NEW DEVELOPMENTS OF AN OLD IDENTITY

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Abstract. We give a direct combinatorial proof of a famous identity,

$$\sum_{i+j=n} \binom{2i}{i}\binom{2j}{j} \;=\; 4^n \tag{1}$$

by actually counting pairs of ℓ-subsets of 2ℓ-sets. Then we discuss two different generalizations of the identity, and end the paper by presenting in explicit form the ordinary generating function of the sequence $\left(\binom{2n+\ell}{n}\right)_{n\in\mathbb{N}_0}$, where $\ell \in \mathbb{R}$.

1. Introduction and definitions

In [10, Exercise 1.4(a)], Richard Stanley asks for a proof of the identity:

$$(1-4x)^{-1/2} = \sum_{n\geq 0} \binom{2n}{n} x^n$$

from which (1) above follows immediately for all nonnegative integers i, j and n [10, Exercise 1.2(c)]. In [10, Exercise 1.4(b)], it is asked for the value of $\sum_{n\geq 0}\binom{2n-1}{n}x^n$. The answer is one half plus one half of the previous sum, and thus

$$\sum_{\substack{i\geq 1\\ i+j=n}} \binom{2i-1}{i}\binom{2j-1}{j} \;=\; 4^{n-1} \quad (n>1). \tag{2}$$

By replacing i with $i+1$ and n with $n+1$ in (2), we may see that this identity, as well as (1), is a special case of the new identity that we will prove — in the notation of [10], where, in particular, for every real ℓ and every nonnegative integer i, $\left(\ell\right)_i = \ell(\ell-1)\cdots(\ell-i+1)$ (the falling factorial) and $\binom{\ell}{i} = \frac{(\ell)_i}{i!}$,

$$\sum_{i+j=n} \binom{2i-\ell}{i}\binom{2j+\ell}{j} \;=\; 4^n. \tag{3}$$

Our proof of the new identity does not use (1). On the contrary, it proves the initial identity by using the inclusion-exclusion principle for positive values of ℓ. Thus, in particular, we give a new solution to the problem of finding a combinatorial proof of (1), that was posed and solved a long time ago, since, according to Paul Erdős, "Hungarian mathematicians tackled it in the thirties: P. Veress proposing and G. Hajos solving it" [12]. This is told by Marta Sved, who, after having posed the problem of providing a combinatorial proof of the identity in a previous article [11], describes a number of answers received meanwhile as follows: "All solutions are based, with some variations, on the count of lattice paths, or equivalently (1, 0) sequences". Although this is neither the case of the cited articles [1, 3], for example, nor the case of the short proof of Lemma 3.2,

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below, a combinatorial proof is still missing where $\binom{2i}{i}$ really stands for the number of i-subsets of a $(2i)$-set.

We give here this proof, based on the following idea: suppose we associate to a given summand on the left-hand side of (1) the pair (A, B), where $A \subseteq [2i]$, $|A| = i$, $B \subseteq [2j]$ and $|B| = j$, being $[\ell] = \{1, 2, \ldots, \ell\}$ for a nonnegative integer ℓ, as usual. We view the two intervals $[2i]$ and $[2j]$ as the sets of slots of two rectangles, of size $2 \times i$ and $2 \times j$, respectively, ordered row by row and joined in a $(2 \times n)$-rectangle R, the first on the left-hand side and the second on the right-hand side. We color the elements of A with one color and the elements of B with another color. Hence, for example, we represent $\big(\{2,3,4,8,9\}, \{2,3,4,7,8,10\}\big)$ by

$$R \;=\; \begin{array}{|c|c|c|c|c||c|c|c|c|c|c|}\hline 1 & \bigcirc & \bigcirc & \bigcirc & 5 & 1 & \blacksquare & \blacksquare & \blacksquare & 5 & 6 \\ \hline 6 & 7 & \bigcirc & \bigcirc & 10 & \blacksquare & \blacksquare & 9 & \blacksquare & 11 & 12 \\ \hline \end{array}\,.$$

