A method for deriving hypergeometric and related identities from the H^2 Hardy norm of conformal maps

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Abstract

We explore a method which is implicit in a paper of Burkholder of identifying the H^2 Hardy norm of a conformal map with the explicit solution of Dirichlet's problem in the complex plane. Using the series form of the Hardy norm, we obtain an identity for the sum of a series obtained from the conformal map. We use this technique to evaluate several hypergeometric sums, as well as several sums that can be expressed as convolutions of the terms in a hypergeometric series. The most easily stated of the identities we obtain are Euler's famous Basel sum, as well as the sum

$$1^2 + \left(1 \times \frac{1}{3} + \frac{1}{3} \times 1\right)^2 + \left(1 \times \frac{1}{5} + \frac{1}{3} \times \frac{1}{3} + \frac{1}{5} \times 1\right)^2 + \left(1 \times \frac{1}{7} + \frac{1}{3} \times \frac{1}{5} + \frac{1}{5} \times \frac{1}{3} + \frac{1}{7} \times 1\right)^2 + \cdots = \frac{\pi^4}{32}.$$

We will be able to obtain the following hypergeometric reduction:

$$_4F_3(1/3, 1/3, 2/3, 2/3; 4/3, 4/3, 1; 1) = \frac{\mathrm{B}(1/3, 1/3)^2}{27}.$$

A related identity is

$$_4F_3(1/4, 1/4, 1/2, 1/2; 5/4, 5/4, 1; 1) \times \frac{1}{\mathrm{B}(1/4, 1/2)^2}$$
$$= \frac{8}{\pi^4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{m+n}}{(2m-1)(2n-1)((2m-1)^2 + (2n-1)^2)}.$$

We will obtain two families of identities depending on a parameter, representative examples of which are

$$\sum_{n=1}^{\infty} \left(\sum_{j=0}^{n} \binom{2/3}{2j}\binom{1/3 + n - j - 1}{n - j}\right)^2 = \frac{1}{4},$$

1

and

$$\sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \binom{j-3/4}{j} \frac{C(2(n-j))}{16^{n-j}} \right)^2 = \sqrt{2},$$

where $C(k)$ is the k-th Catalan number. We will also sum two series whose terms are defined by certain recurrence relations, and discuss an extension of the method to maps which are not conformal.

1   Introduction

Suppose that V is a simply connected domain properly contained in the complex plane $\mathbb{C}$. Given a point $a \in V$, the Riemann Mapping Theorem guarantees a conformal map f_a from the unit disc $\mathbb{D}$ onto V with $f_a(0) = a$. The Hardy norm $|| \cdot ||_{H^2}$ of f_a is defined as

$$(1.1) \qquad ||f_a||_{H^2} := \left(\sup_{r<1} \frac{1}{2\pi} \int_0^{2\pi} |f_a(re^{i\theta})|^2 d\theta \right)^{1/2}.$$

The map f_a is not uniquely determined, however any two such maps differ only by precomposition with a rotation, so $||f_a||_{H^2}$ is well defined. For holomorphic functions on $\mathbb{D}$, Parseval's Identity takes the form (see [16, Thm. 10.22])

$$(1.2) \qquad \frac{1}{2\pi} \int_0^{2\pi} |f_a(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n} \quad \text{for } r < 1,$$

where f_a admits the Taylor series expansion $f_a(z) = \sum_{n=0}^{\infty} a_n z^n$. It follows that

$$(1.3) \qquad ||f_a||_{H^2}^2 = \sum_{n=0}^{\infty} |a_n|^2.$$

It is straightforward to verify by precomposing f_a with an appropriate Möbius transformation that if $||f_a||_{H^2}^2 < \infty$ for some $a \in V$, then $||f_b||_{H^2}^2 < \infty$ for all $b \in V$. If V is a domain for which $||f_a||_{H^2}^2 < \infty$, we define

$$(1.4) \qquad \tilde{\mathcal{H}}_V(a) := ||f_a||_{H^2}^2 = \sum_{n=0}^{\infty} |a_n|^2.$$

If $||f_a||_{H^2}^2 = \infty$ on V, we will say that $\tilde{\mathcal{H}}_V$ does not exist. The following result gives sufficient conditions for $\tilde{\mathcal{H}}_V$ to exist and be determinable.

Proposition 1. *Suppose that a nonnegative function $\tilde{\mathcal{G}}_V$ exists on the simply connected domain V with the following properties:*

2

(i) $\tilde{\mathcal{G}}_V$ is harmonic on V.

(ii) $\tilde{\mathcal{G}}_V$ extends continuously to the boundary of V, and $\tilde{\mathcal{G}}_V(a) = |a|^2$ on δV.

(iii) There is a constant $C > 1$ such that $|a|^2 \leq \tilde{\mathcal{G}}_V(a) \leq C|a|^2 + C$.

