TWO NEW TRIANGLES OF q-INTEGERS VIA q-EULERIAN POLYNOMIALS OF TYPE A AND B

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ABSTRACT. The classical Eulerian polynomials can be expanded in the basis $t^{k-1}(1+t)^{n+1-2k}$ $(1 \le k \le \lfloor (n+1)/2 \rfloor)$ with positive integral coefficients. This formula implies both the symmetry and the unimodality of the Eulerian polynomials. In this paper, we prove a q-analogue of this expansion for Carlitz's q-Eulerian polynomials as well as a similar formula for Chow-Gessel's q-Eulerian polynomials of type B. We shall give some applications of these two formulae, which involve two new sequences of polynomials in the variable q with positive integral coefficients. An open problem is to give a combinatorial interpretation for these polynomials.

1. Introduction

The Eulerian polynomials $A_n(t) := \sum_{k=1}^n A_{n,k} t^{k-1}$ (see [FS70, Fo09, St97]) may be defined by

$$\sum_{k>1} k^n t^k = \frac{A_n(t)}{(1-t)^{n+1}} \qquad (n \in \mathbb{N}).$$

It is well known (see [FS70,SGW83]) that there are nonnegative integers $a_{n,k}$ such that

$$A_n(t) = \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} a_{n,k} t^{k-1} (1+t)^{n+1-2k}.$$
 (1.1)

For example, for n = 1, ..., 4, the identity reads

$$A_1(t) = 1$$
, $A_2(t) = 1 + t$, $A_3(t) = (1+t)^2 + 2t^2$, $A_4(t) = (1+t)^3 + 8t(1+t)$.

In particular, this formula implies both the *symmetry* and the *unimodality* (see for instance [Br08] for the definitions) of the Eulerian numbers $(A_{n,k})_{1 \leq k \leq n}$ for any fixed n. The coefficients $a_{n,k}$ defined by (1.1) satisfy the following recurrence relation:

$$a_{n,k} = ka_{n-1,k} + 2(n+2-2k)a_{n-1,k-1}$$
(1.2)

for $n \ge 2$ and $1 \le k \le \lfloor (n+1)/2 \rfloor$, with $a_{1,1} = 1$, and $a_{n,k} = 0$ for $k \le 0$ or $k > \lfloor (n+1)/2 \rfloor$.

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$n \backslash k$	1	2	3	4		$n \backslash k$	0	1	2	3
1					•	1	1			
2	1					2	1	4		
3	1	2				3	1	20		
4	1	8	16			4	1	72	80	
5	1	22	16					232		
6	1	52	136			6	1	716	766	3904

Table 1. The first values of $(a_{n,k})$ and $(b_{n,k})$

The classical Eulerian polynomials are the descent polynomials of the symmetric group or Coxeter group of type A. Analogues of Eulerian polynomials for other Coxeter groups were introduced and studied from a combinatorial point of view in the last three decades. For instance, the Eulerian polynomials $B_n(t)$ of type B are defined by

$$\sum_{n\geq 0} (2k+1)^n t^n = \frac{B_n(t)}{(1-t)^{n+1}}.$$
(1.3)

The type B version of (1.1) appeared quite recently (see [Pe07, St08, Ch08]) and reads as follows

$$B_n(t) = \sum_{k=0}^{\lfloor n/2 \rfloor} b_{n,k} t^k (1+t)^{n-2k}, \tag{1.4}$$

where $b_{n,k}$ are positive integers satisfying the recurrence relation

$$b_{n,k} = (2k+1)b_{n-1,k} + 4(n+1-2k)b_{n-1,k-1}, (1.5)$$

for $n \geq 2$ and $0 \leq k \leq \lfloor n/2 \rfloor$, with $b_{1,0} = 1$, and $b_{n,k} = 0$ for $k \leq 0$ or $k > \lfloor n/2 \rfloor$.

The numbers $a_{n,k}$ and $4^{-k}b_{n,k}$ appear as A101280 and A008971 in The On-Line Encyclopedia of Integer Sequences: http://oeis.org.

