Cantor Primes as Prime-Valued Cyclotomic Polynomials

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Abstract

Cantor primes are primes p such that $1/p$ belongs to the middle-third Cantor set. One way to look at them is as containing the base-3 analogues of the famous Mersenne primes, which encompass all base-2 repunit *primes, i.e., primes consisting of a contiguous sequence of 1's in base 2 and satisfying an equation of the form $p+1=2^q$. The Cantor primes encompass all base-3 repunit primes satisfying an equation of the form $2p+1=3^q$, and I show that in general all Cantor primes > 3 satisfy a closely related equation of the form $2pK+1=3^q$, with the base-3 repunits being the special case $K=1$. I use this to prove that the Cantor primes > 3 are exactly the prime-valued cyclotomic polynomials of the form $\Phi_s(3^{s^j}) \equiv 1 \pmod 4$. Significant open problems concern the infinitude of these, making Cantor primes perhaps more interesting than previously realised.*

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1 Introduction

Any base-N repunit prime p is a cyclotomic polynomial evaluated at N, $\Phi_q(N)$, with q also prime, i.e.,

$$p = \Phi_q(N) = \frac{N^q - 1}{N - 1} = \sum_{k=0}^{q-1} N^k \tag{1}$$

It is therefore expressible as a contiguous sequence of 1's in base N. For example, $p = 31$ satisfies (1) for $N = 2$ and $q = 5$ and can be expressed as 11111 in base 2. The term *repunit* was coined by A. H. Beiler [1] to indicate that numbers like these consist of repeated units.

The case $N = 2$ corresponds to the famous Mersenne primes on which there is a vast literature [6]. They are sequence number A000668 in The Online Encyclopedia of Integer Sequences [7] and are exactly the prime-valued cyclotomic polynomials of the form $\Phi_s(2) \equiv 3 \pmod 4$.

In this note I show that *Cantor primes* can be characterised in a similar way as being exactly the prime-valued cyclotomic polynomials of the form $\Phi_s(3^{s^j}) \equiv 1 \pmod 4$. They are primes whose reciprocals belong to the middle-third Cantor set $\mathcal{C}_3$.

It is easily shown that $\mathcal{C}_3$ contains the reciprocals of all base-3 repunit primes, i.e., those primes p which satisfy an equation of the form $2p + 1 = 3^q$ with q prime. $\mathcal{C}_3$ is a fractal consisting of all the points in $[0, 1]$ which have non-terminating base-3 representations involving only the digits 0 and 2. Rerranging (1) to get the infinite series

$$\frac{1}{p} = \frac{N-1}{N^q - 1} = \sum_{k=1}^{\infty} \frac{N-1}{N^{qk}} \tag{2}$$

and putting $N = 3$ shows that those primes p which satisfy $2p+1 = 3^q$ are such that $\frac{1}{p}$ can be expressed in base 3 using only zeros and the digit 2. This single digit 2 will appear periodically in the base-3 representation of $\frac{1}{p}$ at positions which are multiples of q. Since only zeros and the digit 2 appear in the ternary representation of $\frac{1}{p}$, $\frac{1}{p}$ is never removed in the construction of $\mathcal{C}_3$, so $\frac{1}{p}$ must belong to $\mathcal{C}_3$.

Base-3 repunit primes are sequence number A076481 in The Online Encyclopedia of Integer Sequences and the exact analogues of the Mersenne primes, i.e., they are the case $N = 3$ in (1). In the next section I show that Cantor primes > 3 more generally satisfy a closely related equation of the form $2pK + 1 = 3^q$, with the base-3 repunits being the special case $K = 1$. A subsequent section proves that the Cantor primes > 3 are exactly the prime-valued cyclotomic polynomials of the form $\Phi_s(3^{s^j}) \equiv 1 \pmod 4$, and a final section considers related open problems.

2 An Exponential Equation Characterising All Cantor Primes

Theorem 2.1. *A prime number $p > 3$ is a Cantor prime if and only if it satisfies an equation of the form $2pK + 1 = 3^q$ where q is the order of 3 modulo p and K is a sum of non-negative powers of 3 each smaller than 3^q.*

Comment. The base-3 repunit primes are then the special case in which $K = 3^0 = 1$. An example is 13, which satisfies $2p + 1 = 3^3$. A counterexample which shows that not all Cantor primes are base-3 repunit primes is 757, which satisfies $26p + 1 = 3^9$ with $K = 3^0 + 3^1 + 3^2 = 13$ and $q = 9$.

Proof. Each $x \in \mathcal{C}_3$ can be expressed in ternary form as

$$x = \sum_{k=1}^{\infty} \frac{a_k}{3^k} = 0.a_1 a_2 \ldots \tag{3}$$

where all the a_k are equal to 0 or 2. The construction of $\mathcal{C}_3$ amounts to systematically removing all the points in $[0, 1]$ which cannot be expressed in ternary form with only 0's and 2's, i.e., the removed points all have $a_k = 1$ for one or more $k \in \mathbb{N}$ [4].