Then $\tilde{\mathcal{H}}_V(a)$ exists and $\tilde{\mathcal{H}}_V(a) = \tilde{\mathcal{G}}_V(a)$.

The purpose of this paper is to explore a method of summing series which is contained in this proposition. For certain domains we will see that we can find an explicit formula for $\tilde{\mathcal{G}}_V$, which then forms an identity with the sum of the squares of the norms of the coefficients of the power series of f_a. By subtracting $|a|^2$ from both $\tilde{\mathcal{G}}_V$ and $\tilde{\mathcal{H}}_V$ we obtain the following equivalent proposition, which will be somewhat easier to use in our application.

Proposition 2. *Suppose that a continuous function $\mathcal{G}_V$ exists on the simply connected domain V with the following properties:*

(i') $\triangle \mathcal{G}_V = -4$ on V.

(ii') $\mathcal{G}_V$ extends continuously to the boundary of V, and $\mathcal{G}_V(a) = 0$ on δV.

(iii') There is a constant $C > 0$ such that $0 \leq \mathcal{G}_V(a) \leq C|a|^2 + C$.

Then $\mathcal{H}_V(a) := \tilde{\mathcal{H}}_V(a) - |a|^2$ exists and $\mathcal{H}_V(a) = \mathcal{G}_V(a)$.

We remark that the upper bound on the growth of $\tilde{\mathcal{G}}_V$ and $\mathcal{G}_V$ given in (iii) and (iii'), which are superfluous when V is bounded, are quite important when V is unbounded. This is because in unbounded domains harmonic functions are not uniquely determined by their boundary values; for instance, $u_1(x, y) = 0$ and $u_2(x, y) = y$ are harmonic functions in the upper half-plane $\{y > 0\}$ and agree on the real axis. The role of (iii) and (iii') in unbounded domains is to allow us to conclude that the displayed functions are the correct choices for $\mathcal{H}_V$ and $\tilde{\mathcal{H}}_V$.

These propositions (and much more) were proved by Burkholder in the elegant paper [5] (see Theorem 3.2 and Remark 3.1 of [5], together with Lemma 1.1 of [4] and/or Lemma 1 of [15]). There, the quantities arose naturally in terms of the exit time of Brownian motion from domains. Burkholder's proof is ingenious, but relies heavily on martingale theory. For the benefit of the reader unfamiliar with such arguments, we include a short analytic proof of Proposition 2 in Section 2. In Section 3 we will use this method to derive a number of identities involving infinite sums. The method is clearly limited by our ability to find f_a and u_V, which is no small difficulty, as conformal maps in particular are notoriously difficult to explicitly display in many cases. Nevertheless, by using symmetric and simple domains we will see that we are able to evaluate a number of sums.

2 Proof of Proposition 1

Suppose that $\mathcal{G}_V$ satisfies the conditions of the proposition. The function $\mathcal{G}_V(f_a(z))$ is harmonic on $\mathbb{D}$, so

(2.1) $$\mathcal{G}_V(a) = \mathcal{G}_V(f_a(0)) = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{G}_V(f_a(re^{i\theta}))d\theta, \quad \text{for } r \in (0, 1).$$

Thus,

$$(2.2) \qquad \frac{1}{2\pi} \int_0^{2\pi} |f_a(re^{i\theta})|^2 d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \mathcal{G}_V(f_a(re^{i\theta})) d\theta = \mathcal{G}_V(a).$$

We conclude that $||f_a||_{H^2} < \infty$. This implies that the functions $f(re^{i\theta})$ approach a limit function $f(e^{i\theta})$ in $L^2([0, 2\pi])$ as $r \nearrow 1$, and $f(e^{i\theta})$ takes values in δV for almost every θ (see [16, Thm. 17.10]). Furthermore, we can apply a fundamental inequality of Hardy and Littlewood (see [9]) to conclude that the function $M_f(\theta) := \sup_{0 < r < 1} |f(re^{i\theta})|$ lies in $L^2([0, 2\pi])$. The conditions on $\mathcal{G}_V$ imply that

$$(2.3) \qquad \mathcal{G}_V(f_a(re^{i\theta})) \leq (C|f_a(re^{i\theta})|^2 + C) \leq CM_f(\theta)^2 + C \in L^1([0, 2\pi]).$$

We may therefore apply the Dominated Convergence Theorem, recalling that $\mathcal{G}_V(z) = |z|^2$ on δV, to conclude that

$$(2.4) \qquad \mathcal{G}_V(a) = \lim_{r \nearrow 1} \frac{1}{2\pi} \int_0^{2\pi} \mathcal{G}_V(f_a(re^{i\theta})) d\theta = \frac{1}{2\pi} \int_0^{2\pi} |f_a(e^{i\theta})|^2 d\theta = ||f_a||_{H^2}^2.$$

This completes the proof of the proposition.