The aim of this paper is to prove a q-analogue of (1.1) with a refinement of the triangle $(a_{n,k})$ for Carlitz's q-Eulerian polynomials [Ca75], and also a q-analogue of (1.4) with a refinement of the triangle $(b_{n,k})$ for Chow-Gessel's q-Eulerian polynomials of type B [CG07]. Note that some other extensions of (1.1) are discussed in [Br08,SW10,SZ10].

This paper is organized as follows: we derive in Section 2 a q-analogue of (1.1) using Carlitz's q-Eulerian polynomials and derive some results about the q-tangent number $T_{2n+1}(q)$ studied in [FH09]. In Section 3, we give a q-analogue of (1.4) using Chow-Gessel's q-Eulerian polynomials of type B, which yields new q-analogues of the secant numbers. In Section 4, we apply our constructions to some conjectures on the unimodality from [CG07]. Finally, we will briefly give some concluding remarks in the fifth and last section.

2. A q-analogue for type A

The q-binomial coefficient $\binom{n}{k}_q$ is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, \qquad n \ge k \ge 0,$$

where $(x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1})$ and $(x;q)_0 = 1$. Recall [Ca54] that Carlitz's q-Eulerian polynomials $A_n(t,q) := \sum_{k=1}^{n-1} A_{n,k}(q) t^k$ can be defined by

$$\sum_{k>0} [k+1]_q^n t^k = \frac{A_n(t,q)}{(t;q)_{n+1}},\tag{2.1}$$

where $[n]_q = 1 + q + \cdots + q^{n-1}$. It is easy to see that $A_{n,k}(q)$ satisfy the recurrence:

$$A_{n,k}(q) = [k]_q A_{n-1,k}(q) + q^{k-1}[n+1-k]_q A_{n-1,k-1}(q) \qquad (1 \le k \le n).$$
 (2.2)

The following is our q-analogue of (1.1).

Theorem 1. For any positive integer n, there are polynomials $a_{n,k}(q) \in \mathbb{N}[q]$ such that the q-Eulerian polynomials $A_n(t,q)$ can be written as follows:

$$A_n(t,q) = \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} a_{n,k}(q) t^{k-1} (-tq^k; q)_{n+1-2k}.$$
 (2.3)

Moreover, the polynomials $a_{n,k}(q)$ satisfy the following recurrence relation

$$a_{n,k}(q) = [k]_q a_{n-1,k}(q) + (1+q^{k-1})q^{k-1}[n+2-2k]_q a_{n-1,k-1}(q)$$
(2.4)

for $n \geq 2$ and $1 \leq k \leq \lfloor (n+1)/2 \rfloor$, with $a_{1,1}(q) = 1$ and $a_{n,k}(q) = 0$ for $k \leq 0$ or $k > \lfloor (n+1)/2 \rfloor$.

Proof. Assume that $a_{n,k}(q)$ are coefficients satisfying (2.4). Then, by the q-binomial formula (cf. [An98, Theorem 3.3]),

$$(z;q)_N = \sum_{j=0}^N {N \brack j}_q (-z)^j q^{j(j-1)/2}, \qquad (2.5)$$

we see that (2.3) is equivalent to:

$$A_{n,k}(q) = \sum_{s>1} {n+1-2s \brack k-s}_q q^{(k-s)s+\binom{k-s}{2}} a_{n,s}(q).$$
 (2.6)

Substituting (2.6) in (2.2), and using (2.4), we derive:

$$\sum_{s\geq 1} {n+1-2s \brack k-s}_q q^{(k-s)s+{k-s \choose 2}} \left([s]_q a_{n-1,s}(q) + (1+q^{s-1})q^{s-1}[n+2-2s]_q a_{n-1,s-1}(q) \right)$$

$$= \sum_{s\geq 1} q^{(k-s)s+{k-s \choose 2}} \left([k]_q {n-2s \brack k-s}_q + [n+1-k]_q {n-2s \brack k-1-s}_q \right) a_{n-1,s}(q).$$

Extracting the coefficients of $a_{n-1,s}(q)$ we obtain:

$$\begin{bmatrix} n+1-2s \\ k-s \end{bmatrix}_q [s]_q + \begin{bmatrix} n-1-2s \\ k-s-1 \end{bmatrix}_q (1+q^s)[n-2s]_q$$

$$= [k]_q \begin{bmatrix} n-2s \\ k-s \end{bmatrix}_q + [n+1-k]_q \begin{bmatrix} n-2s \\ k-1-s \end{bmatrix}_q.$$

Canceling the common factors we get:

$$[n+1-2s]_q[s]_q+[n-k-s+1]_q(1+q^s)[k-s]_q=[k]_q[n-k-s+1]_q+[n+1-k]_q[k-s]_q.$$
 The last identity is easy to verify, and this shows that (2.3) is satisfied.

