

# PERFECT CUBOIDS AND MULTISYMMETRIC POLYNOMIALS.

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ABSTRACT. A perfect Euler cuboid is a rectangular parallelepiped with integer edges and integer face diagonals whose space diagonal is also integer. The problem of finding such parallelepipeds or proving their non-existence is an old unsolved mathematical problem. The Diophantine equations of a perfect Euler cuboid have an explicit  $S_3$  symmetry. In this paper the cuboid equations are factorized with respect to their  $S_3$  symmetry in terms of multisymmetric polynomials. Some factor equations are calculated explicitly.

## 1. INTRODUCTION.

The search for perfect cuboids has the long history since 1719 (see [1–39]). This history is presented as an adventure story in [37]. Let  $x_1, x_2, x_3$  be the edges of a cuboid and let  $d_1, d_2, d_3$  be its face diagonals. Then we have the equations

$$\begin{aligned} (x_1)^2 + (x_2)^2 - (d_3)^2 &= 0, & (d_3)^2 + (x_3)^2 - L^2 &= 0, \\ (x_2)^2 + (x_3)^2 - (d_1)^2 &= 0, & (d_1)^2 + (x_1)^2 - L^2 &= 0, \\ (x_3)^2 + (x_1)^2 - (d_2)^2 &= 0, & (d_2)^2 + (x_2)^2 - L^2 &= 0, \end{aligned} \quad (1.1)$$

where  $L$  is the space diagonal of the cuboid. In the case of a perfect Euler cuboid the equations (1.1) constitute a system of Diophantine equations with respect to seven variables  $x_1, x_2, x_3, d_1, d_2, d_3$ , and  $L$ . In [40] the equations (1.1) were reduced to a single Diophantine equation with respect to four especially introduced parameters  $a, b, c$ , and  $u$ . On the base of this equation in [41] three cuboid conjectures were formulated. These conjectures are studied (but not yet proved) in [42–44].

In the present paper we apply a quite different approach to the equations (1.1). The equations (1.1) possess a natural  $S_3$  symmetry. Indeed, the symmetric group  $S_3$  is composed by transformations of the set of three numbers  $\{1, 2, 3\}$ :

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ \sigma 1 & \sigma 2 & \sigma 3 \end{pmatrix}. \quad (1.2)$$

The transformation (1.2) is applied to the equations (1.1) as follows:

$$\sigma(x_i) = x_{\sigma i}, \quad \sigma(d_i) = d_{\sigma i}, \quad \sigma(L) = L. \quad (1.3)$$

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Looking at (1.1) and (1.3), one can easily see that the system of equations (1.1) in whole is invariant with respect to the transformations  $\sigma \in S_3$ .

The main goal of this paper is to factorize the equations (1.1) with respect to the  $S_3$  symmetry (1.3). We reach this goal by deriving some new equations from (1.1). These new equations are written in terms of the values of so-called multisymmetric polynomials (they generalize well-known symmetric polynomials).

## 2. MULTISYMMETRIC POLYNOMIALS.

Multisymmetric polynomials, which are also known as vector symmetric polynomials, diagonally symmetric polynomials, McMahon polynomials etc, were initially studied in [45–51] (see also later publications [52–65]). Let's consider a set of variables arranged into some  $m \times n$  matrix as follows:

$$M = \begin{vmatrix} x_{11} & \dots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \dots & x_{mn} \end{vmatrix} \quad (2.1)$$

The symmetric group  $S_n$  acts upon the matrix (2.1) by permuting its columns:

$$\sigma(x_{ij}) = x_{i\sigma j}. \quad (2.2)$$

**Definition 2.1.** A polynomial  $p \in \mathbb{Q}[x_{11}, \dots, x_{mn}]$  is called *multisymmetric* if it is invariant with respect to the action (2.2) of the symmetric group  $S_n$ , i. e. if

$$p(x_{1\sigma 1}, \dots, x_{m\sigma n}) = p(x_{11}, \dots, x_{mn}) \quad \text{for all } \sigma \in S_n.$$

Let  $q(x) = q(x_{11}, \dots, x_{mn})$  be an arbitrary polynomial of the variables composing the matrix (2.1). Then we can produce a multisymmetric polynomial by applying the symmetrization operator  $S$  to the polynomial  $q(x_{11}, \dots, x_{mn})$ :

$$S(q(x_{11}, \dots, x_{mn})) = \frac{1}{n!} \sum_{\sigma \in S_n} q(x_{1\sigma 1}, \dots, x_{m\sigma n}). \quad (2.3)$$

