On the number of permutations with bounded run lengths

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Abstract

In this work we obtain recurrent formulae for the number of permutations with either increasing or monotonic (i.e., both increasing and decreasing) runs of bounded length. Our formulae allow one to efficiently compute the number of such permutations. In particular, we use the formulae to find and correct a few miscalculations in the classic 1966 book by David, Kendall, and Barton.

We further use our formulae to derive differential equations for the corresponding exponential generating functions. In the case of increasing runs, we solve these equations and obtain closed-form expressions for the generating functions.

1 Introduction

A (monotonic) *run* in a permutation $p = (p_1, p_2, \ldots, p_n)$ is a maximal increasing or decreasing subsequence of consecutive elements in p. Similarly, an *increasing* (resp. *decreasing*) *run* in p is a maximal increasing (resp. decreasing) subsequence of consecutive elements in p. David et al. [2] in Tables 7.4.1 and 7.4.2 give counts[1] for

- the number $I^k(n)$ of order n permutations whose longest increasing run length equals k (for $k, n \leq 18$);
- the number $A^k(n)$ of order n permutations whose longest monotonic run length equals k (for $k, n \leq 14$).

It turns out that their counts for $I^k(n)$ and $A^k(n)$ are incorrect for $n \geq 16$ and $n \geq 13$ respectively.

We notice that $I^k(n) = U^k(n) - U^{k-1}(n)$ and $A^k(n) = B^k(n) - B^{k-1}(n)$ where $U^k(n)$ and $B^k(n)$ is the number of permutations of order n whose runs length does not exceed k.

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[1] Colin Mallows and Neil Sloane suggest that these counts were almost certainly the result of hand calculations.

In this note we derive recurrent formulae for $U^k(n)$ and $B^k(n)$ as well as differential equations for their exponential generating functions. These formulae allowed us to compute $B^k(n)$ and $A^k(n)$ accurately and correct miscalculations in [2].

We remark that another way to obtain differential equations for these generating functions was described by Elizalde and Noy [3] who studied a more general problem of counting permutations with forbidden subpermutations using symbolic methods. In contrast, we use only elementary observations to obtain recurrent formulae for $U^k(n)$ and $B^k(n)$ and describe algorithms for computing them. We also derive an explicit closed-form expressions for the generating functions for $U^k(n)$.

2 Recurrent formulae for $U^k(n)$ and $B^k(n)$

Let $p = (p_1, p_2, \ldots, p_n)$ be a permutation of order $n > 1$ with runs length not exceeding k. It is easy to see that p_t may be a maximum element of p (i.e., $p_t = n$) only in one of the following three cases:

1. $t = 1$ and $p_1 > p_2$;

2. $t = n$ and $p_{n-1} < p_n$;

3. $1 < t < n$, $p_{t-1} < p_t$, and $p_t > p_{t+1}$.

In case 1, we remove the first element from p to obtain a permutation $p' = (p_2, p_3, \ldots, p_n)$ of order $n-1$. The permutation p' here can be any permutation whose (monotonic or increasing) runs length does not exceed k with an additional restriction (in the case of bounded monotonic runs) that if the initial run in p' is decreasing, then its length does not exceed $k-1$.

Similarly, in case 2, we remove the last element from p to obtain a permutation $p' = (p_1, p_2, \ldots, p_{n-1})$ of order $n-1$. The permutation p' here can be any permutation whose (monotonic or increasing) runs length does not exceed k with an additional restriction that if the final run in p' is increasing then its length does not exceed $k-1$.

In case 3, removing of the element p_t splits p into two vectors $(p_1, p_2, \ldots, p_{t-1})$ and $(p_{t+1}, p_{t+2}, \ldots, p_n)$. We relabel their elements with integers $1, 2, \ldots, t-1$ and $1, 2, \ldots, n-t$ (preserving the order relationship) to obtain permutations p' and p'' of order $t-1$ and $n-t$ respectively. By construction, the permutation p' has the same length and order of runs as the prefix of length $t-1$ of p, while the permutations p'' has the same length and order of runs as the suffix of length $n-t$ of p. Therefore, p' and p'' can be any permutations whose runs length does not exceed k with additional restrictions that

- if the final run in p' in increasing, then its length is at most $k-1$;

- (in case of bounded monotonic runs) if the initial run in p'' is decreasing, then its length is at most $k-1$.

