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Counting false entries in truth tables of bracketed formulae connected by m-implication

Volkan Yildiz

ah05146@qmul.ac.uk
or
vo1kan@hotmail.co.uk

Abstract

In this paper we count the number of rows y_n with the value "false" in the truth tables of all bracketed formulae with n distinct variables connected by the binary connective of "modified-implication". We find a recurrence and an asymptotic formulae for y_n. We also determine the parity of y_n.

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1 Introduction

In this paper we study enumerative and asymptotic questions on formulae of propositional logic which are correctly bracketed chains of m-implications, where the letter 'm' stands for 'modified'.

For brevity, we represent truth values of propositional variables and formulae by 1 for "true" and "0" for "false".

For background information on propositional logic the reader can refer to the following books, [6], and [3], or to the introduction page of, [4]. In-fact this paper is an extension of [4]. In [4], we have shown that the following results are true:

Theorem 1.1 *Let f_n be the number of rows with the value "false" in the truth tables of all bracketed formulae with n distinct propositions $p_1, \ldots, p_n$ connected by the binary connective of implication. Then*

$$f_n = \sum_{i=1}^{n-1} (2^i C_i - f_i) f_{n-i}, \quad \textit{with} \;\; f_1 = 1 \tag{1}$$

and for large n, $f_n \sim \left(\frac{3-\sqrt{3}}{6}\right) \frac{2^{3n-2}}{\sqrt{\pi n^3}}$. *Where C_i is the ith Catalan number.*

A number of new enumerative problems arise if we modify the binary connective of implication as in below cases.

Case(i) Use $\rightharpoonup$ instead of $\rightarrow$, where $\rightharpoonup$ defined as follows

$$\phi \rightharpoonup \psi \equiv \phi \rightarrow \neg\psi$$

For any valuation ν,

$$\nu(\phi \rightharpoonup \psi) = \begin{cases} 0 & \text{if } \nu(\phi) = 1 \text{ and } \nu(\psi) = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Case(ii) Use $\leftharpoondown$ instead of $\rightarrow$, where $\leftharpoondown$ defined as follows

$$\phi \leftharpoondown \psi \equiv \neg\phi \rightarrow \psi$$

For any valuation ν,

$$\nu(\phi \leftharpoondown \psi) = \begin{cases} 0 & \text{if } \nu(\phi) = 0 \text{ and } \nu(\psi) = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Case(iii) Use $\rightleftharpoons$ instead of $\rightarrow$, where $\rightleftharpoons$ defined as follows

$$\phi \rightleftharpoons \psi \equiv \neg\phi \rightarrow \neg\psi$$

For any valuation ν,

$$\nu(\phi \rightleftharpoons \psi) = \begin{cases} 0 & \text{if } \nu(\phi) = 0 \text{ and } \nu(\psi) = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Let s_n, h_n be the number of rows with the value "false" in the truth tables of all bracketed formulae with n distinct propositions $p_1, \ldots, p_n$ connected by the binary connective of m-implication, in the case (iii) and (ii), respectively.

1.1 Case(iii)

A row with the value false comes from an expression $\psi \rightleftharpoons \chi$ where $\nu(\psi) = 0$ and $\nu(\chi) = 1$. If ψ contains i variables, then χ contains $n-i$, and the number of choices is given by the summand:

$$s_n = \sum_{i=1}^{n-1} s_i(2^{n-i}C_{n-i} - s_{n-i}), \ \textit{where } s_0 = 0, s_1 = 1. \tag{2}$$

The recurrence relation (2) is equivalent to the recurrence relation (1), so all the results we have in [4], and [8] hold for the case(iii) too.

1.2 Case(ii)

A row with the value false comes from an expression $\psi \leftharpoonup \chi$ where $\nu(\psi) = 0$ and $\nu(\chi) = 0$. If ψ contains i variables, then χ contains $n-i$, and the number of choices is given by the summand:

$$h_n = \sum_{i=1}^{n-1} h_i h_{n-i}, \ \textit{where } h_0 = 0, h_1 = 1. \tag{3}$$

The recurrence relation (3) is very well known; it is the recurrence relation for Catalan numbers.

