ENUMERATION OF PERMUTATIONS BY NUMBER OF CYCLIC OCCURRENCE OF PEAKS AND VALLEYS

SHI-MEI MA AND CHAK-ON CHOW

Abstract. In this paper, we focus on the enumeration of permutations by number of cyclic occurrence of peaks and valleys. We find several recurrence relations involving the number of permutations with a prescribed number of cyclic peaks, cyclic valleys, fixed points and cycles. Several associated permutation statistics and the corresponding generating functions are also studied. In particular, we establish a connection between cyclic valleys and Pell numbers as well as cyclic peaks and alternating runs.

Keywords: Cyclic peaks; Cyclic valleys; Fixed points; Cycles; Alternating runs

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1. Introduction

Let $[n] = \{1, 2, \ldots, n\}$, and let $\mathfrak{S}_n$ denote the the set of permutations of $[n]$. A permutation $\pi \in \mathfrak{S}_n$ can be written in *one-line notation* as the word $\pi = \pi(1)\pi(2)\cdots\pi(n)$. Another way of writing the permutation is given by the *standard cycle decomposition*, where each cycle is written with its smallest entry first and the cycles are written in increasing order of their smallest entry. For example, the permutation $\pi = 64713258 \in \mathfrak{S}_8$ has the standard cycle decomposition $(1, 6, 2, 4)(3, 7, 5)(8)$.

A permutation $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$ is *alternating* if $\pi(1) > \pi(2) < \pi(3) > \pi(4) < \cdots$. Similarly, π is *reverse alternating* if $\pi(1) < \pi(2) > \pi(3) < \pi(4) > \cdots$. It is well known [1] that the *Euler numbers* E_n defined by

$$\sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = \tan x + \sec x$$

count alternating permutations in $\mathfrak{S}_n$. The first few values of E_n are $1, 1, 1, 2, 5, 16, 61, 272, \ldots$. The bijection $\pi \mapsto \pi^c$ on $\mathfrak{S}_n$ defined by $\pi^c(i) = n + 1 - \pi(i)$ shows that E_n is also the number of reverse alternating permutations in $\mathfrak{S}_n$. The study of the Euler numbers is a topic in combinatorics (see [22]). For example, Elizalde and Deutsch [9] recently studied *cycle up-down* permutations. A cycle is said to be up-down if, when written in standard cycle form, say $(b_1, b_2, b_3, \ldots)$, we have $b_1 < b_2 > b_3 < \cdots$. We say that π is a cycle up-down permutation if it is a product of up-down cycles. Elizalde and Deutsch [9, Proposition 2.1] found that the number of cycle up-down permutations of $[n]$ is E_{n+1} .

There is a wealth of literature on peak statistics of permutations (see [3, 8, 11, 12, 16, 17, 23] for instance). For example, Kitaev [11] found that there are 2^{n-1} permutations of $[n]$ without interior peaks. Let $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$. An *interior peak* (resp. *interior valley*) in π is an

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index $i \in \{2, 3, \ldots, n-1\}$ such that $\pi(i-1) < \pi(i) > \pi(i+1)$ (resp. $\pi(i-1) > \pi(i) < \pi(i+1)$). Clearly, interior peaks and interior valleys are equidistributed on $\mathfrak{S}_n$. Let $\mathrm{pk}\,(\pi)$ (resp. $\mathrm{val}\,(\pi)$) denote the number of interior peaks (resp. the number of interior valleys) in π. An *left peak* in π is an index $i \in [n-1]$ such that $\pi(i-1) < \pi(i) > \pi(i+1)$, where we take $\pi(0) = 0$. Let $\mathrm{pk}^l\,(\pi)$ denote *the number of left peaks* in π. For example, the permutation $\pi = 64713258 \in \mathfrak{S}_8$ has $\mathrm{pk}\,(\pi) = 2, \mathrm{val}\,(\pi) = 3$ and $\mathrm{pk}^l\,(\pi) = 3$.

For $n \geqslant 1$, we define

$$W_n(q) = \sum_{\pi \in \mathfrak{S}_n} q^{\mathrm{pk}\,(\pi)} \quad and \quad \overline{W}_n(q) = \sum_{\pi \in \mathfrak{S}_n} q^{\mathrm{pk}^l\,(\pi)}.$$

It is well known that the polynomials $W_n(q)$ satisfy the recurrence relation

$$W_{n+1}(q) = (nq - q + 2)W_n(q) + 2q(1-q)W_n'(q),$$

with initial values $W_1(q) = 1$, $W_2(q) = 2$ and $W_3(q) = 4 + 2q$, and the polynomials $\overline{W}_n(q)$ satisfy the recurrence relation

$$\overline{W}_{n+1}(q) = (nq + 1)\overline{W}_n(q) + 2q(1-q)\overline{W}'_n(q),$$

with initial values $\overline{W}_1(q) = 1$, $\overline{W}_2(q) = 1 + q$ and $\overline{W}_3(q) = 1 + 5q$ (see [18, A008303, A008971]). The exponential generating functions of the polynomials $W_n(q)$ and $\overline{W}_n(q)$ are respectively given as follows (see [12]):

$$W(q,z) = \sum_{n \geq 1} W_n(q) \frac{z^n}{n!} = \frac{\sinh(z\sqrt{1-q})}{\sqrt{1-q}\cosh(z\sqrt{1-q}) - \sinh(z\sqrt{1-q})},$$

$$\overline{W}(q,z) = 1 + \sum_{n \geq 1} \overline{W}_n(q) \frac{z^n}{n!} = \frac{\sqrt{1-q}}{\sqrt{1-q}\cosh(z\sqrt{1-q}) - \sinh(z\sqrt{1-q})}. \tag{1}$$

An occurrence of a *pattern* τ in a permutation π is defined as a subsequence in π whose letters are in the same relative order as those in τ. For example, the permutation $\pi = 64713258 \in \mathfrak{S}_8$ has two occurrences of the pattern 1–2–3–4, namely the subsequences 1358 and 1258. In [2], Babson and Steingrímsson introduced *generalized permutation patterns* that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation. Thus, an occurrence of an interior peak in a permutation is an occurrence of the pattern 132 or 231. Similarly, an occurrence of interior valley is an occurrence of the pattern 213 or 312. Recently, Parviainen [14, 15] explored *cyclic occurrence of patterns* over $\mathfrak{S}_n$ via continued fractions.