Definition 1.1. We call *configuration* (or *n-configuration*) to a $(2 \times n)$-rectangle where exactly n of the $2n$ slots are painted with one of two colors, with the restriction that columns with both colors do not exist. The pair (i, j) for a $(i + j)$-configuration R with i slots colored with color one and j slots colored with color two (or i slots filled with $\bigcirc$ and j slots filled with $\blacksquare$) is the *type* of R. The columns with no colored slots are *empty columns* and the columns with two colored slots *towers*. Both are called *even columns* and the columns with exactly one colored slot are called *odd*. A type (i, j) configuration is *ordered* if the i slots with color one belong to the leftmost $(2 \times i)$-subrectangle and the j slots of color two belong to the complementary $(2 \times j)$-subrectangle. Finally, the set of ordered n-configurations is denoted $\mathcal{O}_n$ and the set of n-configurations without towers is denoted $\mathcal{T}_n$.

Note that the ordered configurations are exactly the configurations that represent the pairs (A, B) as defined above. By definition, the number of towers and the number of empty columns in each of the two original subrectangles are equal. In the example above, the type is $(5, 6)$, the empty columns have *order* 1, 5, 10 and 11 and the towers 3, 4, 7 and 9.

Note that 4^n is the total number of n-configurations *without* towers. Hence, what (1) says is that the number of ordered configurations equals the number of configurations without towers, and in fact we define (recursively) a bijection φ between these two sets that leaves the ordered configurations without towers invariant. More precisely, if R is an ordered configuration with k towers then $\varphi(R)$ is a configuration where exactly in k cases a column of color two is followed by a column of color one. The algorithm beneath the proof was implemented in *Mathematica* and can be obtained in [4]. For example, for the configuration R above,

$$\varphi(R) \;=\; \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline \blacksquare & & \bigcirc & & & & \blacksquare & & \bigcirc & & \bigcirc \\ \hline & \blacksquare & & \blacksquare & \bigcirc & \blacksquare & & \bigcirc & & \blacksquare & \\ \hline \end{array}\,.$$

After this new proof, in Section 3 we prove (3) and another generalization of (1): given integers $n \geq 0$ and $t > 0$, define

$$S_t(n) = \sum_{i_1 + \cdots + i_t = n} \binom{2i_1}{i_1}\binom{2i_2}{i_2} \cdots \binom{2i_t}{i_t},$$

where $i_1, \ldots, i_n$ are any nonnegative integers (note that (1) states that $S_2(n) = 4^n$). Only very recently [3], Guisong Chang and Chen Xu showed, with a probabilistic proof, that $S_t(n)$ depends only on n and t. We give here a combinatorial proof of this fact and obtain a generalization that also includes (3) as a special case. Finally, in Section 4 we obtain explicitly the generating functions of the sequences involved in these identities.

2. New proof of the main identity

Given an n-configurations R with k towers, let R' be $(2k)$-configuration obtained from R by removing all the odd columns. For a $(2k)$-configuration S without odd columns, if we delete one of the two equal rows of S and then rearrange the remaining $2k$ slots by placing, for every $1 \leq i \leq k$, slot $2i$ under slot $2i-1$, we obtain a k-rectangle $S_{\downarrow}$ called the *compression of* S. In the opposite direction, given an k-configuration T, by doubling each slot of T column by column, downwards and from left to right, we obtain the *expansion* of T, denoted $T^{\uparrow}$. The *tower-configuration of* R is $R'_{\downarrow}$.

Note that all these three operations, when applied to an ordered configuration, still give an ordered configuration. For example, for the 11-configuration R above,

$$R' = \begin{array}{|c|c|c|c||c|c|c|c|}\hline 1 & \bigcirc & \bigcirc & 4 & \blacksquare & \blacksquare & 3 & 4 \\ \hline & \bigcirc & \bigcirc & & \blacksquare & \blacksquare & & \\ \hline \end{array}, \quad R'_{\downarrow} = \begin{array}{|c|c||c|c|}\hline 1 & \bigcirc & \blacksquare & 3 \\ \hline \bigcirc & 4 & \blacksquare & 4 \\ \hline \end{array} \text{ and } (R'_{\downarrow})'_{\downarrow} = \begin{array}{||c|}\hline \blacksquare \\ \hline \\ \hline \end{array}.$$

In the proof of Theorem 2.2, we will need the following definition:

Definition 2.1. Let R be a n-configuration where all but the first and the last columns are odd, and where either:

(a) The first column of R is a tower of color c and the last one is empty.
(b) The first column of R is empty and the last one is a tower of color c.