Remark 1. The quantities $\mathcal{H}_V(a)$ and $\tilde{\mathcal{H}}_V(a)$ arise naturally in probability theory in the following way. Suppose that B_t is a planar Brownian motion starting at a, and let τ be the first time $B_t \in \delta V$. Then $\mathcal{H}_V(a) = E[|B_\tau|^2]$ and $\tilde{\mathcal{H}}_V(a) = 2E[\tau]$. The intuition gained by these equivalences can be very helpful; the interested reader is referred again to [5].

Remark 2. Another way to see that the function $a \longrightarrow ||f_a||_{H^2}^2$ must be harmonic on V and approach $|a|^2$ on δV is to express it in terms of the Greens potential. Let the Greens function on V with pole at a be denoted $G_V(a, z)$; there are various normalizations, but we will use the one presented in [2], so that $G_{\mathbb{D}}(0, z) = \log(1/|z|)$. We then have, using the conformal invariance of the Greens function and the fact that the Jacobian of the map $f_a(z)$ is $|f_a'(z)|^2$,

$$
\begin{aligned}
(2.5) \qquad \frac{2}{\pi} \iint_V G_V(a, x + iy) dx dy &= \frac{2}{\pi} \iint_{\mathbb{D}} G_{\mathbb{D}}(0, x + iy)|f_a'(x + iy)|^2 dx dy \\
&= 4 \int_0^1 r \log \frac{1}{r} \left(\frac{1}{2\pi} \int_0^{2\pi} |f_a'(re^{i\theta})|^2 d\theta \right) dr \\
&= 4 \int_0^1 r \log \frac{1}{r} \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-2} dr \\
&= 4 \sum_{n=1}^{\infty} n^2 |a_n|^2 \int_0^1 r^{2n-1} \log \frac{1}{r} dr \\
&= 4 \sum_{n=1}^{\infty} n^2 |a_n|^2 \left(\frac{1}{4n^2} \right) = \sum_{n=1}^{\infty} |a_n|^2 = \mathcal{H}_V(a).
\end{aligned}
$$

4

Note that Parseval's Identity was applied to obtain the third equality. The function

$$(2.6) \qquad a \longrightarrow \frac{2}{\pi} \iint_V G_V(a, x + iy) dx dy$$

is well known to satisfy (i′) and (ii′); see for instance [8, Thm. 8.4] or any number of standard texts on partial differential equations.

3 Examples

In what follows we will write $a = x + iy$, and any references to x and y will always refer to the real and imaginary parts of a. Examples 1 through 3 appear in [15], but since the approach was somewhat different in that paper it seems as well to briefly review them. In each case $\mathcal{G}_V(z)$ will be a nonnegative function satisfying (i′)-(iii′) on V, which by Proposition 2 is unique if it exists.

Example 1 Let V be the infinite vertical strip $\{\frac{-\pi}{4} < x < \frac{\pi}{4}\}$. The conformal map from $\mathbb{D}$ to V which fixes 0 is given by (see [15] for a quick proof of this standard fact)

$$(3.1) \qquad \tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} - \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{2n + 1}.$$

By inspection, we have $\mathcal{G}_V(x + iy) = \frac{\pi^2}{8} - 2x^2$. Proposition 2 with $a = 0$ yields

$$(3.2) \qquad \sum_{n=0}^{\infty} \frac{1}{(2n + 1)^2} = \frac{\pi^2}{8}.$$

This is easily seen to be equivalent to Euler's classical result that

$$(3.3) \qquad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Example 2 Let V be the equilateral triangle with vertices at $1, e^{2\pi i/3}, e^{4\pi i/3}$. The conformal map from $\mathbb{D}$ to V sending 0 to 0 can be achieved via the Schwarz-Christoffel transformation (see [7]). A method for determining this function precisely is present in [14], although the explicit formula is not stated. Later, in [15], the explicit formula was given as

$$(3.4) \qquad f(z) = \frac{3z}{B(1/3, 1/3)} \, _2F_1(1/3, 2/3; 4/3; z^3),$$

where

$$(3.5) \qquad {}_pF_q(a_1, \ldots, a_p; b_1, \ldots b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{z^n}{n!}$$

is the hypergeometric function with $(a)_0 = 1, (a)_n = a(a+1) \ldots (a+n-1)$ denoting the Pochhammer symbol, and $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the beta function. We obtain

$$(3.6) \qquad \mathcal{H}_V(0) = \frac{9}{B(1/3, 1/3)^2} \, {}_4F_3(1/3, 1/3, 2/3, 2/3; 4/3, 4/3, 1; 1).$$

