Abstract

We prove a formal power series identity, relating the arithmetic sum-of-divisors function to commuting triples of permutations. This establishes a conjecture of Franklin T. Adams-Watters.

Submitted exclusively to the London Mathematical Society
doi:10.1112/0000/000000

A formal identity involving commuting triples of permutations

John R. Britnell

The object of this note is to establish the following formal identity:

$$\prod_{j=1}^{\infty} \left(1 - u^j\right)^{-\sigma(j)} = \sum_{n=0}^{\infty} \frac{T(n)}{n!} u^n, \tag{1}$$

where σ is the arithmetic sum-of-divisors function, and $T(n)$ is the number of triples of pairwise-commuting elements of the symmetric group S_n. (Here S_0 is the trivial group.) This is a surprising fact, as there seems no obvious reason for any connection between the function σ and commuting permutations.

The power series expansion of the left-hand side of this identity has coefficients which are listed on the Online Encyclopedia of Integer Sequences (OEIS) [**3**] as sequence A061256. The coefficients on the right-hand side are listed as sequence A079860. The identity of the two sequences has been stated conjecturally on OEIS. This conjecture (from 2006) is due to Franklin T. Adams-Watters; he informs me that it was based empirically on the numerical evidence.

For a finite group G, we shall write $k(G)$ for the number of conjugacy classes of G. The following simple fact seems first to have been stated by Erdős and Turán [**1**].

Lemma 1. *The number of pairs of commuting elements of G is $|G|\, k(G)$.*

Let $g \in G$. It follows from Lemma 1 that the number of commuting triples of G whose first element is g, is given by $|\operatorname{Cent}_G(g)|\, k(\operatorname{Cent}_G(g))$. So if $T(G)$ is the total number of commuting triples, then

$$\frac{T(G)}{|G|} = \sum_{g \in G} \frac{|\operatorname{Cent}_G(g))|}{|G|} k(\operatorname{Cent}_G(g)) = \sum_{i=1}^{r} k(\operatorname{Cent}_G(g_i)), \tag{2}$$

where $\{g_1, \ldots, g_r\}$ is a set of conjugacy class representatives for G.

In the case that G is the symmetric group S_n, the conjugacy classes are parameterized by partitions of n, whose parts correspond to cycle lengths. Let $g \in S_n$ have m_t cycles of length t

2000 *Mathematics Subject Classification* 05A15; Secondary 05A05, 11A25, 20B30.

for all t. Then the centralizer of g in S_n is given (up to isomorphism) by

$$\mathrm{Cent}_{S_n}(g) \cong \prod_{t=1}^{n} W(t, m_t),$$

where $W(t,m)$ is the wreath product $\mathbf{Z}_t \wr S_m$. (Here $\mathbf{Z}_t$ is used as a shorthand for $\mathbf{Z}/t\mathbf{Z}$, the integers modulo t.) It follows that

$$k(\mathrm{Cent}_{S_n}(g)) = \prod_{t=1}^{n} k(W(t, m_t)). \tag{3}$$

We may regard an element of $W(t,m)$ as a pair (A,e), where $A \in {\mathbf{Z}_t}^m$ and $e \in S_m$. There is a natural action of S_m on the coordinates of ${\mathbf{Z}_t}^m$ given by $(B^e)_i = B_{ie^{-1}}$. The group multiplication $*$ in $W(t,m)$ is defined by

$$(A,e) * (B,f) = (A + B^e, ef).$$

Conjugacy in groups of the form $H \wr S_m$ is described in [**2**, Section 4.2]; the case that $H = \mathbf{Z}_t$ is relatively straightforward. Let (A,e) be an element of $W(t,m)$, where $A = (a_1, \ldots, a_m)$. Let c be a cycle of the permutation e, and let $\mathrm{supp}(c)$ be the support of c (i.e. the elements of $\{1, \ldots, m\}$ moved by c). We shall write $|c|$ for $|\mathrm{supp}(c)|$, the length of the cycle. Define the *cycle sum* $A[c] \in \mathbf{Z}_t$ by

$$A[c] = \sum_{i \in \mathrm{supp}(c)} a_i.$$

The *cycle sum invariant* of (A,e) corresponding to the cycle c is defined to be the pair $(A[c], |c|)$. The element (A,e) has one such invariant for each cycle of e.

