$ has integer roots if and only if the system of Diophantine equations (4.14), (4.15), (4.16), and (4.21) is solvable with respect to the integer variables C_8, D_8, C_6, D_6, $A_0 > 0$, and $B_0 > 0$.*

Note that the equations (4.15) can be explicitly resolved with respect to the variables C_8 and D_8. As a result we get

$$C_8 = A_0^2 + 6\,u^2 - 2\,a^2 - 2\,b^2,$$
$$D_8 = B_0^2 + 6\,\tilde{u}^2 - 2\,\tilde{a}^2 - 2\,\tilde{b}^2. \qquad (4.22)$$

Similarly, the equations (4.16) can be explicitly resolved with respect to the variables C_6 and D_6. Resolving them, we get

$$C_6 = A_0^2\,C_8 + a^4 + b^4 + u^4 + 4\,a^2\,u^2 + 4\,b^2\,u^2 - 12\,b^2\,a^2,$$
$$D_6 = B_0^2\,D_8 + \tilde{a}^4 + \tilde{b}^4 + \tilde{u}^4 + 4\,\tilde{a}^2\,\tilde{u}^2 + 4\,\tilde{b}^2\,\tilde{u}^2 - 12\,\tilde{b}^2\,\tilde{a}^2. \qquad (4.23)$$

Then we can substitute the expressions (4.22) for C_8 and D_8 into the equations (4.23). As a result we get the expressions for C_6 and D_6 directly through A_0 and

B_0. We write these expressions in the following way:

$$C_6 = A_0^2 \left(A_0^2 + 6\,u^2 - 2\,a^2 - 2\,b^2 \right) + a^4 + \\ + b^4 + u^4 + 4\,a^2\,u^2 + 4\,b^2\,u^2 - 12\,b^2\,a^2, \tag{4.24}$$

$$D_6 = B_0^2 \left(B_0^2 + 6\,\tilde{u}^2 - 2\,\tilde{a}^2 - 2\,\tilde{b}^2 \right) + \tilde{a}^4 + \\ + \tilde{b}^4 + \tilde{u}^4 + 4\,\tilde{a}^2\,\tilde{u}^2 + 4\,\tilde{b}^2\,\tilde{u}^2 - 12\,\tilde{b}^2\,\tilde{a}^2. \tag{4.25}$$

The next step is to substitute (4.24) and (4.25) into the equation (4.21). As a result we obtain the following equation for the variables A_0 and B_0:

$$\tilde{a}^2\,\tilde{b}^2\,\tilde{u}^2\,A_0^2 \left(A_0^2 \left(A_0^2 + 6\,u^2 - 2\,a^2 - 2\,b^2 \right) + a^4 + b^4 + u^4 + 4\,a^2\,u^2 + \\ + 4\,b^2\,u^2 - 12\,b^2\,a^2 \right) + a^2\,b^2\,u^2\,B_0^2 \left(B_0^2 \left(B_0^2 + 6\,\tilde{u}^2 - 2\,\tilde{a}^2 - 2\,\tilde{b}^2 \right) + \\ + \tilde{a}^4 + \tilde{b}^4 + \tilde{u}^4 + 4\,\tilde{a}^2\,\tilde{u}^2 + 4\,\tilde{b}^2\,\tilde{u}^2 - 12\,\tilde{b}^2\,\tilde{a}^2 \right) - Z^4 \left(8\,Z^2 - \\ - 6\,\tilde{b}^2\,a^2 - 6\,b^2\,\tilde{a}^2 + 2\,\tilde{a}^2\,u^2 + 2\,\tilde{b}^2\,u^2 + 2\,\tilde{u}^2\,a^2 + 2\,\tilde{u}^2\,b^2 \right) = 0. \tag{4.26}$$

With the use of the equation (4.26) now the lemma 4.2 is reformulated as follows.

Lemma 4.3. *For any three positive coprime integer numbers a, b, and u the polynomial $P_{abu}(t)$ has integer roots if and only if the system of Diophantine equations (4.14) and (4.26) is solvable with respect to the integer variables $A_0 > 0$, and $B_0 > 0$.*

5. THE PRIME FACTORS STRUCTURE.

Below we continue studying the equations (4.14) and (4.26) implicitly assuming a, b, and u to be three positive coprime integer numbers. Assuming p_1, ..., p_n to be the prime factors of a, b, and u, we apply the formulas (3.1) with the multiplicities α_i, β_i, and ω_i obeying the inequalities (3.2). For the least common multiple Z of the numbers a, b, and u in (4.8) we use the formula (3.4), where the exponents θ_i are given by the formula (3.3).

The equation (4.14) combined with the formula (3.4) means that the numbers A_0 and B_0 cannot have prime factors other than p_1, ..., p_n. Therefore we write

$$A_0 = p_1^{\mu_1} \cdot \ldots \cdot p_n^{\mu_m}, \qquad\qquad B_0 = p_1^{\eta_1} \cdot \ldots \cdot p_n^{\eta_m}. \tag{5.1}$$

In terms of (5.1) and (3.3) the equation (4.14) yields the equalities

$$\mu_i + \eta_i = \theta_i \tag{5.2}$$

for each particular value of the index $i = 1$, ..., n. The coprimality condition for the numbers a, b, u is $\gcd(a, b, u) = 1$. It leads to (3.8) and (3.9). Due to (3.9) at least one of the three options is fulfilled for each particular $i = 1$, ..., n:

$$\alpha_i = 0, \qquad \text{or} \qquad \beta_i = 0, \qquad \text{or} \qquad \omega_i = 0. \tag{5.3}$$

Note that the multiplicities α_i, β_i, and ω_i in (5.3) cannot vanish simultaneously. For this reason, applying the formula (3.3), we derive

$$\theta_i = \max(\alpha_i, \beta_i, \omega_i) > 0. \tag{5.4}$$

The inequality (5.4) means that the multiplicities μ_i and η_i in (5.2) cannot vanish simultaneously either.

In order to investigate the equation (4.26) we introduce the notation $\mathrm{mult}_p(N)$ for the multiplicity of the prime number p in the prime factors expansion of N:

$$\mathrm{mult}_p(N) = k \quad \text{means} \quad N = N' \cdot p^k, \quad \text{where} \quad N' \not\equiv 0 \ (\mathrm{mod}\ p). \tag{5.5}$$

Assume that an integer number N is a sum of some other integer numbers:

$$N = N_1 + \ldots + N_m. \tag{5.6}$$

Let's denote through $k_i = \mathrm{mult}_p(N_i)$ the multiplicities of the summands in (5.6) and denote through $k_{\min}$ the minimum of these multiplicities:

$$k_{\min} = \min(k_1, \ldots, k_m). \tag{5.7}$$

In terms of the notations (5.5), (5.6), and (5.7) we can formulate the following three simple lemmas. Their proofs are obvious.

Lemma 5.1. *If exactly one term N_s in the sum (5.6) has the minimal multiplicity $k_s = k_{\min}$, then the multiplicity of the sum in whole is equal to this minimal multiplicity, i. e. $\mathrm{mult}_p(N) = k_s = k_{\min}$.*

Lemma 5.2. *If more than one term in the sum (5.6) has the minimal multiplicity $k_{\min}$, then $\mathrm{mult}_p(N) \geqslant k_{\min}$.*

Lemma 5.3. *If exactly one term N_s in the sum (5.6) has the minimal multiplicity $k_s = k_{\min}$, then the sum N cannot vanish, i. e. $N \neq 0$.*

Using the notation (5.5) and the above three lemmas 5.1, 5.2, and 5.3, below we study several options derived from (5.3).

The case $\alpha_i > \beta_i > \omega_i = 0$ **and** $p_i \neq 2$. In this case the multiplicities of the parameters a, b, and u obey the following equalities and inequalities:

$$\mathrm{mult}_{p_i}(a) = \alpha_i > \mathrm{mult}_{p_i}(b) = \beta_i > \mathrm{mult}_{p_i}(u) = \omega_i = 0. \tag{5.8}$$

Applying the formulas (3.3), (3.4), and (4.20) to (5.8), we derive

$$\begin{aligned} \theta_i &= \mathrm{mult}_{p_i}(Z) = \alpha_i, & \tilde{\alpha}_i &= \mathrm{mult}_{p_i}(\tilde{a}) = 0, \\ \tilde{\beta}_i &= \mathrm{mult}_{p_i}(\tilde{b}) = \alpha_i - \beta_i, & \tilde{\omega}_i &= \mathrm{mult}_{p_i}(\tilde{u}) = \alpha_i. \end{aligned} \tag{5.9}$$

Combining (5.9) with the inequalities (5.8), we get

$$\mathrm{mult}_{p_i}(\tilde{u}) = \tilde{\omega}_i > \mathrm{mult}_{p_i}(\tilde{b}) = \tilde{\beta}_i > \mathrm{mult}_{p_i}(\tilde{a}) = \tilde{\alpha}_i = 0. \tag{5.10}$$

In order to continue studying the equation (4.26), we write this equation as

$$\tilde{a}^2\, \tilde{b}^2\, \tilde{u}^2\, A_0^2\, C_6 + a^2\, b^2\, u^2\, B_0^2\, D_6 - Z^4\, E = 0, \tag{5.11}$$

where C_6 and D_6 are given by the formulas (4.23), while E is a new parameter:

$$E = 8\, Z^2 - 6\, \tilde{b}^2\, a^2 - 6\, b^2\, \tilde{a}^2 + 2\, \tilde{a}^2\, u^2 + 2\, \tilde{b}^2\, u^2 + 2\, \tilde{u}^2\, a^2 + 2\, \tilde{u}^2\, b^2. \tag{5.12}$$

The right hand side of the formula (5.12) is a sum of seven terms. Applying the formulas (5.8), (5.9), and (5.10) and taking into account that $p_i \neq 2$, we derive

$$\begin{aligned}
&\text{mult}_{p_i}(8\,Z^4) = 2\,\alpha_i, && \text{mult}_{p_i}(-6\,\tilde{b}^2\,a^2) \geqslant 4\,\alpha_i - 2\,\beta_i, \\
&\text{mult}_{p_i}(-6\,b^2\,\tilde{a}^2) \geqslant 2\,\beta_i, && \text{mult}_{p_i}(2\,\tilde{b}^2\,u^2) = 2\,\alpha_i - 2\,\beta_i, \\
&\text{mult}_{p_i}(2\,\tilde{u}^2\,a^2) = 4\,\alpha_i, && \text{mult}_{p_i}(2\,\tilde{u}^2\,b^2) = 2\,\alpha_i + 2\,\beta_i,
\end{aligned} \tag{5.13}$$

The only term with the minimal multiplicity in the sum (5.12) is the term $2\,\tilde{a}^2\,u^2$:

$$\text{mult}_{p_i}(2\,\tilde{a}^2\,u^2) = 0. \tag{5.14}$$

Applying the lemma 5.1, from (5.13) and (5.14) we derive the multiplicity of E:

$$\text{mult}_{p_i}(E) = 0. \tag{5.15}$$

Using (5.15), we can calculate the multiplicity of the last term in (5.11):

$$\text{mult}_{p_i}(-Z^4\,E) = 4\,\alpha_i. \tag{5.16}$$

Assume that both multiplicities μ_i and η_i in (5.2) are nonzero. Under this assumption for the terms in the right hand side of the formulas (4.23) we have

$$\begin{aligned}
&\text{mult}_{p_i}(A_0^2\,C_8) \geqslant 2\,\mu_i, && \text{mult}_{p_i}(a^4) = 4\,\alpha_i, \\
&\text{mult}_{p_i}(b^4) = 4\,\beta_i, && \text{mult}_{p_i}(u^4) = 0, \\
&\text{mult}_{p_i}(4\,a^2\,u^2) = 2\,\alpha_i, && \text{mult}_{p_i}(4\,b^2\,u^2) = 2\,\beta_i, \\
&\text{mult}_{p_i}(-12\,b^2\,a^2) = 2\,\alpha_i + 2\,\beta_i, && \text{mult}_{p_i}(B_0^2\,D_8) \geqslant 2\,\eta_i, \\
&\text{mult}_{p_i}(\tilde{a}^4) = 0, && \text{mult}_{p_i}(\tilde{b}^4) = 4\,\alpha_i - 4\,\beta_i, \\
&\text{mult}_{p_i}(\tilde{u}^4) = 4\,\alpha_i, && \text{mult}_{p_i}(4\,\tilde{a}^2\,\tilde{u}^2) = 2\,\alpha_i, \\
&\text{mult}_{p_i}(4\,\tilde{b}^2\,\tilde{u}^2) = 4\,\alpha_i - 2\,\beta_i, && \text{mult}_{p_i}(-12\,\tilde{b}^2\,\tilde{a}^2) = 2\,\alpha_i - 2\,\beta_i.
\end{aligned} \tag{5.17}$$

Applying the formulas (5.17) and the lemma 5.1 to (4.22), we find

$$\text{mult}_{p_i}(C_6) = 0, \qquad\qquad \text{mult}_{p_i}(D_6) = 0. \tag{5.18}$$

Now we apply (5.18) to the equation (5.11). As a result we get

$$\begin{aligned}
&\text{mult}_{p_i}(\tilde{a}^2\,\tilde{b}^2\,\tilde{u}^2\,A_0^2\,C_6) = 4\,\alpha_i - 2\,\beta_i + 2\,\mu_i, \\
&\text{mult}_{p_i}(a^2\,b^2\,u^2\,B_0^2\,D_6) = 2\,\alpha_i + 2\,\beta_i + 2\,\eta_i.
\end{aligned} \tag{5.19}$$

Due to (5.19) and (5.16) the lemma 5.3 applied to the equation (5.11) means that at least one of the following three conditions should be fulfilled:

$$4\,\alpha_i - 2\,\beta_i + 2\,\mu_i = 4\,\alpha_i \leqslant 2\,\alpha_i + 2\,\beta_i + 2\,\eta_i, \tag{5.20}$$

$$2\,\alpha_i + 2\,\beta_i + 2\,\eta_i = 4\,\alpha_i \leqslant 4\,\alpha_i - 2\,\beta_i + 2\,\mu_i, \tag{5.21}$$

$$2\,\alpha_i + 2\,\beta_i + 2\,\eta_i = 4\,\alpha_i - 2\,\beta_i + 2\,\mu_i \leqslant 4\,\alpha_i. \tag{5.22}$$

The equality in (5.20) is easily resolvable. Resolving this equality, we obtain

$$\mu_i = \beta_i, \qquad\qquad\qquad \eta_i = \alpha_i - \beta_i. \qquad\qquad (5.23)$$

Substituting (5.23) back into (5.20), we find that the inequality (5.20) turns to the equality and the condition (5.20) in whole appears to be fulfilled.

The equality in (5.21) is also easily resolvable. The solution of this equality coincides with (5.23). Substituting (5.23) into (5.20), we again find that the inequality (5.21) turns to the equality and the condition (5.21) in whole appears to be fulfilled. Similarly, the solution of the equality (5.22) coincides with (5.23) and the condition (5.22) in whole appears to be fulfilled upon substituting (5.23) into it.

The subcase $\mu_i = 0$ is slightly different. The formula (5.16) remains unchanged, while the formulas (5.19) in this subcase are replaced by the following ones:

$$\begin{aligned}
\mathrm{mult}_{p_i}(\tilde{a}^2\,\tilde{b}^2\,\tilde{u}^2\,A_0^2\,C_6) &= \zeta_i \geqslant 4\,\alpha_i - 2\,\beta_i, \\
\mathrm{mult}_{p_i}(a^2\,b^2\,u^2\,B_0^2\,D_6) &= 4\,\alpha_i + 2\,\beta_i.
\end{aligned} \qquad (5.24)$$

Due to (5.24) and (5.16) the lemma 5.3 applied to the equation (5.11) means that at least one of the following three conditions should be fulfilled:

$$\zeta_i = 4\,\alpha_i \leqslant 4\,\alpha_i + 2\,\beta_i, \qquad\qquad (5.25)$$

$$4\,\alpha_i + 2\,\beta_i = 4\,\alpha_i \leqslant \zeta_i, \qquad\qquad (5.26)$$

$$4\,\alpha_i + 2\,\beta_i = \zeta_i \leqslant 4\,\alpha_i. \qquad\qquad (5.27)$$

The conditions (5.26) and (5.27) are inconsistent since $\beta_i > 0$. However, the subcase $\mu_i = 0$ in whole is consistent because (5.25) is consistent. In this subcase $\eta_i = \alpha_i$ due to the relationships (5.2) and (5.9).