Let U_j^k be the number of permutations p of order n whose increasing runs length does not exceed k, and the final increasing run (if it is present) in p has length at most j. Trivially $U_j^k(1) = 1$ for any $1 \le j \le k$. We find it convenient to define $U_j^k(n) = 0$ whenever $j < 1$.

Similarly, let $B^k_{i,j}(n)$ be the number of permutations p of order n whose runs length does not exceed k, and the initial decreasing run (if it is present) in p has length at most i and the final increasing run (if it is present) in p has length at most j. Trivially $B^k_{i,j}(1) = 1$ for any $1 \le i, j \le k$. We find it convenient to define $B^k_{i,j}(n) = 0$ whenever $i < 1$ or $j < 1$.

The above observations lead to the following formulae:

$$U^k_j(n) = U^k_j(n-1) + U^k_{j-1}(n-1) + \sum_{t=2}^{n-1} \binom{n-1}{t-1} \cdot U^k_{k-1}(t-1) \cdot U^k_j(n-t)$$

$$B^k_{i,j}(n) = B^k_{i-1,j}(n-1) + B^k_{i,j-1}(n-1) + \sum_{t=2}^{n-1} \binom{n-1}{t-1} \cdot B^k_{i,k-1}(t-1) \cdot B^k_{k-1,j}(n-t)$$

which hold for $n > 1$ and any $1 \le i, j \le k$. Here the binomial coefficient $\binom{n-1}{t-1}$ stands for the number of ways to distribute elements $1, 2, \ldots, n-1$ of p between the prefix and suffix corresponding to the permutations p' and p'' in case 3.

We also remark that the involution $(p_1, p_2, \ldots, p_n) \mapsto (p_n, p_{n-1}, \ldots, p_1)$ on the set of all permutations of order n implies that $B^k_{i,j}(n) = B^k_{j,i}(n)$.

3 Computing $U^k(n)$ and $B^k(n)$

Suppose that k is fixed. It is easy to see that $U^k(n) = U^k_k(n)$ and $B^k(n) = B^k_{k,k}(n)$ which allows one to compute them efficiently using the recurrent formulae for $U^k_j(n)$ and $B^k_{i,j}(n)$. In particular, $U^k(n) = U^k_k(n)$ for all $n \le N$ can be computed as follows:

```
U ← an array of size k
for j ← 1 to k do
    U[j] ← an array of integers of size N
    U[j][1] ← 1
end for
for n ← 2 to N do
    for j ← 1 to k do
        U[j][n] ← U[j][n − 1]
        if j > 1 then
            U[j][n] ← U[j][n] + U[j − 1][n − 1]
        end if
        if n > 2 and k > 1 then
            U[j][n] ← U[j][n] + ∑_{t=2}^{n−1} C(n−1, t−1) · U[k − 1][t − 1] · U[j][n − t]
        end if
    end for
end for
return U[k]
```

Similarly, $B^k(n) = B^k_{k,k}(n)$ for all $n \le N$ can be computed as follows:

```
B ← an array of size k × k
```

```
for i ← 1 to k do
    for j ← 1 to k do
        B[i, j] ← an array of integers of size N
        B[i, j][1] ← 1
    end for
end for
for n ← 2 to N do
    for i ← 1 to k do
        for j ← 1 to k do
            B[i, j][n] ← 0
            if i > 1 then
                B[i, j][n] ← B[i, j][n] + B[i − 1, j][n − 1]
            end if
            if j > 1 then
                B[i, j][n] ← B[i, j][n] + B[i, j − 1][n − 1]
            end if
            if n > 2 and k > 1 then
                B[i, j][n] ← B[i, j][n] + ∑_{t=2}^{n−1} binom(n−1, t−1) · B[i, k − 1][t − 1] · B[k − 1, j][n − t]
            end if
        end for
    end for
end for
return B[k, k].
```