The first values of the coefficients $a_{n,k}(q)$ read as follows:

In [FH09] Foata and Han defined a new sequence of q-tangent numbers $T_{2n+1}(q)$ by

$$T_{2n+1}(q) = (-1)^n q^{\binom{n}{2}} A_{2n+1}(-q^{-n}, q).$$
(2.7)

We derive easily the following result from Theorem 1, which is the most difficult part of the main result in [FH09, Theorem 1.1].

Corollary 2. The q-tangent number $T_{2n+1}(q)$ is a polynomial with positive integral coefficients.

Proof. Let
$$a_{n,k}^*(q) = q^{-k(k+1)/2} a_{n,k}(q)$$
. Then (2.4) becomes
$$a_{n,k}^*(q) = [k]_q a_{n-1,k}^*(q) + (1+q^{k-1})[n+2-2k]_q a_{n-1,k-1}^*(q)$$

with the same initial conditions as $a_{n,k}(q)$. This proves that $a_{n,k}^*(q)$ is a polynomial in q with nonnegative integral coefficients. Now we show that $T_{2n+1}(q) = a_{2n+1,n+1}^*(q)$, which

is sufficient to conclude. Replacing n by 2n + 1, k by n + 1, and t by $-q^{-n}$ in (2.3), we get

$$A_{2n+1}(-q^{-n},q) = \sum_{k=1}^{n+1} a_{2n+1,k}(q)(-q^{-n})^{k-1}(q^{k-n};q)_{2n+2-2k} = a_{2n+1,n+1}(q)(-q^{-n})^n,$$

since $(q^{k-n};q)_{2n+2-2k}=0$ for $k=1,2,\ldots,n$. The result follows then from (2.7).

We can also derive straightforwardly the following result, which was proved in [FH09] using combinatorics of the so-called *doubloons*.

Corollary 3. The quotient $A_{2n}(t,q)/(1+tq^n)$ is a polynomial in t and q with positive integral coefficients.

Proof. Note that

$$A_{2n}(t,q) = \sum_{k=1}^{n} a_{2n,k}(q)t^{k-1}(-tq^{k};q)_{2n+1-2k}.$$

The result follows then from the fact that for $k=1,\ldots,n$, the coefficient $(-tq^k;q)_{2n+1-2k}=(1+tq^k)\cdots(1+tq^{2n-k})$ contains the factor $1+tq^n$.

For any nonnegative integer n, set

$$f_n(q) := \sum_{k=0}^{2n+1} {2n+1 \choose k} \frac{(-1)^k}{1+q^{k-n}}.$$
 (2.8)

Using the doubloon model, Foata-Han [FH09] proved that

$$d_n(q) := \frac{T_{2n+1}(q)}{(1+q)(1+q^2)\dots(1+q^n)} = \frac{(-1)^{n+1}(-1;q)_{n+2}}{(1-q)^{2n+1}} f_n(q)$$

is a polynomial in $\mathbb{N}[q]$. Actually we can prove the integrality of $d_n(q)$ without using the combinatorial device.

Proposition 4. We have $d_n(q) \in \mathbb{Z}[q]$.

Proof. Let $g_n(q) = (-1)^{n+1}(-1;q)_{n+2}$. Then $f_n(q)g_n(q)$ is clearly a polynomial in $\mathbb{Z}[q]$. We must show that 1 is a zero of order 2n+1 of the polynomial $f_n(q)g_n(q)$ or

$$d^{p}(f_{n}(q)g_{n}(q))/dq^{n}|_{q=1} = 0$$
 for $p = 0, ..., 2n$.