The subcase $\eta_i = 0$ is another option. In this subcase the formula (5.16) remains unchanged, while the formulas (5.19) are replaced by the following ones:

$$\begin{aligned}
\mathrm{mult}_{p_i}(\tilde{a}^2\,\tilde{b}^2\,\tilde{u}^2\,A_0^2\,C_6) &= 6\,\alpha_i - 2\,\beta_i, \\
\mathrm{mult}_{p_i}(a^2\,b^2\,u^2\,B_0^2\,D_6) &= \xi_i \geqslant 2\,\alpha_i + 2\,\beta_i.
\end{aligned} \qquad (5.28)$$

Due to (5.28) and (5.16) the lemma 5.3 applied to the equation (5.11) means that at least one of the following three conditions should be fulfilled:

$$6\,\alpha_i - 2\,\beta_i = 4\,\alpha_i \leqslant \xi_i, \qquad\qquad (5.29)$$

$$\xi_i = 4\,\alpha_i \leqslant 6\,\alpha_i - 2\,\beta_i, \qquad\qquad (5.30)$$

$$\xi_i = 6\,\alpha_i - 2\,\beta_i \leqslant 4\,\alpha_i. \qquad\qquad (5.31)$$

The conditions (5.29) and (5.31) are inconsistent since $\alpha_i > \beta_i$. However, the subcase $\eta_i = 0$ in whole is consistent because (5.30) is consistent. In this subcase $\mu_i = \alpha_i$ due to the relationships (5.2) and (5.9).

The cases and subcases are too numerous. In order to describe them we use tables. For this purpose let's introduce the following notations:

$$m_C = \mathrm{mult}_{p_i}(\tilde{a}^2\,\tilde{b}^2\,\tilde{u}^2\,A_0^2\,C_6), \qquad m_D = \mathrm{mult}_{p_i}(a^2\,b^2\,u^2\,B_0^2\,D_6). \qquad (5.32)$$

Besides (5.32), let's denote through m_E the multiplicity of the last term in (5.11):

$$m_E = \text{mult}_{p_i}(-Z^4\,E). \tag{5.33}$$

In terms of (5.32) and (5.33) we build the table for the first case considered above.

Table 5.1

$\alpha_i > \beta_i > \omega_i = 0,\ p_i \neq 2$	$\mu_i > 0$ and $\eta_i > 0 \ \Rightarrow\ \mu_i = \beta_i$ and $\eta_i = \alpha_i - \beta_i$	
$m_C = 4\,\alpha_i - 2\,\beta_i + 2\,\mu_i,$	$m_D = 2\,\alpha_i + 2\,\beta_i + 2\,\eta_i,\qquad m_E = 4\,\alpha_i$	
$4\,\alpha_i - 2\,\beta_i + 2\,\mu_i = 4\,\alpha_i \leqslant 2\,\alpha_i + 2\,\beta_i + 2\,\eta_i$		$\checkmark$
$2\,\alpha_i + 2\,\beta_i + 2\,\eta_i = 4\,\alpha_i \leqslant 4\,\alpha_i - 2\,\beta_i + 2\,\mu_i$		$\checkmark$
$2\,\alpha_i + 2\,\beta_i + 2\,\eta_i = 4\,\alpha_i - 2\,\beta_i + 2\,\mu_i \leqslant 4\,\alpha_i$		$\checkmark$

The first row of the table 5.1 is a general information. The second row of this table reflects the formulas (5.19) and (5.16). The rest of this table is the conditions (5.20), (5.21), and (5.22). Check marks say that all of these conditions are consistent.

Table 5.2

$\alpha_i > \beta_i > \omega_i = 0,\ p_i \neq 2$	$\mu_i = 0$ and $\eta_i = \alpha_i$	
$m_C = \zeta_i \geqslant 4\,\alpha_i - 2\,\beta_i,$	$m_D = 4\,\alpha_i + 2\,\beta_i,\qquad m_E = 4\,\alpha_i$	
$\zeta_i = 4\,\alpha_i \leqslant 4\,\alpha_i + 2\,\beta_i$		$\checkmark$
$4\,\alpha_i + 2\,\beta_i = 4\,\alpha_i \leqslant \zeta_i$		
$4\,\alpha_i + 2\,\beta_i = \zeta_i \leqslant 4\,\alpha_i$		

The last two rows in the table 5.2 are not check marked. This means that the corresponding conditions (5.26) and (5.27) are not consistent.

Table 5.3

$\alpha_i > \beta_i > \omega_i = 0,\ p_i \neq 2$	$\mu_i = \alpha_i$ and $\eta_i = 0$	
$m_C = 6\,\alpha_i - 2\,\beta_i,$	$m_D = \xi_i \geqslant 2\,\alpha_i + 2\,\beta_i,\qquad m_E = 4\,\alpha_i$	
$6\,\alpha_i - 2\,\beta_i = 4\,\alpha_i \leqslant \xi_i$		
$\xi_i = 4\,\alpha_i \leqslant 6\,\alpha_i - 2\,\beta_i$		$\checkmark$
$\xi_i = 6\,\alpha_i - 2\,\beta_i \leqslant 4\,\alpha_i$		

Below we list other cases and subcases in a tabular form without any detailed comments for them.

Table 5.4

$\alpha_i > \beta_i > \omega_i = 0,\ p_i = 2$	$\mu_i > 0$ and $\eta_i > 0 \ \Rightarrow\ \mu_i = \beta_i$ and $\eta_i = \alpha_i - \beta_i$	
$m_C = 4\,\alpha_i - 2\,\beta_i + 2\,\mu_i,$	$m_D = 2\,\alpha_i + 2\,\beta_i + 2\,\eta_i,\qquad m_E = 4\,\alpha_i + 1$	
$4\,\alpha_i - 2\,\beta_i + 2\,\mu_i = 4\,\alpha_i + 1 \leqslant 2\,\alpha_i + 2\,\beta_i + 2\,\eta_i$		
$2\,\alpha_i + 2\,\beta_i + 2\,\eta_i = 4\,\alpha_i + 1 \leqslant 4\,\alpha_i - 2\,\beta_i + 2\,\mu_i$		
$2\,\alpha_i + 2\,\beta_i + 2\,\eta_i = 4\,\alpha_i - 2\,\beta_i + 2\,\mu_i \leqslant 4\,\alpha_i + 1$		$\checkmark$

Table 5.5

$\alpha_i > \beta_i > \omega_i = 0,\, p_i = 2$	$\mu_i = 0$ and $\eta_i = \alpha_i$		
$m_C = \zeta_i \geqslant 4\,\alpha_i - 2\,\beta_i,$	$m_D = 4\,\alpha_i + 2\,\beta_i,$	$m_E = 4\,\alpha_i + 1$	
$\zeta_i = 4\,\alpha_i + 1 \leqslant 4\,\alpha_i + 2\,\beta_i$			$\checkmark$
$4\,\alpha_i + 2\,\beta_i = 4\,\alpha_i + 1 \leqslant \zeta_i$			
$4\,\alpha_i + 2\,\beta_i = \zeta_i \leqslant 4\,\alpha_i + 1$			

Table 5.6

$\alpha_i > \beta_i > \omega_i = 0,\, p_i = 2$	$\mu_i = \alpha_i$ and $\eta_i = 0$		
$m_C = 6\,\alpha_i - 2\,\beta_i,$	$m_D = \xi_i \geqslant 2\,\alpha_i + 2\,\beta_i,$	$m_E = 4\,\alpha_i + 1$	
$6\,\alpha_i - 2\,\beta_i = 4\,\alpha_i + 1 \leqslant \xi_i$			
$\xi_i = 4\,\alpha_i + 1 \leqslant 6\,\alpha_i - 2\,\beta_i$			$\checkmark$
$\xi_i = 6\,\alpha_i - 2\,\beta_i \leqslant 4\,\alpha_i + 1$			

Table 5.7

$\alpha_i > \beta_i = \omega_i = 0,\, p_i \neq 2, 3$	$\eta_i > 0 \Rightarrow \eta_i = \alpha_i$ and $\mu_i = 0$		
$m_C = \zeta_i \geqslant 4\,\alpha_i + 2\,\mu_i,$	$m_D = 2\,\alpha_i + 2\,\eta_i,$	$m_E = \varkappa_i \geqslant 4\,\alpha_i$	
$\varkappa_i = \zeta_i \leqslant 2\,\alpha_i + 2\,\eta_i$			$\checkmark$
$\varkappa_i = 2\,\alpha_i + 2\,\eta_i \leqslant \zeta_i$			$\checkmark$
$\zeta_i = 2\,\alpha_i + 2\,\eta_i \leqslant \varkappa_i$			$\checkmark$

Table 5.8

$\alpha_i > \beta_i = \omega_i = 0,\, p_i \neq 2, 3$	$\eta_i = 0$ and $\mu_i = \alpha_i$		
$m_C = \zeta_i \geqslant 6\,\alpha_i,$	$m_D = \xi_i \geqslant 2\,\alpha_i,$	$m_E = \varkappa_i \geqslant 4\,\alpha_i$	
$\varkappa_i = \zeta_i \leqslant \xi_i$			$\checkmark$
$\varkappa_i = \xi_i \leqslant \zeta_i$			$\checkmark$
$\zeta_i = \xi_i \leqslant \varkappa_i$			$\checkmark$

The following tables correspond to the special values of the prime factor p_i, i. e. to $p_i = 2$ and to $p_i = 3$.

Table 5.9

$\alpha_i > \beta_i = \omega_i = 0,\, p_i = 2$	$\eta_i > 0 \Rightarrow \eta_i = \alpha_i$ and $\mu_i = 0$		
$m_C = \zeta_i \geqslant 4\,\alpha_i + 2\,\mu_i,$	$m_D = 2\,\alpha_i + 2\,\eta_i,$	$m_E = \varkappa_i \geqslant 4\,\alpha_i + 1$	
$\varkappa_i = \zeta_i \leqslant 2\,\alpha_i + 2\,\eta_i$			
$\varkappa_i = 2\,\alpha_i + 2\,\eta_i \leqslant \zeta_i$			
$\zeta_i = 2\,\alpha_i + 2\,\eta_i \leqslant \varkappa_i$			$\checkmark$

Table 5.10

$\alpha_i > \beta_i = \omega_i = 0,\ p_i = 2$	$\eta_i = 0$ and $\mu_i = \alpha_i$		
$m_C = \zeta_i \geqslant 6\,\alpha_i,$	$m_D = \xi_i \geqslant 2\,\alpha_i,$	$m_E = \varkappa_i \geqslant 4\,\alpha_i + 1$	
	$\varkappa_i = \zeta_i \leqslant \xi_i$		$\checkmark$
	$\varkappa_i = \xi_i \leqslant \zeta_i$		$\checkmark$
	$\zeta_i = \xi_i \leqslant \varkappa_i$		$\checkmark$

Table 5.11

$\alpha_i > \beta_i = \omega_i = 0,\ p_i = 3$	$\eta_i > 0 \Rightarrow \eta_i = \alpha_i$ and $\mu_i = 0$		
$m_C = \zeta_i \geqslant 4\,\alpha_i + 2\,\mu_i,$	$m_D = 2\,\alpha_i + 2\,\eta_i,$	$m_E = 4\,\alpha_i$	
	$4\,\alpha_i = \zeta_i \leqslant 2\,\alpha_i + 2\,\eta_i$		$\checkmark$
	$4\,\alpha_i = 2\,\alpha_i + 2\,\eta_i \leqslant \zeta_i$		$\checkmark$
	$\zeta_i = 2\,\alpha_i + 2\,\eta_i \leqslant 4\,\alpha_i$		$\checkmark$

Table 5.12

$\alpha_i > \beta_i = \omega_i = 0,\ p_i = 3$	$\eta_i = 0$ and $\mu_i = \alpha_i$		
$m_C = \zeta_i \geqslant 6\,\alpha_i,$	$m_D = \xi_i \geqslant 2\,\alpha_i,$	$m_E = 4\,\alpha_i$	
	$4\,\alpha_i = \zeta_i \leqslant \xi_i$		
	$4\,\alpha_i = \xi_i \leqslant \zeta_i$		$\checkmark$
	$\zeta_i = \xi_i \leqslant 4\,\alpha_i$		

In the following cases the multiplicity β_i is not zero. It is equal to the multiplicity α_i and, according to (3.3), we have $\theta_i = \max(\alpha_i, \beta_i, \omega_i) = \alpha_i$.

Table 5.13

$\alpha_i = \beta_i > \omega_i = 0,\ p_i \neq 2$	$\mu_i > 0\ \Rightarrow\ \mu_i = \alpha_i$ and $\eta_i = 0$		
$m_C = 2\,\alpha_i + 2\,\mu_i,$	$m_D = \xi_i \geqslant 4\,\alpha_i + 2\,\eta_i,$	$m_E = \varkappa_i \geqslant 4\,\alpha_i$	
	$2\,\alpha_i + 2\,\mu_i = \varkappa_i \leqslant \xi_i$		$\checkmark$
	$\xi_i = \varkappa_i \leqslant 2\,\alpha_i + 2\,\mu_i$		$\checkmark$
	$\xi_i = 2\,\alpha_i + 2\,\mu_i \leqslant \varkappa_i$		$\checkmark$

Table 5.14

$\alpha_i = \beta_i > \omega_i = 0,\ p_i \neq 2$	$\mu_i = 0$ and $\eta_i = \alpha_i$		
$m_C = \zeta_i \geqslant 2\,\alpha_i,$	$m_D = \xi_i \geqslant 6\,\alpha_i,$	$m_E = \varkappa_i \geqslant 4\,\alpha_i$	
	$\zeta_i = \varkappa_i \leqslant \xi_i$		$\checkmark$
	$\xi_i = \varkappa_i \leqslant \zeta_i$		$\checkmark$
	$\xi_i = \zeta_i \leqslant \varkappa_i$		$\checkmark$

The case $p_i = 3$ for $\alpha_i = \beta_i > \omega_i = 0$ leads to the same equalities and inequalities as the other cases in the tables 5.13 and 5.14.

Table 5.15

$\alpha_i = \beta_i > \omega_i = 0,\ p_i = 2$	$\mu_i > 0\ \Rightarrow\ \mu_i = \alpha_i$ and $\eta_i = 0$		
$m_C = 2\,\alpha_i + 2\,\mu_i,$	$m_D = \xi_i \geqslant 4\,\alpha_i + 2\,\eta_i,$	$m_E = \varkappa_i \geqslant 4\,\alpha_i + 1$	
	$2\,\alpha_i + 2\,\mu_i = \varkappa_i \leqslant \xi_i$		
	$\xi_i = \varkappa_i \leqslant 2\,\alpha_i + 2\,\mu_i$		
	$\xi_i = 2\,\alpha_i + 2\,\mu_i \leqslant \varkappa_i$		$\checkmark$

Table 5.16

$\alpha_i = \beta_i > \omega_i = 0,\ p_i = 2$	$\mu_i = 0$ and $\eta_i = \alpha_i$		
$m_C = \zeta_i \geqslant 2\,\alpha_i,$	$m_D = \xi_i \geqslant 6\,\alpha_i,$	$m_E = \varkappa_i \geqslant 4\,\alpha_i + 1$	
	$\zeta_i = \varkappa_i \leqslant \xi_i$		$\checkmark$
	$\xi_i = \varkappa_i \leqslant \zeta_i$		$\checkmark$
	$\xi_i = \zeta_i \leqslant \varkappa_i$		$\checkmark$

In the following cases the multiplicity β_i is zero, while ω_i is nonzero.