We used these algorithms to compute values $U^k(n)$ and $B^k(n)$ for $k, n \leq 18$ and listed them in Tables 1 and 3 respectively. Subtracting from each row the previous one, we obtain Tables 2 and 4 listing values of $I^k(n)$ and $A^k(n)$. We remark that Table 2 is present (column-wise) in the OEIS [4] as sequence A008304 with its rows (for $2 \leq k \leq 6$) given by sequences A008303, A000402, A000434, A000456, and A000467. Table 4 (column-wise) is present in the OEIS as sequence A211318 with its rows (for $2 \leq k \leq 5$) given by sequences A001250, A001251, A001252, and A001253.

4 Exponential generating function for $U^k(n)$

For fixed integers k, j, let $\mathcal{U}^k_j(x)$ be the exponential generating function for $U^k_j(n)$:

$$\mathcal{U}^k_j(x) = \sum_{n=0}^{\infty} U^k_j(n) \cdot \frac{x^n}{n!}.$$

The recurrent formula for $U^k_j(n)$ implies the following system of differential equations:

$$\left\{ \frac{d}{dx}\mathcal{U}^k_j(x) = 1 + \mathcal{U}^k_j(x) + \mathcal{U}^k_{j-1}(x) + \mathcal{U}^k_{k-1}(x) \cdot \mathcal{U}^k_j(x), \qquad j = 1, 2, \ldots, k. \right.$$

In particular, for $j = k$ we have

$$\frac{d}{dx}\mathcal{U}_k^k(x) = (1 + \mathcal{U}_k^k(x)) \cdot (1 + \mathcal{U}_{k-1}^k(x))$$

or

$$1 + \mathcal{U}_{k-1}^k(x) = \frac{1}{1 + \mathcal{U}_k^k(x)} \frac{d}{dx}\mathcal{U}_k^k(x).$$

Plugging this into the j-th equation, we conclude

$$\frac{d}{dx}\left(\frac{\mathcal{U}_j^k(x)}{1 + \mathcal{U}_k^k(x)}\right) = \frac{1}{1 + \mathcal{U}_k^k(x)} + \frac{\mathcal{U}_{j-1}^k(x)}{1 + \mathcal{U}_k^k(x)}$$

implying that for $y(x) = \frac{1}{1+\mathcal{U}_k^k(x)}$ and any $j = 0, 2, \ldots, k-1$, we have

$$(1 - y)^{(k-j)} = y^{(k-1-j)} + y^{(k-2-j)} + \cdots + y' + y + y \cdot \mathcal{U}_j^k(x).$$

In particular, for $j = 0$ we get the following linear differential equation:

$$y^{(k)} + y^{(k-1)} + y^{(k-2)} + \cdots + y' + y = 0.$$

General solution to this equation is

$$y(x) = \sum_{i=1}^{k} c_i \cdot e^{r^i \cdot x}$$

where r is a primitive $(k + 1)$-st degree root of 1 and c_i are constant coefficients such that $\sum_{i=1}^{k} c_i = y(0) = 1$.

For $j = 1, 2, \ldots, k - 1$, we expressing $\mathcal{U}_j^k(x)$ in terms of y as follows:

$$\mathcal{U}_j^k(x) = -\frac{1}{y} \cdot \sum_{m=0}^{k-j} y^{(m)}.$$

Since $\mathcal{U}_j^k(0) = 0$, this expression implies

$$\sum_{m=0}^{k-j} \sum_{i=1}^{k} c_i \cdot r^{im} = 0 \qquad \text{or} \qquad \sum_{i=1}^{k} c_i \cdot \frac{r^{i(k-j+1)} - 1}{r^i - 1} = 0.$$