In this paper, we focus on the enumeration of permutations by number of cyclic occurrence of peaks and valleys. The paper is organised as follows. In Section 2, we collect some notation, definitions and results that will be needed in the rest of the paper. In Section 3, we present several recurrence relations. In Section 4, we discuss two classes of triangular arrays. In Section 5, we compute two exponential generating functions of generating functions of permutations by their numbers of cyclic peaks/valleys, cycles and fixed points. In Section 6, we establish a connection between cyclic valleys and the famous Pell numbers. In Section 7, we establish a connection between cyclic peaks and alternating runs.

2. Notation, definitions and preliminaries

In the following discussion we always write a permutation $\pi \in \mathfrak{S}_n$ in standard cycle decomposition. Let

$$\pi = (c_1^1, c_2^1, \dots c_{i_1}^1)(c_1^2, c_2^2, \dots c_{i_2}^2) \dots (c_1^k, c_2^k, \dots c_{i_k}^k),$$

and let σ be a generalised pattern. Following Parviainen [14, 15], the pattern σ occurs cyclically in π if it occurs in the permutation

$$\Psi(\pi) = c_1^1, c_2^1, \dots c_{i_1}^1, c_1^2, c_2^2, \dots c_{i_2}^2, \dots c_1^k, c_2^k, \dots c_{i_k}^k,$$

with the further restriction that $c_{i_j}^j$ and c_1^{j+1} are not adjacent, where $1 \le j \le k-1$. For example, 132 does not occur in $(1)(2,4,5)(3)$, but 13–2 occurs exactly twice. An entry c_m^j in the cycle $(c_1^j, c_2^j, \dots c_{i_j}^j)$ is called a *cyclic peak* (resp. *cyclic valley*) of π if $c_{m-1}^j < c_m^j > c_{m+1}^j$ (resp. $c_{m-1}^j > c_m^j < c_{m+1}^j$), where $2 \leqslant m \leqslant i_j - 1$ and $1 \leqslant j \leqslant k$. For example, the permutation $(1,6,2,4)(3,7,5)(8)$ has the cyclic peaks 6 and 7 and the cyclic valley 2.

Let $\mathrm{cpk}\,(\pi)$ (resp. $\mathrm{cval}\,(\pi)$) denote *the number of cyclic peaks* (resp. *the number of cyclic valleys*) of π. The number of *fixed points* of π is $\mathrm{fix}\,(\pi) = \#\{1 \leqslant i \leqslant n : \pi(i) = i\}$. A fixed-point-free permutation is called a *derangement*. Let $\mathcal{D}_n$ denote the set of derangements of $[n]$. Denote by $\mathrm{cyc}\,(\pi)$ the number of *cycles* of π. For $n \geqslant 1$, we introduce the following generating functions:

$$\begin{aligned}
P_n(q,x,y) &= \sum_{\pi \in \mathfrak{S}_n} q^{\mathrm{cpk}\,(\pi)} x^{\mathrm{cyc}\,(\pi)} y^{\mathrm{fix}\,(\pi)};\\
V_n(q,x,y) &= \sum_{\pi \in \mathfrak{S}_n} q^{\mathrm{cval}\,(\pi)} x^{\mathrm{cyc}\,(\pi)} y^{\mathrm{fix}\,(\pi)};\\
M_n(q) &= \sum_{\pi \in \mathfrak{S}_n} q^{\mathrm{cpk}\,(\pi)} = \sum_{k \geqslant 0} M_{n,k} q^k;\\
\overline{M}_n(q) &= \sum_{\pi \in \mathfrak{S}_n} q^{\mathrm{cval}\,(\pi)} = \sum_{k \geqslant 0} \overline{M}_{n,k} q^k;\\
D_n(q) &= \sum_{\pi \in \mathcal{D}_n} q^{\mathrm{cpk}\,(\pi)} = \sum_{k \geqslant 0} D_{n,k} q^k;\\
\overline{D}_n(q) &= \sum_{\pi \in \mathcal{D}_n} q^{\mathrm{cval}\,(\pi)} = \sum_{k \geqslant 0} \overline{D}_{n,k} q^k.
\end{aligned}$$

Let

$$p(n,t,s,r) = \#\{\pi \in \mathfrak{S}_n : \mathrm{cpk}\,(\pi) = t, \mathrm{cyc}\,(\pi) = s, \mathrm{fix}\,(\pi) = r\},$$

and let

$$v(n,t,s,r) = \#\{\pi \in \mathfrak{S}_n : \mathrm{cval}\,(\pi) = t, \mathrm{cyc}\,(\pi) = s, \mathrm{fix}\,(\pi) = r\}.$$

The *Stirling numbers of the second kind* $\left\{ n \atop k \right\}$ is the number of partitions of $[n]$ into k blocks. Let $S_n(x) = \sum_{k=1}^n \left\{ n \atop k \right\} x^k$. It is well known (see [18, A008277]) that

$$\left\{ n+1 \atop k \right\} = \left\{ n \atop k-1 \right\} + k \left\{ n \atop k \right\},$$

which is equivalent to

$$S_{n+1}(x) = xS_n(x) + xS_n'(x).$$

The *associated Stirling numbers of second kind* $T(n,k)$ is the number of partitions of $[n]$ into k blocks of size at least 2. Let $T_n(x)=\sum_{k\geqslant 1}T(n,k)x^k$. It is well known (see [18, A008299]) that

$$T(n+1,k)=kT(n,k)+nT(n-1,k-1),$$

which is equivalent to

$$T_{n+1}(x)=xT_n'(x)+nxT_{n-1}(x).$$

Clearly, $P_n(0,x,1)=S_n(x)$ and $P_n(0,x,0)=T_n(x)$. For example, take a permutation

$$\pi=(\pi(i_1),\ldots)(\pi(i_2),\ldots)\cdots(\pi(i_s),\ldots)$$

counted by $p(n,0,s,r)$. Recall that

$$p(n,0,s,r)=\#\{\pi\in\mathfrak{S}_n:\operatorname{cpk}(\pi)=0,\operatorname{cyc}(\pi)=s,\operatorname{fix}(\pi)=r\}.$$

Erasing the parentheses, we get a partition of $[n]$ with s blocks. Hence

$$\sum_{r\geqslant 0}p(n,0,s,r)=\left\{{n\atop s}\right\}.$$

We say that a permutation π is called a *circular permutation* if $\operatorname{cyc}(\pi)=1$. Denote by $\mathcal{C}_n$ the set of circular permutations of $[n]$. Each $\pi\in\mathcal{C}_{n+1}$ can be written uniquely as a cycle of the form $\pi=(1,a_1,a_2,\ldots,a_n)$. Let $\varphi(\pi)=b_1b_2\cdots b_n$, where $b_i=a_i-1$ for $1\leqslant i\leqslant n$. The correspondence $\varphi:\mathcal{C}_{n+1}\mapsto\mathfrak{S}_n$ is clearly a bijection. Using the bijection φ, it is clear that the coefficient of x in the polynomial $P_{n+1}(q,x,y)$ (resp. $V_{n+1}(q,x,y)$) is $\overline{W}_n(q)$ (resp. $W_n(q)$). In the next section, we present recurrence relations for the polynomials $P_n(q,x,y)$ and $V_n(q,x,y)$.

