POLYNOMIALS WITH ONLY REAL ZEROS AND THE EULERIAN POLYNOMIALS OF TYPE D

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Abstract. A remarkable identity involving the Eulerian polynomials of type D was obtained by Stembridge (Adv. Math. 106 (1994), p. 280, Lemma 9.1). In this paper we explore an equivalent form of this identity. We prove Brenti's real-rootedness conjecture for the Eulerian polynomials of type D.

1. Introduction

Let $\mathcal{S}_n$ denote the symmetric group of all permutations of $[n]$, where $[n] = \{1, 2, \ldots, n\}$. For a permutation $\pi \in \mathcal{S}_n$, we define a *descent* to be a position i such that $\pi(i) > \pi(i+1)$. Denote by $\operatorname{des}(\pi)$ the number of descents of π. Let

$$A_n(x) = \sum_{\pi \in \mathcal{S}_n} x^{\operatorname{des}(\pi)+1} = \sum_{k=1}^{n} A(n,k)x^k.$$

The polynomial $A_n(x)$ is called an *Eulerian polynomial*, while $A(n,k)$ is called an *Eulerian number*. Denote by B_n the Coxeter group of type B. Elements π of B_n are signed permutations of $\pm[n]$ such that $\pi(-i) = -\pi(i)$ for all i, where $\pm[n] = \{\pm 1, \pm 2, \ldots, \pm n\}$. Let

$$B_n(x) = \sum_{\pi \in B_n} x^{\operatorname{des}_B(\pi)} = \sum_{k=0}^{n} B(n,k)x^k,$$

where $\operatorname{des}_B = |\{i \in [n] : \pi(i-1) > \pi(i)\}|$ with $\pi(0) = 0$. The polynomial $B_n(x)$ is called an *Eulerian polynomial of type B*, while $B(n,k)$ is called an *Eulerian number of type B*. Denote by D_n the Coxeter group of type D. The Coxeter group D_n is the subgroup of B_n consisting of signed permutations $\pi = \pi(1)\pi(2)\cdots\pi(n)$ with an even number of negative entries. Let

$$D_n(x) = \sum_{\pi \in D_n} x^{\operatorname{des}_D(\pi)} = \sum_{k=0}^{n} D(n,k)x^k,$$

where $\operatorname{des}_D = |\{i \in [n] : \pi(i-1) > \pi(i)\}|$ with $\pi(0) = -\pi(2)$. The polynomial $D_n(x)$ is called an *Eulerian polynomial of type D*, while $D(n,k)$ is called an *Eulerian number of type D* (see [17, A066094] for details). Below are the polynomials $D_n(x)$ for $n \le 3$:

$$D_0(x) = 1, D_1(x) = 1, D_2(x) = 1 + 2x + x^2, D_3(x) = 1 + 11x + 11x^2 + x^3.$$

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In 1994, Stembridge [18, Lemma 9.1] obtained the following remarkable identity:

$$D_n(x) = B_n(x) - n2^{n-1}A_{n-1}(x) \quad \textit{for} \quad n \geq 2. \tag{1.1}$$

Let $P_n(x) = A_n(x)/x$. It is well known that

$$\sum_{n=0}^{\infty} P_n(-1)\frac{x^n}{n!} = 1 + \tanh(x)$$

and

$$\sum_{n=0}^{\infty} B_n(-1)\frac{x^n}{n!} = \text{sech }(2x)$$

(see [13] for instance). For $n \geq 3$, Chow [6, Corollary 6.10] obtained that

$$\operatorname{sgn} D_n(-1) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ (-1)^{\frac{n}{2}} & \text{if } n \text{ is even.} \end{cases} \tag{1.2}$$

This paper is organized as follows. Section 2 is devoted to an equivalent form of the identity (1.1). In Section 3, we prove Brenti's [2, Conjecture 5.1] real-rootedness conjecture for the Eulerian polynomials of type D.