It is easy to see that solution to this system of linear equations is $c_i = \frac{r^i - 1}{(k+1)r^i}$, implying that

$$y(x) = \sum_{n=0}^{\infty} q_n \frac{x^n}{n!} \qquad \text{and} \qquad \mathcal{U}_k^k(x) = \left(\sum_{n=0}^{\infty} q_n \frac{x^n}{n!}\right)^{-1} - 1$$

where

$$q_n = \begin{cases} 1 & \text{if } n \equiv 0 \pmod{k+1}, \\ -1 & \text{if } n \equiv 1 \pmod{k+1}, \\ 0 & \text{if } n \not\equiv 0, 1 \pmod{k+1}. \end{cases}$$

4.1 Examples

Exponential generating functions of $U^k(n)$ for $k = 1,2,3,4$ are:

$$\mathcal{U}_1^1(x) = \left(1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\right)^{-1} - 1 = 1 \cdot \frac{x}{1} + 1 \cdot \frac{x^2}{2!} + 1 \cdot \frac{x^3}{3!} + 1 \cdot \frac{x^4}{4!} + 1 \cdot \frac{x^5}{5!} + \dots$$

$$\mathcal{U}_2^2(x) = \left(1 - \frac{x}{1!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \dots\right)^{-1} - 1 = 1 \cdot \frac{x}{1} + 2 \cdot \frac{x^2}{2!} + 5 \cdot \frac{x^3}{3!} + 17 \cdot \frac{x^4}{4!} + 70 \cdot \frac{x^5}{5!} + \dots$$

$$\mathcal{U}_3^3(x) = \left(1 - \frac{x}{1!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \dots\right)^{-1} - 1 = 1 \cdot \frac{x}{1} + 2 \cdot \frac{x^2}{2!} + 6 \cdot \frac{x^3}{3!} + 23 \cdot \frac{x^4}{4!} + 111 \cdot \frac{x^5}{5!} + \dots$$

$$\mathcal{U}_4^4(x) = \left(1 - \frac{x}{1!} + \frac{x^5}{5!} - \frac{x^6}{6!} + \dots\right)^{-1} - 1 = 1 \cdot \frac{x}{1} + 2 \cdot \frac{x^2}{2!} + 6 \cdot \frac{x^3}{3!} + 24 \cdot \frac{x^4}{4!} + 119 \cdot \frac{x^5}{5!} + \dots$$

The difference $\mathcal{U}_k^k(x) - \mathcal{U}_{k-1}^{k-1}(x)$ gives the exponential generating function of the numbers $I^k(n)$:

$$\mathcal{U}_2^2(x) - \mathcal{U}_1^1(x) = 0 \cdot \frac{x}{1} + 1 \cdot \frac{x^2}{2!} + 4 \cdot \frac{x^3}{3!} + 16 \cdot \frac{x^4}{4!} + 69 \cdot \frac{x^5}{5!} + \dots$$

$$\mathcal{U}_3^3(x) - \mathcal{U}_2^2(x) = 0 \cdot \frac{x}{1} + 0 \cdot \frac{x^2}{2!} + 1 \cdot \frac{x^3}{3!} + 6 \cdot \frac{x^4}{4!} + 41 \cdot \frac{x^5}{5!} + \dots$$

$$\mathcal{U}_4^4(x) - \mathcal{U}_3^3(x) = 0 \cdot \frac{x}{1} + 0 \cdot \frac{x^2}{2!} + 0 \cdot \frac{x^3}{3!} + 1 \cdot \frac{x^4}{4!} + 8 \cdot \frac{x^5}{5!} + \dots$$

5 Exponential generating function for $B^k(n)$

For fixed integers k, i, j, let $\mathcal{B}_{i,j}^k(x)$ be the exponential generating function for $B_{i,j}^k(n)$:

$$\mathcal{B}_{i,j}^k(x) = \sum_{n=0}^{\infty} B_{i,j}^k(n) \cdot \frac{x^n}{n!}.$$

The recurrent formula for $B_{i,j}^k(n)$ implies the following system of differential equations:

$$\left\{ \frac{d}{dx}\mathcal{B}_{i,j}^k(x) = 1 + \mathcal{B}_{i-1,j}^k(x) + \mathcal{B}_{i,j-1}^k(x) + \mathcal{B}_{i,k-1}^k(x) \cdot \mathcal{B}_{k-1,j}^k(x), \qquad i, j = 1, 2, \dots, k. \right.$$

Because of the symmetry $\mathcal{B}_{i,j}^k(x) = \mathcal{B}_{j,i}^k(x)$, this system contains $\frac{k(k+1)}{2}$ distinct functions and equations.