The construction of the Cantor set suggests some simple conditions which a prime number must satisfy in order to be a Cantor prime. If a prime number $p > 3$ is to be a Cantor prime, the first non-zero digit a_{k_1} in the ternary expansion of $\frac{1}{p}$ must be 2. This means that for some $k_1 \in \mathbb{N}$, p must satisfy

$$\frac{2}{3^{k_1}} < \frac{1}{p} < \frac{1}{3^{k_1-1}} \tag{4}$$

or equivalently

$$3^{k_1} \in (2p, 3p) \tag{5}$$

Prime numbers for which there is no power of 3 in the interval $(2p, 3p)$, e.g., 5, 7, 17, 19, 23, 41, 43, 47, ..., can therefore be excluded immediately from further consideration. Note that there cannot be any other power of 3 in the interval (2p, 3p) since 3^{k_1-1} and 3^{k_1+1} lie completely to the left and completely to the right of $(2p, 3p)$ respectively.

If the next non-zero digit after a_{k_1} is to be another 2 rather than a 1, it must be the case for some $k_2 \in \mathbb{N}$ that

$$\frac{2}{3^{k_1+k_2}} < \frac{1}{p} - \frac{2}{3^{k_1}} < \frac{1}{3^{k_1+k_2-1}} \tag{6}$$

or equivalently

$$3^{k_2} \in \left(\frac{2p}{3^{k_1} - 2p}, \frac{3p}{3^{k_1} - 2p} \right) \tag{7}$$

Thus, any prime numbers for which there is a power of 3 in the interval $(2p, 3p)$ but for which there is no power of 3 in the interval $(\frac{2p}{3^{k_1}-2p}, \frac{3p}{3^{k_1}-2p})$ can again be excluded, e.g., 37, 113, 331, 337, 353, 991, 997, 1009.

Continuing in this way, the condition for the third non-zero digit to be a 2 is

$$3^{k_3} \in \left(\frac{2p}{3^{k_2}(3^{k_1} - 2p) - 2p}, \frac{3p}{3^{k_2}(3^{k_1} - 2p) - 2p} \right) \tag{8}$$

and the condition for the nth non-zero digit to be a 2 is

$$3^{k_n} \in \left(\frac{2p}{3^{k_{n-1}}(\cdots(3^{k_2}(3^{k_1}-2p)-2p)\cdots)-2p}, \frac{3p}{3^{k_{n-1}}(\cdots(3^{k_2}(3^{k_1}-2p)-2p)\cdots)-2p}\right) \tag{9}$$

The ternary expansions under consideration are all non-terminating, so at first sight it seems as if an endless sequence of tests like these would have to be applied to ensure that $a_k \neq 1$ for any $k \in \mathbb{N}$. However, this is not the case. Let p be a Cantor prime and let 3^{k_1} be the smallest power of 3 that exceeds $2p$. Since p is a Cantor prime, both (5) and (9) must be satisfied for all n. Multiplying (9) through by $3^{k_1-k_n}$ we get

$$3^{k_1} \in \left(\frac{3^{k_1-k_n}\cdot 2p}{3^{k_{n-1}}(\cdots(3^{k_2}(3^{k_1}-2p)-2p)\cdots)-2p}, \frac{3^{k_1-k_n}\cdot 3p}{3^{k_{n-1}}(\cdots(3^{k_2}(3^{k_1}-2p)-2p)\cdots)-2p}\right) \tag{10}$$

Since all ternary representations of prime reciprocals $\frac{1}{p}$ for $p > 3$ have a repeating cycle which begins immediately after the point, it must be the case that $k_n = k_1$ for some n in (10). Setting $k_n = k_1$ in (10) we can therefore deduce from the fact that $3^{k_1} \in (2p, 3p)$ and the fact that (10) must be consistent with this for all values of n, that all Cantor primes must satisfy an equation of the form

$$3^{k_{n-1}}(\cdots(3^{k_2}(3^{k_1}-2p)-2p)\cdots)-2p = 1 \tag{11}$$

where $k_1 + k_2 + \cdots + k_{n-1} = q$ is the cycle length in the ternary representation of $\frac{1}{p}$. In other words, q is the order of 3 modulo p. By successively considering the cases in which there is only one non-zero term in the repeating cycle, two non-zero terms, three non-zero terms, etc., in (11), and defining

$$\begin{aligned}
d_1 &= q - k_1\\
d_2 &= q - k_1 - k_2\\
d_3 &= q - k_1 - k_2 - k_3\\
&\vdots\\
d_n &= q - k_1 - k_2 - \cdots - k_n = 0
\end{aligned}$$

it is easy to see that (11) can be rearranged as

$$2p\sum_{i=1}^{n} 3^{d_i} + 1 = 3^q \tag{12}$$

Setting $K = \sum_{i=1}^{n} 3^{d_i}$, we conclude that every Cantor prime must satisfy an equation of the form $2pK + 1 = 3^q$ as claimed.

