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Notes on higher-dimensional partitions

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Abstract

We show the existence of a series of transforms that capture several structures that underlie higher-dimensional partitions. These transforms lead to a sequence of triangles whose entries are given combinatorial interpretations as the number of particular types of skew Ferrers diagrams. The end result of our analysis is the existence of a triangle, that we denote by F, which implies that the data needed to compute the number of partitions of a given positive integer is reduced by a factor of half. The number of spanning rooted forests appears intriguingly in a family of entries in the F. Using modifications of an algorithm due to Bratley-McKay, we are able to directly enumerate entries in some of the triangles. As a result, we have been able to compute numbers of partitions of positive integers ≤ 25 in any dimension.

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1   Introduction

An unrestricted d-dimensional partition of n is a collection of n points (nodes) in $\mathbb{Z}_+^{d+1}$ satisfying the following property: if the collection contains a node $\mathbf{a} = (a_1, a_2, \ldots, a_{d+1})$, then all nodes $\mathbf{x} = (x_1, x_2, \ldots, x_{d+1})$ with $0 \leq x_i \leq a_i \; \forall \; i = 1, \ldots, d+1$ also belong to it [1,2]. Let $p_d(n)$ denote the number of distinct such partitions. Denote by $P_d(q)$, the generating function of unrestricted d-dimensional partitions: $(p_d(0) \equiv 1)$

$$P_d(q) = \sum_{n=0}^{\infty} p_d(n) \; q^n \; . \tag{1.1}$$

There exist explicit formulae for the generating functions for $d = 1$ and $d = 2$ due to Euler and MacMahon respectively [3]. However, no such formulae exist for $d > 2$ as an inspired guess of MacMahon was subsequently proven to be false [1]. It appears that there is no simple formula and one has to take recourse to brute force enumeration. Given that asymptotically one has [4–6]

$$\log p_d(n) \sim n^{d/d+1} \; , \tag{1.2}$$

it is easy to see that the numbers grow exponentially fast and naive enumeration is not the way to go.

   The first serious attempt at direct enumeration of partitions in any dimension is due to Atkin et. al. [1] based on an algorithm due to Bratley and McKay [7]. Knuth provided another algorithm that enumerates numbers of topological sequences which can be used, in principle, to generate numbers of partitions in any dimension [8]. Both algorithms are highly recursive and easily implemented on a computer.

   This paper attempts to find structures in the enumeration of partitions and come up with refinements in their enumeration. Such refinements when cleverly combined with computer-based enumeration should in principle enable one to enumerate partitions of integers below some maximum value in any dimension. The maximum value turns out be 25 in our case though we believe that, with some effort, this number can be pushed to around 30.

   Our refinements begin with the result of of Atkin et. al. who showed that the binomial transform of $p_d(n)$ leads to a lower-triangular matrix that we denote by $A = (a_{n,r})$.

$$p_d(n) = \sum_{r=0}^{d+1} \binom{d+1}{r} a_{n,r} \; . \tag{2.4}$$

This transform implies that in order to compute partitions of a positive integer n in any dimension, we need to only compute $(n-1)$ numbers that make up a particular row of the triangle A. We show the existence another triangular matrix, that we denote by $F = (f_{n,x})$, as a transform of the matrix A with fewer

entries.

$$a_{m+r+1,r} = \sum_{x=0}^{r}\sum_{p=x}^{m}\binom{r}{x}\binom{\binom{r-x}{2}}{m-p}\, f_{p+x+1,\ x}\,. \qquad (2.20)$$

Our result is that we need only $[(n-1)/2]$ independent numbers i.e., roughly half of the initial estimate to determine partitions of n in any dimension. We illustrate the gain by explicitly displaying the first eleven rows of the A and F-matrices.

$$
A = \begin{pmatrix}
1 \\
0 & 1 \\
0 & 1 & 1 \\
0 & 1 & 3 & 1 \\
0 & 1 & 5 & 6 & 1 \\
0 & 1 & 9 & 18 & 10 & 1 \\
0 & 1 & 13 & 44 & 49 & 15 & 1 \\
0 & 1 & 20 & 97 & 172 & 110 & 21 & 1 \\
0 & 1 & 28 & 195 & 512 & 550 & 216 & 28 & 1 \\
0 & 1 & 40 & 377 & 1370 & 2195 & 1486 & 385 & 36 & 1 \\
0 & 1 & 54 & 694 & 3396 & 7603 & 7886 & 3514 & 638 & 45 & 1
\end{pmatrix}
\,,\quad
F = \begin{pmatrix}
1 \\
0 \\
0 & 1 \\
0 & 1 \\
0 & 1 & 3 \\
0 & 1 & 7 \\
0 & 1 & 11 & 16 \\
0 & 1 & 18 & 58 \\
0 & 1 & 26 & 135 & 125 \\
0 & 1 & 38 & 293 & 618 \\
0 & 1 & 52 & 574 & 1927 & 1296
\end{pmatrix}\,.
$$

The F-triangle is, in a sense, the end-point of a sequence of transforms and triangles that we introduce. We also provide combinatorial interpretations for the various triangles that appear as a result of these transforms. This enables use to modify the Bratley-McKay(BM) algorithm to directly enumerate the matrix A that we mentioned earlier and a second triangle, C that we define in the sequel. As we discuss in the appendix, similar refinements can be carried out for partitions restricted in a box.

1.1 Summary of results

1. Given a partition in any dimension, we have introduced two new attributes: its intrinsic dimension (i.d.) - see definition 2.1 and its reduced dimension (r.d.) - see definition 2.7.

2. These two attributes lead to two new triangles, the A and C-matrices (see Eq. (2.4) and (2.11)) whose entries admit combinatorial interpretations. We propose a further refinement in the form of two other triangles, the D matrix(see Eq. (2.15)).

3. We show that the C/D triangles are the first in a series of transforms, the end-point of which leads to a triangle F (see Eq. (2.20)). The n-th row of this matrix has only $[(n-1)/2]$ entries (where $[x]$ is the integral part of x) and these entries determine the partitions of n in any dimension. This constitutes the main result of this paper.

4. We see an intriguing relationship between the numbers of spanning rooted forests on m vertices and α components and a family of entries in the F-matrix. This is Proposition 2.16.

5. We conjecture the existence of two other triangles, the α- and the β-matrices with integer entries.

6. We prove a conjecture of Hanna on the existence of a triangle that determines all higher-dimensional partitions.

7. We propose a modification to an algorithm of Bratley and McKay that enables us to directly compute the A and C matrices. We compute the first 25 rows of the F-matrix thereby obtaining partitions in all dimensions for integers ≤ 25.

8. Tables 1-8 provide the numerical results that we have obtained.

2 Structures in higher-dimensional partitions

2.1 Ferrers diagrams and permutation symmetry

A Ferrers diagram represents the partition as a $(d+1)$-dimensional arrangement of nodes. For instance, the following one-dimensional partition of 4

$$\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\} \text{ or } \left(\begin{smallmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{smallmatrix} \right) \text{ in compressed form },$$

is represented by the following two-dimensional Ferrers diagram or as a Young diagram where we replace the nodes by squares(more generally, hypercubes).
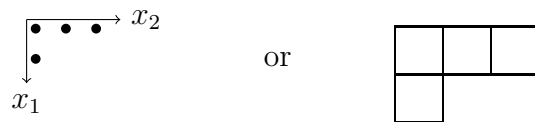
There is a natural action of S_{d+1} on the $(d+1)$-dimensional Ferrers diagram – this corresponds to permuting the $(d+1)$ coordinates. For one-dimensional partitions, this is referred to as conjugation. The symmetry group of a d-dimensional partition is the *largest* sub-group of S_{d+1} that acts trivially on the corresponding Ferrers diagram.

2.2 The intrinsic dimension

Typically, one is interested in the asymptotic behavior of $p_d(n)$ for large number of nodes n while keeping the dimension d fixed. However, one may ask about what happens to $p_d(n)$ if we keep the number of nodes. i.e., n, fixed and keep increasing d. It is easy to see that when $d > n + 1$, all the nodes of the Ferrers diagram necessarily lie in some r-dimensional hyperplane with $r < d$. This motivates the following definition (implicitly present in Atkin et. al. [1]).

Definition 2.1 *Given a Ferrers diagram, let it be contained in a r-dimensional hyperplane but not in any $(r-1)$-dimensional hyperplane. The intrinsic dimension(i.d.) of the Ferrers diagram is defined to be r.*

Note that such a r-dimensional hyperplane is given by setting $(d + 1 - r)$ coordinates to zero. Any permutation of the $(d + 1 - r)$ coordinates (that are set to zero to obtain the hyperplane containing the nodes) does not change the Ferrers diagram. It is thus easy to see that the symmetry of a Ferrers diagram in $(d+1)$-dimensions of i.d. r is necessarily of the form $H \times S_{d+1-r}$ where $H \subseteq S_r$. We shall (somewhat loosely) call H, the symmetry of the Ferrers diagram.

Let two d-dimensional partitions be equivalent if their Ferrers diagram are related by an S_{d+1} action. It is easy to see that all d-dimensional partitions belonging to such an equivalence class have the same intrinsic dimension. Further, given a $d + 1$-dimensional Ferrers diagram with symmetry H and i.d. r, the number of Ferrers diagrams in its equivalence class is given by the order of the coset $S_{d+1}/(H \times S_{d+1-r})$ i.e.,

$$\frac{(d+1)!}{(d+1-r)! \times \text{ord}(H)} = \binom{d+1}{r} \times \frac{r!}{\text{ord}(H)} \ .$$

Definition 2.2 *A Ferrers diagram is said to be strict when its intrinsic dimension equals its dimension.*

Given a $d + 1$-dimensional Ferrers diagram of i.d. r, it is useful to drop the $(d + 1 - r)$ dimensions that are orthogonal to the hyperplane containing the nodes thus obtaining a strict FD. The symmetry of the strict Ferrers diagram is now $H \subseteq S_r$.

Definition 2.3 *A generalized Ferrers diagram (gFD) refers to the equivalence class of strict Ferrers diagrams obtained by the action of S_r on a given strict Ferrers diagram of i.d. r.*

The number of strict FD's in a gFD of i.d. r and symmetry group H is $\frac{r!}{\text{ord}(H)}$.

Definition 2.4 *The weight of a gFD of i.d. r and symmetry $H \subseteq S_r$ is defined to be $\frac{r!}{\text{ord}(H)}$.*

Since $H \subseteq S_r$, Lagrange's theorem implies that the weight, $\frac{r!}{\text{ord}(H)}$, is a positive non-zero integer. Note that the weight is independent of the dimension of the Ferrers diagram and is the same for all elements in an equivalence class. Thus, to an equivalence class of a given Ferrers diagram, we associate three numbers: the number of nodes n, the i.d. r, and the weight, w. An important observation is that there exist *no* Ferrers diagram with n nodes and i.d. $r \geq n$ – this follows from noting that one needs at least $r + 1$ nodes to create a Ferrers diagram of

4

i.d. r. We see that the number of d-dimensional partitions is thus given by

$$p_d(n) = \sum_{r=0}^{n-1} \binom{d+1}{r} \sum_{\lambda \vdash (n,r)} 1 \qquad (2.1)$$

$$= \sum_{r=0}^{n-1} \binom{d+1}{r} \sum_{[\lambda] \vdash (n,r)} w(\lambda) \qquad (2.2)$$

$$:= \sum_{r=0}^{n-1} \binom{d+1}{r} a_{n,r} \ , \qquad (2.3)$$

where the second line defines $a_{n,r}$ as the sum over all strict FD's with n nodes and i.d. r. In the second line, the sum over $[\lambda]$ indicates that we sum over equivalence classes of strict Ferrers diagrams(gFD). Note that $a_{n,r}$ has no dependence on d and counts the numbers of strict Ferrers diagrams with n nodes and i.d. r. We shall provide a second, and more useful, combinatorial description of $a_{n,r}$ later.