3. Recurrence relations

Theorem 1. *The numbers $p(n,t,s,r)$ satisfy the recurrence relation*

$$\begin{aligned}p(n+1,t,s,r)=(2t+s-r)p(n,t,s,r)+p(n,t,s-1,r-1)+\\(r+1)p(n,t,s,r+1)+(n+2-2t-s)p(n,t-1,s,r).\end{aligned}$$

Proof. Let n be a fixed positive integer. Let $\sigma_i\in\mathfrak{S}_{n+1}$ be the permutation obtained from $\sigma\in\mathfrak{S}_n$ by inserting the entry $n+1$ either to the left or to the right of $\sigma(i)$ if $i\in[n]$ or as a new cycle $(n+1)$ if $i=n+1$. Then

$$\operatorname{cyc}(\sigma_i)=\begin{cases}\operatorname{cyc}(\sigma) & \text{if } i\in[n],\\ \operatorname{cyc}(\sigma)+1 & \text{if } i=n+1;\end{cases}$$

and

$$\operatorname{fix}(\sigma_i)=\begin{cases}\operatorname{fix}(\sigma)-1 & \text{if } i\in[n] \text{ and } \sigma(i)=i,\\ \operatorname{fix}(\sigma) & \text{if } i\in[n] \text{ and } \sigma(i)\neq i,\\ \operatorname{fix}(\sigma)+1 & \text{if } i=n+1.\end{cases}$$

Recall that

$$p(n+1,t,s,r)=\#\{\sigma_i\in\mathfrak{S}_{n+1}:\operatorname{cpk}(\sigma_i)=t,\operatorname{cyc}(\sigma_i)=s,\operatorname{fix}(\sigma_i)=r\}.$$

It is evident that $\operatorname{cpk}(\sigma_i)=\operatorname{cpk}(\sigma)$ or $\operatorname{cpk}(\sigma_i)=\operatorname{cpk}(\sigma)+1$. There are four cases to consider.

(a) If σ has t cyclic peaks, s cycles and r fixed points, then we can put the entry $n+1$ on either side of some cyclic peak or at the end of a cycle of length greater than or equal to 2. This means we have $2t+s-r$ choices for the positions of $n+1$. As we have $p(n,s,t,r)$ choices for σ, the first term of the recurrence relation is obtained.

(b) If σ has t cyclic peaks, $s-1$ cycles and $r-1$ fixed points, then we can insert the entry $n+1$ at the end of σ to form a new cycle $(n+1)$. As we have $p(n,t,s-1,r-1)$ choices for σ, the second term of the recurrence relation is obtained.

(c) If σ has t cyclic peaks, s cycles and $r+1$ fixed points, then we can insert the entry $n+1$ to the right of a fixed points. This means we have $r+1$ choices for the positions of $n+1$. This case applies to the third term in the recurrence relation.

(d) If σ has $t-1$ cyclic peaks, s cycles and r fixed points, then we can insert the entry $n+1$ in one of the $n-2(t-1)-s=n+2-2t-s$ middle positions and the last term of the recurrence relation is obtained.

This completes the proof. □

We can express Theorem 1 in terms of differential operators.

Corollary 2. *For $n \geqslant 1$, we have*

$$\begin{aligned} P_{n+1}(q,x,y) = (nq+xy)P_n(q,x,y) + 2q(1-q)\frac{\partial P_n(q,x,y)}{\partial q} + x(1-q)\frac{\partial P_n(q,x,y)}{\partial x} \\ + (1-y)\frac{\partial P_n(q,x,y)}{\partial y}. \end{aligned}$$

In particular,

$$P_{n+1}(q,x,1) = (nq+x)P_n(q,x,1) + 2q(1-q)\frac{\partial P_n(q,x,1)}{\partial q} + x(1-q)\frac{\partial P_n(q,x,1)}{\partial x}. \tag{2}$$

By Corollary 2, we can easily compute the first few polynomials $P_n(q,x,y)$:

$$\begin{aligned} P_1(q,x,y) &= xy, \\ P_2(q,x,y) &= x + x^2y^2, \\ P_3(q,x,y) &= (1+q)x + 3x^2y + x^3y^3, \\ P_4(q,x,y) &= (1+5q)x + (3+4y+4qy)x^2 + 6x^3y^2 + x^4y^4. \end{aligned}$$

Recall that $M_n(q) = P_n(q,1,1)$. For $1 \leqslant n \leqslant 7$, using (2), the coefficients of $M_n(q)$ can be arranged as follows with $M_{n,k}$ in row n and column k:

1			
2			
5	1		
15	9		
52	63	5	
203	416	101	
877	2741	1361	61

Let $D_{n,k}(q,x)$ be the coefficient of y^k in $P_n(q,x,y)$. Clearly, the polynomial $P_n(q,x,0)$ is the corresponding enumerative polynomial on D_n. Note that

$$\begin{aligned} D_{n,k}(q,x) &= \sum_{\substack{\pi\in\mathfrak{S}_n \\ \mathrm{fix}\,(\pi)=k}} q^{\mathrm{cpk}\,(\pi)}x^{\mathrm{cyc}\,(\pi)} \\ &= \binom{n}{k}x^k \sum_{\sigma\in\mathcal{D}_{n-k}} q^{\mathrm{cpk}\,(\sigma)}x^{\mathrm{cyc}\,(\sigma)} \\ &= \binom{n}{k}x^k P_{n-k}(q,x,0). \end{aligned}$$

Thus

$$\left(\frac{\partial P_n(q,x,y)}{\partial y}\right)_{y=0} = D_{n,1}(q,x) = nxP_{n-1}(q,x,0).$$

Therefore, we get the following result.