Conversely, every prime which satisfies an equation of this form must be a Cantor prime. To see this, note that we can rearrange (12) to get

$$\frac{1}{p} = \frac{2\sum_{i=1}^{n} 3^{d_i}}{3^q - 1} = 2\sum_{i=1}^{n} 3^{d_i}\left\{\frac{1}{3^q} + \frac{1}{3^{2q}} + \frac{1}{3^{3q}} + \cdots\right\} \tag{13}$$

Since $2\sum_{i=1}^{n} 3^{d_i}$ involves only products of 2 with powers of 3 which are each less than 3^q, (13) is an expression for $\frac{1}{p}$ which corresponds to a ternary representation involving only 2s. Thus, $\frac{1}{p}$ must be in the Cantor set if $2pK+1 = 3^q$.

3 Cantor Primes as Cyclotomic Polynomials

Let n be a positive integer and let ζ_n be the complex number $e^{2\pi i/n}$. The n^{th} cyclotomic polynomial is defined as

$$\Phi_n(x) = \prod_{\substack{1\le k<n \\ \gcd(k,n)=1}} (x - \zeta_n^k)$$

The degree of $\Phi_n(x)$ is $\varphi(n)$ where φ is the Euler totient function. There is now a powerful body of theory relating to cyclotomic polynomials and discussions of their basic properties can be found in any textbook on abstract algebra.

Lemma 3.1. *$x^{(n-1)a} + x^{(n-2)a} + \cdots + x^{2a} + x^a + 1$ is irreducible in $\mathbb{Z}[x]$ if and only if $n = p$ and $a = p^k$ for some prime p and non-negative integer k.*

Proof. This is proved as Theorem 4 in [5].

Theorem 3.2. *A prime number $p > 3$ is a Cantor prime if and only if $p = \Phi_s(3^{s^j}) \equiv 1$ (mod 4) where s is an odd prime and j is a non-negative integer.*

Proof. Assume p is a Cantor prime. By Theorem 2.1 we then have

$$pK = \frac{3^q - 1}{2} = R_q^{(3)} \tag{14}$$

where $R_q^{(3)}$ denotes the base-3 repunit consisting of q contiguous units, and q and K are as defined in that theorem. If q is composite, say $q = rs$, we obtain the factorisation

$$R_q^{(3)} = R_r^{(3)} \cdot (3^{(s-1)r} + 3^{(s-2)r} + \cdots + 3^{2r} + 3^r + 1) \tag{15}$$

If q is prime we can take $r = 1$. Therefore in both cases at least one factor of pK must be a base-3 repunit.

If $K = 1$ then $p = R_q^{(3)} = \Phi_s(3)$, since q must be prime in this case. ($R_q^{(3)}$ is composite if q is). If $K > 1$, p is not a base-3 repunit and by Theorem 2.1 K is a sum of powers of 3, so p must be of the general form

$$p = 3^{(s-1)r} + 3^{(s-2)r} + \cdots + 3^{2r} + 3^r + 1 \tag{16}$$

for some s and r, and K must be a corresponding base-3 repunit $R_r^{(3)}$, otherwise their product could not be $R_{rs}^{(3)}$. But the polynomial in (16) can only be prime if it is irreducible in $\mathbb{Z}[x]$. By Lemma 3.1, this requires s to be a prime number and $r = s^j$ for some non-negative integer j, and we therefore have $p = \Phi_s(3^{s^j})$ in this case. We conclude that in all cases we must have $p = \Phi_s(3^{s^j})$ if p is a Cantor prime. Note that s must be an *odd* prime as $\Phi_s(3^{s^j})$ is even for $s = 2$.

Conversely, suppose that $p = \Phi_s(3^{s^j})$ is a prime number. Then we can multiply it by the base-3 repunit $R_r^{(3)}$ where $r = s^j$ to get the repunit $R_q^{(3)}$ as in (15). Thus, p must satisfy (14) and must therefore be a Cantor prime.

Base-3 repunits are congruent to 0 modulo 4 when they consist of an even number of digits, and to 1 modulo 4 otherwise. Therefore if $p > 3$ is a base-3 repunit prime it must be of the form $4k + 1$.

If p is prime but not a base-3 repunit, both $r = s^j$ and $q = rs$ in (15) are odd, so both $R_q^{(3)}$ and $R_r^{(3)}$ are base-3 repunits with odd numbers of digits, and thus of the form $4k+1$. It follows that p is also of the form $4k+1$ in this case.

4 Open Problems

The infinitude of Cantor primes is currently an open problem shown to be significant in this paper because of the equivalence of Cantor primes and prime-valued cyclotomic polynomials of the form $\Phi_s(3^{s^j})$.

In the case $j = 0$, it is known that $\Phi_s(3)$ is prime for s = 7, 13, 71, 103, 541, 1091, 1367, 1627, 4177, 9011, 9551, 36913, 43063, 49681, 57917, 483611, and 877843. It seems plausible that there are infinitely many such values of s but this remains to be proved.

The Cantor prime $757 = \Phi_3(3^3)$ is an example with $j > 0$. It is again an open problem to prove there are infinitely many integers $j > 0$ for which $\Phi_s(3^{s^j})$ is prime given a prime s, though all such cyclotomic polynomials must be irreducible.

Previous studies have considered the infinitude of prime-valued cyclotomic polynomials of other types. For example, primes of the form $\Phi_s(1)$ and $\Phi_s(2)$ are studied in [3], and other cases are discussed in [2].

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