2.3   The first transform

We extend a_{nr} into a lower-triangular matrix, that we denote by A, by setting $a_{nr} = 0$ when $r \geq n$. Thus, we obtain the matrix $A = (a_{nr})$ for $n = 1, 2, \ldots$ and $r = 0, 1, 2, \ldots$. With this definition, we an rewrite the above equation as

$$\boxed{p_d(n) = \sum_{r=0}^{d+1} \binom{d+1}{r} a_{nr}} \ . \qquad (2.4)$$

To our knowledge, the above observation first appeared in a paper by Atkin et. al. [1]. Thus the $p_d(n)$, for a fixed value of n, corresponds to the Binomial Transform of the n-th row of the matrix A. It is easy to see that $a_{n,0} = \delta_{n,1}$. The lower triangular nature of A implies that only $(n-1)$ numbers, $(a_{n,1}, a_{n,2}, \ldots, a_{n,(n-1)})$ determine $p_d(n)$ for any d. The matrix A appears in the OEIS as sequence number A119271 [9]. The inverse Binomial transform is given by

$$\boxed{a_{nr} = \sum_{d=0}^{r-1} (-1)^{d+r+1} \binom{r}{d+1} p_d(n) \quad \text{for } n \geq r+1} \ , \qquad (2.5)$$

with $p_0(n) \equiv 1$. Of course, $a_{nr} = 0$ when $n < r+1$ reflecting the lower-triangular nature of the matrix. Suppose we know all partitions of n_{max} up to d_{max}. This determines the first n_{max} rows and $(d_{max} + 1)$ columns of the matrix A.

For low values of n, we can explicitly compute the entries in the A-matrix by

listing the gFD's and working out their weights as we do below.

$$p_d(2) = \binom{d+1}{1} w\left(\text{▢▢}\right) = \binom{d+1}{1} \tag{2.6}$$

$$p_d(3) = \binom{d+1}{1} w\left(\text{▢▢▢}\right) + \binom{d+1}{2} w\left(\text{▢}\right) = \binom{d+1}{1} + \binom{d+1}{2} . \tag{2.7}$$

$$p_d(4) = \binom{d+1}{1} w\left(\text{▢▢▢▢}\right) + \binom{d+1}{2} w\left(\text{▢}\right) + \binom{d+1}{2} w\left(\text{▢}\right) + \binom{d+1}{3} w\left(\text{◈}\right)$$

$$= \binom{d+1}{1} + 3\binom{d+1}{2} + \binom{d+1}{3} . \tag{2.8}$$

The first few rows of the A-matrix are as follows (see also [10])

$$A = \begin{pmatrix}
1 \\
0 & 1 \\
0 & 1 & 1 \\
0 & 1 & 3 & 1 \\
0 & 1 & 5 & 6 & 1 \\
0 & 1 & 9 & 18 & 10 & 1 \\
0 & 1 & 13 & 44 & 49 & 15 & 1 \\
0 & 1 & 20 & 97 & 172 & 110 & 21 & 1 \\
0 & 1 & 28 & 195 & 512 & 550 & 216 & 28 & 1 \\
0 & 1 & 40 & 377 & 1370 & 2195 & 1486 & 385 & 36 & 1 \\
0 & 1 & 54 & 694 & 3396 & 7603 & 7886 & 3514 & 638 & 45 & 1 \\
0 & 1 & 75 & 1251 & 7968 & 23860 & 35115 & 24318 & 7484 & 999 & 55 & 1
\end{pmatrix} .$$

Definition 2.5 *Consider a pair of FD's (λ, μ) such that $\mu \subseteq \lambda$. Then, a skew Ferrers diagram is the set of nodes $\lambda \setminus \mu$.*

One can think of the entries in the A-matrix as counting skew Ferrers diagrams obtained by deleting the node at the origin $(0, 0, \ldots, 0)^T$ that is contained in any Ferrers diagram. Then, $a_{n,r}$ is the number of strict FD's of dimension r obtained by adding $(n-1)$ nodes to the node at the origin. One sets $a_{1,0} \equiv 1$.

2.4 A combinatorial interpretation

We will now provide another combinatorial interpretation for the numbers a_{nr} that make up the lower-triangular matrix A. We begin with the observation that $a_{r+1,r} = 1$ – this follows because there is a unique FD of i.d. r containing $r+1$ nodes. The coordinates are given in the following $r \times (r+1)$ matrix[1]

$$\mu_r := \begin{pmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & & 1 \end{pmatrix} . \tag{2.9}$$

---

[1]Recall that each column is the coordinate of a node and thus there are $(r+1)$ columns and r rows.

This FD has maximal symmetry S_r and weight 1.

Remark: Every FD with intrinsic dimension r necessarily contains μ_r. This implies that an FD with n nodes and i.d. r can be obtained by adding $m = n - r - 1$ additional nodes to μ_r. This leads to the following combinatorial interpretation for $a_{m+r+1,r}$.

Proposition 2.6 $a_{m+r+1,r}$ *is the number of strict Ferrers diagrams with i.d.* r *obtained by adding m nodes to the standard Ferrers diagram,* μ_r.

Let λ be an FD that contributes to $a_{n,r}$. Its symmetry group $H \subseteq S_r$ – this implies that there will be $r!/\text{odd}(H) = wt(\lambda)$ distinct FD's obtained from it by the action of S_r. It is easy to see that the process of adding m nodes to μ_r will generate the same number of FD's that belong to the equivalence class (gFD) $[\lambda]$.

So far we have completely determined the first 25 rows of the A-matrix (see Table 2). The entries have been determined by combining several methods: (i) taking the inverse Binomial transform of known numbers for higher-dimensional partitions, (ii) by direct enumeration using the combinatorial interpretation and (iii) by determining another triangle, the C-matrix, that we introduce later. It is important to note that the numbers, when available, from the different methods agree. Further, none of the conjectural formulae are used in determining the entries.

2.5 The second transform

Definition 2.7 *Let λ be an FD of i.d. r and consider the skew FD $\lambda \setminus \mu_r$. Let the nodes of the skew FD be contained in a x-dimensional hyperplane (obtained by setting $r - x$ coordinates to zero) but not in any $(x-1)$-dimensional hyperplane. The reduced dimension (r.d.) of the FD λ is said to be x.*

Clearly the reduced dimension of an FD is always less than or equal to its intrinsic dimension. The symmetry of a FD with i.d. r and r.d. x is necessarily of the form $H \times S_{r-x} \subset S_r$. Then, one has

$$a_{m+r+1,r} = \sum_{x=0}^{r} \binom{r}{x} c_{m,x} , \qquad (2.10)$$

where the binomial term $\binom{r}{x}$ takes into account the situation with maximal symmetry and $c_{0,0} \equiv 1$ and $c_{m,0} = c_{0,m} \equiv 0$ for $m > 0$.

1. The coefficients $c_{m,x}$ are clearly independent of the the i.d. (r) as they are related to the skew FD's with m nodes and r.d. x.

2. We say that a skew FD is *strict* if its dimension and r.d. are the same.

3. Let us denote the equivalence class of strict skew Ferrers diagrams, $\lambda \setminus \mu_x$, under the S_x action as an sFD. All skew FD's in an sFD will have identical reduced and intrinsic dimensions. Thus, given such a skew Ferrers diagram with symmetry $H \subseteq S_x$, its equivalence class will contain $\frac{x!}{\mathrm{ord}(H)}$ distinct skew Ferrrers diagrams.

4. The $c_{m,x}$ are non-negative integers since they count the number of strict skew FD's with m nodes and r.d. x.

5. For fixed m, one can see that the maximum value of r.d. with m nodes is $2m$. This enables us to convert the above equation into a second binomial transform

$$
\boxed{a_{m+r+1,r} = \sum_{x=0}^{2m} \binom{r}{x} c_{m,x}} \,, \tag{2.11}
$$

where we extend $c_{m,x}$ into a triangle, $C = (c_{m,x})$, by setting $c_{m,x} = 0$ for $x > 2m$. We usually do not write out the zeroth row and column of the C-matrix.

6. For fixed m, we can consider $a_{m+r+1,r}$ as a function of r. The function $g_m(r) := 2m!!\, a_{m+r+1,r}$ is a polynomial of degree $2m$, conjecturally with integer coefficients, in the variable r and $g_m(0) = 0$ for $m > 0$.

7. We have directly determined eleven rows ($m \in [0,10]$) of the C-matrix (see Table 3). The first few rows of the C-matrix are:

$$
C = \begin{pmatrix}
1 \\
0 & 1 & 1 \\
0 & 1 & 3 & 6 & 3 \\
0 & 1 & 7 & 20 & 46 & 45 & 15 \\
0 & 1 & 11 & 61 & 198 & 480 & 645 & 420 & 105 \\
0 & 1 & 18 & 138 & 706 & 2508 & 6441 & 10395 & 9660 & 4725 & 945
\end{pmatrix}.
$$

It is easy to see that there is only one sFD with m nodes and r.d. $2m$. In the picture below, the m nodes of the sFD are indicated by open circles. The filled circles indicate the nodes of μ_{2m} that must be added to the sFD to obtain an FD.

$$\tag{2.12}$$

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The symmetry of the skew FD is $(S_m \ltimes \mathbb{Z}_2^m)$ and thus $c_{m,2m}$ is the dimension of the coset i.e.,

$$c_{m,2m} = \frac{\dim(S_{2m})}{\operatorname{ord}(S_m \ltimes \mathbb{Z}_2^m))} = \frac{2m!}{2m!!} = (2m-1)!! \ .$$

Definition 2.8 *A skew FD of i.d. r is said to be reducible if a proper subset of its nodes are contained in a d-dimensional hyperplane (obtained by setting $r - d$ coordinates to zero) with $d < r$ and the nodes not in the proper subset lie in the orthogonal $(r - d)$-dimensional hyperplane (obtained by setting the other d coordinates to zero).*

Definition 2.9 *We say that an FD, λ, of i.d. r is reducible if the skew FD, $\lambda \setminus \mu_r$ is reducible.*

Thus a reducible sFD has multiple *components* consisting of non-intersecting proper subsets of its nodes lying in mutually orthogonal hyperplanes. Thus the sFD given in Eq. (2.12) is reducible with m components each of which is isomorphic to the irreducible sFD σ_2 defined as follows:

$$\sigma_2 \equiv \quad \begin{array}{c} \end{array} = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) \ . \tag{2.13}$$

We can thus write the sFD (2.12) as $\sigma_2 \times \sigma_2 \times \cdots \times \sigma_2 = \sigma_2^m$.