By Leibniz's rule it suffices to show that $f_n^{(p)}(1) = 0$ for $p = 0, \dots, 2n$.

For any $k \in \mathbb{Z}$ and $m \in \mathbb{N}$, we define the Laurent polynomial $P_{m,k}(x)$ by the relation:

$$h_k^{(m)}(x) = \left(\frac{d}{dx}\right)^m (1+x^k)^{-1} = \frac{P_{m,k}(x)}{(1+x^k)^{m+1}}.$$

Thus $P_{0,k} = 1$, $P_{1,k} = -kx^{k-1}$, and for $m \ge 0$, we have

$$P_{m+1,k}(x) = (1+x^k)P'_{m,k}(x) - k(m+1)x^{k-1}P_{m,k}(x).$$

Therefore the $P_{m,k}$ can, for $m \geq 1$, be written as follows:

$$P_{m,k}(x) = \sum_{l=1}^{m} \alpha_{l,m} x^{lk-m},$$

where $\alpha_{1,1} = -k$ and for $m \ge 1$, $\alpha_{1,m+1} = (k-m)\alpha_{1,m}$, $\alpha_{m+1,m+1} = (m-k)\alpha_{m,m}$,

$$\alpha_{l,m+1} = (lk - m)\alpha_{l,m} + (lk - mk - 2k - m)\alpha_{l-1,m}, \quad 2 \le l \le m.$$

This shows that for $m \geq 1$ and $1 \leq l \leq m$, the coefficient $\alpha_{l,m}$ is a polynomial in the variable k, with degree less than or equal to m. We deduce that $P_{m,k}(1) = \sum_{l=1}^{m} \alpha_{l,m}$ is also a polynomial in the variable k, with degree less than or equal to m, therefore we can write for some rational coefficients $a_i(m)$ only depending on m:

$$h_k^{(m)}(1) = \frac{P_{m,k}(1)}{2^{m+1}} = \sum_{j=0}^m a_j(m)k^j.$$

Thus, differentiating (2.8) m times $(m \ge 0)$ and then setting q = 1, we get

$$f_n^{(m)}(1) = \sum_{k=0}^{2n+1} {2n+1 \choose k} (-1)^k \sum_{j=0}^m a_j(m)(k-n)^j$$
$$= \sum_{j=0}^{2n} a_j(m) \sum_{k=0}^{2n+1} {2n+1 \choose k} (-1)^k (k-n)^j.$$

Now, applying 2n+1 times the finite difference operator Δ (defined by $\Delta f(x) := f(x+1) - f(x)$) to the polynomial $(n+1-x)^j$ $(0 \le j \le 2n)$ and setting x=0 we get

$$\Delta^{2n+1}(n+1-x)^j\Big|_{x=0} = \sum_{k=0}^{2n+1} {2n+1 \choose k} (-1)^k (k-n)^j,$$

which should vanish because $(n+1-x)^j$ is a polynomial in x of degree j < 2n+1. \square

3. A q-analogue for type B

A B_n -analogue of Carlitz's q-Eulerian polynomials are introduced by Chow and Gessel [CG07]. These polynomials $B_n(t,q)$ are defined by

$$\sum_{k\geq 0} [2k+1]_q^n t^k = \frac{B_n(t,q)}{(t;q^2)_{n+1}}.$$
(3.1)

Let $B_n(t,q) := \sum_{k=0}^n B_{n,k}(q) t^k$. Then, the coefficients $B_{n,k}(q)$ satisfy the recurrence relation [CG07, Prop. 3.2]:

$$B_{n,k}(q) = [2k+1]_q B_{n-1,k}(q) + q^{2k-1} [2n-2k+1]_q B_{n-1,k-1}(q) \qquad 1 \le k \le n.$$
 (3.2)

We have the following q-analogue of (1.4).