Table 5.17

$\alpha_i > \omega_i > \beta_i = 0,\ p_i \neq 2,3$	$\mu_i > 0$ and $\eta_i > 0\ \Rightarrow\ \mu_i = \omega_i$ and $\eta_i = \alpha_i - \omega_i$		
$m_C = 4\,\alpha_i - 2\,\omega_i + 2\,\mu_i,$	$m_D = 2\,\alpha_i + 2\,\omega_i + 2\,\eta_i,$	$m_E = 4\,\alpha_i$	
$4\,\alpha_i - 2\,\omega_i + 2\,\mu_i = 4\,\alpha_i \leqslant 2\,\alpha_i + 2\,\omega_i + 2\,\eta_i$			$\checkmark$
$2\,\alpha_i + 2\,\omega_i + 2\,\eta_i = 4\,\alpha_i \leqslant 4\,\alpha_i - 2\,\omega_i + 2\,\mu_i$			$\checkmark$
$2\,\alpha_i + 2\,\omega_i + 2\,\eta_i = 4\,\alpha_i - 2\,\omega_i + 2\,\mu_i \leqslant 4\,\alpha_i$			$\checkmark$

Table 5.18

$\alpha_i > \omega_i > \beta_i = 0,\ p_i \neq 2,3$	$\mu_i = 0$ and $\eta_i = \alpha_i$		
$m_C = \zeta_i \geqslant 4\,\alpha_i - 2\,\omega_i,$	$m_D = 4\,\alpha_i + 2\,\omega_i,$	$m_E = 4\,\alpha_i$	
	$\zeta_i = 4\,\alpha_i \leqslant 4\,\alpha_i + 2\,\omega_i$		$\checkmark$
	$4\,\alpha_i + 2\,\omega_i = 4\,\alpha_i \leqslant \zeta_i$		
	$4\,\alpha_i + 2\,\omega_i = \zeta_i \leqslant 4\,\alpha_i$		

Table 5.19

$\alpha_i > \omega_i > \beta_i = 0,\ p_i \neq 2,3$	$\mu_i = \alpha_i$ and $\eta_i = 0$		
$m_C = 6\,\alpha_i - 2\,\omega_i,$	$m_D = \xi_i \geqslant 2\,\alpha_i + 2\,\omega_i,$	$m_E = 4\,\alpha_i$	
	$6\,\alpha_i - 2\,\omega_i = 4\,\alpha_i \leqslant \xi_i$		
	$\xi_i = 4\,\alpha_i \leqslant 6\,\alpha_i - 2\,\omega_i$		$\checkmark$
	$\xi_i = 6\,\alpha_i - 2\,\omega_i \leqslant 4\,\alpha_i$		

Table 5.20

$\alpha_i > \omega_i > \beta_i = 0$, $p_i = 2,3$	$\mu_i > 0$ and $\eta_i > 0 \;\Rightarrow\; \mu_i = \omega_i$ and $\eta_i = \alpha_i - \omega_i$	
$m_C = 4\,\alpha_i - 2\,\omega_i + 2\,\mu_i,$	$m_D = 2\,\alpha_i + 2\,\omega_i + 2\,\eta_i,$ $\qquad$ $m_E = 4\,\alpha_i + 1$	
	$4\,\alpha_i - 2\,\omega_i + 2\,\mu_i = 4\,\alpha_i + 1 \leqslant 2\,\alpha_i + 2\,\omega_i + 2\,\eta_i$	
	$2\,\alpha_i + 2\,\omega_i + 2\,\eta_i = 4\,\alpha_i + 1 \leqslant 4\,\alpha_i - 2\,\omega_i + 2\,\mu_i$	
	$2\,\alpha_i + 2\,\omega_i + 2\,\eta_i = 4\,\alpha_i - 2\,\omega_i + 2\,\mu_i \leqslant 4\,\alpha_i + 1$	$\checkmark$

Table 5.21

$\alpha_i > \omega_i > \beta_i = 0$, $p_i = 2,3$	$\mu_i = 0$ and $\eta_i = \alpha_i$	
$m_C = \zeta_i \geqslant 4\,\alpha_i - 2\,\omega_i,$	$m_D = 4\,\alpha_i + 2\,\omega_i,$ $\qquad$ $m_E = 4\,\alpha_i + 1$	
	$\zeta_i = 4\,\alpha_i + 1 \leqslant 4\,\alpha_i + 2\,\omega_i$	$\checkmark$
	$4\,\alpha_i + 2\,\omega_i = 4\,\alpha_i + 1 \leqslant \zeta_i$	
	$4\,\alpha_i + 2\,\omega_i = \zeta_i \leqslant 4\,\alpha_i + 1$	

Table 5.22

$\alpha_i > \omega_i > \beta_i = 0$, $p_i = 2,3$	$\mu_i = \alpha_i$ and $\eta_i = 0$	
$m_C = 6\,\alpha_i - 2\,\omega_i,$	$m_D = \xi_i \geqslant 2\,\alpha_i + 2\,\omega_i,$ $\qquad$ $m_E = 4\,\alpha_i + 1$	
	$6\,\alpha_i - 2\,\omega_i = 4\,\alpha_i + 1 \leqslant \xi_i$	
	$\xi_i = 4\,\alpha_i + 1 \leqslant 6\,\alpha_i - 2\,\omega_i$	
	$\xi_i = 6\,\alpha_i - 2\,\omega_i \leqslant 4\,\alpha_i + 1$	$\checkmark$

In the following cases the multiplicity ω_i coincides with the multiplicity α_i and, according to (3.3), we have $\theta_i = \max(\alpha_i, \beta_i, \omega_i) = \alpha_i$.

Table 5.23

$\alpha_i = \omega_i > \beta_i = 0$, $p_i \neq 2,3$	$\mu_i > 0 \;\Rightarrow\; \mu_i = \alpha_i$ and $\eta_i = 0$	
$m_C = 2\,\alpha_i + 2\,\mu_i,$	$m_D = \xi_i \geqslant 4\,\alpha_i + 2\,\eta_i,$ $\qquad$ $m_E = \varkappa_i \geqslant 4\,\alpha_i$	
	$2\,\alpha_i + 2\,\mu_i = \varkappa_i \leqslant \xi_i$	$\checkmark$
	$\xi_i = \varkappa_i \leqslant 2\,\alpha_i + 2\,\mu_i$	$\checkmark$
	$\xi_i = 2\,\alpha_i + 2\,\mu_i \leqslant \varkappa_i$	$\checkmark$

Table 5.24

$\alpha_i = \omega_i > \beta_i = 0$, $p_i \neq 2,3$	$\mu_i = 0$ and $\eta_i = \alpha_i$	
$m_C = \zeta_i \geqslant 2\,\alpha_i,$	$m_D = \xi_i \geqslant 6\,\alpha_i,$ $\qquad$ $m_E = \varkappa_i \geqslant 4\,\alpha_i$	
	$\zeta_i = \varkappa_i \leqslant \xi_i$	$\checkmark$
	$\xi_i = \varkappa_i \leqslant \zeta_i$	$\checkmark$
	$\xi_i = \zeta_i \leqslant \varkappa_i$	$\checkmark$

Table 5.25

$\alpha_i = \omega_i > \beta_i = 0$, $p_i = 2$	$\mu_i > 0 \;\Rightarrow\; \mu_i = \alpha_i$ and $\eta_i = 0$		
$m_C = 2\,\alpha_i + 2\,\mu_i,$	$m_D = \xi_i \geqslant 4\,\alpha_i + 2\,\eta_i,$	$m_E = \varkappa_i \geqslant 4\,\alpha_i + 1$	
	$2\,\alpha_i + 2\,\mu_i = \varkappa_i \leqslant \xi_i$		
	$\xi_i = \varkappa_i \leqslant 2\,\alpha_i + 2\,\mu_i$		
	$\xi_i = 2\,\alpha_i + 2\,\mu_i \leqslant \varkappa_i$		$\checkmark$

Table 5.26

$\alpha_i = \omega_i > \beta_i = 0$, $p_i = 2$	$\mu_i = 0$ and $\eta_i = \alpha_i$		
$m_C = \zeta_i \geqslant 2\,\alpha_i,$	$m_D = \xi_i \geqslant 6\,\alpha_i,$	$m_E = \varkappa_i \geqslant 4\,\alpha_i + 1$	
	$\zeta_i = \varkappa_i \leqslant \xi_i$		$\checkmark$
	$\xi_i = \varkappa_i \leqslant \zeta_i$		$\checkmark$
	$\xi_i = \zeta_i \leqslant \varkappa_i$		$\checkmark$

Table 5.27

$\alpha_i = \omega_i > \beta_i = 0$, $p_i = 3$	$\mu_i > 0 \;\Rightarrow\; \mu_i = \alpha_i$ and $\eta_i = 0$		
$m_C = 2\,\alpha_i + 2\,\mu_i,$	$m_D = \xi_i \geqslant 4\,\alpha_i + 2\,\eta_i,$	$m_E = 4\,\alpha_i$	
	$2\,\alpha_i + 2\,\mu_i = 4\,\alpha_i \leqslant \xi_i$		$\checkmark$
	$\xi_i = 4\,\alpha_i \leqslant 2\,\alpha_i + 2\,\mu_i$		$\checkmark$
	$\xi_i = 2\,\alpha_i + 2\,\mu_i \leqslant 4\,\alpha_i$		$\checkmark$

Table 5.28

$\alpha_i = \omega_i > \beta_i = 0$, $p_i = 3$	$\mu_i = 0$ and $\eta_i = \alpha_i$		
$m_C = \zeta_i \geqslant 2\,\alpha_i,$	$m_D = \xi_i \geqslant 6\,\alpha_i,$	$m_E = 4\,\alpha_i$	
	$\zeta_i = 4\,\alpha_i \leqslant \xi_i$		$\checkmark$
	$\xi_i = 4\,\alpha_i \leqslant \zeta_i$		
	$\xi_i = \zeta_i \leqslant 4\,\alpha_i$		

In the following cases the multiplicity ω_i is greater than the multiplicity α_i. Therefore, according to (3.3), we have $\theta_i = \max(\alpha_i, \beta_i, \omega_i) = \omega_i$.

Table 5.29

$\omega_i > \alpha_i > \beta_i = 0$, $p_i \neq 2$	$\mu_i > 0$ and $\eta_i > 0 \;\Rightarrow\; \mu_i = \alpha_i$ and $\eta_i = \omega_i - \alpha_i$		
$m_C = 4\,\omega_i - 2\,\alpha_i + 2\,\mu_i,$	$m_D = 2\,\omega_i + 2\,\alpha_i + 2\,\eta_i,$	$m_E = 4\,\omega_i$	
	$4\,\omega_i - 2\,\alpha_i + 2\,\mu_i = 4\,\omega_i \leqslant 2\,\omega_i + 2\,\alpha_i + 2\,\eta_i$		$\checkmark$
	$2\,\omega_i + 2\,\alpha_i + 2\,\eta_i = 4\,\omega_i \leqslant 4\,\omega_i - 2\,\alpha_i + 2\,\mu_i$		$\checkmark$
	$2\,\omega_i + 2\,\alpha_i + 2\,\eta_i = 4\,\omega_i - 2\,\alpha_i + 2\,\mu_i \leqslant 4\,\omega_i$		$\checkmark$

Table 5.30

$\omega_i > \alpha_i > \beta_i = 0,\ p_i \neq 2$	$\mu_i = 0$ and $\eta_i = \omega_i$	
$m_C = \zeta_i \geqslant 4\,\omega_i - 2\,\alpha_i,$	$m_D = 4\,\omega_i + 2\,\alpha_i,\qquad m_E = 4\,\omega_i$	
	$\zeta_i = 4\,\omega_i \leqslant 4\,\omega_i + 2\,\alpha_i$	
	$4\,\omega_i + 2\,\alpha_i = 4\,\omega_i \leqslant \zeta_i$	$\checkmark$
	$4\,\omega_i + 2\,\alpha_i = \zeta_i \leqslant 4\,\omega_i$	

Table 5.31

$\omega_i > \alpha_i > \beta_i = 0,\ p_i \neq 2$	$\mu_i = \omega_i$ and $\eta_i = 0$	
$m_C = 6\,\omega_i - 2\,\alpha_i,$	$m_D = \xi_i \geqslant 2\,\omega_i + 2\,\alpha_i,\qquad m_E = 4\,\omega_i$	
	$6\,\omega_i - 2\,\alpha_i = 4\,\omega_i \leqslant \xi_i$	
	$\xi_i = 4\,\omega_i \leqslant 6\,\omega_i - 2\,\alpha_i$	$\checkmark$
	$\xi_i = 6\,\omega_i - 2\,\alpha_i \leqslant 4\,\omega_i$	

Table 5.32

$\omega_i > \alpha_i > \beta_i = 0,\ p_i = 2$	$\mu_i > 0$ and $\eta_i > 0\ \Rightarrow\ \mu_i = \alpha_i$ and $\eta_i = \omega_i - \alpha_i$	
$m_C = 4\,\omega_i - 2\,\alpha_i + 2\,\mu_i,$	$m_D = 2\,\omega_i + 2\,\alpha_i + 2\,\eta_i,\qquad m_E = 4\,\omega_i + 1$	
	$4\,\omega_i - 2\,\alpha_i + 2\,\mu_i = 4\,\omega_i + 1 \leqslant 2\,\omega_i + 2\,\alpha_i + 2\,\eta_i$	
	$2\,\omega_i + 2\,\alpha_i + 2\,\eta_i = 4\,\omega_i + 1 \leqslant 4\,\omega_i - 2\,\alpha_i + 2\,\mu_i$	
	$2\,\omega_i + 2\,\alpha_i + 2\,\eta_i = 4\,\omega_i - 2\,\alpha_i + 2\,\mu_i \leqslant 4\,\omega_i + 1$	$\checkmark$

Table 5.33

$\omega_i > \alpha_i > \beta_i = 0,\ p_i = 2$	$\mu_i = 0$ and $\eta_i = \omega_i$	
$m_C = \zeta_i \geqslant 4\,\omega_i - 2\,\alpha_i,$	$m_D = 4\,\omega_i + 2\,\alpha_i,\qquad m_E = 4\,\omega_i + 1$	
	$\zeta_i = 4\,\omega_i + 1 \leqslant 4\,\omega_i + 2\,\alpha_i$	$\checkmark$
	$4\,\omega_i + 2\,\alpha_i = 4\,\omega_i + 1 \leqslant \zeta_i$	
	$4\,\omega_i + 2\,\alpha_i = \zeta_i \leqslant 4\,\omega_i + 1$	

Table 5.34

$\omega_i > \alpha_i > \beta_i = 0,\ p_i = 2$	$\mu_i = \omega_i$ and $\eta_i = 0$	
$m_C = 6\,\omega_i - 2\,\alpha_i,$	$m_D = \xi_i \geqslant 2\,\omega_i + 2\,\alpha_i,\qquad m_E = 4\,\omega_i + 1$	
	$6\,\omega_i - 2\,\alpha_i = 4\,\omega_i + 1 \leqslant \xi_i$	
	$\xi_i = 4\,\omega_i + 1 \leqslant 6\,\omega_i - 2\,\alpha_i$	$\checkmark$
	$\xi_i = 6\,\omega_i - 2\,\alpha_i \leqslant 4\,\omega_i + 1$	

The case $p_i = 3$ for $\omega_i > \alpha_i > \beta_i = 0$ leads to the same equalities and inequalities as the other cases in the tables 5.29, 5.30, and 5.31.