5.1 Examples

5.1.1 $k = 2$

For $k = 2$, we have the following system of differential equations:

$$\begin{cases} \frac{d}{dx}\mathcal{B}^2_{2,2}(x) = 1 + 2\mathcal{B}^2_{1,2}(x) + \mathcal{B}^2_{1,2}(x)^2, \\ \frac{d}{dx}\mathcal{B}^2_{1,2}(x) = 1 + \mathcal{B}^2_{1,1}(x) + \mathcal{B}^2_{1,1}(x) \cdot \mathcal{B}^2_{1,2}(x), \\ \frac{d}{dx}\mathcal{B}^2_{1,1}(x) = 1 + \mathcal{B}^2_{1,1}(x)^2. \end{cases}$$

with the constraints $\mathcal{B}^k_{i,j}(0) = 0$.

The system has the following solution:

$$\begin{cases} \mathcal{B}^2_{1,1}(x) = \tan(x), \\ \mathcal{B}^2_{1,2}(x) = \tan(x) + \sec(x) - 1, \\ \mathcal{B}^2_{2,2}(x) = 2(\tan(x) + \sec(x) - 1) - x. \end{cases}$$

Therefore, the numbers $B^2(n)$ represent the coefficients in the series expansion

$$\mathcal{B}^2_{2,2}(x) = 2(\tan(x) + \sec(x) - 1) - x = x + 2 \cdot \frac{x^2}{2!} + 4 \cdot \frac{x^3}{3!} + 10 \cdot \frac{x^4}{4!} + 32 \cdot \frac{x^5}{5!} + \cdots$$

and form sequence A001250 in [4]. It also counts the number of permutations of order n with exactly $n - 1$ runs [1].

5.1.2 $k = 3$

For $k = 3$, we have the following system of differential equations:

$$\begin{cases} \frac{d}{dx}\mathcal{B}^3_{3,3}(x) = 1 + 2\mathcal{B}^3_{2,3}(x) + \mathcal{B}^3_{2,3}(x)^2 \\ \frac{d}{dx}\mathcal{B}^3_{2,3}(x) = 1 + \mathcal{B}^3_{1,3}(x) + \mathcal{B}^3_{2,2}(x) + \mathcal{B}^3_{2,2}(x) \cdot \mathcal{B}^3_{2,3}(x) \\ \frac{d}{dx}\mathcal{B}^3_{1,3}(x) = 1 + \mathcal{B}^3_{1,2}(x) + \mathcal{B}^3_{1,2}(x) \cdot \mathcal{B}^3_{2,3}(x) \\ \frac{d}{dx}\mathcal{B}^3_{2,2}(x) = 1 + 2\mathcal{B}^3_{1,2}(x) + \mathcal{B}^3_{2,2}(x)^2 \\ \frac{d}{dx}\mathcal{B}^3_{1,2}(x) = 1 + \mathcal{B}^3_{1,1}(x) + \mathcal{B}^3_{1,2}(x) \cdot \mathcal{B}^3_{2,2}(x) \\ \frac{d}{dx}\mathcal{B}^3_{1,1}(x) = 1 + \mathcal{B}^3_{1,2}(x)^2 \end{cases}$$

Although we have not been able to solve this system, we remark that the last three equations form a subsystem involving the functions $\mathcal{B}^3_{2,2}(x)$, $\mathcal{B}^3_{1,2}(x)$, and $\mathcal{B}^3_{1,1}(x)$, which implies the following autonomous ordinary differential equation for $y(x) = \mathcal{B}^3_{2,2}(x)$:

$$2 \cdot y''' - 6 \cdot y \cdot y'' - 7 \cdot y'^2 + 8 \cdot y^2 \cdot y' + 4 \cdot y' - y^4 - 2 \cdot y^2 - 5 = 0.$$

This equation further reduces to the following second order differential equation for $w(y) = y'$ [5]:

$$2 \cdot w^2 \cdot w'' + 2 \cdot w \cdot w'^2 - 6 \cdot y \cdot w \cdot w' - 7 \cdot w^2 + 8 \cdot y^2 \cdot w + 4 \cdot w - y^4 - 2 \cdot y^2 - 5 = 0.$$

Solving this equation would be a step towards obtaining the generating function $\mathcal{B}^3_{3,3}(x)$ for the numbers $B^3(n)$.