Theorem 5. For any positive integer n, there are polynomials $b_{n,k}(q) \in \mathbb{N}[q]$ such that the q-Eulerian polynomials of type B can be written as follows:

$$B_n(t,q) = \sum_{k=0}^{n} B_{n,k}(q)t^k = \sum_{k=0}^{\lfloor n/2 \rfloor} b_{n,k}(q)t^k (-tq^{2k+1}; q^2)_{n-2k}.$$
 (3.3)

Moreover, the coefficients $b_{n,k}(q)$ satisfy the following recurrence relation:

$$b_{n,k}(q) = [2k+1]_q b_{n-1,k}(q) + (1+q)(1+q^{2k-1})q^{2k-1}[n+1-2k]_{q^2} b_{n-1,k-1}(q)$$
(3.4)

for
$$n \ge 2$$
 and $0 \le k \le \lfloor n/2 \rfloor$, with $b_{1,0}(q) = 1$, and $b_{n,k}(q) = 0$ for $k < 0$ or $k > \lfloor n/2 \rfloor$.

Proof. Assume that $b_{n,k}(q)$ are coefficients satisfying (3.4). Then, by applying (2.5) with the substitution $q \leftarrow q^2$, we derive that (3.3) is equivalent to:

$$B_{n,k}(q) = \sum_{s>0} {n-2s \brack k-s}_{q^2} q^{k^2-s^2} b_{n,s}(q).$$
 (3.5)

Substituting (3.5) in (3.2), and using (3.4), we get:

$$\sum_{s>0} {n-2s \brack k-s}_{q^2} q^{k^2-s^2} \left([2s+1]_q b_{n-1,s}(q) + (1+q)(1+q^{2s-1})q^{2s-1} [n+1-2s]_{q^2} b_{n-1,s-1}(q) \right)$$

$$= \sum_{s>0} q^{k^2-s^2} \left([2k+1]_q \begin{bmatrix} n-1-2s \\ k-s \end{bmatrix}_{q^2} + [2n+1-2k]_q \begin{bmatrix} n-1-2s \\ k-1-s \end{bmatrix}_{q^2} \right) b_{n-1,s}(q).$$

Extracting the coefficients of $b_{n-1,s}(q)$ we obtain:

$$\begin{bmatrix} n-2s \\ k-s \end{bmatrix}_{q^2} [2s+1]_q + \begin{bmatrix} n-2-2s \\ k-s-1 \end{bmatrix}_{q^2} (1+q)(1+q^{2s+1})[n-1-2s]_{q^2}
= [2k+1]_q \begin{bmatrix} n-1-2s \\ k-s \end{bmatrix}_{q^2} + [2n+1-2k]_q \begin{bmatrix} n-1-2s \\ k-1-s \end{bmatrix}_{q^2}.$$

Canceling the common factors yields:

$$[n-2s]_{q^2}[2s+1]_q + [n-k-s]_{q^2}(1+q)(1+q^{2s+1})[k-s]_{q^2}$$

= $[2k+1]_q[n-k-s]_{q^2} + [2n+1-2k]_q[k-s]_{q^2}.$

The last identity is easy to verify, and this proves (3.3).

For n = 1, ..., 4, equation (3.3) reads:

$$B_1(t,q) = 1 + qt;$$

$$B_2(t,q) = (-tq;q^2)_2 + (q + 2q^2 + q^3)t;$$

$$B_3(t,q) = (-tq;q^2)_3 + (2q + 5q^2 + 6q^3 + 5q^4 + 2q^5)t(1 + tq^3);$$

$$B_4(t,q) = (-tq;q^2)_4 + (3q + 9q^2 + 15q^3 + 18q^4 + 15q^5 + 9q^6 + 3q^7)t(-tq^3;q^2)_2 + (2q^4 + 7q^5 + 11q^6 + 13q^7 + 14q^8 + 13q^9 + 11q^{10} + 7q^{11} + 2q^{12})t^2.$$

Theorem 5 implies immediately the following result, of which the first was derived in [FH10, Theorem 1.1 (d)] with more work.