Similarly, one has two distinct sFD's with $x = 2m - 1$ and the two sFD's are reducible containing σ_2^n (for some suitable value of n) as one of the components and the other component are the following two irreducible sFD's that contribute to $c_{1,1}$ and $c_{2,3}$ respectively.

$$\sigma_1 \equiv \quad \bullet\!-\!\bullet\!-\!\circ\!\longrightarrow \qquad\qquad \sigma_3 \equiv \quad \begin{array}{c} \end{array} = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \ , \tag{2.14}$$

$$\text{(a)} \qquad\qquad\qquad\qquad \text{(b)}$$

where we have called the second sFD σ_3 – it has two nodes and has r.d. 3. In other words, $c_{2m,2m-1}$ has contributions from two sFD's – one of the form $\sigma_2^{(m-1)} \times \sigma_1$ and the other of the form $\sigma_2^{(m-2)} \times \sigma_3$. Studying the symmetries of these two sFD's with r.d. $(2m - 1)$, one obtains

$$c_{m,2m-1} = \frac{(2m-1)!}{(2m-2)!!} + \frac{(2m-1)!}{2(2m-4)!!} = m \times \frac{(2m-1)!}{(2m-2)!!} \ .$$

Clearly, such a diagrammatic method will enable one to write further formulae (we will provide a few more in an appendix) for $c_{m,x}$. However, it can get tricky to find all possible diagrams. Keeping this in mind, we make the following definition.

Definition 2.10 *The density, ρ, of a sFD with m nodes and r.d. x is $\rho \equiv m/x$.*

The density of a sFD is always greater than or equal to $\frac{1}{2}$ since $c_{m,x} = 0$ when $x > 2m$.

Proposition 2.11 *When its density is in the range $(\frac{1}{2}, \frac{2}{3})$, an sFD with m nodes and r.d. x is necessarily reducible and one of its components is the sFD, $(\sigma_2)^n$, for some $n \geq n_{min} \equiv 2x - 3m$.*

The proof follows from Proposition 2.15 that we prove later. When $\rho < 2/3$, the proposition implies it is impossible to construct an sFD that does not contain σ_2 as a component. The first new sFD, σ_3, appears at $\rho = \frac{2}{3}$. The minimum value of n is fixed by the condition that the density of the sFD goes past or equals $\frac{2}{3}$ after deleting the nodes that appear in $(\sigma_2)^n$ i.e., it is smallest value of n such that

$$\frac{m - n}{x - 2n} \geq \frac{2}{3} \implies n \geq 2x - 3m \ .$$

2.6 The third transform

Proposition 2.11 suggests that in counting the skew FD's that contribute to $c_{m,x}$, we can remove components isomorphic to σ_2 in reducible skew FD's and only count skew FD's that do not contain any σ_2 components. This motivates the next transform where we introduce a new triangle $D = (d_{m,x})$.

$$\boxed{c_{m,x} = \sum_{y=y_{\min}}^{m} \frac{x!}{(2y)!!(x - 2y)!} \, d_{m-y,x-2y}} \ , \tag{2.15}$$

with $d_{0,0} = 1$, $d_{m,0} = d_{0,m} = 0$ for $m > 0$ and $y_{\min} = 2x - 3m$. The pre-factor in the transform is determined by the order of the symmetry of σ_2^y which is $2^y y! = (2y)!!$.

1. $d_{m,x}$ counts the number of skew FD's with m nodes and r.d. x not containing σ_2 as its components. Thus it is positive definite.

2. Proposition 2.11 implies that $d_{m,x} = 0$ when $m/x > 2/3$. This is stronger than the condition $m/x > 1/2$ implied by the property of the C-matrix.

3. It is useful to rewrite the transform as follows:

$$c_{m,2m-z} = \sum_{y=\lceil z/2 \rceil}^{2z} \frac{(2m - z)!}{(2m - 2y)!!(2y - z)!} \, d_{y,2y-z} \ . \tag{2.16}$$

   In this form, one sees that completely determining row z of the D-matrix leads to a nice compact formula for $c_{m,2m-z}$. The D-matrix clearly contains fewer terms than the C-matrix since $d_{m,x} = 0$ when $\rho < 2/3$.

4. To illustrate the transform, consider $c_{m,2m-1}$ which we have already computed. One sees that

$$c_{2m,2m-1} = \sum_{y=1}^{2} \frac{(2m-1)!}{(2m-2y)!!(2y-1)!} \, d_{y,2y-1}$$

$$= \frac{(2m-1)!}{(2m-2)!!} \, d_{1,1} + \frac{(2m-1)!}{3!(2m-4)!!} \, d_{2,3} \; . \tag{2.17}$$

It is easy to see that $d_{1,1} = 1$ as there is precisely one sFD ((a) in Eq. (2.14)) and $d_{2,3} = 3$ as there are three inequivalent diagrams under the action of S_3 on the sFD, σ_3.

5. When $\rho = 2/3$, there is only one sFD, σ_3^m, that contributes to $d_{2m,3m}$. This implies that

$$d_{2m,3m} = \frac{3m!}{m! \, 2^m} \; , \quad m = 1, 2, 3, \ldots \tag{2.18}$$

$$D = \begin{pmatrix} 1 \\ 0 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 1 & 7 & 17 & 28 \\ 0 & 1 & 11 & 58 & 156 & 295 & 90 \\ 0 & 1 & 18 & 135 & 640 & 1913 & 3786 & 2310 \end{pmatrix}$$

2.7 The final transform

The main advantage of the D-matrix is that it contains fewer terms than the C-matrix. Using it, we have arrived at formulae for $c_{m,2m-z}$ for $z = 2, 3, 4, 5$ analogous to the one in Eq. (2.17) that can be obtained, in principle, from the table that gives the D-matrix. Can we do better? We saw that as the density increased from $1/2$ to $2/3$, only one irreducible diagram appears. At $\rho = \frac{3}{4}$, two new sFD's appear. They are

$$\sigma_{4a} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad , \quad \sigma_{4b} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \; . \tag{2.19}$$

In fact, one can define another transform that removes reducible components of type σ_3 from sFD;s that contribute to the D-matrix for $\rho \in (2/3, 3/4)$. The next proposition will enable to do this and a lot more by removing a whole family of reducible components that necessarily appear when in sFD's with $\rho < 1$.

Definition 2.12 *Let $\mathcal{D} \equiv \cup_r \mathcal{D}_r$, where $\mathcal{D}_r$ denotes the set of strict Ferrers diagrams of dimension r consisting only of nodes of the form $(1, 1, 0, \ldots, 0)^T$ or its S_r images in addition to the nodes present in μ_r.*

We say, somewhat loosely, that a strict skew FD, σ of r.d. x is in $\mathcal{D}$ if the FD $\mu_x \cup \sigma \in \mathcal{D}$. One can show that σ_2, σ_3 and $\sigma_{4a/b}$ are the only irreducible strict skew Ferrers diagrams at dimensions 2, 3 and 4 respectively that appear in $\mathcal{D}$.

Let $e_{m,r}$ denote the number of Ferrers diagrams in $\mathcal{D}$ obtained by adding m nodes to μ_r. It is easy to see that $e_{m,x} = \dbinom{\binom{x}{2}}{m}$ as there are $\binom{x}{2}$ possible nodes from which we need to choose m nodes. We define a new transform that removes reducible components that are in $\mathcal{D}$.

$$a_{m+r+1,r} = \sum_{x=1}^{r} \sum_{p=0}^{m} \binom{r}{x} e_{m-p,r-x} \, f_{p+x+1,\, x}$$

$$= \sum_{x=1}^{r} \sum_{p=0}^{m} \binom{r}{x} \binom{\binom{r-x}{2}}{m-p} f_{p+x+1,\, x} \, , \qquad (2.20)$$

where in the second line we use the explicit formula for $e_{m,x}$ and $f_{1,0} \equiv 1$, $f_{n,0} = f_{1,n-1} = 0$ for $n > 1$. In the first line, a typical term in the summation on the right hand side consists of reducible strict FD's with the component in $\mathcal{D}$ having i.d. $r - x$ and $(m - p)$ nodes added to μ_{r-x} and the other component consisting of an strict FD with no reducible component in $\mathcal{D}$, i.d. and r.d. x and p nodes added to μ_x – their number is counted by $f_{p+x+1,x}$. The binomial factor $\binom{r}{x}$ is the number of ways one can choose x dimensions occupied by the FD's contributing to $f_{p+x+1,x}$. The above formula defines a new triangle $F = (f_{n,r})$. The entry $f_{r+m+1,r}$ is the *the number of strict FD's of i.d. r obtained by adding m nodes to μ_r and does not contain any reducible components that are in $\mathcal{D}$*. Such an FD must necessarily have r.d. also equal to r, else it will necessarily have a reducible component isomorphic to μ_{r-x} if its r.d. is x.

It is easy to see that $f_{r+1,r} = 0$. The only contribution to $a_{r+1,r}$ is the unique FD μ_r which is $\mathcal{D}$. Similarly, $f_{r+2,r} = 0$ when $r > 1$ as the only contribution to $a_{r+2,r}$ is of the form $\sigma_1 \times \sigma_2^{r-1}$. One also has $f_{3,1} = 1$ with σ_1 being the unique FD contributing to it. The next proposition shows the advantage of defining the F-matrix.

Proposition 2.13 $f_{m+r+1,r} = 0$ *when $r > m$.*

Proof: Let λ be an FD of i.d. r with $m + r + 1$ nodes that contributes to $f_{m+r+1,r}$. Consider the skew FD, $\lambda \setminus \mu_r$ – it has m nodes. It must be a strict skew FD else it has a irreducible component isomorphic to μ_x for some $x < r$. Thus, the proposition implies that there are no strict skew FD's with density $\rho = m/r < 1$.

We can also assume that the skew FD is irreducible – if it is reducible, it must necessarily have at least one irreducible component with density < 1 and we can focus on (proving the non-existence) such irreducible components. Our goal is thus reduced to proving that there are no irreducible strict skew FD's with density < 1.

Definition 2.14 *Let us call the nodes obtained by all permutations of the coordinates of the node $(1, 1, 0, \ldots, 0)^T$ as nodes of type 1. Similarly, call the nodes obtained by permuting coordinates of $(2, 0, \ldots, 0)^T$ as type 2. Nodes of type 3 are nodes that are not of type 1 or 2.*

Examples of type 3 nodes include $(1, 1, 1, 0, \ldots, 0)^T$ and $(3, 0, \ldots, 0)$. Such nodes cannot be added to the FD μ_r without including supporting nodes of type 1 and 2. The addition of nodes of type 3, when possible, never increase the r.d. of an FD thus increasing the density. Thus, given a FD λ (of i.d. r and r.d. r) containing type 3 nodes, we can form a new FD λ' with the same r.d. but lower density. Further, if $\lambda \setminus \mu_r$ is irreducible, $\lambda' \setminus \mu_r$ is also irreducible. The skew FD $\lambda' \setminus \mu_r$ thus consists of nodes of type 1 and type 2. If it consists of only nodes of type 1, then $\lambda' \in \mathcal{D}$. Thus, we need to only consider irreducible strict skew FD's containing at least one node of type 2.

