On sequences with $\{-1, 0, 1\}$ Hankel transforms

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Abstract

We study Hankel transforms of sequences, where the transform elements are members of the set $\{-1, 0, 1\}$. We relate these Hankel transforms to special continued fraction expansions. In particular, we posit a conjecture relating the distribution of non-zero terms in the Hankel transform to the distribution of powers of the variable in the defining continued fractions.

1 Introduction

Given a sequence a_n, we denote by h_n the general term of the sequence with $h_n = |a_{i+j}|_{0 \le i,j \le n}$. The sequence h_n is called the Hankel transform of a_n [2]. For a given sequence a_n, it can be shown that the sequence

$$\sum_{k=0}^{n} \binom{n}{k} r^{n-k} a_k$$

will also have the same Hankel transform. Similarly, if the sequence a_n has generating function $f(x)$, where $f(0) \ne 0$, then the sequence with generating function $\frac{f(x)}{1-rxf(x)}$ will also have same Hankel transform. In both cases, $r \in \mathbb{R}$ is arbitrary. Thus many sequences may have the same Hankel transform, and therefore the problem of inverting the Hankel transform is not an elementary one. In this note, we show, subject to a deep conjecture, that in one instance, the relationship between a sequence h_n and its pre-image is more easily determined.

Although in the sequel we will exhibit Hankel transforms with ostensibly more than one pre-image, we introduce the notion of element multiplicity to confer a degree of uniqueness in the transform.

In the sequel, we will work with integer sequences. We will refer to some known sequences by their "*Annnnnn*" number in the On-Line Encyclopedia of Integer Sequences [4].

1

2 A conjecture concerning special continued fractions

Conjecture 1. *Consider a sequence of natural numbers p_n, where $p_0 = 1$ and the sequence b_n defined by*

$$b_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} p_{n-2k} - \frac{1 + (-1)^n}{2} \tag{1}$$

is non-negative and non-decreasing, with $b_n \in \mathbb{N}_0$, $b_0 = 0, b_1 \neq 0$. Then the sequence a_n with generating function expressed as the continued fraction

$$\cfrac{1}{1 \pm \cfrac{x^{p_0}}{1 \pm \cfrac{x^{p_1}}{1 \pm \cfrac{x^{p_2}}{1 \pm \cdots}}}}$$

has a Hankel transform consisting solely of the numbers -1, 0, and 1. Moreover, the non-zero terms occur at locations indexed by the sequence b_n.

Conversely, given a sequence of numbers $b_n \in \mathbb{N}_0$, $b_0 = 0, b_1 \neq 0, b_2, b_3, \ldots$ with $b_i \leq b_{i+1}$, then the sequence whose generating function is given by the above continued fraction where the sequence p_n is given by

$$
\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ \vdots \end{pmatrix}
=
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
0 & -1 & 0 & 1 & 0 & 0 & \cdots \\
0 & 0 & -1 & 0 & 1 & 0 & \cdots \\
0 & 0 & 0 & -1 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\begin{pmatrix} 1 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ \vdots \end{pmatrix}
$$

has a $\{-1, 0, 1\}$ Hankel transform whose non-zero elements are indexed by the sequence b_n.

Equation (1) can be visualized in the following way.

$$
\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ \vdots \end{pmatrix}
=
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 1 & 0 & \cdots \\
0 & 1 & 0 & 1 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ \vdots \end{pmatrix},
$$

where column k of the above matrix (except for column 0) has generating function $\frac{x^k}{1-x^2}$. The sequence of numbers given by $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} p_{n-2k}$ corresponds to the degrees of the denominator polynomials in the partial continued fractions.

2

In the case that $b_i = b_{i+1} = \cdots = b_{i+r}$ we say that the corresponding non-zero term occurs with *multiplicity* $r + 1$.