Regular symmetric polynomials (see [66]) correspond to the special case  $m = 1$  in the definition 2.1. Like in the case  $m = 1$ , in general case  $m > 1$  there are elementary symmetric polynomials. However, in this general case elementary symmetric polynomials are enumerated not by a single index, but by a multiindex:

$$\alpha = [\alpha_1, \dots, \alpha_m], \quad \text{where } \alpha_i \geq 0 \quad \text{and} \quad |\alpha| = \alpha_1 + \dots + \alpha_m \leq n. \quad (2.4)$$

Let's denote through  $x^\alpha$  the following monomial:

$$x^\alpha = \underbrace{x_{11} \dots x_{1\alpha_1}}_{\alpha_1} \cdot \underbrace{x_{2\alpha_1+1} \dots x_{2\alpha_1+\alpha_2}}_{\alpha_2} \cdot \dots \cdot \underbrace{x_{mn-\alpha_m+1} \dots x_{mn}}_{\alpha_m}. \quad (2.5)$$

The variables in the product (2.5) are taken from  $n$  consecutive columns of the matrix (2.1). The initial group of  $\alpha_1$  of them is taken from the first row of this matrix, the next group of  $\alpha_2$  of these variables is taken from the second row and so on. The last group of  $\alpha_m$  variables is taken from the last  $m$ -th row of the matrix

(2.1). If  $\alpha_i = 0$ , then the corresponding  $i$ -th group in (2.5) is empty and hence the variables of  $i$ -th row do not enter the monomial (2.5) at all.

**Definition 2.2.** An elementary multisymmetric polynomial  $e_\alpha(x_{11}, \dots, x_{mn})$  corresponding to the multiindex (2.4) is produced from the monomial (2.5) by means of the symmetrization operator (2.3) according to the formula

$$e_\alpha(x) = \frac{n!}{\alpha!} S(x^\alpha), \text{ where } \alpha! = \alpha_1! \cdot \dots \cdot \alpha_m!. \quad (2.6)$$

Note that the ratio  $n!/\alpha!$  in (2.6) is always an integer number and  $e_\alpha(x)$  is the sum of exactly  $n!/\alpha!$  monomials produced from the monomial (2.5) by means of the permutations of variables (2.2). In the case of the trivial multiindex  $0 = [0, \dots, 0]$  the formulas (2.5) and (2.6) reduce to the following ones:

$$x^0 = 1, \quad e_0 = 1.$$

Like in the case of regular symmetric polynomials, there is the following fundamental theorem for multisymmetric polynomials.

**Theorem 2.1.** *The elementary multisymmetric polynomials (2.6) with multiindices  $0 < |\alpha| \leq n$  generate the ring of all multisymmetric polynomials, i. e. each multisymmetric polynomial  $p \in \mathbb{Q}[x_{11}, \dots, x_{mn}]$  can be expressed as a polynomial with rational coefficients through these elementary multisymmetric polynomials.*

The proof of the fundamental theorem 2.1 can be found in [51]. Unfortunately the elementary multisymmetric polynomials (2.6) are not algebraically independent over  $\mathbb{Q}$  for  $m > 1$  (see [60]). For this reason the expression of  $p$  as a polynomial with rational coefficients through the elementary multisymmetric polynomials  $e_\alpha(x)$ , which is claimed by the fundamental theorem 2.1, is not unique.

## 2. MULTISYMMETRIC POLYNOMIALS ASSOCIATED WITH A CUBOID.

Note that the formulas (1.3) can be treated as a special case of the formulas (2.2). Indeed, let's compose the  $2 \times 3$  matrix

$$M = \begin{vmatrix} x_1 & x_2 & x_3 \\ d_1 & d_2 & d_3 \end{vmatrix}. \quad (3.1)$$

Due to (1.3) the transformations  $\sigma \in S_3$  act as permutations of columns upon the matrix (3.1). Applying the definition 2.1 to the matrix (3.1), we get the concept of a multisymmetric polynomial of six variables  $x_1, x_2, x_3$  and  $d_1, d_2, d_3$ . Now we calculate the elementary multisymmetric polynomials corresponding to the matrix (3.1). The first three of these polynomials are

$$\begin{aligned} e_{[1,0]} &= x_1 + x_2 + x_3, \\ e_{[2,0]} &= x_1 x_2 + x_2 x_3 + x_3 x_1, \\ e_{[3,0]} &= x_1 x_2 x_3. \end{aligned} \quad (3.2)$$