For the rest of the discussion, let λ' be an FD such that $\lambda' \setminus \mu_r$ is an irreducible strict skew FD containing only nodes of type 1 and at least one node of type 2. It is easy to see that removing of node of type 2 does not affect the irreducibility of the skew FD. Further, it does not reduce the r.d. as the only way a type 2 node can reduce the r.d. of a skew FD is when it appears as a part of a reducible component isomorphic to σ_1. Thus, we can delete all type 2 nodes to obtain a new FD λ'' that is irreducible and contains only type 1 nodes. Again, it is easy to see that $\rho(\lambda'') \leq \rho(\lambda')$. Further $\lambda'' \in \mathcal{D}$. Thus, one has the sequence

$$\rho(\lambda'') \leq \rho(\lambda') \leq \rho(\lambda) \ . \tag{2.21}$$

Let $\lambda' \setminus \mu_r$ have $(r-1)$-nodes so that its density is just below one and contain z nodes of type 2. Then, $\lambda'' \setminus \mu_r$ will have $(r-1-z)$ nodes and be irreducible. The next proposition shows that such a λ'' does not exist. Hence, there exists *no* FD λ'' and hence no FD λ' with density < 1. $\qquad\qquad \square$

Proposition 2.15 *The only strict FD's in $\mathcal{D}$ of i.d. r such that the skew FD $\lambda \setminus \mu_r$ is strict and irreducible with density less than 1 necessarily have $\rho = \frac{r-1}{r}$.*

Proof: Let us assume that $\lambda \setminus \mu_r$ has $(r-2)$ nodes and is irreducible. Let us try to construct such a strict skew FD and we will see that there are not enough nodes. Start by putting the first type 1 node in the $x^1 x^2$ plane. The irreducibility condition implies that the second node must be either in the $x^1 x^\alpha$ or $x^2 x^\alpha$ plane where α is not 1 or 2. The key point is that the additional node must contain one of the used up coordinates, x^1 or x^2 in this case and a new coordinate so that irreducibility is maintained. Clearly, such a process needs $(r-1)$ nodes to get an irreducible skew FD $\lambda \setminus \mu_r$ with r.d. r. This is impossible. Hence, there exists *no* irreducible skew FD λ with density $\frac{r-2}{r}$. it is easy to extend the argument to exclude even lower densities. Thus, the only possibility that is not ruled out is

to have strict skew FD's with $(r-1)$ nodes with r.d. r – these have density $\frac{r-1}{r}$.

$\square$

Remark: σ_2, σ_3 and $\sigma_{4a/b}$ are the only irreducible strict skew FD's with r.d. $2, 3, 4$ respectively.

2.7.1 Properties of the F-matrix

1. The most important property is the one implied by Proposition 2.13 which says that the F matrix is lower triangular with $f_{n,r} = 0$ when $r < [(n-1)/2]$. For fixed value of n, the F-matrix has far fewer terms (roughly half) than the corresponding row in the A-matrix. We have determined the first 25 rows of the F-matrix (see Table 4).

2. It turns out that there are other transforms that also lead to matrices with fewer entries like the F-matrix. See for instance, the box transform that we consider in the appendix. However, their relationship to A is not as simple as Eq. (2.20). The simplicity of Eq. (2.20) is what picks out the F-matrix as special.

3. We can also use this idea to refine the counting problem associated with the C-matrix. Let $C^{\mathcal{D}} = (c^{\mathcal{D}}_{m,x})$ denote the contribution to the C-matrix that arise from FD's that are in $\mathcal{D}$. Since the set $\mathcal{D}_r$ is invariant under S_r, it is easy to see that $C^{\mathcal{D}}$ is given by the transform

$$\binom{\binom{x}{2}}{m} = \sum_{x=0}^{2m} \binom{r}{x} c^{\mathcal{D}}_{m,x} \ . \tag{2.22}$$

Then, we can define $\widetilde{C} = (\widetilde{c}_{m,x})$ by removing contributions that arise from reducible parts that are isomorphic to contributions to $\mathcal{C}^{\mathcal{D}}$. Then, one has

$$c_{m,x} = \widetilde{c}_{m,x} + c^{\mathcal{D}}_{m,x} + \sum_{y=1}^{x-1}\sum_{p=1}^{m-1} \binom{x}{y} c^{\mathcal{D}}_{m-p,x-y} \widetilde{c}_{p,y} \ . \tag{2.23}$$

Given a strict skew FD that contributes to $\widetilde{c}_{m,x}$, it is easy to see that there is a unique FD obtained by adding nodes in μ_x to the skew FD. Further, this FD must contribute to the entry $f_{m+x+1,x}$ in F. Since the converse also holds i.e, given a strict FD of i.d. x that contributes to the F-matrix, the skew FD obtained by deleting nodes in μ_x gives a skew FD that contributes to $\widetilde{C}$. Thus, one has

$$\widetilde{c}_{m,x} = f_{m+x+1,x} \ . \tag{2.24}$$

We observe numerically that $f_{2m+1,m} = (m+1)^{m-2}$ for $m = 0, 1, 2, \ldots, 12$. We will show that it holds for all m. These numbers appear in the sequence numbered A000272 in the OEIS [9]. The next proposition presents a further

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refinement. We need a few definitions which we briefly state. A graph, consisting of vertices and undirected edges, with no cycles is called an acyclic graph or a forest. A forest may consist of disconnected components and is called a tree if it has only one connected component. A rooted tree is one with a marked/special vertex (called the root) while a rooted forest is one in which every component is rooted. A spanning forest is any subgraph that is both a forest (contains no cycles) and spanning (includes every vertex) [11, 12].

Proposition 2.16 *Let α be the number of nodes of type 2 contained in an FD that contributes to $f_{2m+1,m}$. Let $f_{2m+1,m}(\alpha)$ denote the total number of such Ferrers diagrams. Then, $f_{2m+1,m}(\alpha)$ is the number of spanning rooted forests on m vertices and α components. It follows from a result due to Cayley on the numbers of spanning rooted forests that [13]*

$$f_{2m+1,m}(\alpha) = \binom{m-1}{\alpha-1} m^{m-\alpha} . \tag{2.25}$$

Proof: We will provide a bijective map relating FD's that contribute to $f_{2m+1,m}(\alpha)$ to spanning rooted forests on m vertices and α components. There is a natural action of S_m on both sides – on the FD side, it corresponds to permuting the m coordinates and on the rooted forest side, it corresponds to relabeling the m nodes. We identify these two groups.

Given a skew FD that contributes to $f_{2m+1,m}(\alpha)$, we can construct a graph with m vertices labeled from $(1, \ldots, m)$ as follows. The type 2 nodes become root vertices carrying the label of the non-vanishing coordinate. Thus if a type 2 node has non-vanishing j-th coordinate, assign it the label j. Add $(m - \alpha)$ vertices and label them with the unused labels. Every type 1 vertex has two non-vanishing coordinates, say the j-th and k-th coordinates. Assign an edge that connects vertex j to vertex k. Repeat for all type 1 nodes. In this process, there are as many components as there are type 2 nodes. Thus the graph is a spanning rooted forest on m vertices and α components. The following example illustrates the map for $m = 4$ and $\alpha = 1$. The root vertex is shown by a filled circle.

To prove the converse statement, given a spanning rooted forest with m vertices and α components, we need to construct an FD that contributes to $f_{2m+1,m}(\alpha)$. This is easy to do . Pick the root vertices and assign them to type 2-nodes whose non-vanishing coordinate decided by the label of the vertex. Next assign to all edges a type 1 node that has non-vanishing coordinates at precisely the locations decided by the labels of the vertices it connects. We thus recover

the skew FD. $\qquad\square$

An example: We know that $f_{5,2} = 3$. The three skew FD's are:

$$\sigma_1^2 = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right) \quad ; \quad \left(\begin{smallmatrix} 2 & 1 \\ 0 & 1 \end{smallmatrix}\right) \quad ; \quad \left(\begin{smallmatrix} 1 & 0 \\ 1 & 2 \end{smallmatrix}\right) \quad ,$$

where the nodes are listed by the ordering: $(a_1, a_2) > (b_1, b_2)$ if $a_1 > b_1$ or $a_1 = b_1$ and $a_2 > b_2$. Note that there are two equivalence classes of skew FD's. Under S_2 action as the second and third skew FD's get mapped to each other.

Remark: Given a skew FD, it is possible to uniquely label the nodes by ordering them by a choice of ordering as illustrated above.

3 Other triangles

3.1 New triangles

So far, we have considered transforms that lead to new triangles $(A/C/D/F)$ all of which have positive definite entries since we they all count numbers of skew Ferrers diagrams. We will now provide two other transforms that are partly conjectural and lead to triangles that are not positive definite – we denote the entries with Greek letters to remind us of this. We begin by expanding the entries in the A-matrix as follows. Let

$$a_{m+r+1,r} = \sum_{z=0}^{2m} \alpha_{m,z} \, \frac{r^{2m-z}}{2m!!} \, , \tag{3.1}$$

with $\alpha_{m,0} = 1$ for $m \geq 0$ and $\alpha_{m,2m} = 0$ for $m > 0$. The above transform provides the entries for another triangular matrix, $\alpha_{m,z}$, that we call the α-triangle by setting $\alpha_{m,z} = 0$ for $z > 2m$. One can explicitly relate the $\alpha_{m,z}$ to the entries in the C-matrix using Stirling numbers of the first kind. Thus the above formula is not conjectural. However the following is conjectural:

Conjecture 3.1 *The entries of the α-triangle, i.e., $\alpha_{m,z}$, are all integers.*

This is true for the first ten rows and appears to hold for the first eleven rows which have been determined using conjectures.

The second conjecture introduces a new triangle, that we call the β-triangle, and its associated transform. It has been determined experimentally and verified to hold to the extent possible.

Conjecture 3.2 *The α-matrix admits the following decomposition.*

$$\alpha_{m,z} = \sum_{y=0}^{\lfloor z/2 \rfloor} \binom{m}{z-y} \beta_{z,y} , \tag{3.2}$$

with $\beta_{0,0} = 1$ and $\beta_{2y,y} = 0$ for all $y > 0$.

By setting $\beta_{z,y} = 0$ for $y > \lfloor z/2 \rfloor$, this becomes the binomial transform

$$\alpha_{m,z} = \sum_{y=0}^{m} \binom{m}{z-y} \beta_{z,y} . \tag{3.3}$$

The inverse transform is

$$\beta_{z,y} = \sum_{m=0}^{z-y} (-1)^{m+z-y} \binom{z-y}{m} \alpha_{m,z} . \tag{3.4}$$

We now state a conjecture of Meeussen that fixes one of the coefficients.

Conjecture 3.3 (Meeussen)

$$\beta_{n,0} = H_n(\tfrac{1}{2}) ,$$

where $H_n(x)$ is the n-th Hermite polynomial.

Recall that the α-matrix has $2m$ non-zero entries in the m-th row. The β-matrix has fewer terms, roughly half the entries in the α-matrix. We were able to determine eleven rows of the α and C-matrices using the β-matrix of which 10 were verified through other means. This was our main motivation in searching for and find the combinatorial problem that eventually lead to the F-matrix.