We shall call the sequence of powers $\{p_n\}$ in the continued fraction the *CF power sequence* and we shall call the indexing sequence b_n the *Hankel pattern sequence*. It specifies the pattern of occurrence of the non-zero terms in the Hankel transform. If we let $P(x)$ denote the generating function of the sequence p_n, and $B(x)$ denote the generating function of b_n, then we have the relations

$$B(x) = \frac{1}{1 - x^2}P(x) - \frac{1}{1 - x^2}, \tag{2}$$

and

$$P(x) = (1 - x^2)B(x) - 1. \tag{3}$$

Note that in the continued fraction above, the $\pm$ indicates that an arbitrary choice of "+" or "−" is possible at each stage. Note also that the conjecture is silent on the matter of the distribution of the minus signs in the transform.

Example 2. The Jacobsthal numbers $J_n = \frac{2^n}{3} - \frac{(-1)^n}{3}$ [A001045](#) give the sequence of numbers

$$0, 1, 1, 3, 5, 11, 21, 43, \ldots.$$

We seek to find a sequence a_n whose Hankel transform is composed of the numbers -1, 0 and 1, where the distribution of the non-zero terms is governed (or indexed) by the Jacobsthal numbers. Thus we want a Hankel transform of the form

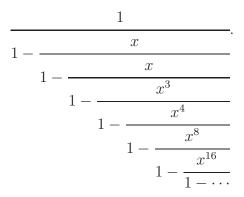$$\alpha, \beta, 0, \gamma, 0, \delta, 0, 0, 0, 0, 0, \epsilon, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \zeta, 0, 0, \ldots$$

where $\alpha, \beta, \ldots \in \{-1, 1\}$. As it is not immediately obvious how to deal with the duplicated 1 ($J_1 = J_2 = 1$), we shall ignore this for the moment and assume that the sequence is

$$0, 1, 3, 5, 11, 21, 43, \ldots.$$

We look for the sequence of powers of x to be used in the defining continued fraction.

$$\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & -1 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & -1 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 3 \\ 5 \\ 11 \\ 21 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 4 \\ 8 \\ 16 \\ \vdots \end{pmatrix}$$

This leads us to consider the sequence a_n with generating function

$$\cfrac{1}{1 - \cfrac{x}{1 - \cfrac{x}{1 - \cfrac{x^3}{1 - \cfrac{x^4}{1 - \cfrac{x^8}{1 - \cfrac{x^{16}}{1 - \cdots}}}}}}}.$$

Thus a_n begins

$$1, 1, 2, 4, 8, 17, 36, 76, 161, 342, 726, 1541, 3272, 6948, 14753, \ldots$$
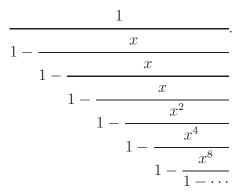
with the desired Hankel transform

$$1, 1, 0, -1, 0, 1, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, \ldots.$$

Now let us see what happens when we retain both J_1 and J_2. We look for the sequence of powers of x to be used in the defining continued fraction.

$$
\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ \vdots \end{pmatrix}
=
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
0 & -1 & 0 & 1 & 0 & 0 & \cdots \\
0 & 0 & -1 & 0 & 1 & 0 & \cdots \\
0 & 0 & 0 & -1 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \\ 5 \\ 11 \\ \vdots \end{pmatrix}
=
\begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 4 \\ 8 \\ \vdots \end{pmatrix}
$$

This leads us to consider the new sequence a_n^* with generating function

$$\cfrac{1}{1 - \cfrac{x}{1 - \cfrac{x}{1 - \cfrac{x}{1 - \cfrac{x^2}{1 - \cfrac{x^4}{1 - \cfrac{x^8}{1 - \cdots}}}}}}}.$$

We obtain the sequence that starts

$$1, 1, 2, 5, 13, 35, 95, 259, 707, 1932, 5281, 14438, 39475, 107933, 295115, 806922, 2206342, \ldots,$$

and which has Hankel transform

$$1, 1, 0, -1, 0, 1, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, \ldots.$$