Table 5.35

$\omega_i > \alpha_i = \beta_i = 0,\ p_i \neq 2$	$\eta_i > 0 \ \Rightarrow\ \eta_i = \omega_i$ and $\mu_i = 0$		
$m_C = \zeta_i \geqslant 4\,\omega_i + 2\,\mu_i,$	$m_D = 2\,\omega_i + 2\,\eta_i,$	$m_E = \varkappa_i \geqslant 4\,\omega_i$	
	$\zeta_i = \varkappa_i \leqslant 2\,\omega_i + 2\,\eta_i$		$\checkmark$
	$2\,\omega_i + 2\,\eta_i = \varkappa_i \leqslant \zeta_i$		$\checkmark$
	$2\,\omega_i + 2\,\eta_i = \zeta_i \leqslant \varkappa_i$		$\checkmark$

Table 5.36

$\omega_i > \alpha_i = \beta_i = 0,\ p_i \neq 2$	$\eta_i = 0$ and $\mu_i = \omega_i$		
$m_C = \zeta_i \geqslant 6\,\omega_i,$	$m_D = \xi_i \geqslant 2\,\omega_i,$	$m_E = \varkappa_i \geqslant 4\,\omega_i$	
	$\zeta_i = \varkappa_i \leqslant \xi_i$		$\checkmark$
	$\xi_i = \varkappa_i \leqslant \zeta_i$		$\checkmark$
	$\xi_i = \zeta_i \leqslant \varkappa_i$		$\checkmark$

Table 5.37

$\omega_i > \alpha_i = \beta_i = 0,\ p_i = 2$	$\eta_i > 0 \ \Rightarrow\ \eta_i = \omega_i$ and $\mu_i = 0$		
$m_C = \zeta_i \geqslant 4\,\omega_i + 2\,\mu_i,$	$m_D = 2\,\omega_i + 2\,\eta_i,$	$m_E = \varkappa_i \geqslant 4\,\omega_i + 1$	
	$\zeta_i = \varkappa_i \leqslant 2\,\omega_i + 2\,\eta_i$		
	$2\,\omega_i + 2\,\eta_i = \varkappa_i \leqslant \zeta_i$		
	$2\,\omega_i + 2\,\eta_i = \zeta_i \leqslant \varkappa_i$		$\checkmark$

Table 5.38

$\omega_i > \alpha_i = \beta_i = 0,\ p_i = 2$	$\eta_i = 0$ and $\mu_i = \omega_i$		
$m_C = \zeta_i \geqslant 6\,\omega_i,$	$m_D = \xi_i \geqslant 2\,\omega_i,$	$m_E = \varkappa_i \geqslant 4\,\omega_i + 1$	
	$\zeta_i = \varkappa_i \leqslant \xi_i$		$\checkmark$
	$\xi_i = \varkappa_i \leqslant \zeta_i$		$\checkmark$
	$\xi_i = \zeta_i \leqslant \varkappa_i$		$\checkmark$

Note that in all of the above cases $\alpha_i \geqslant \beta_i$. The rest of the cases are those where $\alpha_i < \beta_i$. Let's recall that the polynomial $P_{abu}(t)$ in (1.1) is invariant with respect to exchanging parameters a and b (see (2.1)). The same is true for $P_{\sigma(abu)}(t)$ in (2.4). Therefore we can produce the rest of the cases from those already considered.

Table 5.39 (symmetric to the table 5.1)

$\beta_i > \alpha_i > \omega_i = 0,\ p_i \neq 2$	$\mu_i > 0$ and $\eta_i > 0 \ \Rightarrow\ \mu_i = \alpha_i$ and $\eta_i = \beta_i - \alpha_i$		
$m_C = 4\,\beta_i - 2\,\alpha_i + 2\,\mu_i,$	$m_D = 2\,\beta_i + 2\,\alpha_i + 2\,\eta_i,$	$m_E = 4\,\beta_i$	
	$4\,\beta_i - 2\,\alpha_i + 2\,\mu_i = 4\,\beta_i \leqslant 2\,\beta_i + 2\,\alpha_i + 2\,\eta_i$		$\checkmark$
	$2\,\beta_i + 2\,\alpha_i + 2\,\eta_i = 4\,\beta_i \leqslant 4\,\beta_i - 2\,\alpha_i + 2\,\mu_i$		$\checkmark$
	$2\,\beta_i + 2\,\alpha_i + 2\,\eta_i = 4\,\beta_i - 2\,\alpha_i + 2\,\mu_i \leqslant 4\,\beta_i$		$\checkmark$

Table 5.40 (symmetric to the table 5.2)

$\beta_i > \alpha_i > \omega_i = 0,\ p_i \neq 2$	$\mu_i = 0$ and $\eta_i = \beta_i$		
$m_C = \zeta_i \geqslant 4\,\beta_i - 2\,\alpha_i,$	$m_D = 4\,\beta_i + 2\,\alpha_i,$	$m_E = 4\,\beta_i$	
	$\zeta_i = 4\,\beta_i \leqslant 4\,\beta_i + 2\,\alpha_i$		$\checkmark$
	$4\,\beta_i + 2\,\alpha_i = 4\,\beta_i \leqslant \zeta_i$		
	$4\,\beta_i + 2\,\alpha_i = \zeta_i \leqslant 4\,\beta_i$		

Table 5.41 (symmetric to the table 5.3)

$\beta_i > \alpha_i > \omega_i = 0,\ p_i \neq 2$	$\mu_i = \beta_i$ and $\eta_i = 0$		
$m_C = 6\,\beta_i - 2\,\alpha_i,$	$m_D = \xi_i \geqslant 2\,\beta_i + 2\,\alpha_i,$	$m_E = 4\,\beta_i$	
	$6\,\beta_i - 2\,\alpha_i = 4\,\beta_i \leqslant \xi_i$		
	$\xi_i = 4\,\beta_i \leqslant 6\,\beta_i - 2\,\alpha_i$		$\checkmark$
	$\xi_i = 6\,\beta_i - 2\,\alpha_i \leqslant 4\,\beta_i$		

Table 5.42 (symmetric to the table 5.4)

$\beta_i > \alpha_i > \omega_i = 0,\ p_i = 2$	$\mu_i > 0$ and $\eta_i > 0\ \Rightarrow\ \mu_i = \alpha_i$ and $\eta_i = \beta_i - \alpha_i$		
$m_C = 4\,\beta_i - 2\,\alpha_i + 2\,\mu_i,$	$m_D = 2\,\beta_i + 2\,\alpha_i + 2\,\eta_i,$	$m_E = 4\,\beta_i + 1$	
	$4\,\beta_i - 2\,\alpha_i + 2\,\mu_i = 4\,\beta_i + 1 \leqslant 2\,\beta_i + 2\,\alpha_i + 2\,\eta_i$		
	$2\,\beta_i + 2\,\alpha_i + 2\,\eta_i = 4\,\beta_i + 1 \leqslant 4\,\beta_i - 2\,\alpha_i + 2\,\mu_i$		
	$2\,\beta_i + 2\,\alpha_i + 2\,\eta_i = 4\,\beta_i - 2\,\alpha_i + 2\,\mu_i \leqslant 4\,\beta_i + 1$		$\checkmark$

Table 5.43 (symmetric to the table 5.5)

$\beta_i > \alpha_i > \omega_i = 0,\ p_i = 2$	$\mu_i = 0$ and $\eta_i = \beta_i$		
$m_C = \zeta_i \geqslant 4\,\beta_i - 2\,\alpha_i,$	$m_D = 4\,\beta_i + 2\,\alpha_i,$	$m_E = 4\,\beta_i + 1$	
	$\zeta_i = 4\,\beta_i + 1 \leqslant 4\,\beta_i + 2\,\alpha_i$		$\checkmark$
	$4\,\beta_i + 2\,\alpha_i = 4\,\beta_i + 1 \leqslant \zeta_i$		
	$4\,\beta_i + 2\,\alpha_i = \zeta_i \leqslant 4\,\beta_i + 1$		

Table 5.44 (symmetric to the table 5.6)

$\beta_i > \alpha_i > \omega_i = 0,\ p_i = 2$	$\mu_i = \beta_i$ and $\eta_i = 0$		
$m_C = 6\,\beta_i - 2\,\alpha_i,$	$m_D = \xi_i \geqslant 2\,\beta_i + 2\,\alpha_i,$	$m_E = 4\,\beta_i + 1$	
	$6\,\beta_i - 2\,\alpha_i = 4\,\beta_i + 1 \leqslant \xi_i$		
	$\xi_i = 4\,\beta_i + 1 \leqslant 6\,\beta_i - 2\,\alpha_i$		$\checkmark$
	$\xi_i = 6\,\beta_i - 2\,\alpha_i \leqslant 4\,\beta_i + 1$		

In the following cases the multiplicity α_i coincides with the multiplicity $\omega_i = 0$. I this case, according to (3.3), we have $\theta_i = \max(\alpha_i, \beta_i, \omega_i) = \beta_i$.

Table 5.45 (symmetric to the table 5.7)

$\beta_i > \alpha_i = \omega_i = 0,\ p_i \neq 2,3$	$\eta_i > 0 \Rightarrow \eta_i = \beta_i$ and $\mu_i = 0$		
$m_C = \zeta_i \geqslant 4\beta_i + 2\mu_i,$	$m_D = 2\beta_i + 2\eta_i,$	$m_E = \varkappa_i \geqslant 4\beta_i$	
	$\varkappa_i = \zeta_i \leqslant 2\beta_i + 2\eta_i$		$\checkmark$
	$\varkappa_i = 2\beta_i + 2\eta_i \leqslant \zeta_i$		$\checkmark$
	$\zeta_i = 2\beta_i + 2\eta_i \leqslant \varkappa_i$		$\checkmark$

Table 5.46 (symmetric to the table 5.8)

$\beta_i > \alpha_i = \omega_i = 0,\ p_i \neq 2,3$	$\eta_i = 0$ and $\mu_i = \beta_i$		
$m_C = \zeta_i \geqslant 6\beta_i,$	$m_D = \xi_i \geqslant 2\beta_i,$	$m_E = \varkappa_i \geqslant 4\beta_i$	
	$\varkappa_i = \zeta_i \leqslant \xi_i$		$\checkmark$
	$\varkappa_i = \xi_i \leqslant \zeta_i$		$\checkmark$
	$\zeta_i = \xi_i \leqslant \varkappa_i$		$\checkmark$

The following tables correspond to the special values of the prime factor p_i, i.e. to $p_i = 2$ and to $p_i = 3$.

Table 5.47 (symmetric to the table 5.9)

$\beta_i > \alpha_i = \omega_i = 0,\ p_i = 2$	$\eta_i > 0 \Rightarrow \eta_i = \beta_i$ and $\mu_i = 0$		
$m_C = \zeta_i \geqslant 4\beta_i + 2\mu_i,$	$m_D = 2\beta_i + 2\eta_i,$	$m_E = \varkappa_i \geqslant 4\beta_i + 1$	
	$\varkappa_i = \zeta_i \leqslant 2\beta_i + 2\eta_i$		
	$\varkappa_i = 2\beta_i + 2\eta_i \leqslant \zeta_i$		
	$\zeta_i = 2\beta_i + 2\eta_i \leqslant \varkappa_i$		$\checkmark$

Table 5.48 (symmetric to the table 5.10)

$\beta_i > \alpha_i = \omega_i = 0,\ p_i = 2$	$\eta_i = 0$ and $\mu_i = \beta_i$		
$m_C = \zeta_i \geqslant 6\beta_i,$	$m_D = \xi_i \geqslant 2\beta_i,$	$m_E = \varkappa_i \geqslant 4\beta_i + 1$	
	$\varkappa_i = \zeta_i \leqslant \xi_i$		$\checkmark$
	$\varkappa_i = \xi_i \leqslant \zeta_i$		$\checkmark$
	$\zeta_i = \xi_i \leqslant \varkappa_i$		$\checkmark$

Table 5.49 (symmetric to the table 5.11)

$\beta_i > \alpha_i = \omega_i = 0,\ p_i = 3$	$\eta_i > 0 \Rightarrow \eta_i = \beta_i$ and $\mu_i = 0$		
$m_C = \zeta_i \geqslant 4\beta_i + 2\mu_i,$	$m_D = 2\beta_i + 2\eta_i,$	$m_E = 4\beta_i$	
	$4\beta_i = \zeta_i \leqslant 2\beta_i + 2\eta_i$		$\checkmark$
	$4\beta_i = 2\beta_i + 2\eta_i \leqslant \zeta_i$		$\checkmark$
	$\zeta_i = 2\beta_i + 2\eta_i \leqslant 4\beta_i$		$\checkmark$

Table 5.50 (symmetric to the table 5.12)

$\beta_i > \alpha_i = \omega_i = 0,\, p_i = 3$	$\eta_i = 0$ and $\mu_i = \beta_i$		
$m_C = \zeta_i \geqslant 6\,\beta_i,$	$m_D = \xi_i \geqslant 2\,\beta_i,$	$m_E = 4\,\beta_i$	
	$4\,\beta_i = \zeta_i \leqslant \xi_i$		
	$4\,\beta_i = \xi_i \leqslant \zeta_i$		$\checkmark$
	$\zeta_i = \xi_i \leqslant 4\,\beta_i$		

In the following cases the multiplicity ω_i is greater than the multiplicity α_i, but less than β_i. In this case, according to (3.3), we have $\theta_i = \max(\alpha_i, \beta_i, \omega_i) = \beta_i$.