3.2   The B-triangle

We now construct another lower triangular matrix $B = (b_{n,r})$ with $n = 1, 2, \ldots$ and $r = 0, 1, 2, \ldots$ and $b_{n,0} = 1$.

$$p_d(n) = \sum_{r=0}^{n-1} \binom{d}{r} b_{n,r} = 1 + \sum_{r=1}^{n-1} \binom{d}{r} b_{n,r} . \tag{3.5}$$

The matrix B appear in the OEIS as sequence number A096806. Using Pascal's identity

$$\binom{d+1}{r} = \binom{d}{r} + \binom{d}{r-1} , \tag{3.6}$$

we can relate the matrix B to A. Thus, one has the relation

$$b_{n,r} = a_{n,r} + a_{n,r+1} . \tag{3.7}$$

One can easily show that $b_{n,n-1} = 1$ using the above formula and known properties of the matrix A. The first six rows of B have been determined explicitly, for instance, in Andrews' book on Partitions [2]. It is easy to check that the above relation holds for all six rows.

3.3 Hanna's triangle

Conjecture 3.4 (Hanna) *There exists a lower-triangular matrix $T = (\tau_{ij})$ (with $i, j = 0, 1, 2, \ldots$) with integral entries and ones on its diagonal such that*

$$p_d(n) = \sum_{j=0}^{n} (T^d)_{n,j} \ .$$

In other words, the sum of the the n-th row of the d-th power of T give the d-dimensional partition of n. This matrix appears in the OEIS as sequence A096651. Since $p_d(0) = 1$, we can set $\tau_{0,0} = 1$ and $\tau_{j,0} = 0$ for $j > 0$. For the rest of the discussion, we will consider $n > 0$ and can delete the zeroth row and column of the T-matrix as they no longer play a role. We shall however use the same symbol T to denote the modified matrix as it is easy to reconstruct the original T matrix by adding back the zeroth row and column. We shall prove the existence as well as the integrality of the matrix T by constructing an explicit map that relates T to the matrix B (and hence A) that we considered in the previous section.

Proof: For $n \geq 1$, the Hanna conjecture can written as

$$p_d(n) = \sum_{j=1}^{n} (T^d)_{n,j} = \sum_{x_1 \cdots x_d} \tau_{n,x_1} \tau_{x_1,x_2} \cdots \tau_{x_{d-1},x_d} \ , \tag{3.8}$$

where $n \geq x_1 \geq x_2 \geq \cdots \geq x_d \geq 1$. It obviously holds for $n = 1$ since $\tau_{11} = 1$. Using the fact that T has ones in its diagonal, we can simplify the above expression to

$$p_d(n) = 1 + \sum_{r=1}^{n-1} \binom{d}{r} \sum_{x_1 \cdots x_r} \tau_{n,x_1} \tau_{x_1,x_2} \cdots \tau_{x_{r-1},x_r} \ . \tag{3.9}$$

with sum now running over all sequences of r positive non-zero integers $(x_1, \cdots, x_r)$ such that $x_0 \equiv n > x_1 > x_2 > \cdots > x_r \geq 1$. The combinatorial factor expresses the number of ways in which diagonal elements are chosen. Comparing the above equation with Eq. (3.5) implies the (potential) identity for $n > 1$ and $r \geq 1$.

$$\sum_{x_1 \cdots x_r} \tau_{n,x_1} \tau_{x_1,x_2} \cdots \tau_{x_{r-1},x_r} = b_{n,r} \ , \tag{3.10}$$

with $n > x_1 > x_2 > \cdots > x_r \geq 1$. Let us assume that this relation holds for $n < m$ (for some $m > 1$) and that we have determined $(m-1)$ rows of T. Then, we can rewrite the above equation as

$$\sum_{1 \leq x < m} \tau_{m,x} b_{x,(r-1)} = b_{m,r} \quad \text{for } m > r \geq 1 \ . \tag{3.11}$$

The above $(m-1)$ equations are linear equations in $(m-1)$ unknowns: $(\tau_{m,1}, \ldots, \tau_{m,m-1})$ – these are the undetermined entries in the m-th row of T. Hence, they have a solution if the matrix (constructed using $b_{x,(r-1)}$) is invertible. The matrix is upper triangular with ones in its diagonal. Hence it is has determinant one and hence is invertible. This enables us to recursively determine all the entries in the matrix T. This proves the *existence* of T.

We shall inductively prove the integrality of the matrix T using more explicit details of Eq. (3.11). We begin with the equation for $r = m - 1$ and it gives

$$\tau_{m,m-1} b_{m-1,m-2} = b_{m,(m-1)} \implies \boxed{\tau_{m,m-1} = 1}, \tag{3.12}$$

where we have used $b_{m,m-1} = 1$ for $m \geq 1$. Next consider, $r = m - 2$. This equation gives $\tau_{m,m-2} + \tau_{m,m-1} b_{m-1,m-3} = b_{m,m-2}$ which gives

$$\tau_{m,m-2} = b_{m,m-2} - \tau_{m,m-1} b_{m-1,m-3}, \tag{3.13}$$

where we have used the fact that $\tau_{m,m-1}$ has been solved for and shown to be integral in the previous step. Note that this implies that $\tau_{m,m-2}$ is integral. Proceeding in this manner from $r = (m-1)$ to $r = 1$, we thus determine all the unknowns. A typical equation will take the form (reflecting the triangular nature of the equations)

$$\boxed{\tau_{m,m-r} = b_{m,m-r} - \sum_{x=m-r+1}^{m-1} \tau_{m,x} b_{x,m-r}}, \tag{3.14}$$

for $r = 1, 2, \ldots, (m-1)$. We assume that $\tau_{m,m-r'}$ is integral for all $r' < r$. Thus the right hand side is integral as it only contains integral terms. Hence $\tau_{m,m-r}$ is integral. This concludes the proof of integrality of the matrix T.

$\square$

We now state an unproven conjecture of Hanna and Meeussen.

Conjecture 3.5 (Hanna-Meeussen) $m! \, \tau_{m+r+1,m}$ *is a polynomial of degree m in r with integral polynomial coefficients.*

It is easy to show that $\tau_{m+r+1,m}$ is a polynomial of degree $(2m-1)$ in r using the properties of the A-matrix. However, the above conjecture is stronger and seems to consistent with known data for $m = 0, 1, \ldots, 11$.

4   Practical Considerations

This section provides details on the exact enumeration of higher-dimensional partitions as well as the triangles defined in this paper. With access to high-performance computing getting easier in recent times, this is indeed an additional computational aspect that can and must be added to the theoretical discussion of the previous section. We will first discuss the algorithms that we used and then discuss exact enumerations as we carried out.

4.1 Algorithms for higher-dimensional partitions

There are two algorithms in the literature for computing higher-dimensional partitions. The first one is due to Bratley and McKay (the BM algorithm) [7] and the second one is due to Knuth [8] – both are more than 40 years old reflecting the lack of progress in this area. Both are highly recursive and provide distinct ways of exactly enumerating higher dimensional partitions.

The BM algorithm

The partitions in any fixed dimension, say d, form a tree which we call the *partition* tree in $(d+1)$-dimensions[2] and denote by the symbol $\mathcal{T}_{d+1}$. Every node of the tree is the Ferrers diagram associated with a partition. The unique Ferrers diagram containing one point is the root node of the tree. New partitions can be formed by adding or deleting a point from the Ferrers diagram[3]. Add a link to partitions connected this way. The depth of the tree is the number of points in the partition.

The BM algorithm recursively traverses the tree up to some fixed depth, say n, such that each node is visited precisely once. The heart of the algorithm is the routine called *part* that takes three arguments and is recursively called in the algorithm. Every time a node is visited, the partition is stored in an array called *current* and presented to user. If one is interested in only counting the number of partitions of an integer in a given dimension, if the current partition has m points, increment a suitable counter, call it $p_d(m)$, by one. At the end of the program, the counter thus contains the number of partitions of all integers less than or equal to the depth of the traversed tree.

The Knuth algorithm

Let $S_m = \mathbb{N}^m$ denote the set of points in the totally positive orthant in a hyper cubic lattice. Let $d_m(k)$ denote the number of topological sequences with index k (see [6, 8] for definitions). Then a theorem due to Knuth [8] relates the numbers of topological sequences to numbers of partitions. To be precise, one has

$$p_m(n) = \sum_{k=0}^{n} d_m(k) p_1(n-k) \ . \tag{4.1}$$

Since one-dimensional partitions are easily enumerated from the generating function, it is simple to generate $p_m(n)$ given $d_m(k)$ for all $k \leq n$. Knuth provided an algorithm to generate and count all topological sequences – he illustrated this

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[2]Recall that the Ferrers diagram for a d-dimensional partition is a set of points in $d+1$ dimensions.

[3]To avoid confusion, in this section alone, we shall refer to nodes of a partition as points in the Ferrers diagram. This is to avoid confusion with the node of the tree.

method by generating numbers for the numbers of solid partitions for integers ≤ 28. Recently, a parallelized version of this algorithm was used by the author and other collaborators to enumerate solid partitions of integers ≤ 68 [6].

Remark: An important aspect of the BM algorithm is that its memory usage is of the order of nd bytes, where d is the dimension and n is the maximum depth. This is vastly superior to the Knuth algorithm, where a similar problem needs memory of the order of n^{d-1} bytes. However, when memory isn't an issue, our implementation of the Knuth algorithm typically takes less time than our implementation of the Bratley-McKay algorithm.

The modified BM algorithm

We begin with the observation that a suitably chosen sub-tree of the partition tree in r-dimensions, $\mathcal{T}_r$ generates all partitions that contribute to the r-th column of the A-matrix i.e., $a_{n,r}$. The head node of this sub-tree is the Ferrers diagram μ_r defined in Eq. (2.9). The rest of the tree is generated by adding points to μ_r. Let us denote this sub-tree by $\mathcal{V}_r$ and the depth of this tree is clearly m where $m = n - r - 1$.

The BM algorithm was designed to recursively traverse the partition tree visiting each node precisely once. The starting point of the algorithm is the root node whose Ferrers diagram consists of one point. Our idea is to change the initial configuration in the BM algorithm to the Ferrers diagram, μ_r and then call the recursive routine *part* with suitably chosen arguments[4]. For this modification to work correctly, the program should traverse the sub-tree $\mathcal{V}_r$ visiting each node precisely once to the chosen depth. This turned out to be easier as we experimentally observed that the sub-tree $\mathcal{V}_r$ appeared naturally in the original BM algorithm for low values of r. We then checked that the modified BM algorithm correctly generated entries in the A-matrix for $r \leq 10$. However, we have *not* rigorously proved that this is indeed the case.

Thus, once we have the modified BM algorithm correctly traversing the subtree $\mathcal{V}_r$, we can do the following:

- Count the number of nodes at each depth – this gives the number $a_{m+r+r,r}$.

- At each node, numerically compute the reduced dimension, x of the Ferrers diagram. Then organizing the partitions by depth and r.d., we determine $\binom{r}{x} c_{m,x}$. The binomial pre factor is present since all $x \leq r$ will appear. This also implies that the algorithm is inefficient computationally for obtaining entries in the C-matrix.

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[4]We have determined that the correct call is $part\left(r + 2, 0, \binom{r+1}{2}\right)$. For comparison, the BM algorithm begins with the call $part(1, 0, 1)$. We thank Arun K. Jayaraman for implementing the BM algorithm as well as working out this modification.