It is easy to see that the polynomials (3.2) coincide with the regular symmetric polynomials of the three variables  $x_1, x_2, x_3$ . The next three elementary multisym-

metric polynomials are similar to (3.2). They are

$$\begin{aligned} e_{[0,1]} &= d_1 + d_2 + d_3, \\ e_{[0,2]} &= d_1 d_2 + d_2 d_3 + d_3 d_1, \\ e_{[0,3]} &= d_1 d_2 d_3. \end{aligned} \tag{3.3}$$

The polynomials (3.3) coincide with the regular symmetric polynomials of the three variables  $d_1, d_2, d_3$ . The rest of the elementary multisymmetric polynomials are actually multisymmetric. They include variables from both rows of the matrix  $M$ :

$$\begin{aligned} e_{[2,1]} &= x_1 x_2 d_3 + x_2 x_3 d_1 + x_3 x_1 d_2, \\ e_{[1,1]} &= x_1 d_2 + d_1 x_2 + x_2 d_3 + d_2 x_3 + x_3 d_1 + d_3 x_1, \\ e_{[1,2]} &= x_1 d_2 d_3 + x_2 d_3 d_1 + x_3 d_1 d_2. \end{aligned} \tag{3.4}$$

The polynomials (3.2), (3.3), and (3.4) constitute the complete set of elementary multisymmetric polynomials associated with the matrix (3.1).

#### 4. THE FIRST FOUR FACTOR EQUATIONS.

Note that the variables  $x_1, x_2, x_3$  and  $d_1, d_2, d_3$  in the matrix (3.1) are not independent. They are related to each other by means of the polynomial equations (1.1). For this reason the elementary multisymmetric polynomials (3.2), (3.3), and (3.4) produced from the variables  $x_1, x_2, x_3$  and  $d_1, d_2, d_3$  gain more algebraic relations in addition to those present in the case of independent variables  $x_1, x_2, x_3$  and  $d_1, d_2, d_3$  (see comments to fundamental theorem 2.1). These algebraic relations are written as polynomial equations with coefficients in  $\mathbb{Q}$ :

$$p(e_{[1,0]}, e_{[2,0]}, e_{[3,0]}, e_{[0,1]}, e_{[0,2]}, e_{[0,3]}, e_{[2,1]}, e_{[1,1]}, e_{[1,2]}, L) = 0. \tag{4.1}$$

The polynomial equations of the form (4.1) derived from (1.1) as well as those fulfilled identically due to (3.2), (3.3), and (3.4) are called *factor equations* of the cuboid equations (1.1) with respect to their  $S_3$  symmetry. Our present goal is to reveal some of these factor equations explicitly.

The equations (1.1) are quadratic with respect to their variables. For this reason it is quite likely that there are no **linear relationships** between multisymmetric polynomials (3.2), (3.3), and (3.4). As for **higher order relationships**, they do actually exist. In order to reveal them we need to consider squares, cubes, fourth powers etc, and various mutual products of the multisymmetric polynomials (3.2), (3.3), and (3.4). For the square  $(e_{[1,0]})^2$  we have

$$(e_{[1,0]})^2 = x_1^2 + x_2^2 + x_3^2 + 2(x_1 x_2 + x_3 x_1 + x_3 x_2). \tag{4.2}$$

On the other hand, from the cuboid equations (1.1) we derive

$$x_1^2 + x_2^2 + x_3^2 = L^2. \tag{4.3}$$

Applying (4.3) to (4.2) and comparing the result with (3.2), we derive

$$(e_{[1,0]})^2 - 2e_{[2,0]} - L^2 = 0. \tag{4.4}$$

The equation (4.4) is the first and the most simple factor equation produced from the cuboid equations (1.1).

The polynomial  $e_{[0,1]}$  in (3.3) is very similar to  $e_{[1,0]}$ . For its square we have

$$(e_{[1,0]})^2 = d_1^2 + d_2^2 + d_3^2 + 2(d_1 d_2 + d_3 d_1 + d_3 d_2). \quad (4.5)$$

On the other hand, from the cuboid equations (1.1) we derive

$$d_1^2 + d_2^2 + d_3^2 = 2L^2. \quad (4.6)$$

Applying (4.6) to (4.5) and comparing the result with (3.3), we derive

$$(e_{[0,1]})^2 - 2e_{[0,2]} - 2L^2 = 0. \quad (4.7)$$

The equation (4.7) is the second factor equation produced from the cuboid equations (1.1). It is equally simple as the equation (4.4).

