# A Generalized Apéry Series

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**Abstract**

The inverse central binomial series

$$S_k(z)=\sum_{n=1}^{\infty}\frac{n^k z^n}{\binom{2n}{n}}$$

popularized by Apéry and Lehmer is evaluated for positive integers $k$ along with the asymptotic behavior for large $k$. It is found that value $z=2$, as commented on by D. H. Lehmer provides a unique relation to $\pi$.

## 1 Introduction

Since the appearance of $S_{-3}(1)$ in Apéry's famous proof [1] in 1979 that $\zeta(3)$ is irrational, an extensive literature has been devoted to the series

$$S_k(z)=\sum_{n=1}^{\infty}\frac{n^k z^n}{\binom{2n}{n}} \tag{1}$$

For example, in 1985 Lehmer [2] presented a number of special cases which could be obtained from the Taylor series for $f(x)=x^{-1/2}(1-x)^{-1/2}\sin^{-1}x$ using only elementary calculus. In passing, he noted that when $k$ is a positive integer, $S_k(2)$ had the form $a_k-b_k\pi$ and that the rational number $a_k/b_k$ "is a close approximation to $\pi$. This remark was recently taken up by Dyson et al. [3], who proved that $|a_k/b_k-\pi|=O(Q^{-k})$ as $k\to\infty$ where $Q=\sqrt{1+(2\pi/\ln 2)^2}$. Lehmer also showed that for positive integer $k$

$$S_k(z)=\frac{2^{k+z^{5/2}}z^{1/2}}{(4-z)^{k+3/2}}(A_k(z/4)\sin^{-1}(\sqrt{z/4})+\sqrt{z(4-z)}B_k(z/4)) \tag{2}$$

where $A_k$ and $B_k$ are recursively defined polynomials. It was apparently not until 2005 that (2) was evaluated explicitly, for $z=1$, by J. Borwein and P. Girgensohn [4] who showed

$$S_k(1)=\frac{1}{2}(-1)^{k+1}\sum_{j=1}^{k+1}(-1)^j j!S(j+1,j)3^{-j}\binom{2j}{j}\left(\sum_{i=1}^{j-1}\frac{3^i}{(2i+1)\binom{2i}{i}}+\frac{2}{3\sqrt{3}}\pi\right). \tag{3}$$

where the *Stirling numbers of the second kind* are defined by

$$S(k,j) = \frac{(-1)^j}{j!} \sum_{m=0}^{j} (-1)^m m^k \binom{j}{m}. \tag{4}$$

The aim of the present note is to extend (3) to complex $z$ and thus to continue (1) analytically beyond its circle of convergence $|z| = 4$.

# 2 Calculation

We begin with the observation that $\left(m\binom{2m}{m}\right)^{-1} = B(m, m+1)$, where B denotes Euler's beta integral. Hence,

$$S_k(z) = \int_0^1 \frac{dt}{t} \sum_{m=1}^{\infty} m^{k+1} (zt(1-t))^m. \tag{5}$$

Next, equation (21) of Girgensohn and Borwein [4],

$$\sum_{m=1}^{\infty} m^p X^m = \sum_{n=1}^{p} \sum_{m=1}^{n} (-1)^{m+n} \binom{n}{m} m^p X^n (1-X)^{-n-1}, \tag{6}$$

gives

$$S_k(z) = \sum_{n=1}^{k+1} \sum_{m=1}^{n} (-1)^{m+n} \binom{n}{m} m^{k+1} \int_0^1 \frac{dt}{t} \frac{(zt(1-t))^n}{(1-zt(1-t))^{n+1}}. \tag{7}$$

In the appendix it is shown that

$$\int_0^1 \frac{dt}{t} \frac{(zt(1-t))^n}{(1-zt(1-t))^{n+1}} = \frac{\sqrt{\pi}\Gamma(n)}{\Gamma(n+1/2)} X^n \, {}_2F_1(-1/2, n; n+1/2; -X) \tag{8}$$

where $X = z/(4-z)$, so

$$S_k(z) = \sum_{n=1}^{k+1} n! B(n, 1/2) S(k+1, n) X^n \, {}_2F_1(-1/2, n; n+1/2; -X). \tag{9}$$

By induction, starting with the tabulated value for $n = 1$ and using Gauss' contiguity relations we find (some details are given in the appendix)

$$\begin{aligned} &{}_2F_1(-1/2, n; n+1/2; -X) = \\ &\left(\frac{1}{2}\right)_n \left( \frac{1}{n!} + \frac{1}{\sqrt{\pi}\Gamma(n)} \sum_{k=0}^{n-1} \frac{(-1)^k \Gamma(k+1/2)}{(k+1)!} \binom{n-1}{k} \left(\frac{X+1}{X}\right)^{k+1} \times \right. \\ &\left. \left[ \sqrt{X} \sin^{-1} \sqrt{\frac{X}{X+1}} - \frac{1}{2} \sum_{l=1}^{k} \frac{(l-1)!}{(1/2)_l} \left(\frac{X}{X+1}\right)^l \right] \right). \end{aligned} \tag{10}$$