Then define $\phi_1(R)$ and $\phi_2(R)$ as follows:

(1) Color with the second color ($\blacksquare$) one slot of the first column of $\phi_i(R)$, as follows: color the bottom one if $c = 1$ and color the top one if $c = 2$.
(2) Color with the first color ($\bigcirc$) one slot of the n-th and last column of $\phi_i(R)$, as follows: color the bottom one in case (2.1) and color the top one in case (2.1).
(3) Color with the second color one slot of every column in-between of $\phi_1(R)$, the same as in the corresponding column of R; if the $(n-1)$-th column of $\phi_1(R)$ becomes different from the first one, change the position of the colored slot of every column but the last one.
(4) Color with the first color one slot of every column in-between of $\phi_2(R)$, the same as in the corresponding column of R; if the second column of $\phi_2(R)$ becomes different from the last one, change the position of the colored slot of every column but the first one.

We may now present our new proof of the theorem:

Theorem 2.2. *For every natural number n there is a bijection $\varphi = \varphi_n : \mathcal{O}_n \to \mathcal{T}_n$.*

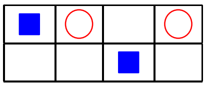
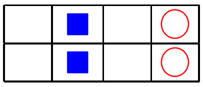
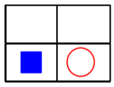
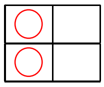
Proof. (By induction on n)

Definition of φ. Let $R \in \mathcal{O}_n$. If R has no towers, define $\varphi(R) = R$. If R has $k > 0$ towers consider $S = R'$ and note that $S_{\downarrow}$, by induction, is in bijection with a configuration without towers, $T = \varphi_k(S)$, say; now, take the $(2k)$-configuration $U = T^{\uparrow}$ and replace every (even) column of R' in R by the corresponding (even) column of U, obtaining thus R^*, say. Since the tower configuration of R^* is still T, which has no towers, the towers and empty columns of R^* appear in pairs, meaning that for every $1 \leq j \leq k$ either the $(2j-1)$-th even column is a tower and the $(2j)$-th even column is empty or vice versa. Note that, by construction, the orders of the $(2j-1)$-th and the $(2j)$-th columns are (a) both smaller or equal to the type t of R or they are (b) both greater than this number.

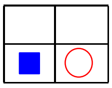
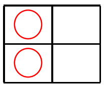
Now, for every $1 \leq j \leq k$, in the cases where (a) holds replace the section R_j between the $(2j-1)$-th even column and the $(2j)$-th even column, included, by $\phi_1(R_j)$ as defined in Definition (2.1), and in the cases where (b) holds replace it by $\phi_2(R_j)$.

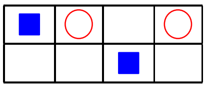
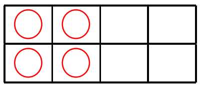
Bijectivity of φ. By considering only the consecutive pairs formed by a column of color two followed by a column of color one of $\varphi(R)$, we obtain a $(2k)$-configuration without towers that encodes, according to Definition (2.1), a unique $(2k)$-configuration S without odd columns. Consider $T = S_\downarrow$ and take, by induction, $U = \left(\varphi_k^{-1}(T)\right)^\uparrow$. Clearly, the towers of U are the towers of R and the column of color one of $\varphi(R)$ corresponding to the last even column of the same color of R has the same order in $\varphi(R)$ as the latter in R, and the same happens with the column of color two of $\varphi(R)$ corresponding to the first even column of the same color in R. Hence, by induction we know the type of R and we may use Definition 2.1 to fully recover R. $\square$