Proposition 3. *For $n \geqslant 2$, the polynomials $P_n(q,x,0)$ satisfy the following recurrence relation*

$$P_{n+1}(q,x,0) = nqP_n(q,x,0)+2q(1-q)\frac{\partial P_n(q,x,0)}{\partial q}+x(1-q)\frac{\partial P_n(q,x,0)}{\partial x}+nxP_{n-1}(q,x,0), \quad (3)$$

with initial values $P_0(q,x,0) = 1, P_1(q,x,0) = 0, P_2(q,x,0) = x$ and $P_3(q,x,0) = (1+q)x$.

Recall that $D_n(q) = P_n(q,1,0)$. For $1 \leqslant n \leqslant 7$, using (3), the coefficients of $D_n(q)$ can be arranged as follows with $D_{n,k}$ in row n and column k:

0			
1			
1	1		
4	5		
11	28	5	
41	153	71	
162	872	759	61

In order to provide a recurrence relation for the numbers $v(n,t,s,r)$, we first define an operation. Assume that π is a permutation in $\mathfrak{S}_n$ with k cycles $C_1, C_2, \ldots, C_k$, where $C_j = (c_1^j, c_2^j, \ldots, c_{i_j}^j)$ and $1 \leqslant j \leqslant k$. Let $\phi : \mathfrak{S}_n \to \mathfrak{S}_n$ be defined as follows:

- $\phi(C_1, C_2, \ldots, C_k) = (\phi(C_1), \phi(C_2), \ldots, \phi(C_k))$.
- For every cycle C_j, we have
$$\phi(c_1^j, c_2^j, \ldots, c_{i_j}^j) = (\phi(c_1^j), \phi(c_2^j), \ldots, \phi(c_{i_j}^j)).$$
- Let $\{a_1, a_2, \ldots, a_{i_j}\}$ be the set of entries of the cycle $C_j = (c_1^j, c_2^j, \ldots, c_{i_j}^j)$, and assume that $a_1 < a_2 < \cdots < a_{i_j}$. Then $\phi(a_m) = a_{i_j+1-m}$, where $1 \leqslant m \leqslant i_j$.

For example, $\phi((1,6,2,4)(3,7,5)(8)) = (6,1,4,2)(7,3,5)(8)$. Following [10], we call this operation a *switching*. Clearly, if π has t cyclic valleys, then $\phi(\pi)$ has t cyclic peaks, and vice versa.

Theorem 4. *The numbers $v(n,t,s,r)$ satisfy the recurrence relation*

$$\begin{aligned}v(n+1,t,s,r) &= (2t+2s-2r)v(n,t,s,r)+v(n,t,s-1,r-1)\\ &\quad +(r+1)v(n,t,s,r+1)+(n+2-2t-2s+r)v(n,t-1,s,r).\end{aligned}$$

Proof. Let n be a fixed positive integer. Let $\sigma_i \in \mathfrak{S}_{n+1}$ be the permutation obtained from $\sigma \in \mathfrak{S}_n$ by inserting the entry $n+1$ either to the left or to the right of $\sigma(i)$ if $i \in [n]$ or as a new cycle $(n+1)$ if $i = n+1$. Recall that

$$v(n+1,t,s,r) = \#\{\sigma_i \in \mathfrak{S}_{n+1} : \mathrm{cval}\,(\sigma_i) = t, \mathrm{cyc}\,(\sigma_i) = s, \mathrm{fix}\,(\sigma_i) = r\}.$$

It is evident that $\mathrm{cval}\,(\sigma_i) = \mathrm{cval}\,(\sigma)$ or $\mathrm{cval}\,(\sigma_i) = \mathrm{cval}\,(\sigma)+1$. There are four cases to consider.

(a) If σ has t cyclic valleys, s cycles and r fixed points, then $\phi(\sigma)$ is a permutation with t cyclic peaks, s cycles and r fixed points. Consider the permutation $\phi(\sigma)$, we can appending $n+1$ either at the beginning or at the end of a cycle of length greater than or equal to 2. We can also put the entry $n+1$ on either side of some cyclic peak of $\phi(\sigma)$. This means we have $2t+2s-2r$ choices for the positions of $n+1$. As we have $v(n,s,t,r)$ choices for σ, the first term of the the recurrence relation is explained.

(b) If σ has t cyclic valleys, $s-1$ cycles and $r-1$ fixed points, then we can insert the entry $n+1$ at the end of σ to form a new cycle $(n+1)$. As we have $v(n,t,s-1,r-1)$ choices for σ, the second term of the recurrence relation is obtained.

(c) If σ has t cyclic valleys, s cycles and $r+1$ fixed points, then we can insert the entry $n+1$ to the right of a fixed points. This means we have $r+1$ choices for the positions of $n+1$. This case applies to the third term of the recurrence relation.

(d) If σ has $t-1$ cyclic valleys, s cycles and r fixed points, then we can insert the entry $n+1$ in one of the $n-s-(s-r)-2(t-1) = n+2-2t-2s+r$ middle positions. As we have $v(n,t-1,s,r)$ choices for σ, the last term of the recurrence relation is obtained.

This completes the proof. □

We can express Theorem 4 in terms of differential operators.

Corollary 5. *For $n \geqslant 1$, we have*

$$\begin{aligned}V_{n+1}(q,x,y) &= (nq+xy)V_n + 2q(1-q)\frac{\partial V_n(q,x,y)}{\partial q} + 2x(1-q)\frac{\partial V_n(q,x,y)}{\partial x}\\ &\quad + (1-2y+qy)\frac{\partial V_n(q,x,y)}{\partial y}.\end{aligned}$$

By Corollary 5, we can easily compute the first few polynomials $V_n(q,x,y)$:

$$\begin{aligned}V_1(q,x,y) &= xy,\\ V_2(q,x,y) &= x+x^2y^2,\\ V_3(q,x,y) &= 2x+3x^2y+x^3y^3,\\ V_4(q,x,y) &= (4+2q)x+(3+8y)x^2+6x^3y^2+x^4y^4,\\ V_5(q,x,y) &= (8+16q)x+(20+20y+10qy)x^2+(15y+20y^2)x^3+10x^4y^3+x^5y^5.\end{aligned}$$

Recall that $\overline{M}_n(q) = V_n(q,1,1)$. For $1 \leqslant n \leqslant 7$, the coefficients of $\overline{M}_n(q)$ can be arranged as follows with $\overline{M}_{n,k}$ in row n and column k:

1		
2		
6		
22	2	
94	26	
460	244	16
2532	2124	384

Let $\overline{D}_{n,k}(q,x)$ be the coefficient of y^k in $V_n(q,x,y)$. Clearly, $\overline{D}_{n,k}(q,x) = \binom{n}{k} x^k V_{n-k}(q,x,0)$. Then

$$\left(\frac{\partial V_n(q,x,y)}{\partial y}\right)_{y=0} = \overline{D}_{n,1}(q,x) = nxV_{n-1}(q,x,0).$$

Hence we get the following result.

