ON A QUESTION OF A. SCHINZEL: OMEGA ESTIMATES FOR A SPECIAL TYPE OF ARITHMETIC FUNCTIONS

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Abstract. The paper deals with lower bounds for the remainder term in asymptotics for a certain class of arithmetic functions. Typically, these are generated by a Dirichlet series of the form $\zeta^2(s)\zeta(2s-1)(\zeta(2s))^M\,H(s)$ where M is an arbitrary integer and $H(s)$ has an Euler product which converges absolutely for $\Re s > \frac{1}{3}$.

To Professor Andrzej Schinzel on his 75th birthday

1. Introduction

1.1. This article is concerned with a special class $\mathfrak{C}$ of arithmetic functions f_H with a generating Dirichlet series[1]

$$F_H(s) = \sum_{n=1}^{\infty} \frac{f_H(n)}{n^s} = \zeta^2(s)\zeta(2s-1)(\zeta(2s))^M\,H(s) \qquad (\Re s > 1)\,, \tag{1.1}$$

where M is an integer, and $H(s)$ has an Euler product which converges absolutely for $\Re s > \frac{1}{3}$.

We mention some examples of special arithmetic interest: Firstly, the function

$$f^*(n) := \sum_{m\,|\,n} \gcd\left(m, \frac{n}{m}\right) \tag{1.2}$$

(see N. Sloane [17]) in a way quantifies the property of n to be not square-free, i.e., to possess non-unitary divisors. (For n square-free, $f^*(n)$ coincides with the number-of-divisors function $d(n)$.) $f^*(n)$ is generated by the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{f^*(n)}{n^s} = \frac{\zeta^2(s)\zeta(2s-1)}{\zeta(2s)} \qquad (\Re(s) > 1)\ . \tag{1.3}$$

This is (1.1) with $M=-1$, $H(s)=1$ identically.

Secondly, consider

$$f_1(n) = \sum_{m\,|\,n} \sigma\left(\gcd\left(m, \frac{n}{m}\right)\right)\,, \tag{1.4}$$

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[1]The dependance of f_H on the integer M will be suppressed in notation throughout.

where σ denotes the sum-of-divisors function: cf. again N. Sloane [19]. The corresponding generating function simply reads

$$F_1(s) = \sum_{n=1}^{\infty} \frac{f_1(n)}{n^s} = \zeta^2(s)\zeta(2s-1) \qquad (\Re s > 1)\,. \tag{1.5}$$

This is (1.1) with $M = 0$, $H(s) = 1$ identically.

As a third example, we mention the *modified Pillai's function* (N. Sloane [18])

$$P^*(n) := \frac{1}{n} \sum_{k=1}^{n} \gcd(k^2, n)\,. \tag{1.6}$$

This is generated by

$$\sum_{n=1}^{\infty} \frac{P^*(n)}{n^s} = \frac{\zeta^2(s)\zeta(2s-1)}{(\zeta(2s))^2}\, H^*(s) \qquad (\Re s > 1)\,, \tag{1.7}$$

where $H^*(s)$ has an Euler product absolutely convergent for $\Re s > \frac{1}{3}$.

1.2. The class of functions $\mathfrak{C}$ has been dealt with in detail in a recent paper by E. Krätzel, W.G. Nowak, and L. Tóth [7]. In that article, the emphasis was on upper bounds for the error term $\mathcal{R}_{f_H}(x)$ in the asymptotic formula

$$\sum_{n \le x} f_H(n) = \operatorname*{Res}_{s=1} \left(F_H(s) \frac{x^s}{s} \right) + \mathcal{R}_{f_H}(x)\,. \tag{1.8}$$