A wish list of algorithms

As we just mentioned, the current algorithm to enumerate entries in the C-matrix is computationally inefficient as we generate $\binom{r}{x}$ partitions for each distinct contribution to $c_{m,x}$. It is also inefficient because we need to compute x for every given partition. Can we create a more efficient algorithm? The problem is that we do not have an elegant characterization of sFD's with r.d. equal to x. This is in contrast to what happened with the A-matrix. In that case, we could show that any FD that has i.d. r necessarily contains the FD μ_r. By using it as our initial configuration, we directly avoided configurations with smaller intrinsic dimension. For the C-matrix, we cannot avoid configurations that have smaller r.d. than the one of interest.

We do not have any algorithms for the α and β matrices as well as the D/F matrices. So far these have been computed only indirectly after the A and C matrices have been computed. However, Proposition 2.16 might be a good starting point to coming up with an algorithm that directly enumerates entries in the F-matrix.

4.2   Exact enumeration of higher-dimensional partitions

In order to evaluate higher-dimensional partitions for integers ≤ 25 and dimensions ≤ 10, we chose to use the Knuth algorithm do carry out our computations. There was no serious memory issues for dimensions ≤ 7 and the Knuth algorithm worked well.

We needed to modify our computation when for dimensions 8, 9 and 10. The reduction in memory was done by counting topological sequences that fit into a box of size b. Then the memory requirement went down from n^{d-1} to b^{d-1}. For instance, when $n = 20$ and $b = 10$ (for $d = 10$), the memory usage went down by a factor of 2^9 and enabled us to keep our memory requirements in the $4 - 8$ GB range as constrained by the IITM supercluster. However, some configurations are missed out as they do not fit into the box. Interestingly, one can show the error due to missed configurations is independent of box size when the index lies in the range $[b + 1, 2b]$. This makes it easy to estimate the errors by comparing with known results at smaller values of b and then slowly increasing the value of b. This method was used, for instance, to determine the ten-dimensional partitions of 20 – this was carried out by using a box of size 11 with errors determined up to $k = b + 9$. This was one of the more difficult computations as it took a several months of computer time to first estimate the errors and then carry out the final run in the box. Table 1 gives the results obtained used the Knuth algorithm for $n \leq 23$ and $d \leq 10$ and represent more than six months of computer time.

4.3 Exact enumeration of the A and C triangles

The modified BM algorithm was used to generate the A and C matrix. The first eight rows of the C-matrix have been completely determined. Two additional rows were determined using additional information from the D-matrix. We obtain

$$c_{m,2m-2} = \frac{(2m-2)!}{6(2m-4)!!}(3m^2 - m - 1)$$
$$c_{m,2m-3} = \frac{(2m-3)!}{6(2m-4)!!}(2m^4 - 6m^3 + 3m^2 + 3m + 4) \tag{4.2}$$
$$c_{m,2m-4} = \frac{(2m-4)!}{180(2m-6)!!}(15m^5 - 75m^4 + 95m^3 + 21m^2 + 88m + 42)$$
$$c_{m,2m-5} = \frac{(2m-5)!}{90(2m-6)!!}(258 - 167m - 80m^2 + 111m^3 - 174m^4 + 116m^5 - 31m^6 + 3m^7)$$

This determines all entries in the A-matrix of the form $a_{m+r+1,r}$ for $m = 0, \ldots 10$ for *all* values of r. We have determined the remaining entries for $a_{n,r}$ for $n \leq 23$ by using the BM algorithm when necessary. The entry $a_{23,11}$ was one of the longest runs and took about 880 hours of CPU time. Tables 2 and 3 provide our results.

Using the β and α matrices as well as the Meeussen conjecture, we have also determined the 11-th row of the C-matrix. While none of these results were used in finally determining the entries in the A-matrix, there doesn't seem to be an inconsistency. This is only to be viewed as evidence for various conjectures.

4.4 Extracting the elements of the other triangles

All other triangles were obtained by using known numbers for the A and C matrices as we do not have an algorithm to enumerate them. The results for the D and the β-matrix are presented in Tables 5 and 6 respectively.

An improved implementation of the Bratley-McKay algorithm was provided to us recently by Prof. Bratley. This enabled us to enumerate a few more terms – in particular, we were able to enumerate rows 24 and 25 up to and including $a_{25,12}$. This enabled us to completely determine 25 rows of the F-matrix. This in turn determines all entries in 25 rows of the A-matrix and hence determines partitions of 25 in any dimension. It also provides a check on the 23 rows of the A-matrix which was independently determined. Table 4 provides our results.

5 Concluding Remarks

We have shown the existence of several structures that lead to simplifications in the exact enumerations of higher-dimensional partitions. The combinatorial interpretations that we have provided have enabled us to come up with an algorithms to evaluate the A and C matrices. A few lines of code in Mathematica/Maple/Maxima/java can be used to store the A matrix and compute $p_d(n)$

for $n \leq 25$ using the Binomial transform in real time. A working implementation of this is provided on the webpage:

<center>http://www.physics.iitm.ac.in/~suresh/partitions.html .</center>

We will be adding these numbers to the OEIS as well as providing modules for SAGE/Mathematica/Maxima.

It appears difficult to improve on our results which have determined all entries for the $n = 25$ row of the A-matrix. In fact, we have determined most of the entries for the $n = 26$ entry and hope to add this row in the future. Further additions to the A-matrix will require new and efficient algorithms to directly enumerate either the C or the F matrix. Another approach would be a naïve parallelization of the BM algorithm. We hope to be able to eventually determine partitions of integers less than 30 in any dimension in the future.

A   Ferrers Diagrams in a symmetric box

Let us consider Ferrers diagrams of i.d. r that fit in a symmetric box of size b – points that lie within the box are such that all their coordinates take values in $(0, 1, \ldots, b-1)$. Let us call them restricted Ferrers Diagrams. It is easy to see that under the action of S_r that permutes the r-axes, FD's that fit in a box get mapped to FD's that also fit in the same box. Due to this property, we can construct analogs of the various triangles $A/C/D/F$ for restricted FD's as well even though the total number of restricted FD's are finite. For instance, we have

$$p_d^{\text{rest}}(n) = \sum_{r=0}^{d+1} \binom{d+1}{r} a_{n,r}^{\text{rest}} , \tag{A.1}$$

where $p_d^{\text{rest}}(n)$ is the number of FD's with n-nodes that fit in a symmetric box of size b^{d+1} and $A^{\text{rest}} = \left(a_{n,r}^{\text{rest}}\right)$. Similarly, we can define C^{rest}. The analog of the A-matrix for restricted partitions not necessarily in a symmetric box has appeared in the work of Destainville et. al. [14].

Let us focus on partitions that fit into a symmetric box of size two and denote the corresponding triangles in obvious notation:

$$A^{\text{box2}} = \left(a_{n,r}^{\text{box2}}\right), \ C^{\text{box2}} = \left(c_{m,x}^{\text{box2}}\right), \ D^{\text{box2}} = \left(d_{m,x}^{\text{box2}}\right) \text{ and } F^{\text{box2}} = \left(f_{n,r}^{\text{box2}}\right).$$

We do not write out their relationships as they exactly mirror the corresponding formulae for unrestricted partitions.

Definition A.1 *Let $\mathcal{B}$ be the set of strict Ferrers diagrams that fit in a symmetric box of size 2.*

We say, somewhat loosely, that a strict skew FD, σ of r.d. x is in $\mathcal{B}$ if the FD $\mu_x \cup \sigma \in \mathcal{B}$. The only irreducible strict skew Ferrers diagrams at dimensions 2, 3 and 4 in $\mathcal{B}$ are σ_2, σ_3 and $\sigma_{4a/b}$ respectively. It is also easy to see that $\mathcal{D} \subset \mathcal{B}$.

The matrix $C^{\text{box2}} = \left(c_{m,x}^{\text{box2}}\right)$ (for $m, x \geq 0$) has non-zero entries when $m \in [0, 2^x - x - 1]$ with $c_{0,0}^{\text{box2}} = 1$. Further, $c_{0,x}^{\text{box2}} = c_{x,0}^{\text{box2}} = 0$ for $x > 0$ and $c_{2^x - x - 1, x}^{\text{box2}} = 1$. Thus, it is a triangle. The maximum value of m, for fixed x, is obtained by considering the FD containing all nodes that are in the box. Such an FD has 2^x nodes in x-dimensions and thus the corresponding skew FD has $2^x - x - 1$ nodes after deleting the nodes that lie in μ_x. Below, we provide the first few rows of the matrix, C^{box2}, for $m \in [1, 6]$.

$$C^{\text{box2}} = \begin{pmatrix} 1 & & & & & & & & \\ 0 & 0 & 1 & & & & & & \\ 0 & 0 & 0 & 3 & 3 & & & & \\ 0 & 0 & 0 & 1 & 16 & 30 & 15 & & \\ 0 & 0 & 0 & 1 & 15 & 135 & 330 & 315 & 105 \\ 0 & 0 & 0 & 0 & 18 & 232 & 1581 & 4410 & 5880 & 3780 & 945 \\ 0 & 0 & 0 & 0 & 13 & 355 & 4000 & 23709 & 71078 & 116550 & 107100 & 51975 & 10395 \end{pmatrix} \tag{A.2}$$

Extending the ideas that were used in defining the D/F triangles, we look to count *only* those skew FD's that do not contain skew FD's in $\mathcal{B}$ are reducible components. Let the matrix $\widehat{C}_{m,x}$ denote this reduced C-matrix that counts strict skew FD's of r.d. x with m-nodes. Such FD's necessarily contain at least one node of type 2. Then one has the following relation that relates $\widehat{C}$ to the C:

$$c_{m,x} = \sum_{y=0}^{x} \sum_{p=0}^{m} \binom{x}{y} c_{m-p,x-y}^{\text{box2}} \, \widehat{c}_{p,y} \, , \tag{A.3}$$

with $\widehat{c}_{0,0} \equiv 1$ and $\widehat{c}_{0,x} = 0$ for $x > 0$. It is better to rewrite the above formula as follows:

$$c_{m,x} = \widehat{c}_{m,x} + c_{m,x}^{\text{box2}} + \sum_{y=1}^{x-1} \sum_{p=1}^{m-1} \binom{x}{y} c_{m-p,x-y}^{\text{box2}} \, \widehat{c}_{p,y} \, . \tag{A.4}$$

The first term in the right hand side of the above equation is the contribution from skew FD's that do no contain any reducible components in $\mathcal{B}$, the second term arise solely from terms that fit into a box of size 2. The last terms runs over terms that contain reducible components in $\mathcal{B}$ but do not fit into a box of size 2. The next proposition shows that $\widehat{C}$ is a lower-triangular matrix with the m-th row containing m non-zero terms.

Proposition A.2 $\widehat{c}_{m,x} = 0$ when $x > m$ or equivalently when the density $\rho < 1$.