Table 5.51 (symmetric to the table 5.17)

$\beta_i > \omega_i > \alpha_i = 0,\, p_i \neq 2,3$	$\mu_i > 0$ and $\eta_i > 0 \;\Rightarrow\; \mu_i = \omega_i$ and $\eta_i = \beta_i - \omega_i$		
$m_C = 4\,\beta_i - 2\,\omega_i + 2\,\mu_i,$	$m_D = 2\,\beta_i + 2\,\omega_i + 2\,\eta_i,$	$m_E = 4\,\beta_i$	
	$4\,\beta_i - 2\,\omega_i + 2\,\mu_i = 4\,\beta_i \leqslant 2\,\beta_i + 2\,\omega_i + 2\,\eta_i$		$\checkmark$
	$2\,\beta_i + 2\,\omega_i + 2\,\eta_i = 4\,\beta_i \leqslant 4\,\beta_i - 2\,\omega_i + 2\,\mu_i$		$\checkmark$
	$2\,\beta_i + 2\,\omega_i + 2\,\eta_i = 4\,\beta_i - 2\,\omega_i + 2\,\mu_i \leqslant 4\,\beta_i$		$\checkmark$

Table 5.52 (symmetric to the table 5.18)

$\beta_i > \omega_i > \alpha_i = 0,\, p_i \neq 2,3$	$\mu_i = 0$ and $\eta_i = \beta_i$		
$m_C = \zeta_i \geqslant 4\,\beta_i - 2\,\omega_i,$	$m_D = 4\,\beta_i + 2\,\omega_i,$	$m_E = 4\,\beta_i$	
	$\zeta_i = 4\,\beta_i \leqslant 4\,\beta_i + 2\,\omega_i$		$\checkmark$
	$4\,\beta_i + 2\,\omega_i = 4\,\beta_i \leqslant \zeta_i$		
	$4\,\beta_i + 2\,\omega_i = \zeta_i \leqslant 4\,\beta_i$		

Table 5.53 (symmetric to the table 5.19)

$\beta_i > \omega_i > \alpha_i = 0,\, p_i \neq 2,3$	$\mu_i = \beta_i$ and $\eta_i = 0$		
$m_C = 6\,\beta_i - 2\,\omega_i,$	$m_D = \xi_i \geqslant 2\,\beta_i + 2\,\omega_i,$	$m_E = 4\,\beta_i$	
	$6\,\beta_i - 2\,\omega_i = 4\,\beta_i \leqslant \xi_i$		
	$\xi_i = 4\,\beta_i \leqslant 6\,\beta_i - 2\,\omega_i$		$\checkmark$
	$\xi_i = 6\,\beta_i - 2\,\omega_i \leqslant 4\,\beta_i$		

Table 5.54 (symmetric to the table 5.20)

$\beta_i > \omega_i > \alpha_i = 0,\, p_i = 2,3$	$\mu_i > 0$ and $\eta_i > 0 \;\Rightarrow\; \mu_i = \omega_i$ and $\eta_i = \beta_i - \omega_i$		
$m_C = 4\,\beta_i - 2\,\omega_i + 2\,\mu_i,$	$m_D = 2\,\beta_i + 2\,\omega_i + 2\,\eta_i,$	$m_E = 4\,\beta_i + 1$	
	$4\,\beta_i - 2\,\omega_i + 2\,\mu_i = 4\,\beta_i + 1 \leqslant 2\,\beta_i + 2\,\omega_i + 2\,\eta_i$		
	$2\,\beta_i + 2\,\omega_i + 2\,\eta_i = 4\,\beta_i + 1 \leqslant 4\,\beta_i - 2\,\omega_i + 2\,\mu_i$		
	$2\,\beta_i + 2\,\omega_i + 2\,\eta_i = 4\,\beta_i - 2\,\omega_i + 2\,\mu_i \leqslant 4\,\beta_i + 1$		$\checkmark$

Table 5.55 (symmetric to the table 5.21)

$\beta_i > \omega_i > \alpha_i = 0$, $p_i = 2, 3$	$\mu_i = 0$ and $\eta_i = \beta_i$		
$m_C = \zeta_i \geqslant 4\,\beta_i - 2\,\omega_i,$	$m_D = 4\,\beta_i + 2\,\omega_i,$	$m_E = 4\,\beta_i + 1$	
$\zeta_i = 4\,\beta_i + 1 \leqslant 4\,\beta_i + 2\,\omega_i$			
$4\,\beta_i + 2\,\omega_i = 4\,\beta_i + 1 \leqslant \zeta_i$			$\checkmark$
$4\,\beta_i + 2\,\omega_i = \zeta_i \leqslant 4\,\beta_i + 1$			

Table 5.56 (symmetric to the table 5.22)

$\beta_i > \omega_i > \alpha_i = 0$, $p_i = 2, 3$	$\mu_i = \beta_i$ and $\eta_i = 0$		
$m_C = 6\,\beta_i - 2\,\omega_i,$	$m_D = \xi_i \geqslant 2\,\beta_i + 2\,\omega_i,$	$m_E = 4\,\beta_i + 1$	
$6\,\beta_i - 2\,\omega_i = 4\,\beta_i + 1 \leqslant \xi_i$			
$\xi_i = 4\,\beta_i + 1 \leqslant 6\,\beta_i - 2\,\omega_i$			$\checkmark$
$\xi_i = 6\,\beta_i - 2\,\omega_i \leqslant 4\,\beta_i + 1$			

In the following cases the multiplicity ω_i coincides with the multiplicity β_i. Then, according to (3.3), we have $\theta_i = \max(\alpha_i, \beta_i, \omega_i) = \beta_i$.

Table 5.57 (symmetric to the table 5.23)

$\beta_i = \omega_i > \alpha_i = 0$, $p_i \neq 2, 3$	$\mu_i > 0 \ \Rightarrow \ \mu_i = \beta_i$ and $\eta_i = 0$		
$m_C = 2\,\beta_i + 2\,\mu_i,$	$m_D = \xi_i \geqslant 4\,\beta_i + 2\,\eta_i,$	$m_E = \varkappa_i \geqslant 4\,\beta_i$	
$2\,\beta_i + 2\,\mu_i = \varkappa_i \leqslant \xi_i$			$\checkmark$
$\xi_i = \varkappa_i \leqslant 2\,\beta_i + 2\,\mu_i$			$\checkmark$
$\xi_i = 2\,\beta_i + 2\,\mu_i \leqslant \varkappa_i$			$\checkmark$

Table 5.58 (symmetric to the table 5.24)

$\beta_i = \omega_i > \alpha_i = 0$, $p_i \neq 2, 3$	$\mu_i = 0$ and $\eta_i = \beta_i$		
$m_C = \zeta_i \geqslant 2\,\beta_i,$	$m_D = \xi_i \geqslant 6\,\beta_i,$	$m_E = \varkappa_i \geqslant 4\,\beta_i$	
$\zeta_i = \varkappa_i \leqslant \xi_i$			$\checkmark$
$\xi_i = \varkappa_i \leqslant \zeta_i$			$\checkmark$
$\xi_i = \zeta_i \leqslant \varkappa_i$			$\checkmark$

Table 5.59 (symmetric to the table 5.25)

$\beta_i = \omega_i > \alpha_i = 0$, $p_i = 2$	$\mu_i > 0 \ \Rightarrow \ \mu_i = \beta_i$ and $\eta_i = 0$		
$m_C = 2\,\beta_i + 2\,\mu_i,$	$m_D = \xi_i \geqslant 4\,\beta_i + 2\,\eta_i,$	$m_E = \varkappa_i \geqslant 4\,\beta_i + 1$	
$2\,\beta_i + 2\,\mu_i = \varkappa_i \leqslant \xi_i$			
$\xi_i = \varkappa_i \leqslant 2\,\beta_i + 2\,\mu_i$			
$\xi_i = 2\,\beta_i + 2\,\mu_i \leqslant \varkappa_i$			$\checkmark$

Table 5.60 (symmetric to the table 5.26)

$\beta_i = \omega_i > \alpha_i = 0$, $p_i = 2$	$\mu_i = 0$ and $\eta_i = \beta_i$		
$m_C = \zeta_i \geqslant 2\,\beta_i,$	$m_D = \xi_i \geqslant 6\,\beta_i,$	$m_E = \varkappa_i \geqslant 4\,\beta_i + 1$	
	$\zeta_i = \varkappa_i \leqslant \xi_i$		$\checkmark$
	$\xi_i = \varkappa_i \leqslant \zeta_i$		$\checkmark$
	$\xi_i = \zeta_i \leqslant \varkappa_i$		$\checkmark$

Table 5.61 (symmetric to the table 5.27)

$\beta_i = \omega_i > \alpha_i = 0$, $p_i = 3$	$\mu_i > 0 \;\Rightarrow\; \mu_i = \beta_i$ and $\eta_i = 0$		
$m_C = 2\,\beta_i + 2\,\mu_i,$	$m_D = \xi_i \geqslant 4\,\beta_i + 2\,\eta_i,$	$m_E = 4\,\beta_i$	
	$2\,\beta_i + 2\,\mu_i = 4\,\beta_i \leqslant \xi_i$		$\checkmark$
	$\xi_i = 4\,\beta_i \leqslant 2\,\beta_i + 2\,\mu_i$		$\checkmark$
	$\xi_i = 2\,\beta_i + 2\,\mu_i \leqslant 4\,\beta_i$		$\checkmark$

Table 5.62 (symmetric to the table 5.28)

$\beta_i = \omega_i > \alpha_i = 0$, $p_i = 3$	$\mu_i = 0$ and $\eta_i = \beta_i$		
$m_C = \zeta_i \geqslant 2\,\beta_i,$	$m_D = \xi_i \geqslant 6\,\beta_i,$	$m_E = 4\,\beta_i$	
	$\zeta_i = 4\,\beta_i \leqslant \xi_i$		$\checkmark$
	$\xi_i = 4\,\beta_i \leqslant \zeta_i$		
	$\xi_i = \zeta_i \leqslant 4\,\beta_i$		

In the following cases the multiplicity ω_i is greater than the multiplicity β_i. Therefore, according to (3.3), we have $\theta_i = \max(\alpha_i, \beta_i, \omega_i) = \omega_i$.

Table 5.63 (symmetric to the table 5.29)

$\omega_i > \beta_i > \alpha_i = 0$, $p_i \neq 2$	$\mu_i > 0$ and $\eta_i > 0 \;\Rightarrow\; \mu_i = \beta_i$ and $\eta_i = \omega_i - \beta_i$		
$m_C = 4\,\omega_i - 2\,\beta_i + 2\,\mu_i,$	$m_D = 2\,\omega_i + 2\,\beta_i + 2\,\eta_i,$	$m_E = 4\,\omega_i$	
$4\,\omega_i - 2\,\beta_i + 2\,\mu_i = 4\,\omega_i \leqslant 2\,\omega_i + 2\,\beta_i + 2\,\eta_i$			$\checkmark$
$2\,\omega_i + 2\,\beta_i + 2\,\eta_i = 4\,\omega_i \leqslant 4\,\omega_i - 2\,\beta_i + 2\,\mu_i$			$\checkmark$
$2\,\omega_i + 2\,\beta_i + 2\,\eta_i = 4\,\omega_i - 2\,\beta_i + 2\,\mu_i \leqslant 4\,\omega_i$			$\checkmark$

Table 5.64 (symmetric to the table 5.30)

$\omega_i > \beta_i > \alpha_i = 0$, $p_i \neq 2$	$\mu_i = 0$ and $\eta_i = \omega_i$		
$m_C = \zeta_i \geqslant 4\,\omega_i - 2\,\beta_i,$	$m_D = 4\,\omega_i + 2\,\beta_i,$	$m_E = 4\,\omega_i$	
	$\zeta_i = 4\,\omega_i \leqslant 4\,\omega_i + 2\,\beta_i$		$\checkmark$
	$4\,\omega_i + 2\,\beta_i = 4\,\omega_i \leqslant \zeta_i$		
	$4\,\omega_i + 2\,\beta_i = \zeta_i \leqslant 4\,\omega_i$		

Table 5.65 (symmetric to the table 5.31)

$\omega_i > \beta_i > \alpha_i = 0,\ p_i \neq 2$	$\mu_i = \omega_i$ and $\eta_i = 0$	
$m_C = 6\,\omega_i - 2\,\beta_i,$	$m_D = \xi_i \geqslant 2\,\omega_i + 2\,\beta_i,$	$m_E = 4\,\omega_i$
$6\,\omega_i - 2\,\beta_i = 4\,\omega_i \leqslant \xi_i$ $\xi_i = 4\,\omega_i \leqslant 6\,\omega_i - 2\,\beta_i$ $\xi_i = 6\,\omega_i - 2\,\beta_i \leqslant 4\,\omega_i$		$\checkmark$

Table 5.66 (symmetric to the table 5.32)

$\omega_i > \beta_i > \alpha_i = 0,\ p_i = 2$	$\mu_i > 0$ and $\eta_i > 0\ \Rightarrow\ \mu_i = \beta_i$ and $\eta_i = \omega_i - \beta_i$	
$m_C = 4\,\omega_i - 2\,\beta_i + 2\,\mu_i,$	$m_D = 2\,\omega_i + 2\,\beta_i + 2\,\eta_i,$	$m_E = 4\,\omega_i + 1$
$4\,\omega_i - 2\,\beta_i + 2\,\mu_i = 4\,\omega_i + 1 \leqslant 2\,\omega_i + 2\,\beta_i + 2\,\eta_i$ $2\,\omega_i + 2\,\beta_i + 2\,\eta_i = 4\,\omega_i + 1 \leqslant 4\,\omega_i - 2\,\beta_i + 2\,\mu_i$ $2\,\omega_i + 2\,\beta_i + 2\,\eta_i = 4\,\omega_i - 2\,\beta_i + 2\,\mu_i \leqslant 4\,\omega_i + 1$		$\checkmark$

Table 5.67 (symmetric to the table 5.33)

$\omega_i > \beta_i > \alpha_i = 0,\ p_i = 2$	$\mu_i = 0$ and $\eta_i = \omega_i$	
$m_C = \zeta_i \geqslant 4\,\omega_i - 2\,\beta_i,$	$m_D = 4\,\omega_i + 2\,\beta_i,$	$m_E = 4\,\omega_i + 1$
$\zeta_i = 4\,\omega_i + 1 \leqslant 4\,\omega_i + 2\,\beta_i$ $4\,\omega_i + 2\,\beta_i = 4\,\omega_i + 1 \leqslant \zeta_i$ $4\,\omega_i + 2\,\beta_i = \zeta_i \leqslant 4\,\omega_i + 1$		$\checkmark$

Table 5.68 (symmetric to the table 5.34)

$\omega_i > \beta_i > \alpha_i = 0,\ p_i = 2$	$\mu_i = \omega_i$ and $\eta_i = 0$	
$m_C = 6\,\omega_i - 2\,\beta_i,$	$m_D = \xi_i \geqslant 2\,\omega_i + 2\,\beta_i,$	$m_E = 4\,\omega_i + 1$
$6\,\omega_i - 2\,\beta_i = 4\,\omega_i + 1 \leqslant \xi_i$ $\xi_i = 4\,\omega_i + 1 \leqslant 6\,\omega_i - 2\,\beta_i$ $\xi_i = 6\,\omega_i - 2\,\beta_i \leqslant 4\,\omega_i + 1$		$\checkmark$

Thus, totally we have 68 cases placed into 68 tables. They describe completely the structure of the prime factors $p_1, \ldots, p_n$ in (5.1).

6. The structural theorem.

Analyzing the whole variety of data in the tables 5.1 through 5.68, we subdivide the set of prime factors $S = \{p_1, \ldots, p_n\}$ from (5.1) into a disjoint union of several sets. The first three of such sets are given by the formulas

$$
\begin{aligned}
S_1 &= \{p_i \in S:\ \alpha_i > \beta_i > \omega_i = 0,\ \mu_i = \beta_i,\ \eta_i = \alpha_i - \beta_i\}, \\
S_2 &= \{p_i \in S:\ \alpha_i > \beta_i > \omega_i = 0,\ \mu_i = 0,\ \eta_i = \alpha_i\}, \\
S_3 &= \{p_i \in S:\ \alpha_i > \beta_i > \omega_i = 0,\ \mu_i = \alpha_i,\ \eta_i = 0\}.
\end{aligned}
\tag{6.1}
$$

The formulas (6.1) correspond to the tables 5.1 through 5.6. Using the formulas (6.1), we define the following integer numbers:

$$b_1 = \prod_{p_i \in S_1} p_i^{\beta_i}, \qquad\qquad \tilde{b}_1 = \prod_{p_i \in S_1} p_i^{\alpha_i - \beta_i},$$
$$b_2 = \prod_{p_i \in S_2} p_i^{\beta_i}, \qquad\qquad \tilde{b}_2 = \prod_{p_i \in S_2} p_i^{\alpha_i - \beta_i}, \qquad (6.2)$$
$$b_3 = \prod_{p_i \in S_3} p_i^{\beta_i}, \qquad\qquad \tilde{b}_3 = \prod_{p_i \in S_3} p_i^{\alpha_i - \beta_i}.$$

The next three sets S_4, S_5, and S_6 are defined by the following formulas:

$$S_4 = \{p_i \in S : \ \omega_i > \beta_i > \alpha_i = 0, \ \mu_i = \beta_i, \ \eta_i = \omega_i - \beta_i\},$$
$$S_5 = \{p_i \in S : \ \omega_i > \beta_i > \alpha_i = 0, \ \mu_i = 0, \ \eta_i = \omega_i\}, \qquad (6.3)$$
$$S_6 = \{p_i \in S : \ \omega_i > \beta_i > \alpha_i = 0, \ \mu_i = \omega_i, \ \eta_i = 0\}.$$

The formulas (6.3) correspond to the tables 5.63 through 5.68. Using the formulas (6.3), we define the following integer numbers:

$$b_4 = \prod_{p_i \in S_4} p_i^{\beta_i}, \qquad\qquad \tilde{b}_4 = \prod_{p_i \in S_4} p_i^{\omega_i - \beta_i},$$
$$b_5 = \prod_{p_i \in S_5} p_i^{\beta_i}, \qquad\qquad \tilde{b}_5 = \prod_{p_i \in S_5} p_i^{\omega_i - \beta_i}, \qquad (6.4)$$
$$b_6 = \prod_{p_i \in S_6} p_i^{\beta_i}, \qquad\qquad \tilde{b}_6 = \prod_{p_i \in S_6} p_i^{\omega_i - \beta_i}.$$

Some of the sets S_1, S_2, S_3, S_4, S_5, S_6 can be empty. Therefore we interpret the formulas (6.2) and (6.4) so that $b_k = \tilde{b}_k = 1$ if the corresponding set S_k is empty. Moreover, $b_k \neq 1$ implies $\tilde{b}_k \neq 1$ and vice versa. If $b_k \cdot \tilde{b}_k \neq 1$, then the prime factors of the number b_k coincide with the prime factors of the number $\tilde{b}_k$.