Corollary 6. For $n \geq 0$, we have

$$B_{2n+1}(-q^{-2n-1},q) = 0, (3.6)$$

$$B_{2n}(-q^{-2n-1},q) = (-1)^n q^{-n(2n+1)} b_{2n,n}(q).$$
(3.7)

Proof. By (3.3) we get

$$B_{2n+1}(-q^{-2n-1},q) = \sum_{k=0}^{n} b_{2n+1,k}(q)(-q^{-2n-1})^k (q^{-2n+2k};q^2)_{2n+1-2k} = 0.$$

Substituting n by 2n and t by $-q^{-2n-1}$ in (3.3) yields

$$B_{2n}(-q^{-2n-1},q) = \sum_{k=0}^{n} b_{2n,k}(q)(-q^{-2n-1})^k (q^{-2n+2k};q^2)_{2n-2k}$$
$$= (-1)^n q^{-n(2n+1)} b_{2n,n}(q).$$

The above result leads to define a q-analogue of $B_{2n}(-1) = (-1)^n 4^n E_{2n}$ (where the E_{2n} 's are the famous secant numbers) by

$$E_{2n}^*(q) := (-1)^n q^{n(n+1)} B_{2n}(-q^{-2n-1}, q).$$
(3.8)

Theorem 7. There is a polynomial $G_{2n}^*(q) \in \mathbb{Z}[q]$ such that $G_{2n}^*(1) = E_{2n}$ and

$$E_{2n}^*(q) = (1+q)(1+q^3)(1+q^5)\cdots(1+q^{2n-1})\cdot(1+q)^n\cdot G_{2n}^*(q).$$

Proof. Recall that $E_{2n}^*(q) = (-1)^n q^{n(n+1)} B_{2n}(-q^{-2n-1}, q)$. From (3.1) we derive

$$\frac{B_n(t,q)}{(t;q^2)_{n+1}} = (1-q)^{-n} \sum_{j\geq 0} (1-q^{2j+1})^n t^j$$

$$= (1-q)^{-n} \sum_{j\geq 0} t^j \sum_{k=0}^n \binom{n}{k} (-q^{2j+1})^k$$

$$= (1-q)^{-n} \sum_{k=0}^n \binom{n}{k} \frac{(-q)^k}{1-tq^{2k}}.$$

Substituting n by 2n and setting $t = -q^{-2n-1}$ we obtain

$$E_{2n}^*(q) = (-1)^n q^{n(n+1)} \frac{(-q^{-2n-1}; q^2)_{2n+1}}{(1-q)^{2n}} \sum_{k=0}^{2n} {2n \choose k} \frac{(-q)^k}{1+q^{2k-2n-1}}.$$

Let

$$G_{2n}^*(q) := \frac{E_{2n}^*(q)}{(1+q)(1+q^3)\dots(1+q^{2n-1})(1+q)^n}$$
$$= (-1)^n q^{-n-1} \frac{(-q;q^2)_{n+1}}{(1+q)^n (1-q)^{2n}} \sum_{k=0}^{2n} {2n \choose k} \frac{(-q)^k}{1+q^{2k-2n-1}}.$$

For any nonnegative integer n, set

$$f_n^*(q) := \sum_{k=0}^{2n} {2n \choose k} \frac{(-q)^k}{1 + q^{2k-2n-1}}.$$
 (3.9)

Let $g_n^*(q) = (-1)^n q^{-n-1} (-q; q^2)_{n+1} / (1+q)^n$. Then $f_n^*(q) g_n^*(q)$ is clearly a polynomial in $\mathbb{Z}[q]$. We must show that 1 is a zero of order 2n of the polynomial $f_n^*(q) g_n^*(q)$ or

$$d^{p}(f_{n}^{*}(q)g_{n}^{*}(q))/dq^{n}|_{q=1} = 0$$
 for $p = 0, \dots, 2n-1$.

By Leibniz's rule it suffices to show that $d^p(f_n^*(q))/dq^p|_{q=1}=0$ for $p=0,\ldots,2n-1$. The rest of the proof is almost the same as that of Proposition 4, and is left to the reader. \square

Conjecture 8. All the coefficients of the polynomials $G_{2n}^*(q)$ are positive.

Since $G_{2n}^*(1) = E_{2n}$, the above conjecture would yield a new refinement of the secant number.