In [3], it was shown that

$$(3.7) \qquad \mathcal{G}_V(a) = \frac{1}{3}(a + \bar{a} + 1)(\omega a + \bar{\omega}\bar{a} + 1)(\omega^2 a + \bar{\omega}^2 \bar{a} + 1),$$

where $\omega = e^{2\pi i/3}$. Verification of (i$'$) may be most easily accomplished by using the complex form of the Laplacian, $\triangle = 4\frac{\partial^2}{\partial a \partial \bar{a}}$, while (ii$'$) is clear since $(a + \bar{a} + 1) = 0$ on $\{x = -\frac{1}{2}\}$ and (3.7) is rotationally invariant. Applying Proposition 2 with $a = 0$ gives

$$(3.8) \qquad \frac{9}{B(1/3, 1/3)^2} \, {}_4F_3(1/3, 1/3, 2/3, 2/3; 4/3, 4/3, 1; 1) = \frac{1}{3}.$$

We conclude that

$$(3.9) \qquad {}_4F_3(1/3, 1/3, 2/3, 2/3; 4/3, 4/3, 1; 1) = \frac{B(1/3, 1/3)^2}{27}.$$

This reduction appears to be nontrivial.

Example 3 Let V be the square with vertices at $\pm\frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}$. Again we will use the Schwarz-Christoffel transformation. It is shown in [14, Ex. 2] that the map

$$(3.10) \qquad f(z) = \frac{4e^{\pi i/2}z}{B(1/4, 1/2)} \, {}_2F_1(1/4, 1/2; 5/4; z^4)$$

maps $\mathbb{D}$ conformally to V, sending 0 to 0 (a proof of this can also be found in [15]). We find that

$$(3.11) \qquad \mathcal{H}_V(0) = \frac{16}{B(1/4, 1/2)^2} \, {}_4F_3(1/4, 1/4, 1/2, 1/2; 5/4, 5/4, 1; 1).$$

The formula for $\mathcal{G}_V$ was presented in [13, Sec. 4.1] (there is a misprint in the book, but the correct formula appears in [10]):

$$(3.12) \qquad \mathcal{G}_V(x+iy) = \frac{128}{\pi^4} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin \frac{(2m-1)\pi(x-\frac{1}{\sqrt{2}})}{\sqrt{2}} \sin \frac{(2n-1)\pi(y-\frac{1}{\sqrt{2}})}{\sqrt{2}}}{(2m-1)(2n-1)((2m-1)^2+(2n-1)^2)}$$

It is clear that this satisfies (ii′), and direct calculation using the Fourier series identity

$$(3.13) \qquad \sum_{m=1}^{\infty} \frac{\sin(2m-1)\pi u}{2m-1} = \frac{\pi}{4} \qquad \text{for } u \in (0,1)$$

shows that (i′) is satisfied as well. Proposition 2 gives us the following identity:

$$(3.14) \qquad \begin{aligned} {}_4F_3(1/4,&1/4, 1/2, 1/2; 5/4, 5/4, 1; 1) \times \frac{1}{\mathrm{B}(1/4, 1/2)^2} \\ &= \frac{8}{\pi^4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{m+n}}{(2m-1)(2n-1)((2m-1)^2+(2n-1)^2)}. \end{aligned}$$

Example 4 Let V be the parabolic region given by $y^2 < 2x - 1$. By inspection,

$$(3.15) \qquad \mathcal{G}_V(x+iy) = 2(2x - y^2 - 1).$$

The conformal map from $\mathbb{D}$ to V mapping 0 to 1 is given in [12](or see [11]) as

$$(3.16) \qquad f(z) = 1 + \frac{2}{\pi^2}\Big(\ln(1+\sqrt{z})/(1-\sqrt{z}) \Big)^2.$$

Note that we may simplify by substituting z for $\sqrt{z}$, since $\|f(z)\|_{H^2} = \|f(z^2)\|_{H^2}$ (or see Proposition 3 below). The coefficients of this series can be found by squaring the series for

$$(3.17) \qquad \ln(1+z)/(1-z) = 2(z + \frac{z^3}{3} + \frac{z^5}{5} + \ldots) = \sum_{n=0}^{\infty} \frac{2z^{2n+1}}{2n+1}.$$

We obtain

$$(3.18) \qquad f(z^2) = 1 + \frac{2}{\pi^2}\Big(\ln(1+z)/(1-z) \Big)^2 = 1 + \frac{8}{\pi^2} \sum_{n=0}^{\infty} \Big(\sum_{j=0}^{n} \frac{1}{(2j+1)(2(n-j)+1)} \Big) z^{2n+2}.$$

We apply Proposition 2 with $a = 1$, using (3.15) and (3.18), to obtain

$$1^2 + \left(1 \times \frac{1}{3} + \frac{1}{3} \times 1\right)^2 + \left(1 \times \frac{1}{5} + \frac{1}{3} \times \frac{1}{3} + \frac{1}{5} \times 1\right)^2 + \left(1 \times \frac{1}{7} + \frac{1}{3} \times \frac{1}{5} + \frac{1}{5} \times \frac{1}{3} + \frac{1}{7} \times 1\right)^2 + \dots$$

$$= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \frac{1}{(2j+1)(2(n-j)+1)}\right)^2 = \frac{\pi^4}{32}.$$