Proposition 6. *For $n \geqslant 2$, the polynomials $V_n(q,x,0)$ satisfy the following recurrence relation*

$$V_{n+1}(q,x,0) = nqV_n(q,x,0)+2q(1-q)\frac{\partial V_n(q,x,0)}{\partial q}+2x(1-q)\frac{\partial V_n(q,x,0)}{\partial x}+nxV_{n-1}(q,x,0), \quad (4)$$

with initial values $V_0(q,x,0)=1, V_1(q,x,0)=0, V_2(q,x,0)=x$ and $V_3(q,x,0)=2x$.

Recall that $\overline{D}_n(q) = V_n(q,1,0)$. For $1 \leqslant n \leqslant 7$, using (4), the coefficients of $\overline{D}_n(q)$ can be arranged as follows with $\overline{D}_{n,k}$ in row n and column k:

0		
1		
2		
7	2	
28	16	
131	118	16
690	892	272

4. On combinations of polynomials and Euler numbers

Recall that $M_n(q) = P_n(q,1,1)$, $\overline{M}_n(q) = V_n(q,1,1)$, $D_n(q) = P_n(q,1,0)$ and $\overline{D}_n(q) = V_n(q,1,0)$. It is easy to verify that $\deg M_n(q) = \deg D_n(q) = \lfloor \frac{n-1}{2} \rfloor$ and $\deg \overline{M}_n(q) = \deg \overline{D}_n(q) = \lfloor \frac{n}{2} \rfloor - 1$ for $n \geqslant 2$.

We define

$$R_n(q) = M_n(q^2) + q\overline{M}_n(q^2) \quad \text{for } n \geqslant 3.$$

Let $R_n(q) = \sum_{k \geqslant 0} R_{n,k} q^k$. Then for $n \geqslant 3$, we have

$$R_{n,k} = \begin{cases} \overline{M}_{n,\frac{k-1}{2}} & \text{if } k \text{ is odd,} \\ M_{n,\frac{k}{2}} & \text{if } k \text{ is even.} \end{cases}$$

The nth *Bell number* B_n counts the number of partitions of $[n]$ into non-empty blocks (see [18, A000110] for details). Recall that $M_{n,0}=\#\{\pi\in\mathfrak{S}_n : \mathrm{cpk}\,(\pi)=0\}$. Take a permutation

$$\pi=(\pi(i_1),\ldots)(\pi(i_2),\ldots)\cdots(\pi(i_j),\ldots)$$

counted by $M_{n,0}$. Erasing the parentheses, we get a partition of $[n]$ with j blocks. Hence

$$R_{n,0}=M_{n,0}=B_n. \tag{5}$$

Set $R_1(x)=1$ and $R_2(x)=2+x$. For $1\leqslant n\leqslant 6$, the coefficients of $R_n(q)$ can be arranged as follows with $R_{n,k}$ in row n and column k:

$$\begin{array}{cccccc}
1 \\
2 & 1 \\
5 & 6 & 1 \\
15 & 22 & 9 & 2 \\
52 & 94 & 63 & 26 & 5 \\
203 & 460 & 416 & 244 & 101 & 16
\end{array}$$

We define

$$I_n(q)=D_n(q^2)+q\overline{D}_n(q^2)\quad \text{for } n\geqslant 2.$$

Let $I_n(q)=\sum_{k\geqslant 0}I_{n,k}q^k$. Then for $n\geqslant 2$, we have

$$I_{n,k}=\begin{cases}\overline{D}_{n,\frac{k-1}{2}} & \text{if } k \text{ is odd},\\ D_{n,\frac{k}{2}} & \text{if } k \text{ is even}.\end{cases}$$

Recall that $D_{n,0}=\#\{\pi\in\mathcal{D}_n : \mathrm{cpk}\,(\pi)=0\}$. Take a permutation

$$\pi=(\pi(p_1),\ldots)(\pi(p_2),\ldots)\cdots(\pi(p_k),\ldots)$$

counted by $D_{n,0}$. When $n\geqslant 2$, erasing the parentheses, we get a partition of $[n]$ into k blocks of size at least 2. Therefore, $D_{n,0}=\sum_{k\geqslant 1}T(n,k)$ for $n\geqslant 2$, where $T(n,k)$ is the associated Stirling numbers of second kind. Set $I_1(q)=1$. For $1\leqslant n\leqslant 6$, the coefficients of $I_n(q)$ can be arranged as follows with $I_{n,k}$ in row n and column k:

$$\begin{array}{cccccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
4 & 7 & 5 & 2 \\
11 & 28 & 28 & 16 & 5 \\
41 & 131 & 153 & 118 & 71 & 16
\end{array}$$

Note that

$$M_{2k+1,k}=D_{2k+1,k}=\#\{\pi\in S_{2k+1} : \mathrm{cpk}\,(\pi)=k\}$$

and

$$\overline{M}_{2k+2,k}=\overline{D}_{2k+2,k}=\#\{\pi\in S_{2k+2} : \mathrm{cval}\,(\pi)=k\}.$$

Then from [9, Proposition 2.1], we get the following result.

Proposition 7. *For $n\geqslant 0$, we have $R_{n+1,n}=I_{n+1,n}=E_n$.*

Let $a_0, a_1, a_2 \ldots, a_n$ be a sequence of nonnegative real numbers. The sequence is said to be *log-concave* if $a_{i-1}a_{i+1} \leqslant a_i^2$ for $i = 1, 2, \ldots, n-1$. We end this section by proposing the following.