Since $F_H(s)$ has a triple pole at $s = 1$, explicitly

$$\operatorname*{Res}_{s=1} \left(F_H(s) \frac{x^s}{s} \right) = x\, p_H(\log x)\,,$$

where p_H is a quadratic polynomial whose coefficients depend on H and M. Using contour integration and properties of the Riemann zeta-function, it has been proved in [7] that

$$\mathcal{R}_{f_H}(x) = O\left(x^{2/3} (\log x)^{16/9} \right)\,. \tag{1.9}$$

Employing Krätzel's method [6], which involves fractional part sums and the theory of (classic) exponent pairs, the slight refinement

$$\mathcal{R}_{f_H}(x) = O\left(x^{925/1392} \right) \tag{1.10}$$

has been obtained ($\frac{925}{1392} = 0.6645\ldots$). Finally, bringing in Martin Huxley's deep and new technique [2], [3], [4] ("Discrete Hardy-Littlewood method"), the authors of [7] deduced the further improvement

$$\mathcal{R}_{f_H}(x) = O\left(x^{547/832} (\log x)^{26947/8320} \right) \qquad \left(\tfrac{547}{832} = 0.65745\ldots\right)\,. \tag{1.11}$$

For the context of the class $\mathfrak{C}$ within the frame of the theory of arithmetic functions, as well as for a wealth of enlightening related results, see also the recent papers by L. Tóth [23], [24].

1.3. The results of [7] have been presented at the 20^{th} Czech and Slovak International Conference on Number Theory in Stará Lesná, September 2011 [8]. At the end of that talk, Professor Andrzej Schinzel raised the following question: "What can be said concerning Omega-estimates for the remainder term?"

The authors of the present paper are very grateful for this valuable stimulation of further research and are pleased to be able to provide the following answer.

Theorem 1. *For any arithmetic function $f_H \in \mathfrak{C}$ with a generating function $F_H(s)$ according to* (1.1)*, it holds true that, as $x \to \infty$,*

$$\sum_{n\le x} f_H(n) = \operatorname*{Res}_{s=1}\left(F_H(s)\frac{x^s}{s}\right) + \Omega\left(\frac{\sqrt{x}(\log x)^2}{(\log\log x)^{|M+1|}}\right).$$

1.4. **Remarks.** For the simplest case (1.5), it is immediate that $f_1(n) = \Omega(\sqrt{n})$, hence also $\mathcal{R}_{f_1}(x) = \Omega(\sqrt{x})$. The achievement of the elaborate analysis to come is thus only an improvement by a logarithmic factor. On the other hand, it is easy to see that, e.g. for the first example mentioned, it follows that $f^*(n) \ll \sqrt{n}(\log\log n)^3$, hence our Ω-bound cannot be deduced by consideration of the individual values of the arithmetic function involved.

The situation may be compared with the sphere problem in $\mathbb{R}^3$: If $r_3(n)$ denotes the number of ways to write the positive integer n as a sum of three squares, then

$$\sum_{n\le x} r_3(n) = \frac{4\pi}{3}\,x^{3/2} + \Omega\left((x\log x)^{1/2}\right)$$

is the best Omega-result known to date [21], while, by the very same asymptotics, $r_3(n) = \Omega(\sqrt{n})$.

The method of proof which turned out to be appropriate in this problem goes back to ideas due to Ramachandra [14] and Balasubramanian, Ramachandra & Subbarao [1]. They have been worked out in papers by Schinzel [16], Kühleitner [9], and the authors [10]. However, in the present situation certain adaptions are necessary: On the one hand, $f_H(n)$ is not as small as $O(n^\epsilon)$. On the other hand, in the last step it will be advantageous to use special properties of the Riemann zeta-function, instead of a general theorem of Ramachandra's on Dirichlet series [14]. It should be mentioned that also in the case $M = 0$, when the generating function does not contain any factor involving $\zeta(2s)$, the present approach seems to give a better result than Soundararajan's method [20] which up to date was most successful in the divisor and circle problems.