Corollary 1.2 *Suppose we have all possible well-formed formulae obtained from $p_1 \leftharpoonup p_2 \leftharpoonup \ldots \leftharpoonup p_n$ by inserting brackets, where $p_1, \ldots, p_n$ are distinct propositions. Then each formula defines the same truth table.*

Example 1.3 *Here are the truth tables, (merged into one), for the bracketed m-implications, in $n = 3$ variables.*

p_1	p_2	p_3	$p_1 \leftharpoonup (p_2 \leftharpoonup p_3)$	$(p_1 \leftharpoonup p_2) \leftharpoonup p_3$
1	1	1	1	1
1	1	0	1	1
1	0	1	1	1
1	0	0	1	1
0	1	1	1	1
0	1	0	1	1
0	0	1	1	1
0	0	0	0	0

1.3 Case(i)

We are interested in *bracketed m-implications, case(i)*, which are formulae obtained from $p_1 \rightharpoonup p_2 \rightharpoonup \ldots \rightharpoonup p_n$ by inserting brackets so that the result is well-formed, where $p_1, \ldots, p_n$ are distinct propositions.

Proposition 1.4 *Let y_n be the number of rows with the value "false" in the truth tables of all brackted m-implications, case(i), with n distinct variables. Then*

$$y_n = \sum_{i=1}^{n-1} \Big((2^i C_i - y_i)(2^{n-i} C_{n-i} - y_{n-i}) \Big), \text{ with } y_0 = 0, \ y_1 = 1. \tag{4}$$

Proof A row with the value false comes from an expression $\phi \rightharpoonup \psi$, where $\nu(\phi) = 1$ and $\nu(\psi) = 1$. If ϕ contains i variables, then ψ contains $n - i$ variables, and the number of choices is given by the summand in the proposition. $\star$

Example 1.5

$$y_1 = 1, y_2 = (2^1 C_1 - y_1)(2^1 C_1 - y_1) = 1$$

and

$$y_3 = (2^1 C_1 - y_1)(2^2 C_2 - y_2) + (2^2 C_2 - y_2)(2^1 C_1 - y_1) = 3 + 3 = 6.$$

Example 1.6 *Here are the truth tables, (merged into one), for the two bracketed m-implications, case(i), in $n = 3$ variables. Where the corresponding rows with the value false are in blue:*

p_1	p_2	p_3	$p_1 \rightharpoonup (p_2 \rightharpoonup p_3)$	$(p_1 \rightharpoonup p_2) \rightharpoonup p_3$
1	1	1	1	1
1	1	0	1	0
1	0	1	0	0
1	0	0	1	0
0	1	1	0	1
0	1	0	1	1
0	0	1	0	1
0	0	0	1	1

which coincides with the result we had from Example 1.5.

Using Proposition 1.4, it is straightforward to calculate the values of y_n for small n. The first 22 values are

$$\begin{aligned}\{y_n\}_{n\geq 1} \;=\;& 1, 1, 6, 29, 162, 978, 6156, 40061, 267338, 1819238,\\ & 12576692, 88079378, 623581332, 4455663876, 32090099352,\\ & 232711721757, 1697799727066, 12452943237342, 91774314536100,\\ & 679234371006982, 5046438870909244, 37623611703611452, \ldots\end{aligned}$$

2 Generating Function

Recall from [4], that the number of bracketings of a product of n terms is the Catalan number with the generating function

$$C_n = \frac{1}{n}\binom{2n-2}{n-1}, \;\; \textit{with } C_0 = 0, \;\; \sum_{n\geq 1} C_n x^n = (1-\sqrt{1-4x})/2$$

respectively (see also [2, page 61]).

Let g_n be the total number of rows in all truth tables for bracketed m-implications, case(i), with n distinct variables. It is clear that $g_n = 2^n C_n$, with $g_0 = 0$. Let $Y(x)$ and $G(x)$ be the generating functions for y_n, and g_n, respectively. That is, $Y(x) = \sum_{n\geq 1} y_n x^n$, and $G(x) = \sum_{n\geq 1} g_n x^n$.