In order to derive the third factor equation from the cuboid equations (1.1) we consider the cube  $(e_{[1,0]})^3$  and apply the formula (3.2):

$$\begin{aligned} (e_{[1,0]})^3 &= x_1^3 + x_2^3 + x_3^3 + 3x_1(x_2^2 + x_3^2) + \\ &+ 3x_2(x_3^2 + x_1^2) + 3x_3(x_1^2 + x_2^2) + 6x_1x_2x_3. \end{aligned} \quad (4.8)$$

Using the cuboid equations (1.1), we derive the following formulas:

$$\begin{aligned} x_1^2 &= L^2 - d_1^2, & x_1^3 &= L^2 x_1 - d_1^2 x_1, \\ x_2^2 &= L^2 - d_2^2, & x_2^3 &= L^2 x_2 - d_2^2 x_2, \\ x_3^2 &= L^2 - d_3^2, & x_3^3 &= L^2 x_3 - d_3^2 x_3. \end{aligned} \quad (4.9)$$

Substituting (4.9) into the equality (4.8), we obtain the formula

$$\begin{aligned} (e_{[1,0]})^3 &= -(x_1 d_1^2 + x_2 d_2^2 + x_3 d_3^2) + 7L^2(x_1 + x_2 + x_3) - \\ &- 3x_1(d_2^2 + d_3^2) - 3x_2(d_3^2 + d_1^2) - 3x_3(d_1^2 + d_2^2) + 6x_1x_2x_3. \end{aligned} \quad (4.10)$$

The right hand side of the formula (4.10) is a multisymmetric polynomial. For this reason we can apply the theorem 2.1 to it. As a result we get

$$\begin{aligned} (e_{[1,0]})^3 &= 2e_{[1,2]} + 6e_{[3,0]} + 4e_{[0,2]}e_{[1,0]} - \\ &- 2e_{[0,1]}e_{[1,1]} - e_{[1,0]}e_{[0,1]}^2 + 7e_{[1,0]}L^2. \end{aligned} \quad (4.11)$$

Note that the equation (4.7) can be resolved with respect to  $e_{[0,2]}$ :

$$e_{[0,2]} = \frac{e_{[0,1]}^2}{2} - L^2. \quad (4.12)$$

Applying (4.12) to (4.11), we can write (4.11) as follows:

$$2e_{[1,2]} + 6e_{[3,0]} - 2e_{[0,1]}e_{[1,1]} + e_{[1,0]}e_{[0,1]}^2 + 3e_{[1,0]}L^2 - e_{[1,0]}^3 = 0. \quad (4.13)$$

The equation (4.13) is the third factor equation derived from the cuboid equations (1.1). It is more complicated than (4.4) and (4.7).

There is another way for transforming the cube  $(e_{[1,0]})^3$  given by the formula (4.8). Indeed, we can resolve the left column of the equations (1.1) with respect to  $(x_1)^2$ ,  $(x_2)^2$ , and  $(x_3)^2$ . As a result we get

$$x_1^2 = \frac{d_2^2 + d_3^2 - d_1^2}{2}, \quad x_2^2 = \frac{d_3^2 + d_1^2 - d_2^2}{2}, \quad x_3^2 = \frac{d_1^2 + d_2^2 - d_3^2}{2}. \quad (4.14)$$

The formulas (4.14) can be used instead of the formulas in the left column of (4.9). Applying these formulas to (4.8), we can get an expression analogous to (4.10) and then we can continue transforming it in a way similar to (4.11) and (4.12), expecting to get some new equation similar to (4.13). But actually we get the equation coinciding with (4.13).

Now let's consider the cube  $(e_{[0,1]})^3$ . It is given by the following formula:

$$\begin{aligned} (e_{[0,1]})^3 &= d_1^3 + d_2^3 + d_3^3 + 3d_1(d_2^2 + d_3^2) + \\ &+ 3d_2(d_3^2 + d_1^2) + 3d_3(d_1^2 + d_2^2) + 6d_1d_2d_3. \end{aligned} \quad (4.15)$$

Note that the equations of the left column of (1.1) can be resolved with respect to  $(d_1)^2$ ,  $(d_2)^2$ , and  $(d_3)^2$ . They yield the equalities

$$\begin{aligned} d_1^2 &= x_2^2 + x_3^2, & d_1^3 &= x_2^2 d_1 + x_3^2 d_1, \\ d_2^2 &= x_3^2 + x_1^2, & d_2^3 &= x_3^2 d_2 + x_1^2 d_2, \\ d_3^2 &= x_1^2 + x_2^2, & d_3^3 &= x_1^2 d_3 + x_2^2 d_3. \end{aligned} \quad (4.16)$$