(We have used the ascending factorial notation $(a)_n = \Gamma(a+n)/\Gamma(a)$). Therefore we have the principal result

$$S_k(z) = \sum_{n=1}^{k+1} n! \left(\frac{z}{4-z}\right)^n S(k+1,n)\times$$

$$\left(\frac{1}{n} + \sum_{p=0}^{n-1}(-1)^p \frac{(1/2)_p}{(p+1)!}\binom{n-1}{p}\left(\frac{4}{z}\right)^{p+1}\left(\sqrt{\frac{z}{4-z}}\sin^{-1}\frac{\sqrt{z}}{2} - \frac{1}{2}\sum_{l=1}^{p}\frac{\Gamma(l)}{(1/2)_l}\left(\frac{z}{4}\right)^l\right)\right) \quad (11)$$

Equation (11) is rather condensed; in unpacking it, sums with upper limit less than the lower limit are to be interpreted as 0. It is clear from (11) that for *rational* $z$

$$\sum_{m=1}^{\infty} \frac{n^k z^n}{\binom{2n}{n}} = R_1(z,k) + R_2(z,k)\sqrt{\frac{z}{4-z}}\sin^{-1}\frac{\sqrt{z}}{2}, \quad (12)$$

where $R_j$ is a rational number.

One sees from (11) that $S_k(z)$ is analytic on the two-sheeted Riemann surface formed by two planes cut and rejoined along the real half-line $x > 4$. The numbers in (12) have the explicit expressions

$$R_1(z,k) = \quad (13)$$

$$\sum_{n=1}^{k+1} n! S(k+1,n)\left(\frac{z}{4-z}\right)^n \left(\frac{1}{n} - \frac{1}{2}\sum_{p=1}^{n-1}\sum_{l=1}^{p}\frac{(-1)^p (1/2)_p}{(p+1)!(1/2)_l}\binom{n-1}{p}\Gamma(l)\left(\frac{4}{z}\right)^{p-l+1}\right),$$

$$R_2(z,k) = \sum_{n=1}^{k+1} n! S(k+1,n)\sum_{p=0}^{n-1}\frac{(-1)^p}{(p+1)!}\binom{n-1}{p}\left(\frac{4}{z}\right)^{p+1}. \quad (14)$$

# 3 Asymptotics

It is convenient to work in terms of the exponential generating function

$$G(z,t) := \sum_{k=0}^{\infty} S_k(z)\frac{t^k}{k!} = S_0(ze^t) = \frac{z}{4-ze^t} + \frac{4\sqrt{z}e^{t/2}}{(4-ze^t)^{3/2}}\sin^{-1}\frac{\sqrt{z}e^{t/2}}{2} \quad (15)$$

To find the generating functions $\rho_j(z,t) := \sum R_j(z,k)t^k/k!$, it would be simplest to start with a series $D_k(z) = R_1(z,k) - R_2(z,k)\sqrt{\frac{z}{4-z}}\sin^{-1}\sqrt{z}/2$, work out its generating function $D(z,t)$ and by taking the sum and difference identify $\rho_1$ and $\rho_2$. However, this series has not been found and there is nothing to guarantee its existence in tractable form. Therefore, the $\rho_j$ were evaluated directly from (13) and (14). The details are omitted as the results

$$\begin{aligned}\rho_1(z,t) \quad &= \quad \frac{ze^t}{4-ze^t} + \\ &\frac{8}{\pi}\sqrt{\frac{ze^t}{(4-ze^t)^{3/2}}}\left(\sin^{-1}\frac{\sqrt{z}e^{t/2}}{2}\cos^{-1}\frac{\sqrt{z}}{2} - \cos^{-1}\frac{\sqrt{z}e^{t/2}}{2}\sin^{-1}\frac{\sqrt{z}}{2}\right), \quad (16)\end{aligned}$$

$$\rho_2(z,t) = 4\sqrt{\frac{(4-z)e^t}{(4-ze^t)^3}} \tag{17}$$

are easily verified. In the case $z = 2$, (15) and (16) are identical to Dyson's formulas [3, 5] obtained empirically.