Since,

$$y_n = \sum_{i=1}^{n-1}\Big((2^i C_i - y_i)(2^{n-i}C_{n-i} - y_{n-i})\Big), \quad \textit{where } \; y_0 = 0, \; y_1 = 1.$$

Then,

$$\begin{aligned}\sum_{n\geq 1} y_n x^n \;=\;& x + \sum_{n\geq 1}\sum_{i=1}^{n-1} 2^i C_i 2^{n-i} C_{n-i} x^n - \sum_{n\geq 1}\sum_{i=1}^{n-1} 2^i C_i y_{n-i} x^n - \\ & \sum_{n\geq 1}\sum_{i=1}^{n-1} y_i 2^i C_{n-i} x^{n-i} + \sum_{n\geq 1}\sum_{i=1}^{n-1} y_i y_{n-i} x^n\end{aligned}$$

Now it is straightforward to get the following result:

$$Y(x) = x + (G(x) - Y(x))^2 \tag{5}$$

where $G(x)$ can be obtained from the generating function of C_n by replacing x by $2x$: that is,

$$G(x) = (1 - \sqrt{1-8x})/2. \tag{6}$$

Substituting (6) into (5) gives the following quadratic equation:

$$2Y(x)^2 + 2Y(x)(\sqrt{1-8x} - 2) + (1 - \sqrt{1-8x} - 2x) = 0 \tag{7}$$

Solving equation (7) gives the following proposition:

Proposition 2.1 *The generating function for the sequence $\{y_n\}_{n\geq 1}$ is given by*

$$Y(x) = \frac{2 - \sqrt{1-8x} - \sqrt{3 - 4x - 2\sqrt{1-8x}}}{2}.$$

(As with the Catalan numbers, the choice of sign in the square root is made to ensure that $Y(0) = 0$.) With the help of Maple we can obtain the first 22 terms of the above series, and hence give the first 22 values of y_n; these agree with the values found from the recurrence relation.

3 Asymptotic Analysis

In this section we want to get an asymptotic formula for the coefficients of the generating function $Y(x)$ from Proposition 2.1. We use the following result [1, page 389]:

Proposition 3.1 *Let a_n be a sequence whose terms are positive for sufficiently large n. Suppose that $A(x) = \sum_{n\geq 0} a_n x^n$ converges for some value of $x > 0$. Let $f(x) = (-\ln(1 - x/r))^b (1 - x/r)^c$, where c is not a positive integer, and we do not have $b = 0$ and $c = 0$. Suppose that $A(x)$ and $f(x)$ each have a singularity at $x = r$ and that $A(x)$ has no singularities in the interval $[-r, r)$. Suppose further that $\lim_{x\to r} \frac{A(x)}{f(x)}$ exists and has nonzero value γ. Then*

$$a_n \sim \begin{cases} \gamma\binom{n-c-1}{n}(\ln n)^b r^{-n}, & \text{if } c \neq 0, \\ \frac{\gamma b (\ln n)^{b-1}}{n}, & \text{if } c = 0. \end{cases}$$

Note 3.2 *We also have*

$$\binom{n-c-1}{n} \sim \frac{n^{-c-1}}{\Gamma(-c)},$$

where the standard gamma-function

$$\Gamma(x) = \int_0^\infty t^{x-1} \mathrm{e}^{-t}\, \mathrm{d}t, \quad \textit{with} \;\; \Gamma(x+1) = x\Gamma(x),\; \Gamma(1/2) = \sqrt{\pi}.$$

It follows that $\Gamma(-1/2) = -\sqrt{\pi}/2$.

Recall that $G(x) = (1 - \sqrt{1-8x})/2$, therefore

$$Y(x) = \frac{(1+2G(x)) - \sqrt{(1+4G(x)) - 4x}}{2}.$$

As in [4], before studying $Y(x)$, we first study $G(x)$. This $G(x)$ could easily be studied by using the explicit formula for its coefficients, which is $2^n\binom{2n-2}{n-1}/n$. But our aim is to understand how to handle the square root singularity. A square root singularity occurs while attempting to raise zero to a power which is not a positive integer. Clearly the square root, $\sqrt{1-8x}$, has a singularity at $1/8$. Therefore by Proposition 3.1, $r = 1/8$. We have $G(1/8) = 1/2$, so we would not be able to divide $G(x)$ by a suitable $f(x)$ as required in Proposition 3.1. To create a function which vanishes at $\frac{1}{8}$, we simply look at $A(x) = G(x) - 1/2$ instead. That is, let