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Table 1: Values $U^k(n)$ for $k, n \leq 18$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$U^1(n)$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$U^2(n)$	1	2	5	17	70	349	2017	13358	99377	822041	7477162	74207209	797771521	9236662346	114579019469	1516103040833	21314681315998	317288088082405
$U^3(n)$	1	2	6	23	111	642	4326	33333	288901	2782082	29471046	340568843	4263603891	57482264322	830335952166	12793889924553	209449977967081	3630626729775362
$U^4(n)$	1	2	6	24	119	709	4928	39144	349776	3472811	37928331	451891992	5832672456	81074690424	1207441809209	19181203110129	323753459184738	5785975294622694
$U^5(n)$	1	2	6	24	120	719	5027	40168	361080	3606480	39623760	474915803	6166512899	86227808578	1291868401830	20645144452320	350547210173280	6302294420371031
$U^6(n)$	1	2	6	24	120	720	5039	40305	362682	3626190	39881160	478490760	6219298800	87055051511	1305598835941	20885951018102	354999461960226	6388879812001704
$U^7(n)$	1	2	6	24	120	720	5040	40319	362863	3628550	39913170	478947480	6226179960	87164597520	1307440134000	20918580896069	355608034188517	6400803479701178
$U^8(n)$	1	2	6	24	120	720	5040	40320	362879	3628781	39916492	478996716	6226941864	87176969880	1307651304960	20922368987520	355679390626560	6402213152423659
$U^9(n)$	1	2	6	24	120	720	5040	40320	362880	3628799	39916779	479001228	6227014404	87178179816	1307672369640	20922752672640	355686706327680	6402359109968640
$U^{10}(n)$	1	2	6	24	120	720	5040	40320	362880	3628800	39916799	479001577	6227020358	87178283010	1307674215120	20922786961440	355687370176320	6402372516146880
$U^{11}(n)$	1	2	6	24	120	720	5040	40320	362880	3628800	39916800	479001599	6227020775	87178290682	1307674357710	20922789683040	355687423926240	6402373618334400
$U^{12}(n)$	1	2	6	24	120	720	5040	40320	362880	3628800	39916800	479001600	6227020799	87178291173	1307674367400	20922789875280	355687427826720	6402373699926240
$U^{13}(n)$	1	2	6	24	120	720	5040	40320	362880	3628800	39916800	479001600	6227020800	87178291199	1307674367971	20922789887312	355687428080496	6402373705380384
$U^{14}(n)$	1	2	6	24	120	720	5040	40320	362880	3628800	39916800	479001600	6227020800	87178291200	1307674367999	20922789887969	355687428095218	6402373705709334
$U^{15}(n)$	1	2	6	24	120	720	5040	40320	362880	3628800	39916800	479001600	6227020800	87178291200	1307674368000	20922789887999	355687428095967	6402373705727118
$U^{16}(n)$	1	2	6	24	120	720	5040	40320	362880	3628800	39916800	479001600	6227020800	87178291200	1307674368000	20922789888000	355687428095999	6402373705727965
$U^{17}(n)$	1	2	6	24	120	720	5040	40320	362880	3628800	39916800	479001600	6227020800	87178291200	1307674368000	20922789888000	355687428096000	6402373705727999
$U^{18}(n)$	1	2	6	24	120	720	5040	40320	362880	3628800	39916800	479001600	6227020800	87178291200	1307674368000	20922789888000	355687428096000	6402373705728000

Table 2: Values $I^k(n)$ for $k, n \leq 18$. Highlighted values indicate disagreements with [2].