Example 5 Let V_p be the hyperbolic region defined by

(3.19)
$$\left\{\left(\frac{\sin^2(\frac{\pi}{2}p)x + \cos^2(\frac{\pi}{2}p)}{\cos(\frac{\pi}{2}p)}\right)^2 - \sin^2(\frac{\pi}{2}p)y^2 > 1, \quad x > 0\right\},$$

where p is a parameter with $0 < p < 1/2$. The reader may check that

(3.20) $\quad \mathcal{G}_V(x + iy) = \dfrac{2\cos^2(\frac{\pi}{2}p)}{\sin^2(\frac{\pi}{2}p)(\cos^2(\frac{\pi}{2}p) - \sin^2(\frac{\pi}{2}p))}\left(\left(\dfrac{\sin^2(\frac{\pi}{2}p)x + \cos^2(\frac{\pi}{2}p)}{\cos(\frac{\pi}{2}p)}\right)^2 - \sin^2(\frac{\pi}{2}p)y^2 - 1\right).$

The conformal map for $\mathbb{D}$ to V_k sending 0 to 1 (which is a focus of the hyperbola) is given in [12] as

(3.21)
$$f(z) = \frac{1}{\sin^2(\frac{\pi}{2}p)}\cosh\left(p\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right) - \cot^2(\frac{\pi}{2}p).$$

As in Example 4, we may simplify by substituting z for $\sqrt{z}$. We calculate

$$f(z^2) = \frac{1}{2\sin^2(\frac{\pi}{2}p)}\left(\left(\frac{1+z}{1-z}\right)^p + \left(\frac{1-z}{1+z}\right)^p\right) - \cot^2(\frac{\pi}{2}p)$$

$$= \frac{1}{2\sin^2(\frac{\pi}{2}p)}\left(\frac{(1+z)^{2p} + (1-z)^{2p}}{(1-z^2)^p}\right) - \cot^2(\frac{\pi}{2}p)$$

(3.22)
$$= \frac{1}{2\sin^2(\frac{\pi}{2}p)}\left(\sum_{j=0}^{\infty}\binom{2p}{j}z^j + \sum_{j=0}^{\infty}\binom{2p}{j}(-z)^j\right)\left(\sum_{j=0}^{\infty}\binom{p+j-1}{j}z^{2j}\right) - \cot^2(\frac{\pi}{2}p)$$

$$= \frac{1}{2\sin^2(\frac{\pi}{2}p)}\left(2\sum_{j=0}^{\infty}\binom{2p}{2j}z^{2j}\right)\left(\sum_{j=0}^{\infty}\binom{p+j-1}{j}z^{2j}\right) - \cot^2(\frac{\pi}{2}p)$$

$$= \frac{1}{\sin^2(\frac{\pi}{2}p)}\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{2p}{2j}\binom{p+n-j-1}{n-j}\right)z^{2n} - \cot^2(\frac{\pi}{2}p),$$

where we have applied the generalized Binomial Theorem in the following forms, where $\binom{a}{b} = \frac{a(a-1)\dots(a-b+1)}{b!}$ is the binomial coefficient:

8

(3.23)
$$(1+z)^{2p} = \sum_{j=0}^{\infty} \binom{2p}{j} z^j, \quad (1-z^2)^{-p} = \sum_{j=0}^{\infty} \binom{p+j-1}{j} z^{2j}.$$

Applying Proposition 2 with $a = 1$ and simplifying gives

(3.24)
$$\sum_{n=1}^{\infty} \left(\sum_{j=0}^{n} \binom{2p}{2j} \binom{p+n-j-1}{n-j} \right)^2 = \frac{2\sin^4(\frac{\pi}{2}p)}{\cos^2(\frac{\pi}{2}p) - \sin^2(\frac{\pi}{2}p)} = \frac{2\sin^4(\frac{\pi}{2}p)}{\cos(\pi p)}$$

It is clear that this approaches ∞ as $p \nearrow \frac{1}{2}$, which means that the sum on the left side of (3.24) diverges for $1/2 \le p \le 1$. This follows also from the results in [5], which show that no function satisfying (i)-(iii) can exist on V_p if $p \in [1/2, 1)$ (see Theorem 3.1 of [5] and the ensuing application). We note that two values of p which give simple expressions on the right side of (3.24) are $p = 1/3$, which yields

(3.25)
$$\sum_{n=1}^{\infty} \left(\sum_{j=0}^{n} \binom{2/3}{2j} \binom{1/3+n-j-1}{n-j} \right)^2 = \frac{1}{4},$$

and $p = 1/4$, which yields

(3.26)
$$\sum_{n=1}^{\infty} \left(\sum_{j=0}^{n} \binom{1/2}{2j} \binom{1/4+n-j-1}{n-j} \right)^2 = \frac{1}{4}(3\sqrt{2}-4).$$