2. Preliminaries

2.1. First of all, we can restrict our analysis to the case that $H(s) = 1$ identically, i.e., to

$$F(s) = \sum_{n=1}^{\infty}\frac{f(n)}{n^s} = \zeta^2(s)\zeta(2s-1)\zeta^M(2s) \qquad (\Re s > 1)\,. \tag{2.1}$$

In fact, assume that for some $f_H \in \mathfrak{C}$ and arbitrarily small $c_0 > 0$,

$$|\mathcal{R}_{f_H}(x)| \le c_0\,\frac{\sqrt{x}(\log x)^2}{(\log\log x)^{|M+1|}} \qquad (x \ge x_0)\,. \tag{2.2}$$

Let

$$\frac{1}{H(s)} = \sum_{n=1}^{\infty} h(n) n^{-s} \qquad \left(\Re s > \frac{1}{3}\right),$$

where the series converges absolutely for $\Re s > \frac{1}{3}$. Since $f = f_H * h$, where $*$ denotes the convolution of arithmetic functions, it readily follows that, for x large[2],

$$\begin{aligned}\sum_{n\le x} f(n) &= \sum_{k\le x} h(k) \sum_{m\le x/k} f_H(m) = \sum_{k\le x} h(k) \left(\frac{x}{k}\, p_H\left(\log\left(\frac{x}{k}\right)\right) + \mathcal{R}_{f_H}\left(\frac{x}{k}\right)\right) \\ &= \operatorname*{Res}_{s=1}\left(F(s)\frac{x^s}{s}\right) + \sum_{k>x} h(k)\,\frac{x}{k}\, p_H\left(\log\left(\frac{x}{k}\right)\right) + \sum_{k\le x} h(k)\mathcal{R}_{f_H}\left(\frac{x}{k}\right). \end{aligned} \tag{2.3}$$

From this it is immediate that

$$\mathcal{R}_f(x) \ll \frac{c_0\sqrt{x}(\log x)^2}{(\log\log x)^{|M+1|}},$$

which yields a contradiction for c_0 sufficiently small, provided that the Theorem has been established for the case of (2.1).

2.2. The assertion will be an easy consequence of the following integral mean result.

Proposition 1. *There exist positive constants B and C_0 with the property that*

$$\int_T^{\infty} \frac{|\mathcal{R}_f(u)|^2}{u^2}\, e^{-u/T^B}\, \mathrm{d}u \ge \frac{C_0\,(\log T)^5}{(\log\log T)^{|2M+2|}} \tag{2.4}$$

for all T sufficiently large.

The conclusion from this result to Theorem 1 is easy and has many analogues in the literature; see [1], [9], [10]. Nevertheless, we supply the details for convenience of the reader. Assume that for any arbitrarily small constant $c_0 > 0$, there exists u_0 so that

$$|\mathcal{R}_f(u)| \le \frac{c_0\sqrt{u}(\log u)^2}{(\log\log u)^{|M+1|}} \qquad \text{for all} \quad u \ge u_0. \tag{2.5}$$

Then, for B as in the Proposition, and T sufficiently large,

$$\begin{aligned} &\int_T^{\infty} \frac{|\mathcal{R}_f(u)|^2}{u^2}\, e^{-u/T^B}\, \mathrm{d}u \\ &\le c_0^2 \int_T^{\infty} \frac{1}{u}\, e^{-u/T^B}\, \frac{(\log u)^4}{(\log\log u)^{|2M+2|}}\, \mathrm{d}u = c_0^2 \left(\int_T^{T^B} + \int_{T^B}^{\infty}\right) \\ &\ll \frac{c_0^2\,(\log T)^4}{(\log\log T)^{|2M+2|}} \int_T^{T^B} \frac{\mathrm{d}u}{u} + c_0^2 \int_1^{\infty} \frac{e^{-v}(\log(T^B v))^4}{(\log\log(T^B v))^{|2M+2|}}\, \frac{\mathrm{d}v}{v} \\ &\ll \frac{c_0^2\,(\log T)^5}{(\log\log T)^{|2M+2|}}. \end{aligned}$$

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[2] It is clear by the asymptotics with O-terms cited, that the main term in this calculation amounts to $\operatorname*{Res}_{s=1}\left(F(s)\frac{x^s}{s}\right)$.