Thus both sequences a_n and a_n* have the same Hankel transform. However, the Hankel transform of a_n corresponds to the pattern sequence $0, 1, 3, 5, 11, \ldots$ while that of a_n* has distribution $0, 1, 1, 3, 5, 11, \ldots$. To distinguish the Hankel transforms, we could accompany each "continued-fraction-derived" $\{-1, 0, 1\}$ Hankel transform with its Hankel pattern sequence (or equivalently its CF power sequence). Using the Hankel pattern sequence for this purpose is equivalent to assigning multiplicities to the non-zero elements of the Hankel transform. Thus in the case of the Hankel transform of a_n*, we assign a multiplicity of two to the second 1 in the Hankel transform, corresponding to the repetition of element $J_1 = J_2$. To emphasize this, we sometimes write this Hankel transform as

$$1, 1_2, 0, -1, 0, 1, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, \ldots,$$

where un-indexed numbers have multiplicity 1.

In subsequent examples we allow for repeated elements in the Hankel pattern sequence. The next case is an extreme case, where all elements (bar the first) are duplicated.

Example 3. The Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ [A000108](#) with generating function $c(x)$ given by

$$c(x) = \frac{1 - \sqrt{1 - 4x}}{2x} = \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{x}{1 - \cfrac{x}{1 - \cdots}}}}$$

is well known to have Hankel transform $1, 1, 1, \ldots$ [1]; that is, each element $h_n = 1$. The CF power sequence for C_n is also $1, 1, 1, \ldots$. This means that the Hankel pattern for C_n is the sequence

$$0, 1, 1, 2, 2, 3, 3, 4, 4, \ldots$$

with general term $\lfloor \frac{n+1}{2} \rfloor$. Thus the first term $h_0 = 1$ has multiplicity 1 while all the other terms $h_n = 1$ have multiplicity 2. Using the index notation, we could thus write the Hankel transform of C_n as

$$1, 1_2, 1_2, 1_2, 1_2, 1_2, 1_2, 1_2, 1_2, 1_2, \ldots.$$

Example 4. We consider the CF power sequence $\{p_n\}$ given by $1, 1, 2, 2, 2, 2, 2, \ldots$. This corresponds to the Hankel pattern

$$0, 1, 2, 3, 4, 5, 6, \ldots.$$

5

The sequence a_n with generating function

$$\cfrac{1}{1 - \cfrac{x}{1 - \cfrac{x}{1 - \cfrac{x^2}{1 - \cfrac{x^2}{1 - \cfrac{x^2}{1 - \cdots}}}}}}$$

has Hankel transform $1, 1, 1, 1, \ldots$ where each "1" has multiplicity 1. Thus we could have written this Hankel transform as

$$1_1, 1_1, 1_1, 1_1, 1_1, 1_1, 1_1, 1_1, \ldots.$$

The sequence a_n begins

$$1, 1, 2, 4, 9, 20, 46, 105, 243, 560, 1299, 3006, \ldots,$$

The generating function of this sequence is

$$\cfrac{1}{1 - \cfrac{x}{1 - xc(x^2)}}.$$

Example 5. It is not the case that every $\{-1, 0, 1\}$ sequence is the Hankel transform of an integer sequence. For example, the sequence

$$1, 0, 1, 0, 1, 0, 1, 0, 1, \ldots$$

is not the Hankel transform of any integer (or real) sequence.

Proof. (Somos) Assume that the sequence $\alpha, \beta, \gamma, \delta, \epsilon, \ldots$ has Hankel transform $1, 0, 1, 0, \ldots$.
We have $h_0 = \alpha$ and hence $\alpha = 1$.
Then $h_1 = \gamma - \beta^2 = 0$ and so $\gamma = \beta^2$, and so the sequence would start $1, \beta, \beta^2, \delta, \ldots$.
Then $h_2 = \beta^6 + 2\beta^3\delta - \delta^2$, and so we must have

$$\beta^6 + 2\beta^3\delta - \delta^2 = 1.$$

The solution of this quadratic in δ is

$$\delta = \beta^3 - i \quad \text{or} \quad \delta = \beta^3 + i.$$

$\square$

Example 6. Consider the CF power sequence p_n given by $1, 1, 3, 3, 3, 3, 3, \ldots$. This gives us the Hankel pattern sequence