In view of the prominent role that the ratio $R_1(z,t)/R_2(z,t)$ plays in Dyson et al. [3] for $z = 2$ it is interesting to examine it for general $z$. From (17) we have

$$R_2(z,k) = \frac{2k!\sqrt{4-z}}{\pi i}\oint \frac{ds}{s^{k+1}}\frac{e^{s/2}}{(4-ze^s)^{3/2}}. \tag{18}$$

The non-zero singularity closest to $s = 0$ is $s_0 = \ln(4/z)$ and it dominates the asymptotic behavior. Ignoring the other singularities, distorting the contour to a small circle about $s_0$ and translating back to the origin by $t = s - s_0$, we have

$$R_2(z,k) \sim -\frac{k!\sqrt{4-z}}{zs_0^{k+1}}\oint \frac{dt}{2\pi i}\frac{e^{t/2}}{(1-e^t))^{3/2}}. \tag{19}$$

The exact value of the integral in (19) is $-(2/\pi)\sqrt{e/(e-1)}$, and so

$$R_2(z,k) \sim \frac{k!}{(\ln(4/z))^{k+1}}\frac{2}{\pi}\sqrt{\frac{e(4-z)}{z(e-1)}}. \tag{20}$$

In the same way we obtain

$$R_1(z,k) \sim \frac{k!}{(\ln(4/z))^{k+1}}\left(\sqrt{2}+\frac{2}{\pi}\left(\sqrt{\frac{e}{e-1}}-\sqrt{2}\right)\cos^{-1}\frac{\sqrt{z}}{2}-\frac{2^{3/2}}{\pi}\sin^{-1}\frac{\sqrt{z}}{2}\right). \tag{21}$$

# 4 Discussion

From (20) and (21) we find

$$\lim_{k\to\infty}\left(\frac{R_1(z,k)}{R_2(z,k)}-\sqrt{\frac{z}{4-z}}\sin^{-1}\frac{\sqrt{z}}{2}\right) = \sqrt{\frac{z}{4-z}}\left(\cos^{-1}\frac{\sqrt{z}}{2}-\sin^{-1}\frac{\sqrt{z}}{2}\right). \tag{22}$$

It thus appears that Lehmer's choice, $z = 2$, is the unique permissible case for which the limit (22) vanishes. (Also the *Lehmer limit*, as defined by Dyson et al. [3], relates to $\pi/4$ here rather than $\pi$). Finally, for negative integer indices, since

$$2S_{-k}(z) = {}_{k+1}F_k(1,\ldots,1;\tfrac{3}{2},2,\ldots,2;\tfrac{1}{4}z), \tag{23}$$

the fact that $S_{-k}(z)$ can be obtained from $S_0(z)$ by successive integrations with respect to $z$ and the explicit evaluations by Lehmer [2], Borwein and Girgensohn [4] and others [6, 7, 8, 9, 10] it should be possible to obtain explicit values for sundry generalized hypergeometric functions.

# 5 Appendix: Derivation of Equations (8) and (10)

Let us consider, for any integrable function $F$,

$$I = \int_0^1 \frac{dt}{t} F(t(1-t))$$

Let $u = t(1-t)$, so $u(0) = u(1) = 0$; $u(1/2) = 1/4$. Then there are two expressions for $t$:

$$t_+ = \frac{1}{2}(1+\sqrt{1-4t}) \quad \text{for } \frac{1}{2} \leq t \leq 1, \text{ with } \frac{dt_+}{t_+} = \left(1 - \frac{1}{\sqrt{1-4u}}\right)\frac{du}{u}$$

and

$$t_- = \frac{1}{2}(1-\sqrt{1-4t}) \quad \text{for } 0 \leq t \leq \frac{1}{2}, \text{ with } \frac{dt_+}{t_+} = \left(1 + \frac{1}{\sqrt{1-4u}}\right)\frac{du}{u}.$$

Consequently,

$$\begin{aligned} I &= \int_0^{1/2} \frac{dt_-}{t_-} F(u) + \int_{1/2}^1 \frac{dt_+}{t_+} F(u) = 2\int_0^{1/4} \frac{du}{u\sqrt{1-4u}} F(u) \\ &= 2\int_0^1 \frac{dx}{x\sqrt{1-x}} F\left(\frac{1}{4}x\right) = 2\int_0^1 \frac{dt}{(1-t)\sqrt{t}} F\left(\frac{1-t}{4}\right) \end{aligned}$$

and, with $t = x^2$,

$$I = 4\int_0^1 \frac{dx}{1-x^2} F\left(\frac{1-x^2}{4}\right).$$

Therefore,

$$L = \int_0^1 \frac{dt}{t} \frac{(zt(1-t))^\alpha}{(1-zt(1-t))^\beta} = 2\left(\frac{z}{4}\right)^{\alpha-\beta} \int_0^1 dx \frac{(1-x^2)^{\alpha-1}}{(a^2+x^2)^\beta},$$

where $a^2 = 1/X = (4-z)/z$.