The next three sets S_7, S_8, and S_9 are defined by the following formulas:

$$S_7 = \{p_i \in S : \ \beta_i > \omega_i > \alpha_i = 0, \ \mu_i = \omega_i, \ \eta_i = \beta_i - \omega_i\},$$
$$S_8 = \{p_i \in S : \ \beta_i > \omega_i > \alpha_i = 0, \ \mu_i = 0, \ \eta_i = \beta_i\}, \qquad (6.5)$$
$$S_9 = \{p_i \in S : \ \beta_i > \omega_i > \alpha_i = 0, \ \mu_i = \beta_i, \ \eta_i = 0\}.$$

The formulas (6.5) correspond to the tables 5.51 through 5.56. Using the formulas (6.5), we define the following integer numbers:

$$u_1 = \prod_{p_i \in S_7} p_i^{\omega_i}, \qquad\qquad \tilde{u}_1 = \prod_{p_i \in S_7} p_i^{\beta_i - \omega_i},$$
$$u_2 = \prod_{p_i \in S_8} p_i^{\omega_i}, \qquad\qquad \tilde{u}_2 = \prod_{p_i \in S_8} p_i^{\beta_i - \omega_i}, \qquad (6.6)$$
$$u_3 = \prod_{p_i \in S_9} p_i^{\omega_i}, \qquad\qquad \tilde{u}_3 = \prod_{p_i \in S_9} p_i^{\beta_i - \omega_i}.$$

The next three sets S_{10}, S_{11}, and S_{12} are defined by the following formulas:

$$S_{10} = \{p_i \in S : \ \alpha_i > \omega_i > \beta_i = 0, \ \mu_i = \omega_i, \ \eta_i = \alpha_i - \omega_i\},$$
$$S_{11} = \{p_i \in S : \ \alpha_i > \omega_i > \beta_i = 0, \ \mu_i = 0, \ \eta_i = \alpha_i\}, \qquad (6.7)$$
$$S_{12} = \{p_i \in S : \ \alpha_i > \omega_i > \beta_i = 0, \ \mu_i = \alpha_i, \ \eta_i = 0\}.$$

The formulas (6.7) correspond to the tables 5.17 through 5.22. Using the formulas (6.7), we define the following integer numbers:

$$u_4 = \prod_{p_i \in S_{10}} p_i^{\omega_i}, \qquad\qquad \tilde{u}_4 = \prod_{p_i \in S_{10}} p_i^{\alpha_i - \omega_i},$$
$$u_5 = \prod_{p_i \in S_{11}} p_i^{\omega_i}, \qquad\qquad \tilde{u}_5 = \prod_{p_i \in S_{11}} p_i^{\alpha_i - \omega_i}, \qquad (6.8)$$
$$u_6 = \prod_{p_i \in S_{12}} p_i^{\omega_i}, \qquad\qquad \tilde{u}_6 = \prod_{p_i \in S_{12}} p_i^{\alpha_i - \omega_i}.$$

Some of the sets S_7, S_8, S_9, S_{10}, S_{11}, S_{12} can be empty. Therefore we interpret the formulas (6.6) and (6.8) so that $u_k = \tilde{u}_k = 1$ if the corresponding set S_k is empty. Moreover, $u_k \neq 1$ implies $\tilde{u}_k \neq 1$ and vice versa. If $u_k \cdot \tilde{u}_k \neq 1$, then the prime factors of the number u_k coincide with the prime factors of the number $\tilde{u}_k$.

The next three sets S_{13}, S_{14}, and S_{15} are defined by the following formulas:

$$S_{13} = \{p_i \in S : \ \omega_i > \alpha_i > \beta_i = 0, \ \mu_i = \alpha_i, \ \eta_i = \omega_i - \alpha_i\},$$
$$S_{14} = \{p_i \in S : \ \omega_i > \alpha_i > \beta_i = 0, \ \mu_i = 0, \ \eta_i = \omega_i\}, \qquad (6.9)$$
$$S_{15} = \{p_i \in S : \ \omega_i > \alpha_i > \beta_i = 0, \ \mu_i = \omega_i, \ \eta_i = 0\}.$$

The formulas (6.9) correspond to the tables 5.29 through 5.34. Using the formulas (6.9), we define the following integer numbers:

$$a_1 = \prod_{p_i \in S_{13}} p_i^{\alpha_i}, \qquad\qquad \tilde{a}_1 = \prod_{p_i \in S_{13}} p_i^{\omega_i - \alpha_i},$$
$$a_2 = \prod_{p_i \in S_{14}} p_i^{\alpha_i}, \qquad\qquad \tilde{a}_2 = \prod_{p_i \in S_{14}} p_i^{\omega_i - \alpha_i}, \qquad (6.10)$$
$$a_3 = \prod_{p_i \in S_{15}} p_i^{\alpha_i}, \qquad\qquad \tilde{a}_3 = \prod_{p_i \in S_{15}} p_i^{\omega_i - \alpha_i}.$$

The next three sets S_{16}, S_{17}, and S_{18} are defined similarly. For this purpose we use the following three formulas analogous to (6.9):

$$S_{16} = \{p_i \in S : \ \beta_i > \alpha_i > \omega_i = 0, \ \mu_i = \alpha_i, \ \eta_i = \beta_i - \alpha_i\},$$
$$S_{17} = \{p_i \in S : \ \beta_i > \alpha_i > \omega_i = 0, \ \mu_i = 0, \ \eta_i = \beta_i\}, \qquad (6.11)$$
$$S_{18} = \{p_i \in S : \ \beta_i > \alpha_i > \omega_i = 0, \ \mu_i = \beta_i, \ \eta_i = 0\}.$$

The formulas (6.11) correspond to the tables 5.39 through 5.44. We use the formulas (6.11) in order to define six numbers a_4, a_5, a_6, $\tilde{a}_4$, $\tilde{a}_5$, $\tilde{a}_6$ similar to the numbers

a_1, a_2, a_3, $\tilde{a}_1$, $\tilde{a}_2$, $\tilde{a}_3$ in (6.10). For this purpose we write

$$a_4 = \prod_{p_i \in S_{16}} p_i^{\alpha_i}, \qquad\qquad \tilde{a}_4 = \prod_{p_i \in S_{16}} p_i^{\beta_i - \alpha_i},$$

$$a_5 = \prod_{p_i \in S_{17}} p_i^{\alpha_i}, \qquad\qquad \tilde{a}_5 = \prod_{p_i \in S_{17}} p_i^{\beta_i - \alpha_i}, \qquad (6.12)$$

$$a_6 = \prod_{p_i \in S_{18}} p_i^{\alpha_i}, \qquad\qquad \tilde{a}_6 = \prod_{p_i \in S_{18}} p_i^{\beta_i - \alpha_i}.$$

Some of the sets S_{13}, S_{14}, S_{15}, S_{16}, S_{17}, S_{18} can be empty. Therefore we interpret the formulas (6.10) and (6.12) so that $a_k = \tilde{a}_k = 1$ if the corresponding set S_k is empty. Moreover, $a_k \neq 1$ implies $\tilde{a}_k \neq 1$ and vice versa. If $a_k \cdot \tilde{a}_k \neq 1$, then the prime factors of the number a_k coincide with the prime factors of the number $\tilde{a}_k$.

The following sets and their associated numbers are defined in a slightly different manner. The sets S_{19}, S_{20}, S_{21}, S_{22} are given by the formulas

$$S_{19} = \{p_i \in S : \alpha_i > \beta_i = \omega_i = 0, \ \mu_i = \alpha_i, \ \eta_i = 0\},$$
$$S_{20} = \{p_i \in S : \alpha_i > \beta_i = \omega_i = 0, \ \mu_i = 0, \ \eta_i = \alpha_i\}, \qquad (6.13)$$

$$S_{21} = \{p_i \in S : \omega_i = \beta_i > \alpha_i = 0, \ \mu_i = \omega_i, \ \eta_i = 0\},$$
$$S_{22} = \{p_i \in S : \omega_i = \beta_i > \alpha_i = 0, \ \mu_i = 0, \ \eta_i = \omega_i\}. \qquad (6.14)$$

The formulas (6.13) and (6.14) correspond to the tables 5.7 through 5.12 and 5.57 through 5.62. Using them, we define the following integer numbers:

$$a_7 = \prod_{p_i \in S_{19}} p_i^{\alpha_i}, \qquad\qquad \tilde{a}_7 = \prod_{p_i \in S_{20}} p_i^{\alpha_i}, \qquad (6.15)$$

$$a_8 = \prod_{p_i \in S_{21}} p_i^{\omega_i}, \qquad\qquad \tilde{a}_8 = \prod_{p_i \in S_{22}} p_i^{\omega_i}, \qquad (6.16)$$

Some of the sets S_{19}, S_{20}, S_{21}, S_{22} can be empty. Therefore we interpret the formulas (6.15) and (6.16) so that $a_k = 1$ or $\tilde{a}_k = 1$ if the corresponding set S_k is empty. Unlike the numbers (6.12), the numbers (6.15) and (6.16) are not correlated within their pairs.

The sets S_{23}, S_{24}, S_{25}, S_{26} are given by the following formulas

$$S_{23} = \{p_i \in S : \beta_i > \alpha_i = \omega_i = 0, \ \mu_i = \beta_i, \ \eta_i = 0\},$$
$$S_{24} = \{p_i \in S : \beta_i > \alpha_i = \omega_i = 0, \ \mu_i = 0, \ \eta_i = \beta_i\}, \qquad (6.17)$$

$$S_{25} = \{p_i \in S : \alpha_i = \omega_i > \beta_i = 0, \ \mu_i = \alpha_i, \ \eta_i = 0\},$$
$$S_{26} = \{p_i \in S : \alpha_i = \omega_i > \beta_i = 0, \ \mu_i = 0, \ \eta_i = \alpha_i\}. \qquad (6.18)$$

The formulas (6.17) and (6.18) correspond to the tables 5.45 through 5.50 and 5.23 through 5.28. Using them, we define the following integer numbers:

$$b_7 = \prod_{p_i \in S_{23}} p_i^{\beta_i}, \qquad\qquad \tilde{b}_7 = \prod_{p_i \in S_{24}} p_i^{\beta_i}, \qquad (6.19)$$

$$b_8 = \prod_{p_i \in S_{25}} p_i^{\alpha_i}, \qquad\qquad \tilde{b}_8 = \prod_{p_i \in S_{26}} p_i^{\alpha_i}, \qquad\qquad (6.20)$$

Some of the sets S_{23}, S_{24}, S_{25}, S_{26} can be empty. Therefore we interpret the formulas (6.19) and (6.20) so that $b_k = 1$ or $\tilde{b}_k = 1$ if the corresponding set S_k is empty. The numbers (6.19) and (6.20) are also not correlated within their pairs.

The sets S_{27}, S_{28}, S_{29}, S_{30} are given by the following formulas:

$$\begin{aligned} S_{27} &= \{p_i \in S : \ \omega_i > \alpha_i = \beta_i = 0, \ \mu_i = \omega_i, \ \eta_i = 0\}, \\ S_{28} &= \{p_i \in S : \ \omega_i > \alpha_i = \beta_i = 0, \ \mu_i = 0, \ \eta_i = \omega_i\}, \end{aligned} \qquad (6.21)$$

$$\begin{aligned} S_{29} &= \{p_i \in S : \ \alpha_i = \beta_i > \omega_i = 0, \ \mu_i = \alpha_i, \ \eta_i = 0\}, \\ S_{30} &= \{p_i \in S : \ \alpha_i = \beta_i > \omega_i = 0, \ \mu_i = 0, \ \eta_i = \alpha_i\}. \end{aligned} \qquad (6.22)$$

The formulas (6.21) and (6.22) correspond to the tables 5.35 through 5.38 and 5.13 through 5.16. Using them, we define the following integer numbers:

$$u_7 = \prod_{p_i \in S_{27}} p_i^{\omega_i}, \qquad\qquad \tilde{u}_7 = \prod_{p_i \in S_{28}} p_i^{\omega_i}, \qquad\qquad (6.23)$$

$$u_8 = \prod_{p_i \in S_{29}} p_i^{\alpha_i}, \qquad\qquad \tilde{u}_8 = \prod_{p_i \in S_{30}} p_i^{\alpha_i}, \qquad\qquad (6.24)$$

Now we can compare the formulas (6.1), (6.3), (6.5), (6.7), (6.9), (6.11), (6.13), (6.14), (6.17), (6.18), (6.21), and (6.22) with the tables 5.1 through 5.68 considered in the previous section. As a result we derive that

$$S = \{p_1, \ \dots \ , p_n\} = \bigcup_{i=1}^{30} S_i. \qquad\qquad (6.25)$$

Then we apply (6.25) to the formulas (6.2), (6.4), (6.2), (6.6), (6.8), (6.10), (6.12), (6.15), (6.16), (6.19), (6.20), (6.23), and (6.24). This yields the following formulas:

$$a = a_7 \, \tilde{a}_7 \, b_8 \, \tilde{b}_8 \, u_8 \, \tilde{u}_8 \prod_{i=1}^{6} a_i \prod_{i=1}^{3} b_i \, \tilde{b}_i \prod_{i=4}^{6} u_i \, \tilde{u}_i, \qquad\qquad (6.26)$$

$$b = b_7 \, \tilde{b}_7 \, u_8 \, \tilde{u}_8 \, a_8 \, \tilde{a}_8 \prod_{i=1}^{6} b_i \prod_{i=1}^{3} u_i \, \tilde{u}_i \prod_{i=4}^{6} a_i \, \tilde{a}_i, \qquad\qquad (6.27)$$

$$u = u_7 \, \tilde{u}_7 \, a_8 \, \tilde{a}_8 \, b_8 \, \tilde{b}_8 \prod_{i=1}^{6} u_i \prod_{i=1}^{3} a_i \, \tilde{a}_i \prod_{i=4}^{6} b_i \, \tilde{b}_i, \qquad\qquad (6.28)$$

$$\tilde{a} = a_8 \, \tilde{a}_8 \, b_7 \, \tilde{b}_7 \, u_7 \, \tilde{u}_7 \prod_{i=1}^{6} \tilde{a}_i \prod_{i=1}^{3} u_i \, \tilde{u}_i \prod_{i=4}^{6} b_i \, \tilde{b}_i, \qquad\qquad (6.29)$$

$$\tilde{b} = b_8 \, \tilde{b}_8 \, u_7 \, \tilde{u}_7 \, a_7 \, \tilde{a}_7 \prod_{i=1}^{6} \tilde{b}_i \prod_{i=1}^{3} a_i \, \tilde{a}_i \prod_{i=4}^{6} u_i \, \tilde{u}_i, \qquad\qquad (6.30)$$

$$\tilde{u} = u_8 \, \tilde{u}_8 \, a_7 \, \tilde{a}_7 \, b_7 \, \tilde{b}_7 \prod_{i=1}^{6} \tilde{u}_i \prod_{i=1}^{3} b_i \, \tilde{b}_i \prod_{i=4}^{6} a_i \, \tilde{a}_i. \qquad\qquad (6.31)$$