4. Application to unimodal problems

A sequence $\{\alpha_0, \ldots, \alpha_d\}$ is unimodal if there exists an index $0 \leq j \leq d$ such that $\alpha_i \leq \alpha_{i+1}$ for $i = 1, \ldots, j-1$ and $\alpha_i \geq \alpha_{i+1}$ for $i = j, \ldots, d$. Chow and Gessel [CG07] studied a kind of unimodality property of the q-Eulerian numbers assuming that q is a real number. In this section, we derive some unimodal properties of the sequences $(A_{n,k}(q))_{1\leq k\leq n}$ and $(B_{n,k}(q))_{1\leq k\leq n}$ from our previous results. From Theorem 1, we are able to deduce the following corollary, which provides a further support to Conjecture 4.8 in [CG07].

Proposition 9. Let $n \ge 2$ be an integer and $j = \lfloor (n+1)/2 \rfloor$. Then for $k = 1, \ldots, j-1$, we have $A_{n,k+1}(q) > A_{n,k}(q)$ if q > 1 and $A_{n,n-k+1}(q) < A_{n,n-k}(q)$ if q < 1.

Proof. We start from (2.6), which can be rewritten

$$A_{n,k}(q) = \sum_{s=1}^{k} {n+1-2s \brack k-s}_q q^{(k-s)(k+s-1)/2} a_{n,s}(q),$$

for k = 1, ..., n, where we assume $a_{n,s}(q) = 0$ for s > j. Thus we can write for k = 1, ..., j - 1:

$$A_{n,k+1}(q) - A_{n,k}(q) = a_{n,k+1}(q)$$

$$+ \sum_{s=1}^{k} {n+1-2s \brack k+1-s}_{q} q^{(k+1-s)(k+s)/2} a_{n,s}(q) \left(1 - q^{-k} \frac{1-q^{k+1-s}}{1-q^{n+1-k-s}}\right).$$

We know that the q-binomial coefficient is a polynomial in q with nonnegative integer coefficients, and from Theorem 1 that this is also true for $a_{n,s}(q)$, s = 1, ..., k + 1. Therefore it is enough to show that the coefficient between brackets is nonegative for $1 \le s \le k \le j - 1$. This coefficient can be rewritten as:

$$\frac{q^{n+1}-q^{k+s}+q^s-q^{k+1}}{q^{n+1}-q^{k+s}}.$$

Assume first that q > 1. As $k+s \le 2j-2 \le n-1 < n+1$, the denominator of this fraction is positive. Moreover, it is not difficult to see that under the conditions $1 \le s \le k \le j-1$, and by using $(n-1)/2 \le j \le (n+1)/2$, we have the following inequalities:

$$\begin{array}{ll} q^{n+1} - q^{k+s} + q^s - q^{k+1} & \geq & q^{n+1} - q^{2k} + q^k - q^{k+1} \\ & \geq & q^{n+1} - q^{2j-2} + q^{j-1} - q^j \\ & \geq & q^{n+1} - q^{n-1} + q^{(n-3)/2} - q^{(n+1)/2}. \end{array}$$

This last expression can be rewritten $(q^{(n+1)/2}-1)(q^{(n+1)/2}-q^{(n-3)/2})$ and is nonnegative, which shows that $A_{n,k+1}(q) \ge A_{n,k}(q)$ for $k=1,\ldots,j-1$.

In the case 0 < q < 1, we only need to use the well-known relation $A_{n,n-k+1}(q) = q^{n(n-1)/2}A_{n,k}(1/q)$ for any $k = 1, \ldots, n$, and the result is obvious from the case q > 1.

In the type B case, it is conjectured in [CG07, Conjecture 4.6] that the sequence $(B_{n,k}(q))_{0 \le k \le n}$ is unimodal. By Theorem 5, we are able to confirm partially this conjecture.

Proposition 10. Let $n \ge 2$ be an integer and $j = \lfloor n/2 \rfloor$. Then for $k = 1, \ldots, j - 1$, we have $B_{n,k+1}(q) > B_{n,k}(q)$ if q > 1 and $B_{n,n-k}(q) < B_{n,n-k-1}(q)$ if q < 1.