$$0, 1, 3, 4, 6, 7, 9, 10, 12, 13, 15, \ldots.$$

The corresponding sequence begins

$$1, 1, 2, 4, 8, 17, 36, 76, 162, 345, 734, 1565, 3336, 7109, 15158, 32318, 68898, \ldots$$

with generating function

$$\cfrac{1}{1 - \cfrac{x}{1 - xc(x^3)}}.$$

Its Hankel transform is equal to the periodic sequence

$$1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, \ldots.$$

Here, the non-zero terms all have multiplicity one.

3   A number theoretic example

We let the Hankel pattern sequence be the sequence $b_n = \binom{n+1}{2}$ of the triangular numbers. We find that the corresponding CF power sequence is the sequence

$$1, 1, 3, 5, 7, 9, 11, \ldots$$

of extended odd numbers. The sequence generated by the continued fraction defined by the power sequence begins

$$1, 1, 2, 4, 8, 17, 36, 76, 161, 341, 723, 1533, 3250, 6891, 14611, 30980, 65688, 139281, \ldots$$

and has a Hankel h_n transform that begins

$$1, 1, 0, -1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, -1, 0, \ldots.$$

In this case, the quantity

$$e_n = \sum_{k=0}^{n} (-1)^{n-k} h_n h_{n-k}$$

is of interest. This is the convolution of h_n with $(-1)^n h_n$. The sequence e_{2n} begins

$$1, -1, 2, 1, 0, 2, 1, 0, 0, 2, 1, 2, \ldots$$

and is related to the so-called "eta quotients".

If we now take the CF power sequence $p_n = \lfloor \frac{n+1}{2} \rfloor + 0^n$, or $1, 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, \ldots$, we obtain the Hankel pattern sequence

$$0, 1, 1, 3, 3, 6, 6, 10, 10, 15, 15, \ldots$$

The sequence a_n now begins

$$1, 1, 2, 5, 13, 35, 95, 260, 713, 1959, 5386, 14815, 40759, 112151, 308609, 849240, 2337009, 6431246, \ldots$$

and has Hankel transform

$$1, 1, 0, -1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, \ldots$$

Here, apart from the first term, the non-zero terms have multiplicity two.

4 More variations on $1, 3, 5, \ldots$

We start this section by noting that the sequence defined by the CF power sequence $1, 3, 5, 7, \ldots$ is [A143951](#), which counts the number of Dyck paths such that the area between the x-axis and the path is n (Emeric Deutsch). The Hankel transform of this sequence begins

$$1, 0, 0, -1, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, -1, \ldots,$$

with Hankel pattern sequence

$$b_n = (n^2 + 3n + 1 + (-1)^n)/2,$$

which is essentially [A176222](#). The sequence b_n thus begins

$$0, 3, 5, 10, 14, 21, 27, 36, 44, 55, 65, \ldots.$$

For the CF power sequence $1, 1, 3, 3, 5, 5, 7, 7, \ldots$ we find that the sequence b_n, which begins

$$0, 1, 3, 4, 8, 9, 15, 16, 24, 25, 35, \ldots,$$

satisfies

$$b_n = (2n^2 + 6n + 1 + (2n - 1)(-1)^n)/8.$$

The corresponding sequence a_n begins

$$1, 1, 2, 4, 8, 17, 36, 76, 162, 345, 734, 1564, 3332, \ldots$$

with Hankel transform

$$1, 1, 0, -1, -1, 0, 0, 0, -1, -1, 0, 0, 0, 0, 0, 1, 1, 0, 0, \ldots.$$

For the CF power sequence $1, 1, 1, 3, 3, 3, 5, 5, 5, 7, 7, 7, \ldots$ we find that the sequence b_n, which begins