$$f(x) = (1 - x/r)^{1/2} = (1-8x)^{1/2}.$$

Then

$$\gamma = \lim_{x \to 1/8} \frac{A(x)}{\sqrt{1-8x}} = -\frac{1}{2}.$$

Now by using Proposition 3.1 and Note 3.2,

$$g_n \sim -\frac{1}{2}\binom{n-\frac{3}{2}}{n}\left(\frac{1}{8}\right)^{-n} \sim -\frac{1}{2}\,\frac{8^n n^{-3/2}}{\Gamma(-1/2)} = \frac{2^{3n-2}}{\sqrt{\pi n^3}}.$$

We are now ready to tackle $Y(x)$, and state the main theorem of the paper.

Theorem 3.3 *Let y_n be number of rows with the value false in the truth tables of all the bracketed m-implications, case(i), with n distinct variables. Then*

$$y_n \sim \left(\frac{10-2\sqrt{10}}{10}\right)\frac{2^{3n-2}}{\sqrt{\pi n^3}}.$$

Proof Recall that

$$Y(x) = \frac{2-\sqrt{1-8x}-\sqrt{3-4x-2\sqrt{1-8x}}}{2}.$$

We find that $r=\frac{1}{8}$, and $f(x)=\sqrt{1-8x}$. Since $Y(1/8) = (2\sqrt{2}-\sqrt{5})/2\sqrt{2} \neq 0$, we need a function which vanishes at $Y(1/8)$, thus we let $A(x) = Y(x) - Y(1/8)$.

$$\lim_{x\to 1/8}\frac{A(x)}{f(x)} = \lim_{x\to 1/8}\frac{-\sqrt{2}\sqrt{1-8x}-\sqrt{2}\sqrt{3-4x-2\sqrt{1-8x}}+\sqrt{5}}{2\sqrt{2}\sqrt{1-8x}}.$$

Let $v=\sqrt{1-8x}$. Then

$$\begin{aligned}\gamma &= \lim_{v\to 0}\frac{-\sqrt{2}v-\sqrt{v^2-4v+5}+\sqrt{5}}{2\sqrt{2}v} = \lim_{v\to 0}\frac{-\sqrt{2}-\frac{1}{2}(2v-4)(v^2-4v+5)^{\frac{-1}{2}}}{2\sqrt{2}}\\ &= \frac{-\sqrt{2}+\frac{2}{\sqrt{5}}}{2\sqrt{2}} = -\frac{10-2\sqrt{10}}{20},\end{aligned}$$

where we have used l'Hôpital's Rule in the penultimate line.

Finally,

$$y_n \sim -\frac{10-2\sqrt{10}}{20}\binom{n-\frac{3}{2}}{n}\left(\frac{1}{8}\right)^{-n} \sim \left(\frac{10-2\sqrt{10}}{10}\right)\frac{2^{3n-2}}{\sqrt{\pi n^3}},$$

and the proof is finished. ⋆

The importance of the constant $\frac{10-2\sqrt{10}}{10} = 0.367544468$ lies in the following fact:

Corollary 3.4 *Let g_n be the total number of rows in all truth tables for bracketed m-implications, case(i), with n distinct variables, and y_n the number of rows with the value "false". Then $\lim_{n\to\infty} y_n/g_n = \frac{10-2\sqrt{10}}{10}$.*

The table below illustrates the convergence.

n	y_n	g_n	y_n/g_n
1	1	2	0.5
2	1	4	0.25
3	6	16	0.25
4	29	80	0.3625
5	162	448	0.36160714286
6	978	2688	0.36383928571
7	6156	16896	0.36434659091
8	40061	109824	0.36477454837
9	267338	732160	0.36513603584
10	1819238	4978688	0.36540510271
100	–	–	0.36735248210

Corollary 3.5 *Let*

$$P(y_n) = \frac{y_n}{g_n} \quad \textit{and} \quad P(f_n) = \frac{f_n}{g_n}$$

then we have the following inequality

$$P(y_n) \geq P(f_n).$$

Where f_n is defined in Theorem 1.1.