Example 6 Let V_θ be the region between the two branches of the hyperbola given by $(\frac{x}{\sin\theta})^2 - (\frac{y}{\cos\theta})^2 = 1$, where $\theta \in (0, \pi/4)$. That is, $V = \{(\frac{x}{\sin\theta})^2 < (\frac{y}{\cos\theta})^2 + 1\}$. It is not hard to see that

(3.27)
$$\mathcal{G}_V(x+iy) = \frac{2\sin^2\theta\cos^2\theta}{\cos^2\theta - \sin^2\theta}\left(1 + (\frac{y}{\cos\theta})^2 - (\frac{x}{\sin\theta})^2\right).$$

It is shown in [11] that, for fixed θ, $f(\theta, z) = \sin(\frac{4\theta}{\pi}\tan^{-1}z)$ is a conformal map from $\mathbb{D}$ to V_θ mapping 0 to 0. If we expand $f(\theta, z) = \sum_{n=1}^{\infty} a_n(\theta)z^n$, then Proposition 2 applied with $a = 0$ shows that

(3.28)
$$\sum_{n=1}^{\infty} a_n(\theta)^2 = \frac{2\sin^2\theta\cos^2\theta}{\cos^2\theta - \sin^2\theta} = \frac{\sin^2 2\theta}{2\cos 2\theta}.$$

A value for θ where we may find an explicit expression for $a_n(\theta)$ is $\theta = \frac{\pi}{8}$. The half-angle formula for sine leads to

9

$$(3.29) \qquad f(\pi/8, z) = \sin(\frac{1}{2}\tan^{-1} z) = \sqrt{\frac{1 - \cos(\tan^{-1} z)}{2}} = \sqrt{\frac{1 - \frac{1}{\sqrt{1+z^2}}}{2}} = \frac{\sqrt{\sqrt{1+z^2} - 1}}{\sqrt{2}(1+z^2)^{1/4}}.$$

Applying the Binomial Theorem in the form of (3.23) gives

$$(3.30) \qquad (1+z^2)^{-1/4} = \sum_{j=0}^{\infty} \binom{j - 3/4}{j}(-1)^j z^{2j}.$$

It is also true that

$$(3.31) \qquad \sqrt{\sqrt{1+z^2} - 1} = \frac{z}{\sqrt{2}} \; {}_2F_1(1/4, 3/4; 3/2; -z^2) = \frac{1}{\sqrt{2}} \sum_{j=0}^{\infty} \frac{(-1)^j C(2j) z^{2j+1}}{16^j},$$

where $C(k) = \frac{1}{k+1}\binom{2k}{k}$ is the k-th Catalan number (see [1] for details). We obtain

$$(3.32) \qquad \sin(\frac{1}{2}\tan^{-1} z) = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \left(\sum_{j=0}^{n} \binom{j - 3/4}{j} \frac{C(2(n-j))}{16^{n-j}} \right) z^{2n+1}.$$

Thus, (3.28) gives the identity

$$(3.33) \qquad \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \binom{j - 3/4}{j} \frac{C(2(n-j))}{16^{n-j}} \right)^2 = \sqrt{2}.$$

Similarly to Case 5, it can be shown that no function satisfying (i)-(iii) can exist on V_θ for $\theta \in [\frac{\pi}{4}, \frac{\pi}{2})$.

Example 7 Let V_ξ be the elliptical region given by $\{\frac{x^2}{\cosh^2 \xi} + \frac{y^2}{\sinh^2 \xi} < 1\}$, where ξ is a parameter greater than 0. The ellipse determining V_ξ has foci at -1 and 1. We may take

$$(3.34) \qquad \mathcal{G}_{V_\xi}(x + iy) = \frac{-2\sinh^2 \xi \cosh^2 \xi}{\sinh^2 \xi + \cosh^2 \xi} \left(\frac{x^2}{\cosh^2 \xi} + \frac{y^2}{\sinh^2 \xi} - 1 \right).$$

Let $K(z, t)$ be the normal elliptic integral of the first kind (following the normalization employed in [11]), that is

$$(3.35) \qquad K(z, t) = \int_0^z \frac{dx}{\sqrt{(1 - x^2)(1 - t^2 x^2)}}$$

10

for $0 < t < 1$. Set

$$(3.36) \qquad \mu(t) = \frac{\pi K(1, \sqrt{1 - t^2})}{2K(1, t)}.$$

Suppose t is chosen such that $\mu(t) = 2\xi$, and let $f_{a,t}(z)$ be the conformal map from $\mathbb{D}$ onto V_ξ mapping 0 to a with $f'_{a,t}(0) > 0$. It is known that

$$(3.37) \qquad f_{0,t}(z) = \sin\left(\frac{\pi}{2K(1,t)} K(z/\sqrt{t}, t)\right);$$

see [11] for details. Let

$$(3.38) \qquad f_{0,t}(z) = \sum_{n=1}^{\infty} a_n(t) z^n.$$

It is shown in [11] that the coefficients $a_n(\xi)$ are positive for odd n and 0 for even n. Proposition 2 then shows that upon setting $A_n(t) = a_{2n+1}(t)$ we have

$$(3.39) \qquad \sum_{n=0}^{\infty} A_n(t)^2 = \frac{2 \sinh^2 \xi \cosh^2 \xi}{\sinh^2 \xi + \cosh^2 \xi} = \frac{\sinh^2 2\xi}{2 \cosh 2\xi} = \frac{\sinh^2 \mu(t)}{2 \cosh \mu(t)}.$$