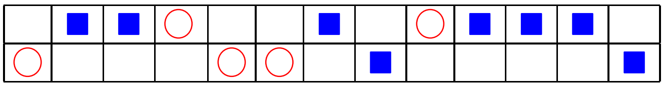
Example 2.3. Since by Definition 2.1

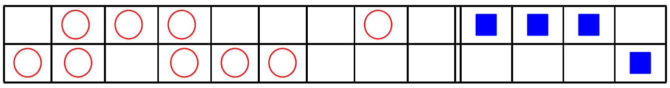
 encodes  and  encodes ,

which contains no ■-towers followed by ○-columns,

$\varphi^{-1}\Big($  $\Big) =$ ,

$\varphi^{-1}\Big($  $\Big) =$  and

$\varphi^{-1}\Big($  $\Big) =$



3. On Chang-Xu generalization of the main identity

The results of this section can be shown using the generating functions obtained in the following section. In the spirit of this article, however, we present combinatorial proofs.

The main result of [3] is a generalization of (1) that we may write as:

$$\sum_{i_1+\cdots+i_t=n} \binom{2i_1}{i_1}\binom{2i_2}{i_2}\cdots\binom{2i_t}{i_t} = 4^n\binom{n+\frac{t}{2}-1}{n}. \tag{4}$$

where $i_1, \ldots, i_n$ are any nonnegative integers.

Denote by $S_t(n)$ the left hand-side of (4). We remark that $S_1(n) = \binom{2n}{n}$ and, by (1), $S_2(n) = 4^n$. Note also that (4) can be obtained using induction and the following lemma. Finally, we observe that, when $t = 2k+1$, $4^n\binom{n+\frac{2k+1}{2}-1}{n} = \frac{\binom{2n+2k}{2n}}{\binom{n+k}{n}}\binom{2n}{n} = \frac{\binom{2n+2k}{n+k}}{\binom{2k}{k}}\binom{n+k}{n}$.

Lemma 3.1. *For every positive integer t and every nonnegative integer n,*

$$S_{t+2}(n+1) = S_t(n+1) + 4\,S_{t+2}(n)$$

Proof. In fact, $S_{t+2}(n+1) = \sum_{j=0}^{n+1} S_2(n+1-j)S_t(j) = S_t(n+1) + 4\sum_{j=0}^{n} 4^{n-j}S_t(j)$. $\square$

We now prove identity (3) by using the inclusion-exclusion principle and then we generalize it in Lemma 3.3, below. Note that the result is valid for every $\ell \in \mathbb{R}$, since

$\sum_{i+j=n} \binom{2i-\ell}{i}\binom{2j+\ell}{j}$ is a polynomial in ℓ (of degree at most n). In particular, this gives us yet another proof of (1), in fact a very short one.

Lemma 3.2. *For every nonnegative integer numbers i, j, n and ℓ such that $\ell > 2n$,*

$$\sum_{i+j=n} \binom{2i-\ell}{i}\binom{2j+\ell}{j} \;=\; 4^n.$$

Proof. First note that:

$$\begin{aligned}
\sum_{i+j=n} \binom{2i-\ell}{i}\binom{2j+\ell}{j} &= \sum_{i+j=n} (-1)^i \binom{\ell-1-i}{i}\binom{2n+\ell-2i}{j} \\
&= \sum_{i+j=n} \left[(-1)^i \binom{\ell-1-i}{i} \sum_{k+m=j} \binom{2n+1}{k}\binom{\ell-1-2i}{m}\right] \\
&= \sum_{k=0}^{n} \left[\binom{2n+1}{k} \sum_{i+m=n-k} (-1)^i \binom{\ell-1-i}{i}\binom{\ell-1-2i}{m}\right]
\end{aligned}$$