Corollary 3.6 *Let d_n be the number of rows with the value "true" in the truth tables of all bracketed formulae with n distinct variables connected by the binary connective of m-implication, case(i). Then*

$$d_n = g_n - y_n, \ \textit{with}\ t_0 = 0,$$

and for large n,

$$d_n \sim \left(\sqrt{\frac{2}{5}}\right) \frac{2^{3n-2}}{\sqrt{\pi n^3}}.$$

Using this Corollary 3.6, it is straightforward to calculate the values of d_n. The table below illustrates this up to $n = 10$.

n	0	1	2	3	4	5	6	7	8	9	10
d_n	0	1	3	10	51	286	1710	10740	69763	464822	3159450

4 Parity

For brevity, we represent the set of even counting numbers by the capital letter E, the set of odd counting numbers by the capital letter O, and the set of natural numbers, $\{1, 2, 3, 4, ...\}$, by $\mathbb{N}$.

We begin by determining the parity of Catalan number C_n, which has the following recurrence relation

$$C_n = \sum_{i=1}^{n-1} C_i C_{n-1}, \quad \textit{with } C_0 = 0, C_1 = 1. \tag{8}$$

From the Segner's recurrence relation, C_n can be expressed as a piecewise function, with respect to the parity of n, (see [7, page 329]).

$$C_n = \begin{cases} 2(C_1 C_{n-1} + C_2 C_{n-2} + \ldots + C_{\frac{n-1}{2}} C_{\frac{n+1}{2}}) & \text{if } n \in O, \\ 2(C_1 C_{n-1} + C_2 C_{n-2} + \ldots + C_{\frac{n-2}{2}} C_{\frac{n+2}{2}}) + C^2_{\frac{n}{2}} & \text{if } n \in E. \end{cases}$$

Lemma 4.1 (Parity of C_n) *[8]*

$$C_n \in O \Longleftrightarrow n = 2^i, \textit{ where } i \in \mathbb{N}.$$

Proof

$$\textit{For } n \geq 2, \; C_n \in O \Longleftrightarrow C^2_{\frac{n}{2}} \in O \Longleftrightarrow C_{\frac{n}{2}} \in O \Longleftrightarrow n = 2^i \;\; \forall i \in \mathbb{N}.$$

Note that $C_1 = 1 \in O$. ⋆

By using Proposition 1.4, we get the following triangular table. Where the left hand side column represents the sum of the corresponding row.

y_2:					1					
y_3:				3		3				
y_4:			10		9		10			
y_5:		51		30		30		51		
y_6:	286		153		100		153		286	

Theorem 4.2 (Parity of y_n) *The sequence $\{y_n\}_{n\geq 1}$ preserves the parity of C_n.*

Proof If an additive partition of y_n, (which is determined by the recurrence relation (4)), is odd, then it comes as a pair; i.e.

$$(2^i C_i - f_i)(2^{n-i} C_{n-i} - y_{n-i}) \in O \Longleftrightarrow y_i, y_{n-i}.$$

Hence, $\Big((2^i C_i - y_i)(2^{n-i} C_{n-i} - y_{n-i}) + (2^{n-i} C_{n-i} - y_{n-i})(2^i C_i - y_i)\Big) \in E$.

Thus, y_n can be expressed as a piecewise function depending on the parity of n:

$$y_n = \begin{cases} 2\sum_{i=1}^{\frac{n-1}{2}} ((2^i C_i - y_i)(2^{n-i} C_{n-i} - y_{n-i})) & \text{if } n \in O, \\ \Big(2\sum_{i=1}^{\frac{n-2}{2}} ((2^i C_i - y_i)(2^{n-i} C_{n-i} - y_{n-i}))\Big) + (2^{\frac{n}{2}} C_{\frac{n}{2}} - y_{\frac{n}{2}})^2 & \text{if } n \in E. \end{cases}$$

Finally,

$$y_n \in O \Longleftrightarrow (2^{\frac{n}{2}} C_{\frac{n}{2}} - y_{\frac{n}{2}})^2 \in O \Longleftrightarrow y_{\frac{n}{2}} \in O \Longleftrightarrow n = 2^i, \ \ \forall i \in \mathbb{N}.$$