Substituting (4.16) into the equality (4.15), we obtain the formula

$$\begin{aligned} (e_{[0,1]})^3 &= 6d_1x_1^2 + 6d_2x_2^2 + 6d_3x_3^2 + 6d_1d_2d_3 + \\ &+ 4d_1(x_2^2 + x_3^2) + 4d_2(x_3^2 + x_1^2) + 4d_3(x_1^2 + x_2^2). \end{aligned} \quad (4.17)$$

The formula (4.17) is analogous to the formula (4.10). Its right hand side is a multisymmetric polynomial. For this reason we can apply the theorem 2.1 and get

$$(e_{[0,1]})^3 = 2e_{[2,1]} + 6e_{[0,3]} - 2e_{[1,0]}e_{[1,1]} - 10e_{[2,0]}e_{[0,1]} + 6e_{[0,1]}e_{[1,0]}^2. \quad (4.18)$$

Note that the equation (4.4) can be resolved with respect to  $e_{[2,0]}$ :

$$e_{[2,0]} = \frac{1}{2}e_{[1,0]}^2 - \frac{1}{2}L^2. \quad (4.19)$$

Applying (4.19) to (4.18), we can write the equality (4.18) as follows:

$$2e_{[2,1]} + 6e_{[0,3]} - 2e_{[1,0]}e_{[1,1]} + e_{[0,1]}e_{[1,0]}^2 + 5e_{[0,1]}L^2 - e_{[0,1]}^3 = 0. \quad (4.20)$$

The equation (4.20) is the fourth factor equation derived from the cuboid equations (1.1). It is similar to the equation (4.13).

The equations of the second column in (1.1) can also be resolved with respect to  $(d_1)^2$ ,  $(d_2)^2$ , and  $(d_3)^2$ . Using them, we can write the formulas

$$\begin{aligned} d_1^2 &= L^2 - x_1^2, & d_1^3 &= d_1 L^2 - d_1 x_1^2, \\ d_2^2 &= L^2 - x_2^2, & d_2^3 &= d_2 L^2 - d_2 x_2^2, \\ d_3^2 &= L^2 - x_3^2, & d_3^3 &= d_3 L^2 - d_3 x_3^2. \end{aligned} \quad (4.21)$$

The formulas (4.21) can be used instead of the equations (4.16). As a result we get another sequence of equations. However, the ultimate result appears to be coinciding with the equation (4.20).

## 5. MORE FACTOR EQUATIONS.

In the next step we consider the square  $(e_{[2,0]})^2$ . Using the formulas (3.2), we get the following explicit expression for this square:

$$(e_{[2,0]})^2 = x_1^2 x_2^2 + x_2^2 x_3^2 + x_3^2 x_1^2 + 2 x_1^2 x_2 x_3 + 2 x_2^2 x_3 x_1 + 2 x_3^2 x_1 x_2. \quad (5.1)$$

In order to transform (5.1) we use the formulas (4.9). This yields

$$\begin{aligned} (e_{[2,0]})^2 &= d_1^2 d_2^2 + d_2^2 d_3^2 + d_3^2 d_1^2 - 2 L^2 (d_1^2 + d_2^2 + d_3^2) - 2 (x_1 x_2 d_3^2 + \\ &\quad + x_2 x_3 d_1^2 + x_3 x_1 d_2^2) + 2 L^2 (x_1 x_2 + x_2 x_3 + x_3 x_1) + 3 L^4. \end{aligned} \quad (5.2)$$

The right hand side of the formula (5.2) is a multisymmetric polynomial. For this reason we can apply the theorem 2.1 to it. As a result we get

$$\begin{aligned} (e_{[2,0]})^2 &= -2 e_{[0,1]} e_{[0,3]} + \frac{2}{3} e_{[1,0]} e_{[1,2]} - \frac{4}{3} e_{[0,1]} e_{[2,1]} - \frac{2}{3} e_{[1,1]}^2 + \\ &\quad + \frac{2}{3} e_{[0,1]} e_{[1,1]} e_{[1,0]} + \frac{8}{3} e_{[2,0]} e_{[0,2]} - \frac{2}{3} e_{[0,1]}^2 e_{[2,0]} + 2 e_{[2,0]} L^2 - \\ &\quad - \frac{2}{3} e_{[1,0]}^2 e_{[0,2]} + e_{[0,2]}^2 + 4 e_{[0,2]} L^2 - 2 e_{[0,1]}^2 L^2 + 3 L^4. \end{aligned} \quad (5.3)$$