Furthermore, it is shown in [11] that

$$(3.40) \qquad f_{-1,2\xi}(z^2) = 2(f_{0,\xi}(z))^2 - 1.$$

Thus,

$$(3.41) \qquad f_{-1,2\xi}(z^2) = -1 + 2 \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} A_j A_{n-j}\right) z^{2n+2}.$$

Applying Theorem 2 with $a = -1$ yields

$$(3.42) \qquad \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} A_j A_{n-j}\right)^2 = \frac{\sinh^2 2\xi \cosh^2 2\xi}{2(\sinh^2 2\xi + \cosh^2 2\xi)} \left(\frac{\cosh^2 2\xi - 1}{\cosh^2 2\xi}\right) = \frac{\sinh^4 \mu(t)}{2 \cosh 2\mu(t)}.$$

Further identities involving the values of $A_n(t)$ are possible using higher order Chebyshev polynomials in order to construct maps from $\mathbb{D}$ to $V_{m\xi}$; see [11, p. 333]. It is also shown in [11] that the values $A_n(t)$ can be characterized as the unique solution to the recurrence relation

11

$$(3.43) \quad (2n+2)(2n+3)A_{n+1}(t) - \left((t+\frac{1}{t})(2n+1)^2 - \frac{\pi^2}{4tK(1,t)^2}\right)A_n(t) + 2n(2n-1)A_{n-1}(t) = 0.$$

$$A_{-1} = 0, A_0 = \frac{\pi}{2\sqrt{t}K(1,t)}.$$

These values can take a relatively simple form when t is chosen so that $K(1,t)$ is a rational multiple of π; for instance, if ω is such that $K(1,\omega) = \pi$, then we have

$$(3.44) \quad A_0(\omega) = \frac{1}{2\sqrt{\omega}}, \quad A_1(\omega) = \frac{3+4\omega^2}{48\omega^{3/2}}, \quad A_2(\omega) = \frac{105+56\omega^2+144\omega^4}{3840\omega^{5/2}},$$

$$A_3(\omega) = \frac{10395 + 4524\omega^2 + 4496\omega^4 + 14400\omega^6}{645120\omega^{7/2}}, \quad \ldots$$

Unfortunately, however, it seems difficult to find an explicit formula for the coefficients $A_n(t)$.

4 Concluding remarks

Remark 3. Cases 4 through 7 contain the formulas for the expectation of the first time that planar Brownian motion hits each of the conics.

Remark 4. An examination of the proof of Proposition 2 will show that the injectivity of f_a is not crucial to the argument. Instead, the important features possessed by conformal maps are analyticity and the mapping of the boundary of $\mathbb{D}$ to the boundary of V. With this in mind, let us define a function f from $\mathbb{D}$ into V to be *B-proper* if f is analytic and $\lim_{r \nearrow 1} f(re^{i\theta})$ exists and lies in δV for almost every θ in $[0, 2\pi)$. A similar definition was made in [15] (the definitions agree if $||f||_{H^2} < \infty$). The proof of Proposition 2 therefore applies to give us the following proposition.

Proposition 3. *Let* $f_a(z) = a + \sum_{n=1}^{\infty} a_n z^n$ *be a B-proper map from* $\mathbb{D}$ *into* V. *Suppose that* u_V *is a continuous function on* V *satisfying (i')-(iii') from Proposition 2. Then* $u_V(a) = \sum_{n=1}^{\infty} |a_n|^2$.

It is difficult to find a really good example to which we can apply this result, but we may give a simple one as follows. Let $f(z) = e^{\tan^{-1} z}$. f is not conformal, but does map $\mathbb{D}$ analytically onto the annulus $\{e^{\frac{-\pi}{4}} < |z| < e^{\frac{\pi}{4}}\}$, taking 0 to 1. f is B-proper, since f extends continuously to map $\{z = e^{i\theta}; -\frac{\pi}{2} < \theta < \frac{\pi}{2}\}$ to $\{|z| = e^{\frac{\pi}{4}}\}$ and $\{z = e^{i\theta}; \frac{\pi}{2} < \theta < \frac{3\pi}{2}\}$ to $\{|z| = e^{\frac{-\pi}{4}}\}$. We may therefore apply Proposition 3. The function $\ln|z|$ is harmonic on V, so it may be directly verified that we may take

$$(4.1) \quad \mathcal{G}(a) = \left(\frac{e^{\pi/2} - e^{-\pi/2}}{\pi/2}\right) \ln|a| + \frac{e^{\pi/2} + e^{-\pi/2}}{2} - |a|^2.$$