For c_0 sufficiently small this contradicts (2.4).

2.3. One special feature of the present situation - as opposed to the cases considered in [1], [16], [9], [10] - is that the function $f(n)$ is not "small", i.e., not $O(n^\epsilon)$ for each $\epsilon > 0$. However, if f is generated by $\zeta^2(s)\zeta(2s-1)(\zeta(2s))^M$, call $\hat{f}$ the arithmetic function generated by $\zeta^2(s)\zeta(2s-1)(\zeta(2s))^{|M|}$. Then,

$$|f(n)| \le \hat{f}(n) = \sum_{m_1 m_2 (m_3 k_1 \dots k_{|M|})^2 = n} m_3 \ll n^{1/2+\epsilon}.$$

Hence, using a trivial version of (1.8), applied to $\hat{f}$, and summation by parts, it follows that

$$\sum_{n \le X} n^\beta \, (f(n))^2 \ll X^{3/2+\beta+\epsilon} \tag{2.6}$$

for each fixed $\beta > -\frac{3}{2}$, large X, and any $\epsilon > 0$.

2.4. The following auxiliary result is classic and provides some information that the factor of $F(s)$ involving $\zeta(2s)$ is "not too harmful" close to the critical line.

Lemma 1. *Let $\epsilon_0 > 0$ be a sufficiently small constant. Then, for $\widehat{T}$ a sufficiently large real parameter, there exists a set $\mathcal{A}(\widehat{T}) \subset [\widehat{T}, 2\widehat{T}]$ with the following properties:*
(i) *$\mathcal{A}(\widehat{T})$ is the union of at most $O(\widehat{T}^{\epsilon_0})$ open intervals, with a total length of $O(\widehat{T}^{\epsilon_0})$.*
(ii)

$$\sup_{t \in [\widehat{T}, 2\widehat{T}] \setminus \mathcal{A}(\widehat{T})} |\zeta(1+2it)|^{\pm 1} \ll \log\log \widehat{T}$$

(iii) *There exist a real number $\delta(\epsilon_0) > 0$ and a certain constant C so that*

$$|\zeta(2s)|^{\pm 1} \ll \widehat{T}^C$$

uniformly in $\Re s \ge \frac{1}{2} - \delta(\epsilon_0)$, $\Im s \in [\widehat{T}, 2\widehat{T}] \setminus \mathcal{A}(\widehat{T})$.

Proof. This result is contained in [14, Theorem 1] and [15, Lemma 3.2]. An extension to Dedekind zeta-functions, along with a neat proof, was given in [11, Lemma and formula (2.6)]. □

2.5. We conclude this section by quoting a deep and celebrated result due to Montgomery and Vaughan which provides a mean-square bound for Dirichlet polynomials.

Lemma 2. *For an arbitrary sequence of complex numbers $(\gamma_n)_{n=1}^\infty$ with the property that $\sum_{n=1}^\infty n\,|\gamma_n|^2$ converges, and a large real parameter X,*

$$\int_0^X \left| \sum_{n=2}^\infty \gamma_n (n+u)^{-it} \right|^2 \mathrm{d}t = \sum_{n=2}^\infty |\gamma_n|^2 \left(X + O(n)\right),$$

uniformly in $-1 \le u \le 1$, and

$$\int_0^X \left| \sum_{n=1}^\infty \gamma_n n^{-it} \right|^2 \mathrm{d}t = \sum_{n=1}^\infty |\gamma_n|^2 \left(X + O(n)\right).$$

Proof. This is an immediate consequence of Montgomery and Vaughan [12, Corollary 2, formula (1.9)]. □