LEMMA 2. *Two elements (A,e) and (B,f) of $W(t,m)$ are conjugate in $W(t,m)$ if and only if they have the same cycle sum invariants—that is, if and only if there is a bijection τ between the cycles of e and the cycles of f, such that for any cycle c of e we have $(A[c], |c|) = (B[c\tau], |c\tau|)$.*

Proof. See [**2**, Theorem 4.2.8], of which this is a particular case. □

Let (A,e) be an element of $W(t,m)$. For each $z \in \mathbf{Z}_t$ we define λ_z to be the partition such that the multiplicity of ℓ as a part of λ_z is equal to the multiplicity of (z, ℓ) as a cycle sum invariant of (A,e). Lemma 2 tells us that the partitions λ_z for $z \in \mathbf{Z}_t$ determine the conjugacy class of (A,e) in $W(t,m)$. Conversely, a collection of t arbitrary partitions $\{\lambda_z \mid z \in \mathbf{Z}_t\}$ determines a conjugacy class of $W(t,m)$ if and only if the total sum of the sizes of the partitions λ_z is equal to m.

Let $p(d)$ denote the number of partitions of d, and let $P(u)$ be the power series

$$P(u) = \sum_{d=0}^{\infty} p(d) u^d.$$

Consider the formal series

$$Q(u) = \prod_{t=1}^{\infty} P(u^t)^t.$$

From the discussion above, it is easily seen that each monomial term of degree tm in the expansion of $P(u^t)^t$ corresponds to a conjugacy class of $W(t,m)$, and that we therefore have

$$P(u^t)^t = \sum_{m=0}^{\infty} k(W(t,m))u^{tm}.$$

Now any single term in the expansion of $Q(u)$ corresponds to a choice, firstly of parameters m_t such that $\sum_t tm_t$ is finite, and secondly of a conjugacy class of $W(t,m_t)$ for each t. It follows from (3) that each term of degree n in this expansion corresponds to a conjugacy class of $\mathrm{Cent}_{S_n}(g)$, where g is an element of S_n with m_t cycles of length t. Now by (2) we have the formal identity

$$Q(u) = \sum_{n=0}^{\infty} \frac{T(n)}{n!} u^n.$$

Thus $Q(u)$ is equal to the right-hand side of (1), and it remains only to show that $Q(u)$ is also equal to the left-hand side.

We use the Eulerian expansion of $P(u)$,

$$P(u) = \prod_{s=1}^{\infty} (1-u^s)^{-1}.$$

From this it follows that

$$Q(u) = \prod_{t=1}^{\infty}\prod_{s=1}^{\infty}(1-u^{st})^{-t} = \prod_{j=1}^{\infty}\prod_{t\mid j}(1-u^j)^{-t} = \prod_{j=1}^{\infty}(1-u^j)^{-\sigma(j)},$$

as required.

Finally, I am indebted to Mark Wildon for the observation that both sides of (1) are convergent in the open unit disc $|u| < 1$, and that they therefore represent a complex function which is analytic in this disc. This can be seen by expressing the formal logarithm of the left-hand side of (1) as

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{\sigma(j)}{k}u^{jk} = \sum_{d=1}^{\infty}\left(\sum_{a\mid d}\frac{a\sigma(a)}{d}\right)u^d,$$

which has radius of convergence 1, since clearly

$$\sum_{a\mid d} a\sigma(a) < d^4.$$

Thus the left-hand side of (1) represents an analytic function on the disc $|u| < 1$, and it follows that the right-hand side is the Taylor series of that function. An immediate consequence of this observation is that the growth of $T(n)/n!$ is subexponential; I do not know of an easy combinatorial proof of this fact.

References

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2. Gordon James and Adalbert Kerber, The representation theory of the symmetric group, Encyclopedia of Mathematics and its Applications 16, Addison-Wesley, 1981.

3. The Online Encyclopedia of Integer Sequences, `http://oeis.org/`.

Heilbronn Institute for Mathematical Research
School of Mathematics
University of Bristol
University Walk
Bristol, BS8 1TW
United Kingdom