Proof: Since $\mathcal{D} \subset \mathcal{B}$ and all the irreducible strict skew FD's with density less than unity lie in $\mathcal{D}$, the above Proposition follows from Proposition 2.13. $\qquad\square$

$$
\widehat{C} = \begin{pmatrix}
1 \\
0 & 1 \\
0 & 1 & 3 \\
0 & 1 & 7 & 16 \\
0 & 1 & 11 & 57 & 125 \\
0 & 1 & 18 & 135 & 602 & 1296 \\
0 & 1 & 26 & 293 & 1911 & 7980 & 16807 \\
0 & 1 & 38 & 574 & 5242 & 31860 & 127977 & 262144 \\
0 & 1 & 52 & 1089 & 12972 & 106505 & 619872 & 2411416 & 4782969
\end{pmatrix}
\tag{A.5}
$$

We observe that $\widehat{c}_{m,m} = (m+1)^{m-1}$.

We can carry out a similar refinement for strict FD's that contribute to the A-triangle. One has

$$
\boxed{a_{n,r} = \widehat{f}_{n,r} + \sum_{s=0}^{r-1} \sum_{p=s+1}^{n-1} \binom{r}{s} a_{n-p+1,r-s}^{\text{box2}} \, \widehat{f}_{p,s}}\,,
\tag{A.6}
$$

with $\widehat{f}_{1,0} \equiv 1$ and $\widehat{f}_{n,0} = 0$ for $n > 0$. In order to interpret the first term, it is better to think of $a_{n,r}$ as the number of skew FD's obtained after removing the node at the origin of a strict FD. Then, the second term is the contribution from such skew FD's that do *not* contain reducible components that fit in a box of size two. A second equivalent definition in terms of m is as follows:

$$
a_{m+r+1,r} = \widehat{f}_{m+r+1,r} + a_{m+r+1,r}^{\text{box2}} + \sum_{s=1}^{r-1} \sum_{p=0}^{m} \binom{r}{s} a_{m-p+r-s+1,r-s}^{\text{box2}} \, \widehat{f}_{p+s+1,s} \,. \tag{A.7}
$$

It is easy to see there is a bijective map that relates skew FD's that contribute to $\widehat{c}_{m,x}$ and those that contribute to $\widehat{a}_{m+x+1,x}$. The bijection follows by observing that if σ is a strict skew FD with m nodes and r.d. x, there is a unique FD (with i.d. and r.d. equal to x) obtained by adding the nodes in μ_x. Thus,

$$
\widehat{f}_{m+x+1,x} = \widehat{c}_{m,x} \,.
$$

It is easy to see using Proposition A.2 that for $\widehat{f}_{n,r} = 0$ when $r > n/2$. We define the matrix $\widehat{F} = \left(\widehat{f}_{n,r}\right)$ for $n = 1, 2, \dots$ and $r = 0, 1, 2, \dots$. Further, we observe that $\widehat{f}_{2x+1,x} = c_{x,x} = (x+1)^{x-1}$. Below we reproduce the first eleven rows

of the $\widehat{F}$-matrix. We reproduce the F-matrix alongside for comparison. The first instance where they differ is when $n = 8$ and $r = 3$ – this is precisely where the node $(1,1,1)^T$ that is not in $\mathcal{D}$ but present in $\mathcal{B}$ appears. As we go to higher values of n, an entry in $\widehat{F}$ will be generically smaller than the corresponding entry in F. Further, entries in both matrices will agree when the density is in the range $[1, 4/3)$ – this is because a node present in $\mathcal{B}$ but not in $\mathcal{D}$ first appears at density $4/3$.

$$\widehat{F} = \begin{pmatrix} 1 & & & & \\ 0 & & & & \\ 0 & 1 & & & \\ 0 & 1 & & & \\ 0 & 1 & 3 & & \\ 0 & 1 & 7 & & \\ 0 & 1 & 11 & 16 & \\ 0 & 1 & 18 & 57 & \\ 0 & 1 & 26 & 135 & 125 \\ 0 & 1 & 38 & 293 & 602 \\ 0 & 1 & 52 & 574 & 1911 & 1296 \end{pmatrix} \quad , \quad F = \begin{pmatrix} 1 & & & & \\ 0 & & & & \\ 0 & 1 & & & \\ 0 & 1 & & & \\ 0 & 1 & 3 & & \\ 0 & 1 & 7 & & \\ 0 & 1 & 11 & 16 & \\ 0 & 1 & 18 & 58 & \\ 0 & 1 & 26 & 135 & 125 \\ 0 & 1 & 38 & 293 & 618 \\ 0 & 1 & 52 & 574 & 1927 & 1296 \end{pmatrix} . \tag{A.8}$$

The second row has only vanishing entries. That is because the only strict FD with two nodes fits in a box of size two. So the first non-vanishing contribution appears at $n = 3, r = 1$ if we ignore the $n = 1, r = 0$ term that is more less part of the definition.

We can now revisit the problem of enumerating partitions of n in any dimension. We see that we need to enumerate the first n rows of the $\widehat{F}$ matrix and A^{box2} in order to obtain row n of the A-matrix. However, from Eq. (A.6) we see that it is sufficient to determine only the first $[n/2]$ elements of row n as that completely determines row n of $\widehat{F}$. However, this reduction is accompanied by the need to evaluate A^{box2} which is yet another computation. Hence, we preferred to work with the F-matrix. However, one should be open to using the $\widehat{F}$-matrix if one has an algorithm to directly compute it. Then, the additional effort to compute A^{box2} might be worth it.

A.1 The box transform

Define the following generating function for the A-matrix

$$A(q, t) = \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} a_{m+r+1, r} \frac{q^m t^r}{r!} , \tag{A.9}$$

along with similar definitions for $A^{\mathrm{box2}}(q, t)$ and $\widehat{F}(q, t)$. Then, Eq. (A.6) implies that the generating functions have a simple relation. One has

$$A(q, t) = A^{\mathrm{box2}}(q, t) \times \widehat{F}(q, t) . \tag{A.10}$$

It is due to this property that we refer to Eq. (A.9) as the *box transform*. Similarly, one defines

$$C(q, t) = \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} c_{m,r} \frac{q^m t^r}{r!} , \tag{A.11}$$

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along with similar definitions for $C^{\text{box2}}(q,t)$ and $\widehat{C}(q,t)$. Again, one has

$$C(q,t) = C^{\text{box2}}(q,t) \times \widehat{C}(q,t) \ . \tag{A.12}$$

There is an obvious extension to our considerations by replacing the symmetric box of size two by one of size b. Again, relations of the kind that we considered between FD's that fit in the box and those that don't appear. For instance, one has

$$A(q,t) = A^{\text{box}b}(q,t) \times \widehat{F}(q,t) \ , \tag{A.13}$$

where $\widehat{A}(q,t)$ is the generating function of FD's that don't fit into a box of size b and do not have reducible parts that fit into the box.

References

[1] A. O. L. Atkin, P. Bratley, I. G. Macdonald, and J. K. S. McKay, "Some computations for m-dimensional partitions," *Proc. Cambridge Philos. Soc.* **63** (1967) 1097–1100.

[2] G. E. Andrews, *The theory of partitions*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1998. Reprint of the 1976 original.

[3] P. A. MacMahon, *Combinatory analysis. Vol. I, II (bound in one volume)*. Dover Phoenix Editions. Dover Publications Inc., Mineola, NY, 2004. Reprint of *An introduction to combinatory analysis* (1920) and *Combinatory analysis. Vol. I, II* (1915, 1916).

[4] F. Y. Wu, G. Rollet, H. Y. Huang, J. M. Maillard, C.-K. Hu, and C.-N. Chen, "Directed Compact Lattice Animals, Restricted Partitions of an Integer, and the Infinite-State Potts Model," *Phys. Rev. Lett.* **76** (1996) 173–176.

[5] D. P. Bhatia, M. A. Prasad, and D. Arora, "Asymptotic results for the number of multidimensional partitions of an integer and directed compact lattice animals," *J. Phys. A* **30** no. 7, (1997) 2281–2285.

[6] S. Balakrishnan, S. Govindarajan, and N. S. Prabhakar, "On the asymptotics of higher-dimensional partitions," *J. Phys.* **A45** (2012) 055001, `arXiv:1105.6231 [cond-mat.stat-mech]`.

[7] P. Bratley and J. K. S. McKay, "Algorithm 313: Multi-dimensional partition generator," *Commun. ACM* (1967) 1–1.

[8] D. E. Knuth, "A note on solid partitions," *Math. Comp.* **24** (1970) 955–961.

[9] "The On-line Encyclopedia of Integer Sequences," 2012. published electronically at http://oeis.org.

[10] S. B. Ekhad, "The number of m-Dimensional Partitions of Eleven and Twelve,". published electronically at http://www.math.rutgers.edu/~zeilberg/pj.html.

[11] Wikipedia, "Spanning tree — Wikipedia, The Free Encyclopedia," 2012. published online at http://en.wikipedia.org/wiki/Spanning_tree; accessed 19-March-2012.

[12] P. Flajolet and R. Sedgewick, *Analytic Combinatorics.* Cambridge University Press, 2009.

[13] I. Pak, "Lectures on bijections at IPAM," 2009. available at http://www.math.ucla.edu/~pak/lectures/lectures-IPAM.htm; accessed 19-March-2012.

[14] N. Destainville, R. Mosseri, and F. Bailly, "Configurational entropy of codimension-one tilings and directed membranes," *Journal of Statistical Physics* **87** (1997) 697–754.

n	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$	$d=9$	$d=10$
1	1	1	1	1	1	1	1	1	1	1
2	2	3	4	5	6	7	8	9	10	11
3	3	6	10	15	21	28	36	45	55	66
4	5	13	26	45	71	105	148	201	265	341
5	7	24	59	120	216	357	554	819	1165	1606
6	11	48	140	326	657	1197	2024	3231	4927	7238
7	15	86	307	835	1907	3857	7134	12321	20155	31548
8	22	160	684	2145	5507	12300	24796	46209	80920	134728
9	30	282	1464	5345	15522	38430	84625	170370	319555	565983
10	42	500	3122	13220	43352	118874	285784	621316	1247780	2350183
11	56	859	6500	32068	119140	362670	953430	2240838	4821737	9661465
12	77	1479	13426	76965	323946	1095430	3151332	8011584	18478640	39401792
13	101	2485	27248	181975	869476	3271751	10314257	28395213	70261505	159527302
14	135	4167	54804	425490	2308071	9673993	33457972	99845553	265266530	641733862
15	176	6879	108802	982615	6056581	28310881	107557792	348333411	994606250	2565774277
16	231	11297	214071	2245444	15724170	82033609	342732670	1205925033	3704360354	10198601886
17	297	18334	416849	5077090	40393693	235359901	1082509680	4142850423	13705110470	40305279454
18	385	29601	805124	11371250	102736274	668779076	3389190112	14122999548	50367905030	158376907546
19	490	47330	1541637	25235790	258790004	1882412994	10518508294	47772540002	183864216415	618742851276
20	627	75278	2930329	55536870	645968054	5249817573	32361863632	160336300356	666612686420	2403142436321
21	792	118794	5528733	121250185	1598460229	14510628853	98711666690	533909133114	2400146830007	
22	1002	186475	10362312	262769080	3923114261	39762851345	298546248070	1763901729589	8581152930795	
23	1255	290783	19295226	565502405	9554122089	108058883583	895425789360			

Table 1: $d \leq 10$-dimensional partitions of $n \leq 23$ as determined by direct enumeration using Knuth's algorithm. This provides an independent cross-check of the entries in the first 11 columns of the A-matrix.