3. Proof of Proposition 1

3.1. For positive real T sufficiently large, we construct a set $\mathcal{U}(T)$ on the real line as follows: Let $J := \left[\frac{\log T}{10\log 2}\right]$, then $2^{-J}T \asymp T^{9/10}$. For $j = 1, \ldots, J$, set $T_j := 2^{-j}T$. For some appropriate small $\epsilon_0 > 0$, consider the sets $\mathcal{A}(T_j)$ furnished by Lemma 1. Let

$$\mathcal{A}(T_j) = \bigcup_{i\in\mathcal{I}_j}]a_i^{(j)}, b_i^{(j)}[$$

be the decomposition of $\mathcal{A}(T_j)$ into $\#(\mathcal{I}_j) = O(T^{\epsilon_0})$ open intervals of total length $O(T^{\epsilon_0})$. Then we define

$$\mathcal{U}(T) := \bigcup_{j=1}^{J} \bigcup_{i\in\mathcal{I}_j}]a_i^{(j)} - (\log T)^2, b_i^{(j)} + (\log T)^2[\,. \tag{3.1}$$

By construction, it is clear that $\mathcal{U}(T)$ consists of $O(T^{2\epsilon_0})$ open intervals of total length $O(T^{2\epsilon_0})$.

3.2. We set $y = T^B$, with a suitably large positive constant B, for throughout what follows. It suffices to consider those values of T for which

$$\int_T^\infty \frac{|\mathcal{R}_f(u)|^2}{u^2} e^{-u/y}\,\mathrm{d}u \le (\log T)^6\,. \tag{3.2}$$

(Otherwise the assertion of Proposition 1 is obvious.) In this subsection, our aim is to deduce the asymptotic representation

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}\, e^{-n/y} \;+\; O(1)\,, \tag{3.3}$$

for

$$s = \tfrac{1}{2} + it\,, \quad t \in [2^{-J}T, T] \setminus \mathcal{U}(T)\,, \tag{3.4}$$

which will be assumed throughout the sequel.

By a version of Perron's formula (see, e.g., [13, p. 380]),

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} e^{-n/y} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} F(s+w) y^w \Gamma(w)\,\mathrm{d}w\,. \tag{3.5}$$

We use Stirling's formula in the crude form

$$\Gamma(\sigma + i\tau) \asymp \exp\left(-\frac{\pi}{2}|\tau|\right) |\tau|^{\sigma-1/2}\,, \qquad (|\tau| \to \infty) \tag{3.6}$$

which holds uniformly in every strip $\sigma_1 \le \sigma \le \sigma_2$. From this it is an immediate consequence that, for every fixed $k \in \mathbb{Z}_+$,

$$\Gamma^{(k)}(\sigma + i\tau) \ll \exp\left(-\frac{\pi}{4}|\tau|\right) \qquad (|\tau| \to \infty) \tag{3.7}$$

again uniformly in any strip of this kind. It is easy to see that we may break off the part of the integration line in (3.5) corresponding to $|w| > (\log T)^2$, with an error of only $O(1)$:

$$\begin{aligned} &\int\limits_{2\pm i(\log T)^2}^{2\pm i\infty} F(s+w)\, y^w\, \Gamma(w)\ \mathrm{d}w \\ &\ll T^{2B} \int\limits_{(\log T)^2}^{\infty} w^{3/2}\, e^{-(\pi/2)w}\ \mathrm{d}w \ll T^{2B}\ \exp\Big(-\frac{\pi}{4}(\log T)^2\Big) \ \ll\ 1\,. \end{aligned} \tag{3.8}$$