The power series for $f(z)$ begins $e^{\tan^{-1} z} = 1 + z + \frac{z^2}{2} - \frac{z^3}{6} - \frac{7z^4}{24} + \frac{z^5}{24} + \ldots$. We conclude from Proposition 3 that

$$(4.2) \qquad 1 + \frac{1}{2^2} + \frac{1}{6^2} + \frac{7^2}{24^2} + \frac{1}{24^2} + \ldots = \frac{e^{\pi/2} + e^{-\pi/2}}{2} - 1.$$

It is unfortunately not easy to find the general expression for the terms in this series. However, we may characterize the terms to be those determined uniquely by the following recurrence relation:

$$(4.3) \qquad a_n - (n+1)a_{n+1} - (n-1)a_{n-1} = 0, \qquad a_1 = 1.$$

This relation comes from equating the coefficients in the power series obtained in the identity

$$(4.4) \qquad (1 + z^2)\frac{d}{dz}e^{tan^{-1}z} = e^{tan^{-1}z}.$$

Remark 5. Suppose we have a domain V, a formula for $\mathcal{G}_V(a)$, and knowledge of a conformal map f_{a_o} from $\mathbb{D}$ to V sending 0 to a_o. Then we obtain not only an identity concerning the power series of f_{a_o} but in fact an entire family of identities, one for each point in V, as setting $w = f_{a_o}^{-1}(a)$ the map $f(\frac{z+w}{1+\bar{w}z})$ is a conformal map sending $\mathbb{D}$ to V and 0 to a. However, in practice the power series of the functions $f(\frac{z+w}{1+\bar{w}z})$ are difficult to explicitly compute, limiting the utility of this observation. We do, however, obtain a bit of intuition on how the method works. We are realizing a specific sum as one in a family of sums, indexed by a complex parameter a. In certain cases it is easier to evaluate the entire family of sums in one operation, using Propositions 1 or 2, than it is to evaluate the specific case we are interested in. We then recover the special case by choosing the correct value for a.

Remark 6. The recent paper [6] contains several more examples to which this method applies.

5   Acknowledgements

References

1. *The On-Line Encyclopedia of Integer Sequences*, http://oeis.org/A048990.

2. L.V. Ahlfors, *Complex analysis: An introduction to the theory of analytic functions of one complex variable*, McGraw-Hill, 1966.

3. A. Alabert, M. Farré, and R. Roy, *Exit times from equilateral triangles*, Applied Mathematics and Optimization **49** (2004), no. 1, 43–53.

4. R. Bañuelos and T. Carroll, *Brownian motion and the fundamental frequency of a drum*, Duke Mathematical Journal **75** (1994), no. 3, 575–602.

5. D.L. Burkholder, *Exit times of Brownian motion, harmonic majorization, and Hardy spaces*, Advances in Mathematics **26** (1977), no. 2, 182–205.

6. M. Coffey, *Expected exit times of Brownian motion from planar domains: Complements to a paper of Markowsky*, arXiv:1203.5142v1.

7. T.A. Driscoll and L.N. Trefethen, *Schwarz-Christoffel mapping*, Cambridge University Press, Cambridge, UK, 2002.

8. P. DuChateau and D.W. Zachmann, *Schaum's outline of theory and problems of partial differential equations*, Schaum's Outline Series, 1986.

9. G.H. Hardy and JE Littlewood, *A maximal theorem with function-theoretic applications*, Acta Mathematica **54** (1930), no. 1, 81–116.

10. K. Helmes, S. Röhl, and R.H. Stockbridge, *Computing moments of the exit time distribution for Markov processes by linear programming*, Operations Research (2001), 516–530.

11. S. Kanas and T. Sugawa, *On conformal representations of the interior of an ellipse*, Annales Academiæ Scientiarum Fennicæ **31** (2006), 329–348.

12. S. Kanas and A. Wisniowska, *Conic regions and k-uniform convexity*, Journal of Computational and Applied Mathematics **105** (1999), no. 1-2, 327–336.

13. F.B. Knight, *Essentials of Brownian motion and diffusion*, American Mathematical Society, 1981.

14. W. Koepf, *Schwarz-Christoffel mappings: Symbolic computation of mapping functions for symmetric polygonal domains*, Proceedings of the Workshop on Functional-Analytic Methods in Complex Analysis and Applications to Partial Differential Equations (1995), 293–305.

15. G. Markowsky, *On the expected exit time of planar Brownian motion from simply connected domains*, Electronic Communications in Probability **16** (2011), 652–663.

16. W. Rudin, *Real and complex analysis*, Tata McGraw-Hill, 2006.