Conjecture 8. *For $n \geqslant 1$, the sequences of coefficients of the polynomials $R_n(x)$ and $I_n(x)$ are both log-concave.*

5. Generating functions

We present in the present section the exponential generating functions of $P_n(q,x,y)$ and $V_n(q,x,y)$. Towards this end, we define

$$P(q,x,y,z) := \sum_{n\geqslant 0} P_n(q,x,y)\frac{z^n}{n!} \qquad \text{and} \qquad V(q,x,y,z) := \sum_{n\geqslant 0} V_n(q,x,y)\frac{z^n}{n!}.$$

Instead of computing $P(q,x,y,z)$ and $V(q,x,y,z)$ directly, we first consider $P(q,x,0,z)$ and $V(q,x,0,z)$ since their recurrence relations involve less partial derivatives.

Theorem 9. *The generating functions $P = P(q,x,0,z)$ and $V = V(q,x,0,z)$ satisfy the following partial differential equations (PDEs)*

$$\begin{aligned}(1-qz)\frac{\partial P}{\partial z} + 2q(q-1)\frac{\partial P}{\partial q} + x(q-1)\frac{\partial P}{\partial x} &= xzP,\\ (1-qz)\frac{\partial V}{\partial z} + 2q(q-1)\frac{\partial V}{\partial q} + 2x(q-1)\frac{\partial V}{\partial x} &= xzV,\end{aligned}$$

and whose solutions are

$$\begin{aligned}P(q,x,0,z) &= e^{-xz}\left[\left(\frac{\sqrt{q}-1}{\sqrt{q}+1}\right)\left(\frac{\sqrt{q}+\cos(z\sqrt{q-1})+\sqrt{q-1}\sin(z\sqrt{q-1})}{\sqrt{q}-\cos(z\sqrt{q-1})-\sqrt{q-1}\sin(z\sqrt{q-1})}\right)\right]^{\frac{x}{2\sqrt{q}}},\\ V(q,x,0,z) &= e^{-xz/q}\left(\frac{\sqrt{q-1}}{\sqrt{q-1}\cos(z\sqrt{q-1})-\sin(z\sqrt{q-1})}\right)^{\frac{x}{q}},\end{aligned}$$

respectively.

Proof. We first prove the assertions for $V = V(q,x,0,z)$. Multiplying (4) by $z^n/n!$, followed by summing over $n \geqslant 1$, and using the fact that $V_1(q,x,0) = 0$, we get

$$\begin{aligned}\sum_{n\geqslant 1} V_{n+1}(q,x,0)\frac{z^n}{n!} = \sum_{n\geqslant 1}\Bigg[& nqV_n(q,x,0) + 2q(1-q)\frac{\partial V_n(q,x,0)}{\partial q} + 2x(1-q)\frac{\partial V_n(q,x,0)}{\partial x}\\ & + nxV_{n-1}(q,x,0)\Bigg]\frac{z^n}{n!}\end{aligned}$$

whence the PDE for V:

$$(1-qz)\frac{\partial V}{\partial z} + 2q(q-1)\frac{\partial V}{\partial q} + 2x(q-1)\frac{\partial V}{\partial x} = xzV. \tag{6}$$

In view of the so-called β-extension [20] (see also [19, §7] for an instance of it), we may assume that $V(q,x,0,z) = v(q,z)^x$ for some function v. Then

$$\frac{\partial V}{\partial z} = xv^{x-1}\frac{\partial v}{\partial z}, \quad \frac{\partial V}{\partial q} = xv^{x-1}\frac{\partial v}{\partial q}, \quad \frac{\partial V}{\partial z} = v^x \ln v$$

so that (6) becomes

$$(1-qz)\frac{\partial v}{\partial z}+2q(q-1)\frac{\partial v}{\partial q}=zv-2(q-1)v\ln v. \tag{7}$$

Next, we let $w=\ln v$. Then

$$\frac{1}{v}\frac{\partial v}{\partial z}=\frac{\partial w}{\partial z}, \qquad \frac{1}{v}\frac{\partial v}{\partial q}=\frac{\partial w}{\partial q}$$

and (7) becomes the following linear PDE:

$$(1-qz)\frac{\partial w}{\partial z}+2q(q-1)\frac{\partial w}{\partial q}=z-2(q-1)w \tag{8}$$

with auxiliary system

$$\frac{dz}{1-qz}=\frac{dq}{2q(q-1)}=\frac{dw}{z-2(q-1)w}.$$

Solving the ordinary differential equation arising from the first equality, namely,

$$\frac{dz}{dq}+\frac{z}{2(q-1)}=\frac{1}{2q(q-1)},$$

we obtain the characteristic defined by

$$c_1=\tan^{-1}\sqrt{q-1}-z\sqrt{q-1},$$

where c_1 is an arbitrary constant. On this characteristic, (8) becomes the following linear ODE:

$$2q(q-1)\frac{dw}{dq}=\frac{\tan^{-1}\sqrt{q-1}-c_1}{\sqrt{q-1}}-2(q-1)w$$

whose solution is

$$\begin{aligned} w(q,z)&=-\frac{z}{q}+\frac{1}{2q}\ln\left(\frac{q-1}{q}\right)+\frac{c_2}{q}\\ &=-\frac{z}{q}+\frac{1}{2q}\ln\left(\frac{q-1}{q}\right)+\frac{f(\tan^{-1}\sqrt{q-1}-z\sqrt{q-1})}{q}\end{aligned}$$

where $c_2=f(c_1)$ and f is a function to be determined.

Since $V(q,x,0,0)=v(q,0)^x=1$ for all x, it follows that $v(q,0)=1$ so that

$$f(\tan^{-1}\sqrt{q-1})=\frac{1}{2}\ln\left(\frac{q}{q-1}\right).$$

Letting $u=\tan^{-1}\sqrt{q-1}$, we then have $q=1+\tan^2 u=\sec^2 u$ and hence

$$f(u)=\frac{1}{2}\ln\left(\frac{\sec^2 u}{\sec^2 u-1}\right)=-\ln\sin u.$$

Since

$$\begin{aligned} f(\tan^{-1}\sqrt{q-1}-z\sqrt{q-1})&=-\ln\sin(z\sqrt{q-1}-z\sqrt{q-1})\\ &=\ln\left(\frac{\sqrt{q}}{\sqrt{q-1}\cos(z\sqrt{q-1})-\sin(z\sqrt{q-1})}\right),\end{aligned}$$

we obtain that

$$V(q,x,0,z)=v(q,z)^x=e^{wx}=e^{-xz/q}\left(\frac{\sqrt{q-1}}{\sqrt{q-1}\cos(z\sqrt{q-1})-\sin(z\sqrt{q-1})}\right)^{\frac{x}{q}}.$$