Next, we replace the remaining line of integration by a broken line segment $\mathfrak{L}$ which joins (in this order) $2-i(\log T)^2$, $-\delta(\epsilon_0)-i(\log T)^2$, $-\delta(\epsilon_0)+i(\log T)^2$, and $2+i(\log T)^2$, where $\delta(\epsilon_0)$ has the meaning as in Lemma 1. By Lemma 1, clause (iii), and known upper bounds for the zeta-function,

$$F(s+w) \ll T^{C'}\,, \tag{3.9}$$

with some positive constant C', as long as w lies on $\mathfrak{L}$ and s satisfies (3.4). Therefore,

$$\int\limits_{-\delta(\epsilon_0)-i(\log T)^2}^{-\delta(\epsilon_0)+i(\log T)^2} F(s+w)y^w\Gamma(w)\ \mathrm{d}w \ll T^{C'-B\delta(\epsilon_0)} \int\limits_0^{\infty} \Gamma(-\delta(\epsilon_0)+iu)\ \mathrm{d}u \ \ll\ 1$$

if we have chosen $B \geq C'/\delta(\epsilon_0)$, and

$$\int_{-\delta(\epsilon_0)\pm i(\log T)^2}^{2\pm i(\log T)^2} F(s+w)y^w\Gamma(w)\ \mathrm{d}w \ll T^{C'+2B}(\log T)^3 \exp\Big(-\frac{\pi}{2}(\log T)^2\Big) \ \ll\ 1\,.$$

Since these integrals are bounded, the main contribution to the right-hand side of (3.5) comes from the residue of the integrand $F(s+w)y^w\Gamma(w)$ at $w=0$, which amounts to $F(s)$. This readily yields (3.3).

3.3. It is an easy consequence of (3.2) that there exists some $T^* \in [T, 2T]$, not an integer, for which

$$\frac{|\mathcal{R}_f(T^*)|\, e^{-T^*/y}}{\sqrt{T^*}} \ll (\log T)^3 \tag{3.10}$$

and

$$\frac{1}{y}\int\limits_{T^*}^{\infty} \frac{|\mathcal{R}_f(u)|}{\sqrt{u}} e^{-u/y}\,\mathrm{d}u \ll (\log T)^3\,. \tag{3.11}$$

This can be readily verified following the example of [1, p. 111, Lemma 4]. With this choice of T^*, we will split up the series on the right-hand side of (3.3). In this subsection, we shall handle

$$\begin{aligned} &\sum_{n>T^*} \frac{f(n)}{n^s} e^{-n/y} = \int\limits_{T^*}^{\infty} u^{-s} e^{-u/y}\ \mathrm{d}\left(\sum_{n\leq u} f(n)\right) \\ &= \int\limits_{T^*}^{\infty} u^{-s}e^{-u/y}\ \mathrm{d}\left(u\, p_H(\log u)\right) + \int\limits_{T^*}^{\infty} u^{-s}e^{-u/y}\ \mathrm{d}\mathcal{R}_f(u) =: I_1 + I_2\,, \end{aligned} \tag{3.12}$$

where we have used Stieltjes integrals. Integrating by parts, we obtain

$$\begin{aligned} I_2 = &\mathcal{R}_f(u)u^{-s}e^{-u/y}\Big|_{u=T^*}^{u=\infty} \\ &+ s\int_{T^*}^{\infty} u^{-s-1}e^{-u/y}\mathcal{R}_f(u)\,\mathrm{d}u + \frac{1}{y}\int_{T^*}^{\infty} u^{-s}e^{-u/y}\mathcal{R}_f(u)\,\mathrm{d}u \\ =&s\int_{\xi}^{\xi+1}\sum_{n>T^*}(n+v)^{-s-1}e^{-(n+v)/y}\mathcal{R}_f(n+v)\,\mathrm{d}v + O\left((\log T)^3\right)\,. \end{aligned} \tag{3.13}$$

Here $\xi := T^* - [T^*] - 1$, and the bounds (3.10), (3.11) have been used.