Now, since $\sum_{i+m=p}(-1)^i\binom{\ell-i}{i}\binom{\ell-2i}{m} = \sum_{i=0}^{p}(-1)^i\binom{\ell-i}{\ell-p}\binom{\ell-p}{i}$, we prove that the value of the latter sum is 1. For this purpose, define $\mathcal{A}$ as the collection of all subsets of $[\ell]$ with $\ell-p$ elements and let $\mathcal{A}_x$ be the collection of those that do not contain x, for $x = 1, \ldots, \ell-p$, so that $\mathcal{A} = \{[\ell-p]\} \cup \mathcal{A}_\varnothing$, where $\mathcal{A}_\varnothing = \bigcup_{x=1}^{\ell-p} \mathcal{A}_x$. Now, let, for $\varnothing \neq T \subset [\ell-p]$, $\mathcal{A}_T = \bigcap_{x\in T} \mathcal{A}_x$. Since, for $i = 0, \ldots, p$, there are $\binom{\ell-p}{i}$ sets of form $\mathcal{A}_T$ with $|T| = i$, and each one has $\binom{\ell-i}{p-i}$ elements, the result follows from the inclusion-exclusion principle. $\square$

Lemma 3.3. *For every nonnegative integers i, j and n and real numbers a, ℓ, we have:*

$$\sum_{i+j=n} \binom{2i+a}{i}\binom{2j}{j} = \sum_{i+j=n} \binom{2i+a-\ell}{i}\binom{2j+\ell}{j}.$$

Proof. We want to prove that, for fixed $n \in \mathbb{N}_0$ and $a \in \mathbb{R}$, $p = \sum_{i+j=n} \binom{2i+a-\ell}{i}\binom{2j+\ell}{j}$ has degree zero as a polynomial (in ℓ). Note that

$$p \;=\; \frac{1}{n!}\sum_{i=0}^{n}(-1)^i\binom{n}{i} q_i \;=\; \frac{(-1)^n}{n!}\Delta^n\, q_0,$$

where, for $i = 0, 1, \ldots, n$, $p_i = \big(\ell - a + i - 1\big)_i\, \big(\ell + 2n\big)_{n-i}$ and q_i is obtained from p_i by replacing ℓ with $\ell - 2i$, so that both are polynomials of degree n in ℓ, and where we write, as usual, $\Delta\, p_i = p_{i+1} - p_i$. Now, induction on m immediately shows that, for $1 \le m \le n - i$,

$$\Delta^m p_i = (-1)^m \big(a+n+m\big)_m\, \big(\ell - a + i - 1\big)_i\, \big(\ell + 2n\big)_{n-i-m}$$

which is either zero or has degree $n - m$. We may rewrite each p_i as

$$p_i = r_n + r_{n-1}\,\ell + \cdots + r_0\,\ell^n,$$

for suitable polynomials r_k in i ($k = 0, \ldots, n$) with $r_0 = 1$, so that

$$\Delta^m p_i = \Delta^m r_n + \Delta^m r_{n-1}\,\ell + \cdots + \Delta^m r_0\,\ell^n.$$

Hence, $\Delta^m r_{m-1} = \cdots = \Delta^m r_1 = \Delta^m r_0 = 0$, which proves that r_k has degree at most k and thus

$$\begin{aligned} q_i &= r_n + r_{n-1}(\ell - 2i) + \cdots + (\ell - 2i)^n, \\ &= s_n + s_{n-1}\,\ell + \cdots + \ell^n, \end{aligned}$$

where each s_k is a polynomial of degree less than or equal to k in i. Therefore, p is constant in ℓ. □

As a clear consequence, we obtain for every nonnegative integers t, n, and $i_1, i_2, \ldots, i_t$ and for every real numbers $\ell_1, \ell_2, \ldots, \ell_t$ such that $\ell = \ell_1 + \ell_2 + \cdots + \ell_t$,

$$\sum_{i_1+\cdots+i_t=n} \binom{2i_1+\ell}{i_1}\binom{2i_2}{i_2}\cdots\binom{2i_t}{i_t} = \sum_{i_1+\cdots+i_t=n} \binom{2i_1+\ell_1}{i_1}\binom{2i_2+\ell_2}{i_2}\cdots\binom{2i_t+\ell_t}{i_t}$$

Whence we obtain the following generalization of both (3) and (4).