By imitating the above calculations, one can obtain the corresponding assertions for $P=P(q,x,0,z)$, whose details are left to the interested readers. $\square$

Corollary 10. *The exponential generating functions $P(q,x,y,z)$ and $V(q,x,y,z)$ have the following explicit expressions*:

$$P(q,x,y,z)=e^{xz(y-1)}\left[\left(\frac{\sqrt{q}-1}{\sqrt{q}+1}\right)\left(\frac{\sqrt{q}+\cos(z\sqrt{q-1})+\sqrt{q-1}\sin(z\sqrt{q-1})}{\sqrt{q}-\cos(z\sqrt{q-1})-\sqrt{q-1}\sin(z\sqrt{q-1})}\right)\right]^{\frac{x}{2\sqrt{q}}},$$
$$V(q,x,y,z)=e^{xz(y-1/q)}\left(\frac{\sqrt{q-1}}{\sqrt{q-1}\cos(z\sqrt{q-1})-\sin(z\sqrt{q-1})}\right)^{\frac{x}{q}},$$

respectively.

Proof. Let $\sigma\in\mathfrak{S}_n$ have k fixed points. There are $\binom{n}{k}$ choices of fixed points. If τ is the partial permutation obtained by deleting all the fixed points of σ, then $\operatorname{cyc}(\sigma)=\operatorname{cyc}(\tau)+k$ and $\operatorname{cval}(\sigma)=\operatorname{cval}(\tau)$ since only ℓ-cycles ($\ell\geqslant 3$) of σ contribute to $\operatorname{cval}(\sigma)$. Thus, $V_n(q,x,y)=\sum_{k=0}^{n}\binom{n}{k}V_{n-k}(q,x,0)(xy)^k$ follows. Hence,

$$\begin{aligned}\sum_{n\geqslant 0}V_n(q,x,y)\frac{z^n}{n!}&=\sum_{n\geqslant 0}\sum_{k=0}^{n}\binom{n}{k}V_{n-k}(q,x,0)(xy)^k\frac{z^n}{n!}\\&=\sum_{k\geqslant 0}\frac{(xyz)^k}{k!}\sum_{n\geqslant k}V_{n-k}(q,x,0)\frac{z^{n-k}}{(n-k)!}\\&=e^{xyz}e^{-xz/q}\left(\frac{\sqrt{q-1}}{\sqrt{q-1}\cos(z\sqrt{q-1})-\sin(z\sqrt{q-1})}\right)^{\frac{x}{q}}.\end{aligned}$$

thus proving the assertion for $V(q,x,y,z)$. The corresponding assertion for $P(q,x,y,z)$ follows from similar consideration. $\square$

In Section 4, we combinatorially prove that the nth Bell number B_n is the constant term of $M_n(0)$, i.e.,

$$M_n(0)=M_{n,0}=B_n.$$

We now present a generating function proof of this result. Since $M_n(q)=P_n(q,1,1)$, we have

$$\begin{aligned}\sum_{n\geqslant 0}M_n(0)\frac{z^n}{n!}&=\lim_{q\to 0}\sum_{n\geqslant 0}P_n(q,1,1)\frac{z^n}{n!}\\&=\lim_{q\to 0}\left[\left(\frac{\sqrt{q}-1}{\sqrt{q}+1}\right)\left(\frac{\sqrt{q}+\cos(z\sqrt{q-1})+\sqrt{q-1}\sin(z\sqrt{q-1})}{\sqrt{q}-\cos(z\sqrt{q-1})-\sqrt{q-1}\sin(z\sqrt{q-1})}\right)\right]^{\frac{1}{2\sqrt{q}}}.\end{aligned}$$

Denote the limit on the right by L. It is easy to see that L is of the indeterminate form 1^∞. So, by l'Hôpital's rule, we have

$$\begin{aligned}\ln L&=\lim_{q\to 0}\frac{1}{2\sqrt{q}}\ln\left(\frac{\sqrt{q}-1}{\sqrt{q}+1}\right)\left(\frac{\sqrt{q}+\cos(z\sqrt{q-1})+\sqrt{q-1}\sin(z\sqrt{q-1})}{\sqrt{q}-\cos(z\sqrt{q-1})-\sqrt{q-1}\sin(z\sqrt{q-1})}\right)\\&=\cosh z+\sinh z-1=e^z-1.\end{aligned}$$

Consequently,

$$\sum_{n\geqslant 0}M_n(0)\frac{z^n}{n!}=e^{e^z-1},$$

the right side being the exponential generating function of B_n, thus proving $M_n(0)=B_n$.

Let $i=\sqrt{-1}$. Note that $\cosh(x)=\cos(ix)$ and $\sinh(x)=-i\sin(ix)$. Combining (1) and Corollary 10, we get the following result.

Theorem 11. *We have*

$$V(q,x,y,z) = e^{xz(y-1/q)}\overline{W}(q,z)^{\frac{x}{q}}.$$

6. A Pell number identity

The Pell numbers P_n are defined by the recurrence relation

$$P_n = 2P_{n-1} + P_{n-2}, \qquad n = 2, 3, \ldots,$$

with initial values $P_0 = 0$ and $P_1 = 1$. Other known facts about the Pell numbers include the Binet formula: for $n = 0, 1, 2, \ldots$,

$$P_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}},$$

and the exponential generating function:

$$\sum_{n=0}^{\infty} P_n \frac{y^n}{n!} = \frac{e^{(1+\sqrt{2})y} - e^{(1-\sqrt{2})y}}{2\sqrt{2}}.$$

The Pell numbers can be expressed as (-1)-evaluation of $V_n(q,1,0)$, as the next theorem shows.