To estimate I_1, we have to deal with integrals of the form

$$\int_{T^*}^{\infty} u^{-s}(\log u)^r e^{-u/y}\ \mathrm{d}u\,,$$

with $r \in \{0, 1, 2\}$, and s satisfying (3.4). We write this as

$$\begin{aligned} &\int_0^{\infty} u^{-s}(\log u)^r e^{-u/y}\ \mathrm{d}u - \int_0^{T^*} u^{-s}(\log u)^r\ \mathrm{d}u \\ &+\int_0^{T^*} u^{-s}(\log u)^r\left(1-e^{-u/y}\right)\ \mathrm{d}u\ \ =:\ J_1 - J_2 + J_3\,. \end{aligned} \tag{3.14}$$

By Taylor expansion,

$$J_3 \ll \frac{1}{y}\int_0^{T^*} u^{1/2}(\log u)^r\ \mathrm{d}u \ll T^{-B+3/2+\epsilon} \ll 1\,.$$

Integrating by parts r times, we infer that

$$J_2 \ll \frac{T^{1/2}(\log T)^r}{|1-s|} \ll \frac{T^{1/2}(\log T)^r}{T^{9/10}} \ll\ 1\,.$$

Finally,

$$\begin{aligned} J_1 &= y^{1-s}\int_0^{\infty} u^{-s}(\log(uy))^r\, e^{-u}\ \mathrm{d}u \\ &= y^{1-s}\sum_{\rho=0}^{r}\binom{r}{\rho}(\log y)^{r-\rho}\,\Gamma^{(\rho)}(1-s)\ \ \ll\ 1\,, \end{aligned}$$

by (3.7). Altogether this shows that $I_1 \ll 1$, hence, together with (3.3), (3.12) and (3.13),

$$\begin{aligned} F(s) = &\sum_{n\le T^*}\frac{f(n)}{n^s}e^{-n/y} \\ &+\ s\int_{\xi}^{\xi+1}\sum_{n>T^*}\frac{\mathcal{R}_f(n+u)}{(n+u)^{s+1}}e^{-(n+u)/y}\ \mathrm{d}u\ +\ O\left((\log T)^3\right) \end{aligned} \tag{3.15}$$

for s satisfying (3.4).

3.4. The next step is to prove that

$$\int\limits_{[2^{-J}T,T]\setminus\mathcal{U}(T)} \left|\frac{F(\frac{1}{2}+it)}{\frac{1}{2}+it}\right|^2 \,\mathrm{d}t \ll 1 + \int\limits_{T}^{\infty} \frac{|\mathcal{R}_f(u)|^2}{u^2}\, e^{-u/y}\,\mathrm{d}u\,. \tag{3.16}$$

By (3.15), the left-hand side of (3.16) is

$$\begin{aligned} &\ll T^{-9/5} \int\limits_{2^{-J}T}^{T} \left|\sum_{n\le T^*} \frac{f(n)}{n^{1/2+it}} e^{-n/y}\right|^2 \,\mathrm{d}t \\ &\quad + \int\limits_{2^{-J}T}^{T} \left|\int\limits_{\xi}^{\xi+1} \sum_{n>T^*} \frac{\mathcal{R}_f(n+u)}{(n+u)^{3/2+it}} e^{-(n+u)/y}\,\mathrm{d}u\right|^2 \,\mathrm{d}t \\ &\quad + O\left(T^{-9/10}(\log T)^6\right)\,. \end{aligned} \tag{3.17}$$

To bound the first integral here, we use Lemma 2. In this way,

$$\begin{aligned} &T^{-9/5} \int\limits_{2^{-J}T}^{T} \left|\sum_{n\le T^*} \frac{f(n)}{n^{1/2+it}} e^{-n/y}\right|^2 \,\mathrm{d}t \\ &\ll T^{-9/5} \sum_{n\le T^*} \frac{(f(n))^2}{n}\,(T+O(n)) \ll T^{-3/10+\epsilon}\,, \end{aligned} \tag{3.18}$$

for any $\epsilon > 0$, by an appeal to (2.6). Similarly, using Cauchy's inequality and Lemma 2 again, we see that the second term of (3.17) is