2. Derivative polynomials

In 1995, Hoffman [14] introduced the derivative polynomials for tangent and secant:

$$\frac{d^n}{d\theta^n}\tan\theta = P_n(\tan\theta) \quad \textit{and} \quad \frac{d^n}{d\theta^n}\sec\theta = \sec\theta\cdot Q_n(\tan\theta).$$

Various refinements of the polynomials $P_n(u)$ and $Q_n(u)$ have been pursued by several authors (see [8, 9, 11, 16] for instance). The derivative polynomials for hyperbolic tangent and secant are defined by

$$\frac{d^n}{d\theta^n}\tanh\theta = \widetilde{P}_n(\tanh\theta) \quad \textit{and} \quad \frac{d^n}{d\theta^n}\text{sech }\theta = \text{sech }\theta\cdot \widetilde{Q}_n(\tanh\theta).$$

It follows from $\tanh\theta = \mathrm{i}\tan(\theta/i)$ and $\text{sech }\theta = \sec(\theta/i)$ that

$$\widetilde{P}_n(x) = \mathrm{i}^{n-1}P_n(\mathrm{i}x) \quad \textit{and} \quad \widetilde{Q}_n(x) = \mathrm{i}^n Q_n(\mathrm{i}x).$$

From the chain rule it follows that the polynomials $\widetilde{P}_n(x)$ satisfy

$$\widetilde{P}_{n+1}(x) = (1-x^2)\widetilde{P}'_n(x) \tag{2.1}$$

with initial values $\widetilde{P}_0(x) = x$. Similarly, $\widetilde{Q}_0(x) = 1$ and

$$\widetilde{Q}_{n+1}(x) = (1-x^2)\widetilde{Q}'_n(x) - x\widetilde{Q}_n(x). \tag{2.2}$$

Let

$$\tan^k(x) = \sum_{n\geq k} T(n,k)\frac{x^n}{n!}$$

and

$$\sec(x)\tan^k(x) = \sum_{n\geq k} S(n,k)\frac{x^n}{n!}.$$

The numbers $T(n,k)$ and $S(n,k)$ are respectively called the *tangent numbers of order* k (see [3, p. 428]) and the *secant numbers of order* k ((see [4, p. 305])). The numbers $T(n,1)$ are sometimes called the *tangent numbers* and $S(n,0)$ are called the *Euler numbers*. Note that the tangent is an odd function and the secant is an even function. Then

$$T(2n,1)=S(2n+1,0)=0,\quad T(2n+1,1)\neq 0 \quad \textit{and} \quad S(2n,0)\neq 0.$$

Recently, Cvijović [8, Theorem 2] showed that

$$\widetilde{P}_n(x)=(-1)^{\frac{n-1}{2}}T(n,1)+\sum_{k=1}^{n+1}\frac{(-1)^{\frac{n+k-1}{2}}}{k}T(n+1,k)x^k$$

and

$$\widetilde{Q}_n(x)=\sum_{k=0}^{n}(-1)^{\frac{n+k}{2}}S(n,k)x^k.$$

In particular, we have

$$\widetilde{P}_{2n-1}(0)=(-1)^{n-1}T(2n-1,1) \quad \textit{and} \quad \widetilde{Q}_{2n}(0)=(-1)^nS(2n,0). \tag{2.3}$$

The first few of the polynomials $\widetilde{P}_n(x)$ and $\widetilde{Q}_n(x)$ are respectively given as follows: $\widetilde{P}_1(x)=-x^2+1, \widetilde{P}_2(x)=2x^3-2x, \widetilde{P}_3(x)=-6x^4+8x^2-2, \widetilde{P}_4(x)=24x^5-40x^3+16x;$

$\widetilde{Q}_1(x)=-x, \widetilde{Q}_2(x)=2x^2-1, \widetilde{Q}_3(x)=-6x^3+5x, \widetilde{Q}_4(x)=24x^4-28x^2+5.$

For $n\geq 2$, we define

$$a_n(x)=(x+1)^{n+1}A_n\left(\frac{x-1}{x+1}\right),\quad b_n(x)=(x+1)^nB_n\left(\frac{x-1}{x+1}\right)$$

and

$$d_n(x)=\left(\frac{x+1}{2}\right)^nD_n\left(\frac{x-1}{x+1}\right). \tag{2.4}$$

Then

$$2^nd_n(x)=b_n(x)-n2^{n-1}a_{n-1}(x) \quad \textit{for} \quad n\geq 2. \tag{2.5}$$

From [11, Theorem 5, Theorem 6], we obtain

$$a_n(x)=(-1)^n\widetilde{P}_n(x) \quad \textit{and} \quad b_n(x)=(-1)^n2^n\widetilde{Q}_n(x). \tag{2.6}$$

Therefore, the polynomials $a_n(x)$ satisfy the recurrence relation

$$a_{n+1}(x)=(x^2-1)a_n'(x) \tag{2.7}$$

with initial values $a_0(x)=x$. The polynomials $b_n(x)$ satisfy the recurrence relation

$$b_{n+1}(x)=2(x^2-1)b_n'(x)+2xb_n(x) \tag{2.8}$$

with initial values $b_0(x)=1$. From (1.1), we get the following result.