$n\backslash r$	0	1	2	3	4	5	6	7	8	9	10	11
1	1											
2	0	1										
3	0	1	1									
4	0	1	3	1								
5	0	1	5	6	1							
6	0	1	9	18	10	1						
7	0	1	13	44	49	15	1					
8	0	1	20	97	172	110	21	1				
9	0	1	28	195	512	550	216	28	1			
10	0	1	40	377	1370	2195	1486	385	36	1		
11	0	1	54	694	3396	7603	7886	3514	638	45	1	
12	0	1	75	1251	7968	23860	35115	24318	7484	999	55	1
13	0	1	99	2185	17910	69580	138155	138075	65997	14667	1495	66
14	0	1	133	3765	38942	191795	495870	677663	471276	161202	26875	2156
15	0	1	174	6354	82338	505640	1657975	2978735	2864408	1424142	360940	46596
16	0	1	229	10607	170265	1285754	5240090	12016809	15354492	10604286	3880561	751696
17	0	1	295	17446	345291	3173220	15821657	45268685	74497870	68869266	34954135	9685709
18	0	1	383	28449	689026	7637795	45999383	161270025	333494972	400292769	272579245	104184949
19	0	1	488	45863	1355253	17996010	129560563	548523528	1397398036	2123894171	1886698315	965585764
20	0	1	625	73400	2632975	41631740	355205608	1794375520	5541288850	10446368715	11819801575	7897875909
21	0	1	790	116421	5058305	94786545	951526108	5678296645	20973892932	48206965521	68073453307	58101011914
22	0	1	1000	183472	9622420	212812255	2498219985	17463026868	76290515426	210725428060	364964576905	390349624764
23	0	1	1253	287021	18139620	471921560	6444739208	52390397612	268136421612	879260678868	1840128105650	2425318710876

Table 2: The first 12 columns and 23 rows of the triangle A. The other 11 columns can be obtained using the ten rows of the C-matrix given below. Thus, one can determine partitions of positive integers ≤ 23 from it.

$m\backslash x$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0	1																			
1	0	1	1																	
2	0	1	3	6	3															
3	0	1	7	20	46	45	15													
4	0	1	11	61	198	480	645	420	105											
5	0	1	18	138	706	2508	6441	10395	9660	4725	945									
6	0	1	26	296	2052	10375	38809	105392	192668	224595	159075	62370	10395							
7	0	1	38	577	5428	36285	184624	713402	2032500	4080195	5580855	5051970	2889810	945945	135135					
8	0	1	52	1092	13226	114220	751639	3854487	15231326	45159822	97613505	150613155	162889650	120270150	57702645	16216200	2027025			
9	0	1	73	1963	30648	332035	2747799	17918432	92357844	370929320	1136808010	2609559315	4427605050	5488733250	4892112225	3047969925	1259458200	310134825	34459425	
10	0	1	97	3471	67868	910729	9268382	74767133	483797592	2498431224	10155656364	31998207087	77214286182	141528086700	195617897475	201837365730	152796603960	82323566325	29876321475	6547290750

Table 3: The second triangle – the first ten rows and nineteen columns of the C-matrix. We have only shown non-zero entries.

$n\backslash x$	0	1	2	3	4	5	6	7	8	9	10	11	12
1	1												
2	0												
3	0	1											
4	0	1											
5	0	1	3										
6	0	1	7										
7	0	1	11	16									
8	0	1	18	58									
9	0	1	26	135	125								
10	0	1	38	293	618								
11	0	1	52	574	1927	1296							
12	0	1	73	1089	5256	8220							
13	0	1	97	1960	12982	32380	16807						
14	0	1	131	3468	30320	107270	131897						
15	0	1	172	5955	67414	319530	633442	262144					
16	0	1	227	10085	145045	888983	2490187	2483096					
17	0	1	293	16759	303101	2346515	8710068	14200018	4782969				
18	0	1	381	27564	619564	5952280	28205459	65151254	53672292				
19	0	1	486	44714	1241845	14617100	86238209	263040064	359302890	100000000			
20	0	1	623	71936	2450043	34962755	252190709	975528302	1899997612	1309707840			
21	0	1	788	114546	4765327	81792100	711409264	3398678150	8749699709	10128660960	2357947691		
22	0	1	998	181102	9157550	187791450	1948153500	11278286646	36739765288	61114773760	35600917115		
23	0	1	1251	284021	17406714	424233500	5203415684	35979941641	144179174632	318163092360	314636749085	61917364224	
24	0	1	1571	442713	32771292	944990470	13605818265	111092074842	536798419714	1499829016296	2148096711540	1066426694784	
25	0	1	1954	685443	61158328	2079070155	34930133300	333670251012	1915118952548	6574308285588	12551603978445	10672681371264	1792160394037

Table 4: The F-matrix as determined using data up to $a_{25,12}$. This determines partitions of all integers ≤ 25. We have only shown non-zero entries.

$m\backslash x$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1															
1	0	1														
2	0	1	3	3												
3	0	1	7	17	28											
4	0	1	11	58	156	295	90									
5	0	1	18	135	640	1913	3786	2310								
6	0	1	26	293	1944	9010	28714	59024	50960	7560						
7	0	1	38	574	5272	33340	154654	509912	1089488	1158192	378000					
8	0	1	52	1089	12998	108465	671389	3123477	10485214	23226165	28428750	14206500	1247400			
9	0	1	73	1960	30336	321130	2551119	15580292	72440912	245511503	561332710	762518790	501491760	102702600		
10	0	1	97	3468	67430	891114	8811002	67908953	409620720	1895816757	6456110604	15166699372	22350118032	17852174340	5864859000	340540200

Table 5: The first eleven rows of the D-matrix. We have only shown non-zero entries.

$z\backslash y$	0	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0	0
2	-1	0	0	0	0	0	0	0	0	0	0
3	-5	6	0	0	0	0	0	0	0	0	0
4	1	74	0	0	0	0	0	0	0	0	0
5	41	252	-40	0	0	0	0	0	0	0	0
6	31	-540	-676	0	0	0	0	0	0	0	0
7	-461	-6470	1180	656	0	0	0	0	0	0	0
8	-895	-4074	69020	10864	0	0	0	0	0	0	0
9	6481	138264	403620	-39016	-9216	0	0	0	0	0	0
10	22591	376872	-961240	-1628984	-191456	0	0	0	0	0	0
11	-107029	-2922930	-21162456	-3687040	463680	195840	0	0	0	0	0
12	$\beta_{12,0}$	-15867390	-40350840	168546560	40336016	7455104	0	0	0	0	0
13	$\beta_{13,0}$	$\beta_{13,1}$	758778240	1656046448	110435472	73922176	-6297600	0	0	0	0
14	$\beta_{14,0}$	$\beta_{14,1}$	$\beta_{14,2}$	-1927766192	-5730022032	-552798336	-382393600	0	0	0	0
15	$\beta_{15,0}$	$\beta_{15,1}$	$\beta_{15,2}$	$\beta_{15,3}$	-44646818832	-10585577760	-7549384960	278906880	0	0	0
16	$\beta_{16,0}$	$\beta_{16,1}$	$\beta_{16,2}$	$\beta_{16,3}$	$\beta_{16,4}$	75450085920	-14753227264	22686050304	0	0	0
17	$\beta_{17,0}$	$\beta_{17,1}$	$\beta_{17,2}$	$\beta_{17,3}$	$\beta_{17,4}$	$\beta_{17,5}$	1603141023616	607200778752	-14729379840	0	0
18	$\beta_{18,0}$	$\beta_{18,1}$	$\beta_{18,2}$	$\beta_{18,3}$	$\beta_{18,4}$	$\beta_{18,5}$	$\beta_{18,6}$	2727351931392	-1449282760704	0	0
19	$\beta_{19,0}$	$\beta_{19,1}$	$\beta_{19,2}$	$\beta_{19,3}$	$\beta_{19,4}$	$\beta_{19,5}$	$\beta_{19,6}$	$\beta_{19,7}$	-47662776674304	873791815680	0
20	$\beta_{20,0}$	$\beta_{20,1}$	$\beta_{20,2}$	$\beta_{20,3}$	$\beta_{20,4}$	$\beta_{20,5}$	$\beta_{20,6}$	$\beta_{20,7}$	$\beta_{20,8}$	101710939668480	0
21	$\beta_{21,0}$	$\beta_{21,1}$	$\beta_{21,2}$	$\beta_{21,3}$	$\beta_{21,4}$	$\beta_{21,5}$	$\beta_{21,6}$	$\beta_{21,7}$	$\beta_{21,8}$	$\beta_{21,9}$	-58358690611200

Table 6: The β-triangle to the extent that we have determined it. The first column is consistent with the Meeussen conjecture. The $\beta_{m+x,x}$ for $x \in [0, m-1]$ completely determine the degree $2m$ polynomial $g_m(r)$. Thus, we have determined all polynomials for $m \leq 11$ albeit assuming the existence of the β-matrix which is conjectural. The polynomials obtained this way agrees with the ones determined by the $C/D/F$ matrices.

m\x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	1																				
1	0	0	1																		
2	0	0	0	3	3																
3	0	0	0	1	16	30	15														
4	0	0	0	1	15	135	330	315	105												
5	0	0	0	0	18	232	1581	4410	5880	3780	945										
6	0	0	0	0	13	355	4000	23709	71078	116550	107100	51975	10395								
7	0	0	0	0	10	450	8075	78725	431460	1353240	2552130	2962575	2079000	810810	135135						
8	0	0	0	0	6	530	14065	204540	1767045	9207945	29811330	62179425	85270185	76621545	43513470	14189175	2027025				
9	0	0	0	0	4	580	22315	456400	5704580	44793784	225211165	746795775	1680747090	2612970360	2812925115	2062160100	983782800	275675400	34459425		
10	0	0	0	0	1	611	33177	918981	15738310	174240318	1268511894	6207749790	20975922462	50107517460	85928953110	106306245045	94166932860	58305347100	23983759800	5892561675	654729075

Table 7: The first eleven rows of the C^{box2}-triangle. We have only shown non-zero entries.

n\r	1	2	3	4	5	6	7	8	9	10	11	12
1	1											
2	0											
3	0											
4	0											
5	0											
6	0											
7	0											
8	0	0	0	1								
9	0											
10	0	0	0	0	12							
11	0	0	0	0	12							
12	0	0	0	0	10	150						
13	0	0	0	0	6	330						
14	0	0	0	0	4	485	2160					
15	0	0	0	0	1	570	7750					
16	0	0	0	0	1	610	17280	36015				
17	0	0	0	0	0	600	30120	185430				
18	0	0	0	0	0	580	45720	574280	688128			
19	0	0	0	0	0	530	63870	1364195	4727520			
20	0	0	0	0	0	470	85325	2751875	19192880	14880348		
21	0	0	0	0	0	387	110625	4994640	59080000	130094748		
22	0	0	0	0	0	310	140322	8480885	152220320	664737850	360000000	
23	0	0	0	0	0	215	174380	13808620	346973284	2557358244	3873139200	
24	0	0	0	0	0	155	212815	21879725	726316080	8167776498	24169328400	9646149645

Table 8: The first 24 rows of the F^{box2}-triangle. We have only shown non-zero entries except for rows which have all zeros where we shown a zero in the first column.