Along with the formulas (6.26), (6.27), (6.28), (6.29), (6.30), (6.31), we derive the following formulas for the parameters Z, A_0, and B_0:

$$Z = \prod_{i=1}^{8} a_i\, \tilde{a}_i \prod_{i=1}^{8} b_i\, \tilde{b}_i \prod_{i=1}^{8} u_i\, \tilde{u}_i, \qquad (6.32)$$

$$A_0 = \prod_{\substack{i=1,3,4 \\ i=6,7,8}} a_i\, b_i\, u_i \prod_{i=3,6} \tilde{a}_i\, \tilde{b}_i\, \tilde{u}_i, \qquad (6.33)$$

$$B_0 = \prod_{\substack{i=1,2,4 \\ i=5,7,8}} \tilde{a}_i\, \tilde{b}_i\, \tilde{u}_i \prod_{i=2,5} a_i\, b_i\, u_i. \qquad (6.34)$$

Thus, the variables a, b, u, $\tilde{a}$, $\tilde{b}$, $\tilde{u}$, Z, A_0, B_0 are expressed through 48 new variables $a_1, \ldots, a_8, \tilde{a}_1, \ldots, \tilde{a}_8, b_1, \ldots, b_8, \tilde{b}_1, \ldots, \tilde{b}_8, u_1, \ldots, u_8, \tilde{u}_1, \ldots, \tilde{u}_8$ by means of the formulas (6.26), (6.27), (6.28), (6.29), (6.30), (6.31), (6.32), (6.33), and (6.34). Most of the new variable are coprime by their definition. Indeed, we have the following coprimality conditions to be fulfilled:

$$\gcd(a_i, a_j) = 1, \ \gcd(b_i, b_j) = 1, \ \gcd(u_i, u_j) = 1 \ \text{ for } i \neq j;$$
$$\gcd(\tilde{a}_i, \tilde{a}_j) = 1, \ \gcd(\tilde{b}_i, \tilde{b}_j) = 1, \ \gcd(\tilde{u}_i, \tilde{u}_j) = 1 \ \text{ for } i \neq j;$$
$$\gcd(a_i, \tilde{a}_j) = \gcd(b_i, \tilde{b}_j) = \gcd(u_i, \tilde{u}_j) = 1 \ \text{ for } i \neq j \text{ or } i > 6 \text{ or } j > 6;$$
$$\gcd(a_i, b_j) = \gcd(a_i, \tilde{b}_j) = \gcd(a_i, u_j) = \gcd(a_i, \tilde{u}_j) = 1; \qquad (6.35)$$
$$\gcd(\tilde{a}_i, b_j) = \gcd(\tilde{a}_i, \tilde{b}_j) = \gcd(\tilde{a}_i, u_j) = \gcd(\tilde{a}_i, \tilde{u}_j) = 1;$$
$$\gcd(b_i, u_j) = \gcd(b_i, \tilde{u}_j) = \gcd(\tilde{b}_i, u_j) = \gcd(\tilde{b}_i, \tilde{u}_j) = 1.$$

The only exception from the above coprimality conditions (6.35) are the numbers within the pairs $(a_i, \tilde{a}_i)$, $(b_i, \tilde{b}_i)$, $(u_i, \tilde{u}_i)$ for $1 \leqslant i \leqslant 6$. In such pairs we have

$$a_i = 1 \Longleftrightarrow \tilde{a}_i = 1 \ \text{ for } \ i = 1, \ldots, 6;$$
$$b_i = 1 \Longleftrightarrow \tilde{b}_i = 1 \ \text{ for } \ i = 1, \ldots, 6; \qquad (6.36)$$
$$u_i = 1 \Longleftrightarrow \tilde{u}_i = 1 \ \text{ for } \ i = 1, \ldots, 6;$$

$$p \mid a_i \Longleftrightarrow p \mid \tilde{a}_i \ \text{ for } p \text{ is prime and } i = 1, \ldots, 6;$$
$$p \mid b_i \Longleftrightarrow p \mid \tilde{b}_i \ \text{ for } p \text{ is prime and } i = 1, \ldots, 6; \qquad (6.37)$$
$$p \mid u_i \Longleftrightarrow p \mid \tilde{u}_i \ \text{ for } p \text{ is prime and } i = 1, \ldots, 6.$$

The conditions (6.36) and (6.37) are called the *cohesion conditions*.

The next step now is to substitute the formulas (6.26), (6.27), (6.28), (6.29), (6.30), (6.31), (6.32), (6.33), and (6.34) into the equation (4.26). As a result we get a polynomial equation with 27 terms. These terms have the common factor

$$C = a_1^4\, a_2^2\, a_3^4\, a_4^4\, a_5^2\, a_6^4\, a_7^2\, a_8^4\, \tilde{a}_1^4\, \tilde{a}_2^4\, \tilde{a}_3^2\, \tilde{a}_4^4\, \tilde{a}_5^4\, \tilde{a}_6^2\, \tilde{a}_7^4\, \tilde{a}_8^2\, b_1^4\, b_2^2\, b_3^4\, b_4^4\, b_5^2\, b_6^4\, b_7^2\, b_8^4 \cdot$$
$$\cdot \tilde{b}_1^4\, \tilde{b}_2^4\, \tilde{b}_3^2\, \tilde{b}_4^4\, \tilde{b}_5^4\, \tilde{b}_6^2\, \tilde{b}_7^4\, \tilde{b}_8^2\, u_1^4\, u_2^2\, u_3^4\, u_4^4\, u_5^2\, u_6^4\, u_7^2\, u_8^4\, \tilde{u}_1^4\, \tilde{u}_2^4\, \tilde{u}_3^2\, \tilde{u}_4^4\, \tilde{u}_5^4\, \tilde{u}_6^2\, \tilde{u}_7^4\, \tilde{u}_8^2. \qquad (6.38)$$

Even upon splitting out the nonzero common factor (6.38) the equation (4.26) appears to be rather huge. It is written as follows:

$$
\begin{aligned}
&a_1^4\,a_2^8\,a_3^4\,a_5^4\,a_7^4\,\tilde a_1^4\,\tilde a_2^4\,\tilde a_3^4\,\tilde a_7^4\,\tilde a_8^4\,b_2^4\,b_5^4\,b_8^4\,\tilde b_1^4\,\tilde b_2^4\,\tilde b_3^4\,\tilde b_4^4\,\tilde b_5^4\,\tilde b_6^4\,\tilde b_8^8\,u_2^4\,u_4^4\,u_5^8\,u_6^4\,u_7^4\,\tilde u_4^4\,\tilde u_5^4\,\tilde u_6^4\,\tilde u_7^4\,\tilde u_8^4 +\\
&+\,6\,a_1^4\,a_2^6\,a_3^4\,a_4^2\,a_5^4\,a_6^2\,a_7^6\,\tilde a_1^2\,\tilde a_2^2\,\tilde a_3^4\,\tilde a_6^2\,\tilde a_7^4\,\tilde a_8^2\,b_1^2\,b_2^4\,b_3^2\,b_5^2\,b_7^2\,b_8^4\,\tilde b_1^4\,\tilde b_2^4\,\tilde b_3^6\,\tilde b_4^2\,\tilde b_5^2\,\tilde b_6^4\,\tilde b_8^6\,u_2^2\,u_4^4\,u_5^6\;\cdot\\
&\cdot\;u_6^4\,u_7^4\,u_8^2\,\tilde u_3^2\,\tilde u_4^4\,\tilde u_5^4\,\tilde u_6^6\,\tilde u_7^2\,\tilde u_8^4 - 2\,a_1^4\,a_2^6\,a_3^4\,a_5^4\,a_7^4\,a_8^2\,\tilde a_1^4\,\tilde a_2^2\,\tilde a_3^4\,\tilde a_6^2\,\tilde a_7^2\,\tilde a_8^4\,b_2^4\,b_4^4\,b_5^4\,b_6^4\,b_7^2\,b_8^4\,\tilde b_1^2\;\cdot\\
&\cdot\;\tilde b_2^2\,\tilde b_3^4\,\tilde b_4^4\,\tilde b_5^4\,\tilde b_6^6\,\tilde b_8^6\,u_1^2\,u_2^4\,u_3^2\,u_4^4\,u_5^6\,u_6^4\,u_7^4\,\tilde u_3^2\,\tilde u_4^2\,\tilde u_5^2\,\tilde u_6^4\,\tilde u_7^4\,\tilde u_8^2 + a_1^4\,a_2^4\,a_3^4\,a_4^4\,a_5^4\,a_6^4\,a_7^8\,\tilde a_3^4\,\tilde a_6^4\;\cdot\\
&\cdot\;\tilde a_7^4\,b_1^4\,b_2^4\,b_3^4\,b_4^4\,b_8^4\,\tilde b_1^4\,\tilde b_2^4\,\tilde b_3^8\,\tilde b_6^4\,\tilde b_8^4\,u_4^4\,u_5^4\,u_6^4\,u_7^4\,u_8^4\,\tilde u_3^4\,\tilde u_4^4\,\tilde u_5^4\,\tilde u_8^8\,\tilde u_8^4 + 4\,a_1^4\,a_2^4\,a_3^4\,a_4^2\,a_5^4\,a_6^2\,a_7^6\;\cdot\\
&\cdot\;a_8^2\,\tilde a_1^2\,\tilde a_2^2\,\tilde a_3^6\,\tilde a_6^4\,\tilde a_7^2\,\tilde a_8^2\,b_1^4\,b_2^2\,b_3^2\,b_4^2\,b_5^2\,b_6^2\,b_7^4\,b_8^4\,\tilde b_1^2\,\tilde b_2^2\,\tilde b_3^6\,\tilde b_4^2\,\tilde b_5^2\,\tilde b_6^6\,\tilde b_8^4\,u_1^2\,u_2^2\,u_3^2\,u_4^4\,u_5^4\,u_6^4\,u_7^4\,u_8^2\;\cdot\\
&\cdot\;\tilde u_3^4\,\tilde u_4^2\,\tilde u_5^2\,\tilde u_6^6\,\tilde u_7^2\,\tilde u_8^2 + a_1^4\,a_2^4\,a_3^4\,a_7^4\,a_8^4\,\tilde a_1^4\,\tilde a_2^2\,\tilde a_3^8\,\tilde a_6^4\,\tilde a_8^4\,b_4^4\,b_5^4\,b_6^4\,b_7^4\,b_8^4\,\tilde b_3^4\,\tilde b_4^4\,\tilde b_5^4\,\tilde b_6^6\,\tilde b_8^4\,u_1^4\,u_2^4\,u_3^4\;\cdot\\
&\cdot\;u_4^4\,u_5^4\,u_6^4\,u_7^8\,\tilde u_3^4\,\tilde u_6^4\,\tilde u_7^4 - 2\,a_1^4\,a_2^4\,a_3^4\,a_4^4\,a_5^4\,a_6^4\,a_7^4\,a_8^2\,\tilde a_3^4\,\tilde a_6^4\,\tilde a_7^2\,b_1^4\,b_2^2\,b_3^4\,b_4^2\,b_6^2\,b_7^6\,b_8^4\,\tilde b_1^2\,\tilde b_2^2\,\tilde b_3^8\;\cdot\\
&\cdot\;\tilde b_6^6\,\tilde b_8^2\,u_1^2\,u_3^2\,u_4^4\,u_5^2\,u_6^4\,u_7^4\,u_8^4\,\tilde u_3^6\,\tilde u_4^2\,\tilde u_5^2\,\tilde u_6^8\,\tilde u_8^2 + 6\,a_1^4\,a_2^4\,a_3^4\,a_4^2\,a_6^4\,a_7^4\,a_8^4\,\tilde a_1^2\,\tilde a_2^2\,\tilde a_3^8\,\tilde a_6^4\,\tilde a_8^2\,b_1^2\,b_3^2\;\cdot\\
&\cdot\;b_4^4\,b_5^2\,b_6^4\,b_7^6\,b_8^4\,\tilde b_3^6\,\tilde b_4^2\,\tilde b_5^2\,\tilde b_6^8\,\tilde b_8^2\,u_1^4\,u_2^2\,u_3^4\,u_4^4\,u_5^2\,u_6^4\,u_7^8\,u_8^2\,\tilde u_3^6\,\tilde u_6^4\,\tilde u_7^2 + a_1^4\,a_3^4\,a_4^4\,a_6^4\,a_7^4\,a_8^4\,\tilde a_3^8\,\tilde a_6^4\;\cdot\\
&\cdot\;b_1^4\,b_3^4\,b_4^4\,b_6^4\,b_7^4\,b_8^4\,\tilde b_3^8\,\tilde b_6^4\,u_1^4\,u_3^4\,u_4^4\,u_6^4\,u_7^8\,u_8^4\,\tilde u_3^8\,\tilde u_6^4 - 2\,a_1^2\,a_2^8\,a_3^2\,a_5^4\,a_7^2\,\tilde a_1^4\,\tilde a_2^2\,\tilde a_3^4\,\tilde a_4^2\,\tilde a_5^2\,\tilde a_7^4\,\tilde a_8^6\;\cdot\\
&\cdot\;b_2^6\,b_5^6\,b_8^2\,\tilde b_1^4\,\tilde b_2^4\,\tilde b_3^2\,\tilde b_4^4\,\tilde b_5^4\,\tilde b_6^2\,\tilde b_7^2\,\tilde b_8^8\,u_2^6\,u_4^2\,u_5^8\,u_6^2\,u_7^2\,\tilde u_1^2\,\tilde u_2^2\,\tilde u_4^4\,\tilde u_5^4\,\tilde u_6^2\,\tilde u_7^4\,\tilde u_8^6 + 4\,a_1^2\,a_2^6\,a_3^2\,a_4^2\,a_5^6\;\cdot\\
&\cdot\;a_6^2\,a_7^4\,\tilde a_1^2\,\tilde a_2^2\,\tilde a_3^2\,\tilde a_4^2\,\tilde a_5^2\,\tilde a_6^2\,\tilde a_7^4\,\tilde a_8^4\,b_1^2\,b_2^6\,b_3^2\,b_5^4\,b_7^2\,b_8^2\,\tilde b_1^4\,\tilde b_2^4\,\tilde b_3^4\,\tilde b_4^2\,\tilde b_5^2\,\tilde b_6^2\,\tilde b_7^2\,\tilde b_8^6\,u_2^2\,u_4^2\,u_5^6\,u_6^2\,u_7^2\,u_8^2\;\cdot\\
&\cdot\;\tilde u_1^2\,\tilde u_2^2\,\tilde u_3^2\,\tilde u_4^4\,\tilde u_5^4\,\tilde u_6^4\,\tilde u_7^2\,\tilde u_8^6 - 12\,a_1^2\,a_2^6\,a_3^2\,a_5^4\,a_7^2\,a_8^2\,\tilde a_1^4\,\tilde a_2^2\,\tilde a_3^2\,\tilde a_4^2\,\tilde a_5^2\,\tilde a_6^2\,\tilde a_7^2\,\tilde a_8^6\,b_2^4\,b_4^2\,b_5^4\,b_6^2\,b_7^4\;\cdot\\
&\cdot\;b_8^2\,\tilde b_1^2\,\tilde b_2^2\,\tilde b_3^2\,\tilde b_4^4\,\tilde b_5^4\,\tilde b_6^4\,\tilde b_7^2\,\tilde b_8^6\,u_1^2\,u_2^6\,u_3^2\,u_4^2\,u_5^6\,u_6^2\,u_7^4\,\tilde u_1^2\,\tilde u_2^2\,\tilde u_3^2\,\tilde u_4^2\,\tilde u_5^2\,\tilde u_6^2\,\tilde u_7^4\,\tilde u_8^4 - 2\,a_1^2\,a_2^4\,a_3^2\,a_4^4\;\cdot\\
&\cdot\;a_5^6\,a_6^4\,a_7^6\,\tilde a_3^2\,\tilde a_4^2\,\tilde a_5^2\,\tilde a_6^4\,\tilde a_7^4\,\tilde a_8^2\,b_1^4\,b_2^6\,b_3^4\,b_5^2\,b_7^4\,b_8^2\,\tilde b_1^4\,\tilde b_2^2\,\tilde b_3^6\,\tilde b_5^2\,\tilde b_6^2\,\tilde b_7^2\,\tilde b_8^4\,u_2^2\,u_4^2\,u_5^4\,u_6^2\,u_7^2\,u_8^4\,\tilde u_1^2\,\tilde u_2^2\,\tilde u_3^4\;\cdot\\
&\cdot\;\tilde u_4^4\,\tilde u_5^4\,\tilde u_6^6\,\tilde u_8^6 - 8\,a_1^2\,a_2^4\,a_3^2\,a_4^2\,a_5^4\,a_6^2\,a_7^4\,a_8^4\,\tilde a_1^2\,\tilde a_2^2\,\tilde a_3^2\,\tilde a_4^2\,\tilde a_5^2\,\tilde a_6^4\,\tilde a_7^2\,\tilde a_8^2\,b_1^4\,b_2^2\,b_3^2\,b_4^2\,b_5^4\,b_6^2\,b_7^4\,b_8^2\;\cdot\\
&\cdot\;\tilde b_1^2\,\tilde b_2^2\,\tilde b_3^4\,\tilde b_4^2\,\tilde b_5^2\,\tilde b_6^4\,\tilde b_7^2\,\tilde b_8^4\,u_1^2\,u_2^4\,u_3^2\,u_4^2\,u_5^4\,u_6^2\,u_7^4\,u_8^2\,\tilde u_1^2\,\tilde u_2^2\,\tilde u_3^4\,\tilde u_4^2\,\tilde u_5^2\,\tilde u_6^4\,\tilde u_7^2\,\tilde u_8^4 - 2\,a_1^2\,a_2^4\,a_3^2\,a_5^2\;\cdot\\
&\cdot\;a_7^2\,a_8^4\,\tilde a_1^4\,\tilde a_2^4\,\tilde a_3^6\,\tilde a_4^4\,\tilde a_5^4\,\tilde a_6^4\,\tilde a_8^6\,b_2^2\,b_4^4\,b_5^6\,b_6^4\,b_7^4\,b_8^2\,\tilde b_3^2\,\tilde b_4^4\,\tilde b_5^4\,\tilde b_6^6\,\tilde b_7^2\,\tilde b_8^4\,u_4^4\,u_5^6\,u_6^4\,u_7^2\,\tilde u_1^4\,\tilde u_2^6\,\tilde u_3^4\;\cdot\\
&\cdot\;\tilde u_4^4\,\tilde u_5^4\,\tilde u_6^4\,\tilde u_7^4\,\tilde u_8^2 - 12\,a_1^2\,a_2^2\,a_3^2\,a_4^4\,a_5^4\,a_6^4\,a_7^6\,a_8^2\,\tilde a_3^4\,\tilde a_4^2\,\tilde a_5^2\,\tilde a_6^4\,\tilde a_7^2\,\tilde a_8^2\,b_1^4\,b_2^4\,b_3^4\,b_4^2\,b_5^2\,b_6^2\,b_8^2\,\tilde b_1^2\,\tilde b_2^2\;\cdot\\
&\cdot\;\tilde b_3^6\,\tilde b_6^4\,\tilde b_7^2\,\tilde b_8^2\,u_1^2\,u_2^2\,u_3^2\,u_4^4\,u_5^2\,u_6^2\,u_7^4\,u_8^4\,\tilde u_1^2\,\tilde u_2^2\,\tilde u_3^6\,\tilde u_4^2\,\tilde u_5^2\,\tilde u_6^6\,\tilde u_8^4 + 4\,a_1^2\,a_2^2\,a_3^2\,a_4^2\,a_5^2\,a_6^2\,a_7^4\,a_8^4\;\cdot\\
&\cdot\;\tilde a_1^2\,\tilde a_2^2\,\tilde a_3^6\,\tilde a_4^2\,\tilde a_5^2\,\tilde a_6^4\,\tilde a_8^8\,b_1^2\,b_2^2\,b_3^2\,b_4^4\,b_5^4\,b_6^4\,b_7^2\,b_8^2\,\tilde b_3^2\,\tilde b_4^2\,\tilde b_5^2\,\tilde b_6^6\,\tilde b_7^2\,\tilde b_8^4\,u_1^4\,u_2^2\,u_3^4\,u_4^2\,u_5^2\,u_6^2\,u_7^4\,u_8^2\,\tilde u_1^2\;\cdot\\
&\cdot\;\tilde u_2^2\,\tilde u_3^6\,\tilde u_6^4\,\tilde u_7^2\,\tilde u_8^2 - 2\,a_1^2\,a_2^2\,a_3^4\,a_4^4\,a_5^2\,a_6^4\,a_7^4\,a_8^4\,\tilde a_3^6\,\tilde a_4^2\,\tilde a_5^2\,\tilde a_6^8\,\tilde a_8^2\,b_1^4\,b_2^2\,b_3^4\,b_4^4\,b_5^2\,b_6^4\,b_7^8\,b_8^2\,\tilde b_3^6\,\tilde b_6^6\,\tilde b_7^2\;\cdot\\
&\cdot\;u_1^4\,u_2^2\,u_3^4\,u_4^2\,u_6^2\,u_7^6\,u_8^4\,\tilde u_1^2\,\tilde u_2^2\,\tilde u_3^8\,\tilde u_6^4\,\tilde u_8^2 + a_2^8\,a_5^8\,\tilde a_1^4\,\tilde a_2^4\,\tilde a_4^4\,\tilde a_5^4\,\tilde a_7^4\,\tilde a_8^8\,b_2^8\,b_5^8\,\tilde b_1^4\,\tilde b_2^4\,\tilde b_4^4\,\tilde b_5^4\,\tilde b_7^4\,\tilde b_8^8\;\cdot\\
&\cdot\;u_2^8\,u_5^8\,\tilde u_1^4\,\tilde u_2^4\,\tilde u_4^4\,\tilde u_5^4\,\tilde u_7^4\,\tilde u_8^8 + 6\,a_2^6\,a_4^2\,a_5^8\,a_6^2\,a_7^2\,\tilde a_1^2\,\tilde a_2^2\,\tilde a_4^4\,\tilde a_5^4\,\tilde a_6^2\,\tilde a_7^4\,\tilde a_8^6\,b_1^2\,b_2^6\,b_3^2\,b_5^6\,b_7^2\,\tilde b_1^2\,\tilde b_2^2\,\tilde b_3^2\;\cdot\\
&\cdot\;\tilde b_4^2\,\tilde b_5^6\,\tilde b_7^4\,\tilde b_8^6\,u_2^6\,u_5^6\,u_8^2\,\tilde u_1^4\,\tilde u_2^4\,\tilde u_3^2\,\tilde u_4^4\,\tilde u_5^4\,\tilde u_6^2\,\tilde u_7^2\,\tilde u_8^8 - 2\,a_2^2\,a_5^6\,a_8^2\,\tilde a_1^2\,\tilde a_2^2\,\tilde a_3^2\,\tilde a_4^4\,\tilde a_5^4\,\tilde a_6^2\,\tilde a_7^8\,\tilde a_8^6\,b_2^6\,b_4^2\;\cdot\\
&\cdot\;b_5^8\,b_6^2\,b_7^2\,\tilde b_1^2\,\tilde b_2^2\,\tilde b_4^4\,\tilde b_5^4\,\tilde b_6^2\,\tilde b_7^4\,\tilde b_8^6\,u_1^2\,u_2^8\,u_3^2\,u_5^6\,u_7^2\,\tilde u_1^4\,\tilde u_2^4\,\tilde u_3^2\,\tilde u_4^2\,\tilde u_5^2\,\tilde u_7^4\,\tilde u_8^6 + a_2^2\,a_4^4\,a_5^8\,a_6^4\,a_7^4\,\tilde a_4^4\;\cdot\\
&\cdot\;\tilde a_5^4\,\tilde a_6^4\,\tilde a_7^4\,\tilde a_8^4\,b_1^4\,b_2^8\,b_3^4\,b_5^4\,b_7^4\,\tilde b_1^4\,\tilde b_2^4\,\tilde b_3^4\,\tilde b_7^4\,\tilde b_8^4\,u_2^4\,u_5^4\,u_8^4\,\tilde u_1^4\,\tilde u_2^4\,\tilde u_3^4\,\tilde u_4^4\,\tilde u_5^4\,\tilde u_6^4\,\tilde u_8^8 + 4\,a_2^4\,a_4^2\,a_5^4\,a_6^2\;\cdot\\
&\cdot\;a_7^2\,a_8^2\,\tilde a_1^2\,\tilde a_2^2\,\tilde a_3^2\,\tilde a_4^4\,\tilde a_5^4\,\tilde a_6^4\,\tilde a_7^2\,\tilde a_8^6\,b_1^2\,b_2^6\,b_3^2\,b_4^2\,b_5^6\,b_6^2\,b_7^4\,\tilde b_1^2\,\tilde b_2^2\,\tilde b_3^2\,\tilde b_4^2\,\tilde b_5^6\,\tilde b_6^2\,\tilde b_7^4\,\tilde b_8^6\,u_1^2\,u_2^6\,u_3^2\,u_5^4\,u_7^2\;\cdot\\
&\cdot\;u_8^2\,\tilde u_1^4\,\tilde u_2^4\,\tilde u_3^4\,\tilde u_4^2\,\tilde u_5^2\,\tilde u_6^4\,\tilde u_7^2\,\tilde u_8^6 + a_2^4\,a_4^4\,a_5^4\,a_6^4\,\tilde a_1^4\,\tilde a_2^4\,\tilde a_3^4\,\tilde a_4^4\,\tilde a_5^4\,\tilde a_6^4\,\tilde a_8^8\,b_2^4\,b_4^4\,b_5^8\,b_6^4\,b_7^4\,\tilde b_4^4\,\tilde b_5^4\,\tilde b_6^4\,\tilde b_7^4\,\tilde b_8^4\;\cdot\\
&\cdot\;u_1^4\,u_2^8\,u_3^4\,u_5^4\,u_7^4\,\tilde u_1^4\,\tilde u_2^4\,\tilde u_3^4\,\tilde u_7^4\,\tilde u_8^4 - 2\,a_2^2\,a_4^4\,a_5^4\,a_6^4\,a_7^4\,a_8^2\,\tilde a_3^4\,\tilde a_4^4\,\tilde a_5^4\,\tilde a_6^6\,\tilde a_7^2\,\tilde a_8^4\,b_1^4\,b_2^6\,b_3^4\,b_4^2\,b_5^6\,b_6^4\;\cdot\\
&\cdot\;b_7^6\,\tilde b_1^2\,\tilde b_2^2\,\tilde b_3^4\,\tilde b_6^2\,\tilde b_7^4\,\tilde b_8^2\,u_1^2\,u_2^4\,u_3^2\,u_5^2\,u_7^4\,u_8^4\,\tilde u_1^4\,\tilde u_2^4\,\tilde u_3^6\,\tilde u_4^2\,\tilde u_5^2\,\tilde u_6^4\,\tilde u_8^6 + 6\,a_2^2\,a_4^2\,a_5^4\,a_6^2\,a_7^2\,a_8^4\,\tilde a_1^2\,\tilde a_2^2
\end{aligned}
$$

$$\cdot \, \tilde{a}_3^4 \, \tilde{a}_4^4 \, \tilde{a}_5^4 \, \tilde{a}_6^6 \, \tilde{a}_8^6 \, b_1^2 \, b_2^4 \, b_3^2 \, b_4^4 \, b_5^6 \, b_6^4 \, b_7^6 \, \tilde{b}_3^2 \, \tilde{b}_4^2 \, \tilde{b}_5^2 \, \tilde{b}_6^4 \, \tilde{b}_7^4 \, \tilde{b}_8^2 \, u_1^4 \, u_2^6 \, u_3^4 \, u_5^2 \, u_7^4 \, u_8^2 \, \tilde{u}_1^4 \, \tilde{u}_2^4 \, \tilde{u}_3^6 \, \tilde{u}_6^2 \, \tilde{u}_7^2 \, \cdot$$

$$\cdot \, \tilde{u}_8^4 + a_4^4 \, a_5^4 \, a_6^4 \, a_7^4 \, a_8^4 \, \tilde{a}_3^4 \, \tilde{a}_4^4 \, \tilde{a}_5^4 \, \tilde{a}_6^8 \, \tilde{a}_8^4 \, b_1^4 \, b_2^4 \, b_3^4 \, b_4^4 \, b_5^4 \, b_6^4 \, b_7^8 \, \tilde{b}_3^4 \, \tilde{b}_6^4 \, \tilde{b}_7^4 \, u_1^4 \, u_2^4 \, u_3^4 \, u_7^4 \, u_8^4 \, \tilde{u}_1^4 \, \cdot$$

$$\cdot \, \tilde{u}_2^4 \, \tilde{u}_3^8 \, \tilde{u}_6^4 \, \tilde{u}_8^4 = 0.$$

The above equation is called the *structural equation.* It is a Diophantine equation with respect to 48 integer variables $a_1, \ldots, a_8, \tilde{a}_1, \ldots, \tilde{a}_8, b_1, \ldots, b_8, \tilde{b}_1, \ldots, \tilde{b}_8,$ $u_1, \ldots, u_8, \tilde{u}_1, \ldots, \tilde{u}_8$. Now, summarizing the results of the sections 5 and 6, then applying the lemma 4.3, we derive the following theorem.

Theorem 6.1. *For a given triple of positive coprime integer numbers a, b, and u such that none of the conditions (1.2) is satisfied the polynomial Diophantine equation (1.3) is resolvable if and only if there are 48 positive integer numbers $a_1, \ldots, a_8, \tilde{a}_1, \ldots, \tilde{a}_8, b_1, \ldots, b_8, \tilde{b}_1, \ldots, \tilde{b}_8, u_1, \ldots, u_8, \tilde{u}_1, \ldots, \tilde{u}_8$ obeying the structural equation on the pages 32 and 33, obeying the cohesion conditions (6.36) and (6.37), obeying the coprimality conditions (6.35), and such that a, b, and u are expressed through them by means of the formulas (6.26), (6.27), (6.28). Under these conditions the equation (1.3) has at least two solutions given by the formulas*

$$t \, = \prod_{\substack{i=1,3,4 \\ i=6,7,8}} a_i \, b_i \, u_i \prod_{i=3,6} \tilde{a}_i \, \tilde{b}_i \, \tilde{u}_i, \qquad\qquad t \, = - \prod_{\substack{i=1,3,4 \\ i=6,7,8}} a_i \, b_i \, u_i \prod_{i=3,6} \tilde{a}_i \, \tilde{b}_i \, \tilde{u}_i. \qquad (6.39)$$

The theorem 6.1 is the required structural theorem for the solutions of the Diophantine equation (1.3). The formulas (6.39) in this structural theorem are immediate from the formulas (6.33) and (4.2).

7. Conclusions.

The structural theorem 6.1 is the main result of this paper. It can be used in computer search for perfect Euler cuboids or maybe in proving their non-existence in the case of the third cuboid conjecture 1.1. The theorem 6.1 is analogous to the structural theorem 4.1 from [3] associated with the second cuboid conjecture.

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