From standard references

$$\int_0^1 dx \cos(xy)(1-x^2)^{\alpha-1} = \sqrt{\frac{\pi y}{8}} \left(\frac{2}{y}\right)^\alpha \Gamma(\alpha) J_{\alpha-1/2}(y),$$

$$\int_0^\infty dx \cos(xy)(a^2+x^2)^{-\beta} = \frac{\sqrt{\pi}}{\Gamma(\beta)} \left(\frac{y}{2a}\right)^{\beta-1/2} K_{\beta-1/2}(ay)$$

so, by the Parseval relation for the Fourier transform

$$\int_0^1 dx \frac{(1-x^2)^{\alpha-1}}{(a^2+x^2)^\beta} = \frac{2^{\alpha-\beta}}{a^{\beta-1/2}} \frac{\Gamma(\alpha)}{\Gamma(\beta)} \int_0^\infty dy y^{\beta-\alpha} J_{\alpha-1/2}(y) K_{\beta-1/2}(ay).$$

This is a tabulated Hankel Transform and yields

$$L = \sqrt{\pi}\left(\frac{4}{z}\right)^{\beta-\alpha} X^\beta \frac{\Gamma(\alpha)}{\Gamma(\alpha+1)}\, {}_2F_1\left(\frac{1}{2}, \beta; \alpha+\frac{1}{2}; -X\right).$$

Consequently

$$\int_0^1 \frac{dx}{x}\frac{(zt(1-t))^n}{(1-zt(1-t))^{n+1}} = \sqrt{\pi}\left(\frac{4}{z}\right) X^{n+1}\ {}_2F_1\left(\frac{1}{2}, n+1; n+\frac{1}{2}; -X\right)$$

However, since ${}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\ {}_2F_1(c-a, c-b; c; z)$,

$${}_2F_1\left(\tfrac{1}{2}, n+1; n+\tfrac{1}{2}; -X\right) = (1+X)^{-1}\ {}_2F_1\left(-\tfrac{1}{2}; n; n+\tfrac{1}{2}; -X\right)$$

Next, we note that [11, p . 590]

$${}_2F_1(-1/2, 1; 3/2; z) = \frac{1}{2}\left(1 + (1-z)\frac{\tanh^{-1}\sqrt{z}}{\sqrt{z}}\right).$$

With $z \to -z$, noting that $-i\tanh^{-1} iw = \sin^{-1}\sqrt{\frac{w}{1+w}}$ one has

$${}_2F_1(-1/2, 1; 3/2; -z) = \frac{1}{2}(1 + (1+z)\frac{\sin^{-1}\sqrt{\frac{z}{1+z}}}{\sqrt{z}}). \tag{24}$$

We next apply Gauss' differentiation formula

$$\frac{d}{dz}((1+z)^k\ {}_2F_1(-1/2, k; k+1/2; -z)) =$$

$$\frac{2k(k+1)}{2k+1}(1+z)\ {}_2F_1(-1/2, k+1; k+3/2; -z). \tag{25}$$

Iteration of (25) starting with (24), after a great deal of tedious algebra, aided by Mathematica, results in (10).

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# References

[1] R. Apéry, Irrationalité de $\zeta(2)$ et $\zeta(3)$, *Asterique* **61** (1979), 11–13.

[2] D. H. Lehmer, Interesting series involving the central binomial coefficient, *Amer. Math. Monthly* **92** (1985), 449–457.

[3] F. J. Dyson, N. E. Frankel and M. L. Glasser, Lehmer's second interesting series, preprint http//:arxiv.org/abs/1009.4274.

[4] J. M. Borwein and R. Girgensohn, Evaluations of binomial series, *Aequ.Math.* **70** (2005), 25–36.

[5] F. J. Dyson, correspondence.

[6] I. J. Zucker, On the series $\sum \binom{2n}{n}^{-1} n^{-k}$ and related series, *J. Number Theory* **20** (1981), 92–102.

[7] N. Batir, On the series $\sum \binom{2k}{k}^{-1} k^{-a} x^k$, *Proc. Ind. Acad. Sci.* **115** (2005), 371–382.

[8] J. -H. Yang and F. -Z. Zhao, Certain sums involving inverses of binomial coefficients and some integrals, *J. Int. Sequences* **10** (2007), 1–11.

[9] R. Sprugnoli, Sums of the reciprocals of the central binomial coefficients, *INTEGERS* **6** (2006), 1–18.

[10] R. J. Mathar, Corrigenda to "Interesting series involving the central binomial coefficient by D. H. Lehmer, preprint http//: arxiv.org/abs/0905.0215v1.

[11] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Tables of Integrals and Series, Vol 3.*, Gordon and Breach Publishers, Amsterdam (1998)

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