$$0, 1, 1, 3, 3, 5, 6, 8, 9, 12, 13, 16, \ldots,$$

has generating function

$$\frac{x(1 + x^2 - x^3)}{(1 - x)^3(1 + 2x + 2x^2 + x^3)}.$$

8

The corresponding sequence a_n begins

$$1, 1, 2, 5, 13, 34, 90, 239, 635, 1689, 4494, 11958, 31823, 84692, 225396, \ldots$$

and has Hankel transform

$$1, 1, 0, 0, -1, 0, 0, 1, 0, -1, 0, 0, 1, 0, -1, 0, \ldots.$$

Some slight variations on this last example are also of interest. For instance, the CF power sequence

$$1, 2, 3, 3, 3, 5, 5, 5, 7, 7, 7, 9, 9, 9, 11, \ldots$$

corresponds to the Hankel pattern sequence b_n which begins

$$0, 2, 3, 5, 6, 10, 11, 15, 18, 22, 25, 31, 34, 40, 45, \ldots,$$

with generating function

$$\frac{x(2 + x - 2x^3 + x^4)}{(1 - x)^3(1 + 2x + 2x^2 + x^3)}.$$

The Hankel transform of a_n is then

$$1, 0, -1, -1, 0, 1, 1, 0, 0, 0, 1, 1, 0, 0, 0, 1, \ldots.$$

5   A pattern avoiding example

The CF power sequence

$$1, 1, 1, 1, 2, 1, 1, 2, 1, 1, 2, 1, 1, 2, 1, 1, 2, 1, 1, 2, 1, \ldots$$

corresponds to the Hankel pattern sequence

$$0, 1, 1, 2, 3, 3, 4, 5, 5, 6, 7, 7, 8, 9, 9, 10, 11, 11, 12, 13, 13, \ldots$$

which gives a $\{-1, 0, 1\}$ Hankel transform where odd indexed non-zero terms have multiplicity two, and the even-indexed non-zero terms have multiplicity one. The corresponding sequence is [A054391](A054391) [3]. It begins

$$1, 1, 2, 5, 14, 41, 123, 374, 1147, 3538, 10958, 34042, 105997, \ldots$$

and has Hankel transform consisting of all 1's (with the multiplicities above). Thus we could write the Hankel transform as

$$1, 1_2, 1, 1_2, 1, 1_2, 1, \ldots.$$

We note that the generating function of this sequence is given by

$$\cfrac{1}{1 - \cfrac{x}{1 - xu}},$$

9

where u satisfies the equation

$$u = \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{x}{1 - x^2 u}}}.$$

Solving, we find that the generating function is equal to

$$\frac{1 - 3x - \sqrt{1 - 2x - 3x^2}}{1 - 3x + 2x^2 + \sqrt{1 - 2x - 3x^2}},$$

confirming that this sequence is the same as A054391.

6 The Motzkin numbers

We consider the CF power sequence

$$1, 1, 2, 1, 1, 2, 1, 1, 2, 1, 1, 2, 1, 1, 2, 1, 1, 2, 1, 1, 2, \ldots.$$

This corresponds to the Hankel pattern sequence

$$0, 1, 2, 2, 3, 4, 4, 5, 6, 6, 7, 8, 8, 9, 10, 10, 11, 12, 12, 13, 14, \ldots$$

so once again we obtain a sequence with an all 1's Hankel transform, but with the multiplicities indicated. Thus we get the Hankel transform

$$1, 1, 1_2, 1, 1_2, 1, 1_2, 1, \ldots.$$

The generating function $u = g(x)$ satisfies the equation

$$u = \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{x}{1 - x^2 u}}},$$

which solves to give

$$g(x) = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2},$$

which coincides with the g.f. of the Motzkin numbers A001006.