Corollary 3.4. *Let $\ell_1, \ldots, \ell_t$ be any real numbers such that $\ell_1 + \cdots + \ell_t = 0$. Then*

$$\sum_{i_1+\cdots+i_t=n} \binom{2i_1+\ell_1}{i_1}\binom{2i_2+\ell_2}{i_2}\cdots\binom{2i_t+\ell_t}{i_t} = \frac{4^n}{n!}\frac{\Gamma(n+\frac{t}{2})}{\Gamma(\frac{t}{2})}$$

4. Generating functions

In what follows, we denote by $f^{(n)}$ the n-th derivative of a function f of one real variable, $g(x) = \sum_{n\geq 0}\binom{2n}{n}x^n$ is the generating function of the central binomial coefficients and $C(x) = \sum_{n\geq 0}\frac{1}{n+1}\binom{2n}{n}x^n$ is the generating function of the Catalan numbers. We remember that $g(x) = \frac{1}{\sqrt{1-4x}}$ and $C(x) = \frac{2}{1+\sqrt{1-4x}}$. Note that $g' = 2g^3$ and $C' = g\,C^2$. In this section we obtain the sequences with generating functions g^t, $g\,C^\ell$ and C^ℓ, for every $t, \ell \in \mathbb{R}$.

Lemma 4.1.

(1) *For every real number t and nonnegative integer n,*

$$\frac{(g^t)^{(n)}}{n!} = 4^n\binom{n+\frac{t}{2}-1}{n}g^{t+2n}.$$

(2) *For every real number ℓ and nonnegative integer n,*

$$\frac{\left(g\,C^\ell\right)^{(n)}}{n!} = \sum_{i=0}^{n}\binom{2n-i}{n-i}\binom{\ell+i-1}{i}g^{1+2n-i}C^{\ell+i}.$$

(3) *For every real number ℓ and positive integer n,*

$$\left(C^\ell\right)^{(n)} = \left(\ell\,g\,C^{\ell+1}\right)^{(n-1)}.$$

Proof. Both (4.1.1) and (4.1.2) are easily shown using induction and (4.1.3) is obvious. □

Our next theorem is the main result of this section. We remark that (4.2.3) was proved by Eugène Catalan in 1876 (V. [2, p. 62 and Errata]).

Theorem 4.2 (Generating functions for g^t, $g\,C^\ell$ and C^ℓ)**.**

(1) *For every real number t,*

$$g(x)^t = \sum_{n\geq 0} 4^n\binom{n+\frac{t}{2}-1}{n}x^n.$$

(2) *For every real number ℓ,*

$$g(x)\, C(x)^{\ell} = \sum_{n\geq 0} \binom{2n+\ell}{n} x^n.$$

(3) *For every real number ℓ,*

$$C(x)^{\ell} = 1 + \sum_{n\geq 1} \frac{\ell}{2n+\ell} \binom{2n+\ell}{n} x^n.$$

Proof. The identity (4.2.1) follows immediately from (4.1.1) whereas (4.2.3) follows from (4.1.3) and (4.2.2). If we show that

$$\sum_{i=0}^{n} \binom{2n-i}{n-i} \binom{\ell+i-1}{i} = \binom{2n+\ell}{n},$$

for all nonnegative integer n and real ℓ, the result follows from Lemma 4.1. Let

$$F(n,i) = \binom{2n-i}{n-i} \binom{\ell+i-1}{i}.$$

By Zeilberger's algorithm[6, 8], as implemented in *Mathematica* by Peter Paule and Markus Schorn [7] and by Christian Krattenthaler [5], we know that $T(n) = \sum_{i=0}^{n} F(n,i)$ verifies

$$(2n+\ell+1)(2n+\ell+2)\, T(n) - (n+\ell+1)(n+1)\, T(n+1) = 0, \tag{5}$$

which is also verified by $T(n) = \binom{2n+\ell}{n}$, and for both it holds $T(0) = 1$. In fact, we can see that for $0 \leq i \leq n+1$,

$$(2n+\ell+1)(2n+\ell+2)\, F(n,i) - (n+\ell+1)(n+1)\, F(n+1,i) = G(n,i+1) - G(n,i)$$

with $G(n,i) = i(i+1)\binom{2n+1-i}{n+1-i}\binom{\ell+i}{i+1}$. Hence, (5) holds since $F(n,n+1) = G(n,n+2) = G(n,0) = 0$. □

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