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$I^1(n)$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$I^2(n)$	0	1	4	16	69	348	2016	13357	99376	822040	7477161	74207208	797771520	9236662345	114579019468	1516103040832	21314681315997	317288088082404
$I^3(n)$	0	0	1	6	41	293	2309	19975	189524	1960041	21993884	266361634	3465832370	48245601976	715756932697	**11277786883720**	**188135296651083**	**3313338641692957**
$I^4(n)$	0	0	0	1	8	67	602	5811	60875	690729	8457285	111323149	1569068565	23592426102	377105857043	**6387313185576**	**114303481217657**	**2155348564847332**
$I^5(n)$	0	0	0	0	1	10	99	1024	11304	133669	1695429	23023811	333840443	5153118154	84426592621	1463941342191	26793750988542	516319125748337
$I^6(n)$	0	0	0	0	0	1	12	137	1602	19710	257400	3574957	52785901	827242933	13730434111	240806565782	4452251786946	86585391630673
$I^7(n)$	0	0	0	0	0	0	1	14	181	2360	32010	456720	6881160	109546009	1841298059	32629877967	608572228291	11923667699474
$I^8(n)$	0	0	0	0	0	0	0	1	16	231	3322	49236	761904	12372360	211170960	3788091451	71356438043	1409672722481
$I^9(n)$	0	0	0	0	0	0	0	0	1	18	287	4512	72540	1209936	21064680	383685120	7315701120	145957544981
$I^{10}(n)$	0	0	0	0	0	0	0	0	0	1	20	349	5954	103194	1845480	34288800	663848640	13406178240
$I^{11}(n)$	0	0	0	0	0	0	0	0	0	0	1	22	417	7672	142590	2721600	53749920	1102187520
$I^{12}(n)$	0	0	0	0	0	0	0	0	0	0	0	1	24	491	9690	192240	3900480	81591840
$I^{13}(n)$	0	0	0	0	0	0	0	0	0	0	0	0	1	26	571	12032	253776	5454144
$I^{14}(n)$	0	0	0	0	0	0	0	0	0	0	0	0	0	1	28	657	14722	328950
$I^{15}(n)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	30	749	17784
$I^{16}(n)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	32	847
$I^{17}(n)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	34
$I^{18}(n)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Table 3: Values $B^k(n)$ for $k, n \leq 18$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$B^1(n)$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$B^2(n)$	1	2	4	10	32	122	544	2770	15872	101042	707584	5405530	44736512	398721962	3807514624	38783024290	419730685952	4809759350882
$B^3(n)$	1	2	6	22	102	564	3652	26986	224458	2073946	21080922	233752052	2807949492	36324988206	503484183878	7443797211854	116931715588046	1944883690208684
$B^4(n)$	1	2	6	24	118	698	4816	37968	336812	3319622	35990262	425668584	5454050314	75257838602	1112621686120	17545752570360	293985178842320	5215578061637498
$B^5(n)$	1	2	6	24	120	718	5014	40016	359280	3584160	39331224	470842102	6106259878	85282508228	1276168085580	20369694217750	345453884789910	6203249305454148
$B^6(n)$	1	2	6	24	120	720	5038	40290	362484	3623580	39845520	477979920	6211578648	86931863566	1303524552206	20849140937272	354312156550056	6375401280887904
$B^7(n)$	1	2	6	24	120	720	5040	40318	362846	3628300	39909540	478893360	6225339120	87150903840	1307205906864	20914372123786	355528646248138	6399233407501172
$B^8(n)$	1	2	6	24	120	720	5040	40320	362878	3628762	39916184	478991832	6226862928	87175648560	1307628241920	20921948087040	355671353182860	6402052600045958
$B^9(n)$	1	2	6	24	120	720	5040	40320	362880	3628798	39916758	479000856	6227008008	87178068432	1307670371280	20922715457280	355685984559360	6402344514209280
$B^{10}(n)$	1	2	6	24	120	720	5040	40320	362880	3628800	39916798	479001554	6227019916	87178274820	1307674062240	20922784034880	355687312256640	6402371326565760
$B^{11}(n)$	1	2	6	24	120	720	5040	40320	362880	3628800	39916800	479001598	6227020750	87178290164	1307674347420	20922789478080	355687419756480	6402373530940800
$B^{12}(n)$	1	2	6	24	120	720	5040	40320	362880	3628800	39916800	479001600	6227020798	87178291146	1307674366800	20922789862560	355687427557440	6402373694124480
$B^{13}(n)$	1	2	6	24	120	720	5040	40320	362880	3628800	39916800	479001600	6227020800	87178291198	1307674367942	20922789886624	355687428064992	6402373705032768
$B^{14}(n)$	1	2	6	24	120	720	5040	40320	362880	3628800	39916800	479001600	6227020800	87178291200	1307674367998	20922789887938	355687428094436	6402373705690668
$B^{15}(n)$	1	2	6	24	120	720	5040	40320	362880	3628800	39916800	479001600	6227020800	87178291200	1307674368000	20922789887998	355687428095934	6402373705726236
$B^{16}(n)$	1	2	6	24	120	720	5040	40320	362880	3628800	39916800	479001600	6227020800	87178291200	1307674368000	20922789888000	355687428095998	6402373705727930
$B^{17}(n)$	1	2	6	24	120	720	5040	40320	362880	3628800	39916800	479001600	6227020800	87178291200	1307674368000	20922789888000	355687428096000	6402373705727998
$B^{18}(n)$	1	2	6	24	120	720	5040	40320	362880	3628800	39916800	479001600	6227020800	87178291200	1307674368000	20922789888000	355687428096000	6402373705728000