7 Euler pentagonal numbers

In this section, we take the Euler pentagonal numbers A001318 [5] as the basis of the CF power sequence. Thus we define

$$p_n = \frac{6n^2 + 6n + 1}{16} - \frac{(2n + 1)(-1)^n}{16} + 0^n$$

10

which begins
$$1, 1, 2, 5, 7, 12, 15, 22, 26, 35, 40, \ldots.$$

The corresponding Hankel pattern sequence is given by

$$b_n = \frac{1}{2} \lfloor \frac{n+1}{2} \rfloor \lfloor \frac{n+2}{2} \rfloor \lfloor \frac{n+3}{2} \rfloor,$$

which is [A028724](#). This begins

$$0, 1, 2, 6, 9, 18, 24, 40, 50, 75, 90, \ldots.$$

The resulting sequence begins

$$1, 1, 2, 4, 9, 20, 45, 101, 227, 511, 1150, 2589, 5828, 13120, 29536, 66492, 149690, \ldots,$$

and has Hankel transform

$$1, 1, 1, 0, 0, 0, 1, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, \ldots$$

Note that if in the continued fraction we take the pattern of signs $-, -, +, +, -, -, +, +, \ldots$ then the resulting sequence begins

$$1, 1, 2, 4, 7, 12, 21, 37, 65, 115, 204, 361, 638, 1128, 1994, 3524, 6230, \ldots$$

and has a Hankel transform that begins

$$1, 1, -1, 0, 0, 0, -1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, \ldots$$

8 A Fibonacci distribution

It is of interest to find a sequence whose Hankel transform has a $\{-1, 0, 1\}$ distribution that follows the Fibonacci numbers [A000045](#). Corresponding to the CF power sequence

$$1, 1, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

we get the Hankel pattern sequence

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots.$$

The sequence defined by the corresponding continued fraction begins

$$1, 1, 2, 5, 14, 41, 123, 373, 1137, 3475, 10634, 32562, 99738, 305546, 936108, 2868084, \ldots$$

and it has a Hankel transform that starts

$$1, 1, 1, 1, 0, -1, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, \ldots.$$

11

9   A gap Hankel transform

Consider the CF power sequence

$$1, 3, 4, 2, 2, 2, 2, 2, \dots.$$

The corresponding Hankel pattern sequence is

$$0, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \dots.$$

Thus there will be a 'gap' at positions 1 and 2 (filled by zeros). In fact, the sequence a_n corresponding to this CF power sequence begins

$$1, 1, 1, 1, 2, 3, 4, 6, 10, 15, 23, 36, 58, 90, 145, 230, 377, 601, 1000, \dots$$

and has Hankel transform h_n given by

$$1, 0, 0, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, \dots.$$

The sequence a_n has some interesting Hankel properties. The sequence a_{n+1} has Hankel transform given by

$$1, 0, -1, 0, 1, 0, -1, 0, 1, 0, \dots.$$

The twice shifted sequence a_{n+2} has Hankel transform

$$1, 1, -1, -1, -2, -2, -3, -3, -4, -4, \dots,$$

while the three-times shifted sequence a_{n+3} has Hankel transform

$$1, -1, -1, 1, 4, -1, -9, 1, 16, \dots.$$

If we prepend $1, 1$ to the sequence then the new sequence has a Hankel transform of

$$1, 0, 0, 0, 0, 1, 1, 2, 2, 3, 3, \dots.$$

Other sequences with "gap" Hankel transforms can be built using the template

$$1, r, r + 1, 2, 2, 2, 2, 2, 2, 2, \dots$$

for the CF power sequence, since this maps to the pattern sequence

$$0, r, r + 1, r + 2, r + 3, r + 4, r + 5, r + 6, r + 7, r + 8, r + 9, \dots.$$

10   Conclusion

The foregoing shows that $\{-1, 0, 1\}$ Hankel transforms are objects worthy of study. The notion of multiplicity seems important, particularly as a way of further characterizing certain sequences that would otherwise have the "same" Hankel transforms. Thus the Catalan numbers C_n have Hankel transform $1, 1, 1, 1, \dots$ but with pattern sequence $0, 1, 1, 2, 2, 3, 3, \dots$, while the Motzkin numbers M_n also have Hankel transform $1, 1, 1, 1, \dots$ but with pattern sequence $0, 1, 2, 2, 3, 4, 4, \dots.$

11 Acknowledgement

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Concerns sequences
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