Note that we can use the equation (4.20) in order to express  $e_{[0,3]}$  through the other elementary multisymmetric polynomials in (4.20):

$$e_{[0,3]} = -\frac{1}{3} e_{[2,1]} + \frac{1}{3} e_{[1,0]} e_{[1,1]} + \frac{1}{6} e_{[0,1]}^3 - \frac{1}{6} e_{[0,1]} e_{[1,0]}^2 - \frac{5}{6} e_{[0,1]} L^2. \quad (5.4)$$

Apart from (5.4), we apply the formulas (4.12) and (4.19) to (5.3). Then we get

$$\begin{aligned} &8 e_{[1,0]} e_{[1,2]} - 8 e_{[0,1]} e_{[2,1]} - 8 e_{[1,1]}^2 + 4 e_{[0,1]}^2 e_{[1,0]}^2 - \\ &\quad - e_{[0,1]}^4 - 3 e_{[1,0]}^4 + 10 e_{[1,0]}^2 L^2 + 4 e_{[0,1]}^2 L^2 + L^4 = 0. \end{aligned} \quad (5.5)$$

The equation (5.5) is the fifth factor equation derived from the cuboid equations (1.1). It is more complicated than all of the previous factor equations.

Now let's consider the other square  $(e_{[0,2]})^2$ . Using the formulas (3.3), we get the following explicit expression for this square:

$$(e_{[0,2]})^2 = d_1^2 d_2^2 + d_2^2 d_3^2 + d_3^2 d_1^2 + 2 d_1^2 d_2 d_3 + 2 d_2^2 d_3 d_1 + 2 d_3^2 d_1 d_2. \quad (5.6)$$

In order to transform (5.6) we use the formulas (4.21). This yields

$$(e_{[0,2]})^2 = x_1^2 x_2^2 + x_2^2 x_3^2 + x_3^2 x_1^2 - 2L^2(x_1^2 + x_2^2 + x_3^2) - 2(d_1 d_2 x_3^2 + d_2 d_3 x_1^2 + d_3 d_1 x_2^2) + 2L^2(d_1 d_2 + d_2 d_3 + d_3 d_1) + 3L^4. \quad (5.7)$$

The right hand side of the formula (5.7) is a multisymmetric polynomial. For this reason we can apply the theorem 2.1 to it. As a result we get

$$(e_{[0,2]})^2 = -2e_{[1,0]}e_{[3,0]} - \frac{4}{3}e_{[1,0]}e_{[1,2]} + \frac{2}{3}e_{[0,1]}e_{[2,1]} - \frac{2}{3}e_{[1,1]}^2 + \frac{2}{3}e_{[1,0]}e_{[0,1]}e_{[1,1]} + \frac{8}{3}e_{[2,0]}e_{[0,2]} - \frac{2}{3}e_{[0,1]}^2e_{[2,0]} + 4e_{[2,0]}L^2 + -\frac{2}{3}e_{[1,0]}^2e_{[0,2]} + e_{[2,0]}^2 + 2e_{[0,2]}L^2 - 2e_{[1,0]}^2L^2 + 3L^4. \quad (5.8)$$

Note that we can use the equation (4.13) in order to express  $e_{[3,0]}$  through the other elementary multisymmetric polynomials in (4.13):

$$e_{[3,0]} = -\frac{1}{3}e_{[1,2]} + \frac{1}{3}e_{[0,1]}e_{[1,1]} + \frac{1}{6}e_{[1,0]}^3 - \frac{1}{6}e_{[1,0]}e_{[0,1]}^2 - \frac{1}{2}e_{[1,0]}L^2. \quad (5.9)$$

We apply the formulas (5.9), (4.12), and (4.19) to (5.8). As a result we get

$$-8e_{[1,0]}e_{[1,2]} + 8e_{[0,1]}e_{[2,1]} - 8e_{[1,1]}^2 + 4e_{[0,1]}^2e_{[1,0]}^2 - e_{[1,0]}^4 - 3e_{[0,1]}^4 + 20e_{[0,1]}^2L^2 - 2e_{[1,0]}^2L^2 - 5L^4 = 0. \quad (5.10)$$

The equation (5.10) is the sixth factor equation derived from the cuboid equations (1.1). It is similar to the equation (5.5).