Note that $y_1 = 1 \in O$. $\star$

Proposition 4.3 (Parity of d_n) *The sequence $\{d_n\}_{n\geq 1}$ preserves the parity of C_n.*

Proof Since

$$d_n = g_n - y_n = 2^n C_n - y_n, \ \textit{with } n \geq 1$$

The sequence $\{g_n\}_{n\geq 1}$ is always even, and the sequence $\{y_n\}_{n\geq 1}$ preserves the parity of C_n by Theorem 4.2. Therefore the sequence $\{d_n\}_{n\geq 1}$ preserves the parity of C_n. $\star$

5 A fruitful tree

We begin by recalling following definitions:

Definition 5.1 *[8], The nth* **Catalan tree***, A_n, is a combinatorical object, characterized by one root, $(n-1)$ main-branches, and C_n sub-branches. Where each main-branch gives rise to a number of sub-branches, and the number of these sub-branches is determined by the additive partition of the corresponding Catalan number, as determined by the recurrence relation (8).*

Definition 5.2 *[8], The Catalan tree A_n is* **fruitful** *iff each sub-branch of A_n has fruits. We denote this new tree by $A_n(\mu_i)$, where $\{\mu_i\}_{i\geq 1}$ is the corresponding fruit sequence.*

Example 5.3 *Let $\{y_n\}_{n\geq 1}$ be the corresponding fruit sequence for the Catalan tree A_n. Then $A_n(y_n)$ has the following symbolic representation,*

$$\begin{gathered}((2^1C_1 - f_1)(2^{n-1}C_{n-1} - y_{n-1}), \ldots, (2^{n-1}C_{n-1} - y_{n-1})(2^1C_1 - f_1))\\ (C_1C_{n-1}, C_2C_{n-2}, \ldots, C_{n-2}C_2, C_{n-1}C_1)\\ (1, 1, \ldots, 1, 1)\\ (1).\end{gathered}$$

Example 5.4 *Let $\{d_n\}_{n\geq 1}$ be the corresponding fruit sequence for the Catalan tree A_n. Then $A_n(d_n)$ has the following symbolic representation,*

$$\begin{gathered}((2^n - (2^1C_1 - f_1)(2^{n-1}C_{n-1} - y_{n-1})), \ldots, (2^n - (2^{n-1}C_{n-1} - y_{n-1})(2^1C_1 - f_1)))\\ (C_1C_{n-1}, C_2C_{n-2}, \ldots, C_{n-2}C_2, C_{n-1}C_1)\\ (1, 1, \ldots, 1, 1)\\ (1).\end{gathered}$$

Proposition 5.5 *For $n > 1$, let $a_n(y_n)$ and $a_n(d_n)$ be the total number of components of the fruitful trees $A_n(y_n)$ and $A_n(d_n)$ respectively. Then*

$$a_n(y_n) = y_n + C_n + n, \quad \text{and} \quad a_n(d_n) = d_n + C_n + n.$$

Using Proposition 5.5, it is straightforward to calculate the values of $a_n(y_n)$, and $a_n(d_n)$. The table below illustrates this up to $n = 10$.

n	0	1	2	3	4	5	6	7	8	9	10
$a_n(y_n)$	0	2	4	11	38	181	1026	6295	40498	268777	1824110
$a_n(d_n)$	0	2	6	15	60	305	1758	10879	70200	466261	3164322

Corollary 5.6 *For $n > 1$, $a_n(y_n)$, and $a_n(d_n)$ are odd iff $n \in O$.*

Proof Since,

$$a_n = (C_n + n) \in O \Longleftrightarrow n \in O \text{ or } n = 2^i, \text{ and } y_n, d_n \in O \Longleftrightarrow n = 2^i.$$

Therefore, $a_n(y_n), a_n(d_n) \in O \Longleftrightarrow n \in O$.

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Onlar ki kurtulamaz ikiyüzlülükten
Canı ayırmaya kalkarlar bedenden;
Horoz gibi tepemde testere olsa
Aklımın kafasını keser atarım ben.

Ö. Hayyam