Theorem 12. *We have for $n \geqslant 1$,*

$$\sum_{\sigma \in \mathcal{D}_n} (-1)^{\mathrm{cval}\,(\sigma)} = P_{n-1}. \tag{9}$$

Proof. By setting $x = 1$ and $q = -1$ in $V(q,x,0,z)$, we see that

$$\begin{aligned}
V(-1,1,0,z) &= e^z\Big(\sqrt{-2}\cos(z\sqrt{-2}) - \sin(z\sqrt{-2})\Big)(-2)^{-1/2} \\
&= \frac{e^z[(\sqrt{2}+1)e^{-\sqrt{2}z} + (\sqrt{2}-1)e^{\sqrt{2}z}]}{2\sqrt{2}} \\
&= 1 + \sum_{n\geqslant 1} P_{n-1}\frac{z^n}{n!},
\end{aligned}$$

since

$$\begin{aligned}
\sum_{n\geqslant 1} P_{n-1}\frac{z^n}{n!} &= \int_0^z \sum_{n\geqslant 0} P_n \frac{s^n}{n!}\, ds \\
&= \frac{1}{2\sqrt{2}}\int_0^z (e^{(1+\sqrt{2})s} - e^{(1-\sqrt{2})s})\, ds \\
&= \frac{e^z[(\sqrt{2}-1)e^{\sqrt{2}z} + (\sqrt{2}+1)e^{-\sqrt{2}z}]}{2\sqrt{2}} - 1.
\end{aligned}$$

Equating the coefficients of z^n on both sides, (9) follows. □

7. Relationship to alternating runs

Let $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$. We say that π changes direction at position i if either $\pi(i-1) < \pi(i) > \pi(i+1)$, or $\pi(i-1) > \pi(i) < \pi(i+1)$, where $i \in \{2, 3, \ldots, n-1\}$. We say that π has k *alternating runs* if there are $k-1$ indices i such that π changes direction at these positions. Let $R(n,k)$ denote the number of permutations in $\mathfrak{S}_n$ with k alternating runs. André [1] found that the numbers $R(n,k)$ satisfy the following recurrence relation

$$R(n,k) = kR(n-1,k) + 2R(n-1,k-1) + (n-k)R(n-1,k-2) \tag{10}$$

for $n, k \geqslant 1$, where $R(1,0) = 1$ and $R(1,k) = 0$ for $k \geqslant 1$.

For $n \geqslant 1$, we define $R_n(q) = \sum_{k=1}^{n-1} R(n,k)q^k$. Then by (10), we obtain

$$R_{n+2}(q) = q(nq+2)R_{n+1}(q) + q\left(1-q^2\right)R'_{n+1}(q), \tag{11}$$

with initial value $R_1(q) = 1$. The first few terms of $R_n(q)$'s are given as follows:

$$\begin{aligned}
R_2(q) &= 2q,\\
R_3(q) &= 2q + 4q^2,\\
R_4(q) &= 2q + 12q^2 + 10q^3,\\
R_5(q) &= 2q + 28q^2 + 58q^3 + 32q^4.
\end{aligned}$$

There is a large literature devoted to the numbers $R(n,k)$ (see [18, A059427]). The reader is referred to [4, 13, 21] for recent progress on this subject. In a series of papers [5, 6, 7], Carlitz studied generating functions for the numbers $R(n,k)$. In particular, Carlitz [5] proved that

$$H(q,z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{k=0}^{n} R(n+1,k)q^{n-k} = \left(\frac{1-q}{1+q}\right)\left(\frac{\sqrt{1-q^2} + \sin(z\sqrt{1-q^2})}{q - \cos(z\sqrt{1-q^2})}\right)^2.$$

Instead of considering $\sum_{k=0}^{n} R(n+1,k)q^{n-k}$ as Carlitz did, we shall study

$$R(q,z) = \sum_{n\geqslant 0} R_{n+1}(q)\frac{z^n}{n!}.$$

It is clear that $R(q,z) = H(\frac{1}{q}, qz)$. Hence

$$R(q,z) = \left(\frac{q-1}{q+1}\right)\left(\frac{\sqrt{q^2-1} + q\sin(z\sqrt{q^2-1})}{1 - q\cos(z\sqrt{q^2-1})}\right)^2.$$

Let $u = \sec^{-1} q$, i.e., $\sec u = q$. Then

$$\begin{aligned}\left(\frac{\sqrt{q^2-1}+q\sin(z\sqrt{q^2-1})}{1-q\cos(z\sqrt{q^2-1})}\right)^2 &= \left(\frac{\sin u+\sin(z\sqrt{q^2-1})}{\cos u-\cos(z\sqrt{q^2-1})}\right)^2\\ &= \frac{\cos^2\left(\frac{u-z\sqrt{q^2-1}}{2}\right)}{\sin^2\left(\frac{u-z\sqrt{q^2-1}}{2}\right)}\\ &= \frac{1+\cos(u-z\sqrt{q^2-1})}{1-\cos(u-z\sqrt{q^2-1})}\\ &= \frac{q+\cos(z\sqrt{q^2-1})+\sqrt{q^2-1}\sin(z\sqrt{q^2-1})}{q-\cos(z\sqrt{q^2-1})-\sqrt{q^2-1}\sin(z\sqrt{q^2-1})},\end{aligned}$$

where in the second, third and fourth equality, we have applied the following results:

$$\begin{aligned}\sin u+\sin(z\sqrt{q^2-1}) &= 2\sin\left(\frac{u+z\sqrt{q^2-1}}{2}\right)\cos\left(\frac{u-z\sqrt{q^2-1}}{2}\right),\\ \cos u-\cos(z\sqrt{q^2-1}) &= -2\sin\left(\frac{u+z\sqrt{q^2-1}}{2}\right)\sin\left(\frac{u-z\sqrt{q^2-1}}{2}\right),\\ \cos(u-z\sqrt{q^2-1}) &= 2\cos^2\left(\frac{u-z\sqrt{q^2-1}}{2}\right)-1 = 1-2\sin^2\left(\frac{u-z\sqrt{q^2-1}}{2}\right),\\ \cos(u-z\sqrt{q^2-1}) &= \frac{\cos(z\sqrt{q^2-1})+\sqrt{q^2-1}\sin(z\sqrt{q^2-1})}{q},\end{aligned}$$

respectively. Thus, we have

$$R(q,z) = \left(\frac{q-1}{q+1}\right)\left(\frac{q+\cos(z\sqrt{q^2-1})+\sqrt{q^2-1}\sin(z\sqrt{q^2-1})}{q-\cos(z\sqrt{q^2-1})-\sqrt{q^2-1}\sin(z\sqrt{q^2-1})}\right). \tag{12}$$

Alternatively, by imitating the proof of Theorem 9, it can be shown that $R = R(q,z)$ satisfies the following linear partial differential equation

$$(1-zq^2)\frac{\partial R}{\partial z} + q(q^2-1)\frac{\partial R}{\partial q} = 2qR,$$

with initial condition $R(q,0) = R_1(q) = 1$ and whose solution is (12).

Combining (12) and Corollary 10, we get the following result.

Theorem 13. *We have*

$$P(q,x,y,z) = e^{xz(y-1)}R(\sqrt{q},z)^{\frac{x}{2\sqrt{q}}}.$$

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