Proof. We start from (3.5), which can be rewritten

$$B_{n,k}(q) = \sum_{s=0}^{k} {n-2s \brack k-s}_{q^2} q^{k^2-s^2} b_{n,s}(q),$$

for k = 0, ..., n, where we assume $b_{n,s}(q) = 0$ for s > j. Thus we can write for k = 0, ..., j - 1:

$$B_{n,k+1}(q) - B_{n,k}(q) = b_{n,k+1}(q)$$

$$+\sum_{s=0}^{k} {n-2s \brack k+1-s}_{q^2} q^{(k+1)^2-s^2} b_{n,s}(q) \left(1-q^{-2k-1} \frac{1-q^{2(k+1-s)}}{1-q^{2(n-k-s)}}\right).$$

We know that the q-binomial coefficient is a polynomial in q with nonnegative integer coefficients, and from Theorem 5 that this is also true for $b_{n,s}(q)$, s = 0, ..., k + 1. Therefore it is enough to show that the coefficient between brackets is nonegative for $0 \le s \le k \le j - 1$. This coefficient can be rewritten as:

$$\frac{q^{2n} - q^{2s+2k} + q^{2s-1} - q^{2k+1}}{q^{2n} - q^{2k+2s}}.$$

Assume first that q > 1. As $k + s \le 2j - 2 \le n - 2 < n$, the denominator of this fraction is positive. Moreover, it is not difficult to see that under the conditions $0 \le s \le k \le j - 1$, and by using $n/2 - 1 \le j \le n/2$, we have the following inequalities:

$$\begin{array}{ll} q^{2n} - q^{2s+2k} + q^{2s-1} - q^{2k+1} & \geq q^{2n} - q^{4k} + q^{2k-1} - q^{2k+1} \\ & \geq q^{2n} - q^{4j-4} + q^{2j-3} - q^{2j-1} \\ & \geq q^{2n} - q^{2n-4} + q^{n-5} - q^{n-1}. \end{array}$$

This last expression can be rewritten $(q^{2n} - q^{n-1})(1 - q^{-4})$ and is nonnegative, which shows that $B_{n,k+1}(q) \ge B_{n,k}(q)$ for $k = 0, \ldots, j-1$.

In the case 0 < q < 1, we only need to use the well-known relation $B_{n,n-k}(q) = q^{n^2}B_{n,k}(1/q)$ for any $k = 0, \ldots, n$, and the result is obvious from the case q > 1.

5. An open problem on the combinatorial interpretations

By Theorems 1 and 5, the polynomials $a_{n,k}(q)$ and $b_{n,k}(q)$ have positive integral coefficients. It is then natural to ask the following question.

Problem 11. What are the combinatorial interpretations for $a_{n,k}(q)$ and $b_{n,k}(q)$?

We can give a combinatorial interpretation for the *odd central terms* $a_{2n+1,n+1}(q)$ by using the *doubloon* model. Recall [FH09] that a *doubloon* of order (2n+1) is defined to be a permutation of the word $012\cdots(2n+1)$, represented as a $2\times(n+1)$ -matrix $\delta=\binom{a_0\cdots a_n}{b_0\cdots b_n}$. Define

$$\operatorname{cmaj}' \delta := \operatorname{maj}(a_0 \cdots a_n b_n \cdots b_0) - (n+1) \operatorname{des}(a_0 \cdots a_n b_n \cdots b_0) + n^2,$$

where "des" and "maj" are the usual number of descents and major index defined for words. A doubloon $\delta = \begin{pmatrix} a_0 \cdots a_n \\ b_0 \cdots b_n \end{pmatrix}$ is said to be interlaced, if for every $k = 1, 2, \ldots, n$ the sequence $(a_{k-1}, a_k, b_{k-1}, b_k)$ or one of its three cyclic rearrangements is monotonic increasing or decreasing. By Theorem 1.5 in [FH09] we have the following result.

Proposition 12. The polynomial $a_{2n+1,n+1}(q)$ is the generating function for the set of interlaced doubloons of order 2n + 1 by the statistic cmaj'.

Another sequence of q-secant numbers is introduced in [FH10] by

$$E_{2n}(q) = (-1)^n q^{n^2} B_{2n}(-q^{-2n}, q).$$

Unfortunately, it seems not easy to relate our coefficients $b_{n,k}(q)$ from Section 3 to the doubloons of type B, even for the central cases.

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