The terms  $e_{[1,2]}$ ,  $e_{[2,1]}$ , and  $e_{[1,1]}$  are mixed in (5.5) and (5.10). We can separate  $e_{[1,2]}$  and  $e_{[2,1]}$  from  $e_{[1,1]}$  in the following two equations:

$$8e_{[1,0]}e_{[1,2]} - 8e_{[0,1]}e_{[2,1]} + e_{[0,1]}^4 - e_{[1,0]}^4 - 8e_{[0,1]}^2L^2 + 6e_{[1,0]}^2L^2 + 3L^4 = 0. \quad (5.11)$$

$$4e_{[1,1]}^2 - 2e_{[0,1]}^2e_{[1,0]}^2 + e_{[0,1]}^4 + e_{[1,0]}^4 - 6e_{[0,1]}^2L^2 - 2e_{[1,0]}^2L^2 + L^4 = 0. \quad (5.12)$$

The above equations (5.11) and (5.12) are just linear combinations of the fifth and the sixth factor equations (5.5) and (5.10).

In order to derive the next factor equation we consider the product  $e_{[2,0]}e_{[3,0]}$ . Using the formulas (3.2), we get the following explicit expression for this product:

$$e_{[2,0]}e_{[3,0]} = x_1x_2^2x_3^2 + x_2x_3^2x_1^2 + x_3x_1^2x_2^2. \quad (5.13)$$

In order to transform (5.13) we use the formulas (4.9). This yields

$$e_{[2,0]}e_{[3,0]} = x_1d_2^2d_3^2 + x_2d_3^2d_1^2 + x_3d_1^2d_2^2 - L^2x_1(d_2^2 + d_3^2) - L^2x_2(d_3^2 + d_1^2) - L^2x_3(d_1^2 + d_2^2) + L^4(x_1 + x_2 + x_3). \quad (5.14)$$



The right hand side of the formula (5.14) is a multisymmetric polynomial. For this reason it can be expressed through elementary multisymmetric polynomials:

$$\begin{aligned} e_{[2,0]} e_{[3,0]} &= -e_{[1,1]} e_{[0,3]} + e_{[1,2]} e_{[0,2]} + e_{[1,2]} L^2 + \\ &+ e_{[0,2]} e_{[1,0]} L^2 - e_{[1,1]} e_{[0,1]} L^2 + e_{[1,0]} L^4. \end{aligned} \quad (5.15)$$

Now we transform (5.15) with the use of the formulas (5.9), (5.4), (4.19), and (4.12):

$$\begin{aligned} -4 e_{[1,0]} e_{[1,1]}^2 + 4 e_{[1,1]} e_{[2,1]} - 2 e_{[1,1]} e_{[0,1]}^3 + 6 e_{[1,2]} e_{[0,1]}^2 + \\ + 2 e_{[1,2]} e_{[1,0]}^2 + e_{[1,0]}^3 e_{[0,1]}^2 - e_{[1,0]}^5 - 2 e_{[1,2]} L^2 + \\ + 5 e_{[1,0]} e_{[0,1]}^2 L^2 + 4 e_{[1,0]}^3 L^2 - 3 e_{[1,0]} L^4 = 0. \end{aligned} \quad (5.16)$$

Then we apply (5.12) to (5.16) in order to eliminate the term with the square  $e_{[1,1]}^2$ :

$$\begin{aligned} 4 e_{[1,1]} e_{[2,1]} - 2 e_{[1,1]} e_{[0,1]}^3 + 6 e_{[1,2]} e_{[0,1]}^2 + \\ + 2 e_{[1,2]} e_{[1,0]}^2 - e_{[1,0]}^3 e_{[0,1]}^2 + e_{[1,0]} e_{[0,1]}^4 - 2 e_{[1,2]} L^2 - \\ - e_{[1,0]} e_{[0,1]}^2 L^2 + 2 e_{[1,0]}^3 L^2 - 2 e_{[1,0]} L^4 = 0. \end{aligned} \quad (5.17)$$

The equation (5.17) is the seventh factor equation derived from the cuboid equations (1.1). Its order is higher than the order of the equations (5.5) and (5.10).