$$\begin{aligned} &\le \int\limits_{\xi}^{\xi+1}\int\limits_{0}^{T} \left|\sum_{n>T^*} \frac{\mathcal{R}_f(n+u)}{(n+u)^{3/2+it}} e^{-(n+u)/y}\right|^2 \,\mathrm{d}t\,\mathrm{d}u \\ &\ll \int\limits_{\xi}^{\xi+1} \left(\sum_{n>T^*} \frac{|\mathcal{R}_f(n+u)|^2}{(n+u)^3} e^{-2(n+u)/y}\,(T+O(n))\right) \mathrm{d}u \\ &\ll \int\limits_{\xi}^{\xi+1} \left(\sum_{n>T^*} \frac{|\mathcal{R}_f(n+u)|^2}{(n+u)^2} e^{-(n+u)/y}\right) \mathrm{d}u \\ &\le \int\limits_{T}^{\infty} \frac{|\mathcal{R}_f(u)|^2}{u^2}\, e^{-u/y}\,\mathrm{d}u\,. \end{aligned} \tag{3.19}$$

Together with (3.17) and (3.18), this readily yields (3.16).

3.5. In order to complete the proof of Proposition 1, it remains to show that

$$\int\limits_{[2^{-J}T,T]\setminus\mathcal{U}(T)} \left|\frac{F(\frac{1}{2}+it)}{\frac{1}{2}+it}\right|^2 \,\mathrm{d}t \gg \frac{(\log T)^5}{(\log\log T)^{|2M+2|}}\,. \tag{3.20}$$

In fact, by the functional equation of the zeta-function (see [22, p. 95]),

$$\begin{aligned}\left|F\left(\tfrac12+it\right)\right| &= \left|\zeta^2\left(\tfrac12+it\right)\zeta(-2it)\left(\zeta\left(1+2it\right)\right)^M\right| \\ &\asymp \left|\zeta^2\left(\tfrac12+it\right)\left(\zeta\left(1+2it\right)\right)^{M+1}\right| |t|^{1/2}\end{aligned}$$

for $|t|$ large. Further, by Lemma 1, clause (iii), and the construction of $\mathcal{U}(T)$ in subsection 3.1, it follows that $|\zeta(1+2it)|^{\pm 1} \ll \log\log T$ for $t \in [2^{-J}T, T] \setminus \mathcal{U}(T)$. Therefore,

$$\begin{aligned}&\int\limits_{[2^{-J}T,T]\setminus\mathcal{U}(T)} \left|\frac{F(\frac12+it)}{\frac12+it}\right|^2 \mathrm{d}t \\ &\gg (\log\log T)^{-2|M+1|} \int\limits_{[2^{-J}T,T]\setminus\mathcal{U}(T)} \left|\zeta\left(\tfrac12+it\right)\right|^4 \frac{\mathrm{d}t}{t}\,.\end{aligned} \tag{3.21}$$

For $t \in \mathcal{U}(T)$, we use the classic pointwise bound $\zeta(\frac12+it) \ll |t|^{1/6+\epsilon}$: See [22, Theorem 5.5]. Since the total length of $\mathcal{U}(T)$ is $O(T^{2\epsilon_0})$, it follows that

$$\int\limits_{\mathcal{U}(T)} \left|\zeta\left(\tfrac12+it\right)\right|^4 \frac{\mathrm{d}t}{t} \ll T^{-7/30+3\epsilon_0}\,. \tag{3.22}$$

On the other hand, by the known asymptotics for the fourth moment of the zeta-function (see [5, p. 129]),

$$\int\limits_{2^{-J}T}^{T} \left|\zeta\left(\tfrac12+it\right)\right|^4 \frac{\mathrm{d}t}{t} \gg (\log T)^5\,. \tag{3.23}$$

Combining (3.21), (3.22), and (3.23), we readily establish (3.20). This completes the proof of Proposition 1 and, by the observation in subsection 2.2, also that of Theorem 1.

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