Table 4: Values $A^k(n)$ for $k, n \leq 18$. Highlighted values indicate disagreements with [2].

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$A^1(n)$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$A^2(n)$	0	2	4	10	32	122	544	2770	15872	101042	707584	5405530	44736512	398721962	3807514624	38783024290	419730685952	4809759350882
$A^3(n)$	0	0	2	12	70	442	3108	24216	208586	1972904	20373338	228346522	**2763212980**	**35926266244**	499676669254	7405014187564	116511984902094	1940073930857802
$A^4(n)$	0	0	0	2	16	134	1164	10982	112354	1245676	14909340	191916532	**2646100822**	**38932850396**	609137502242	10101955358506	177053463254274	3270694371428814
$A^5(n)$	0	0	0	0	2	20	198	2048	22468	264538	3340962	45173518	**652209564**	**10024669626**	163546399460	2823941647390	51468705947590	987671243816650
$A^6(n)$	0	0	0	0	0	2	24	274	3204	39420	514296	7137818	105318770	**1649355338**	27356466626	479446719522	8858271760146	172151975433756
$A^7(n)$	0	0	0	0	0	0	2	28	362	4720	64020	913440	13760472	219040274	3681354658	65231186514	1216489698082	23832126613268
$A^8(n)$	0	0	0	0	0	0	0	2	32	462	6644	98472	1523808	24744720	422335056	7575963254	142706934722	2819192544786
$A^9(n)$	0	0	0	0	0	0	0	0	2	36	574	9024	145080	2419872	42129360	767370240	14631376500	291914163322
$A^{10}(n)$	0	0	0	0	0	0	0	0	0	2	40	698	11908	206388	3690960	68577600	1327697280	26812356480
$A^{11}(n)$	0	0	0	0	0	0	0	0	0	0	2	44	834	15344	285180	5443200	107499840	2204375040
$A^{12}(n)$	0	0	0	0	0	0	0	0	0	0	0	2	48	982	19380	384480	7800960	163183680
$A^{13}(n)$	0	0	0	0	0	0	0	0	0	0	0	0	2	52	1142	24064	507552	10908288
$A^{14}(n)$	0	0	0	0	0	0	0	0	0	0	0	0	0	2	56	1314	29444	657900
$A^{15}(n)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	60	1498	35568
$A^{16}(n)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	64	1694
$A^{17}(n)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	68
$A^{18}(n)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2