The eighth factor equation is derived similarly. In order to derive it we consider the product  $e_{[0,2]} e_{[0,3]}$ . Applying the formulas (3.3) to this product, we get

$$e_{[0,2]} e_{[0,3]} = d_1 d_2^2 d_3^2 + d_2 d_3^2 d_1^2 + d_3 d_1^2 d_2^2. \quad (5.18)$$

In order to transform (5.18) we use the formulas (4.21). This yields

$$\begin{aligned} e_{[0,2]} e_{[0,3]} &= d_1 x_2^2 x_3^2 + d_2 x_3^2 x_1^2 + d_3 x_1^2 x_2^2 - L^2 d_1 (x_2^2 + x_3^2) - \\ &- L^2 d_2 (x_3^2 + x_1^2) - L^2 d_3 (x_1^2 + x_2^2) + L^4 (d_1 + d_2 + d_3). \end{aligned} \quad (5.19)$$

The right hand side of the formula (5.19) is a multisymmetric polynomial. For this reason it can be expressed through elementary multisymmetric polynomials:

$$\begin{aligned} e_{[0,2]} e_{[0,3]} &= -e_{[1,1]} e_{[3,0]} + e_{[2,1]} e_{[2,0]} + e_{[2,1]} L^2 + \\ &+ e_{[2,0]} e_{[0,1]} L^2 - e_{[1,1]} e_{[1,0]} L^2 + e_{[0,1]} L^4. \end{aligned} \quad (5.20)$$

Transforming (5.20) with the use of the formulas (5.9), (5.4), (4.19), (4.12), we get

$$\begin{aligned} -4 e_{[0,1]} e_{[1,1]}^2 + 4 e_{[1,1]} e_{[1,2]} - 2 e_{[1,1]} e_{[1,0]}^3 + 6 e_{[2,1]} e_{[1,0]}^2 + \\ + 2 e_{[2,1]} e_{[0,1]}^2 + e_{[0,1]}^3 e_{[1,0]}^2 - e_{[0,1]}^5 + 2 e_{[2,1]} L^2 - \\ - 2 e_{[1,1]} e_{[1,0]} L^2 + 4 e_{[0,1]} e_{[1,0]}^2 L^2 + 7 e_{[0,1]}^3 L^2 - 4 e_{[0,1]} L^4 = 0. \end{aligned} \quad (5.21)$$

Then we apply (5.12) to (5.21) in order to eliminate the term with the square  $e_{[1,1]}^2$ :

$$\begin{aligned} 4 e_{[1,1]} e_{[1,2]} - 2 e_{[1,1]} e_{[1,0]}^3 + 6 e_{[2,1]} e_{[1,0]}^2 + \\ + 2 e_{[2,1]} e_{[0,1]}^2 - e_{[0,1]}^3 e_{[1,0]}^2 + e_{[0,1]} e_{[1,0]}^4 + 2 e_{[2,1]} L^2 - \\ - 2 e_{[1,1]} e_{[1,0]} L^2 + 2 e_{[0,1]} e_{[1,0]}^2 L^2 + e_{[0,1]}^3 L^2 - 3 e_{[0,1]} L^4 = 0. \end{aligned} \quad (5.22)$$

The equation (5.22) is the eighth factor equation derived from the cuboid equations (1.1). It is similar to the equation (5.17).

## 6. CONCLUDING REMARKS.

One can continue deriving factor equation more and more. In order to reasonably terminate this process we need some theoretical considerations. Note that the left hand sides of the factor equations are polynomials from the ring

$$\mathbb{Q}[e_{[1,0]}, e_{[2,0]}, e_{[3,0]}, e_{[0,1]}, e_{[0,2]}, e_{[0,3]}, e_{[2,1]}, e_{[1,1]}, e_{[1,2]}, L], \quad (6.1)$$

where  $e_{[1,0]}$ ,  $e_{[2,0]}$ ,  $e_{[3,0]}$ ,  $e_{[0,1]}$ ,  $e_{[0,2]}$ ,  $e_{[0,3]}$ ,  $e_{[2,1]}$ ,  $e_{[1,1]}$ ,  $e_{[1,2]}$ , and  $L$  are treated as independent variables. If we continue deriving factor equation endlessly, their left hand sides would generate a certain ideal  $J$  in the ring (6.1). By means of the formulas (3.2), (3.3), and (3.4) this ideal  $J$  is mapped onto some certain ideal  $I_{\text{sym}}$  of the ring of multisymmetric polynomials  $\text{Sym}\mathbb{Q}[x_1, x_2, x_2, d_1, d_2, d_3, L]$ . The ideal  $I_{\text{sym}}$  is produced as the intersection

$$I_{\text{sym}} = I \cap \text{Sym}\mathbb{Q}[x_1, x_2, x_2, d_1, d_2, d_3, L], \quad (6.2)$$

where  $I$  is the ideal of the polynomial ring  $\mathbb{Q}[x_1, x_2, x_2, d_1, d_2, d_3, L]$  generated by the left hand sides of the cuboid equations (1.1). Calculating the intersection (6.2) is an algorithmically solvable computational problem. It will be considered in a separate paper.

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