Proposition 2.1. *For $n\geq 2$, we have*

$$2d_n(x)=(-1)^n(n\widetilde{P}_{n-1}(x)+2\widetilde{Q}_n(x)). \tag{2.9}$$

The first few terms of $d_n(x)$ can be computed directly as follows:

$$\begin{aligned}
d_2(x)&=x^2,\\
d_3(x)&=3x^3-2x,\\
d_4(x)&=12x^4-12x^2+1,\\
d_5(x)&=60x^5-80x^3+21x,\\
d_6(x)&=360x^6-600x^4+254x^2-13.
\end{aligned}$$

It follows from (2.1) and (2.2) that $d_n(-1) = (-1)^n$ for $n \geq 2$.

Corollary 2.2. *For $n \geq 1$, we have $D_{2n-1}(-1) = 0$ and*

$$D_{2n}(-1) = (-4)^n (S(2n, 0) - nT(2n - 1, 1)),$$

where $T(n, 1)$ are the tangent numbers and $S(n, 0)$ are the Euler numbers.

Proof. Note that $D_{2n-1}(-1) = 2^{2n-1} d_{2n-1}(0)$. It is easy to verify that $\widetilde{P}_{2n-2}(0) = \widetilde{Q}_{2n-1}(0) = 0$, $\widetilde{P}_{2n-1}(0) = (-1)^{n-1} T(2n-1, 1)$ and $\widetilde{Q}_{2n}(0) = (-1)^n S(2n, 0)$. Then $D_{2n-1}(-1) = 0$. By (2.4), we obtain $D_{2n}(-1) = 4^n d_{2n}(0)$. From (2.9), we obtain $d_{2n}(0) = n\widetilde{P}_{2n-1}(0) + \widetilde{Q}_{2n}(0)$. Then by (2.3), we get the desired result. □

3. Main results

Polynomials with only real zeros arise often in combinatorics, algebra and geometry. We refer the reader to [1, 5, 6, 10, 15, 19] for various results involving zeros of the polynomials $A_n(x), B_n(x)$ and $D_n(x)$. This Section is devoted to prove Brenti's [2, Conjecture 5.1] real-rootedness conjecture for the Eulerian polynomials of type D.

Let RZ denote the set of real polynomials with only real zeros. Denote by $\mathrm{RZ}(I)$ the set of such polynomials all whose zeros are in the interval I. Suppose that $f, F \in \mathrm{RZ}$. Let $\{s_i\}$ and $\{r_j\}$ be all zeros of F and f in nonincreasing order respectively. Following [7], we say that F *interleaves* f, denoted by $f \preceq F$, if $\deg f \leq \deg F \leq \deg f + 1$ and

$$s_1 \geq r_1 \geq s_2 \geq r_2 \geq s_3 \geq r_3 \geq \cdots . \tag{3.1}$$

If no equality sign occurs in (3.1), then we say that F *strictly interleaves* f. Let $f \prec F$ denote F strictly interleaves f.

The key ingredient of our proof is the following result due to Hetyei [12].

Lemma 3.1 ([12, Proposition 6.5, Theorem 8.6])**.** *For $n \geq 1$, we have $\widetilde{P}_n(x) \in \mathrm{RZ}[-1, 1]$, $\widetilde{Q}_n(x) \in \mathrm{RZ}(-1, 1)$ and $\widetilde{Q}_n(x) \prec \widetilde{P}_n(x)$. Moreover, $\widetilde{P}_{n-1}(x) \preceq \widetilde{P}_n(x)$ and $\widetilde{Q}_{n-1}(x) \preceq \widetilde{Q}_n(x)$ for $n \geq 2$.*

By Lemma 3.1, we obtain $a_{n-1}(x) \preceq a_n(x), b_{n-1}(x) \preceq b_n(x)$ and $b_n(x) \prec a_n(x)$. Let sgn denote the sign function defined on $\mathbb{R}$ by

$$\operatorname{sgn} x = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

We now present the main result of this paper.

Theorem 3.2. *For $n \geq 2$, we have $D_n(x) \in \mathrm{RZ}(-\infty, 0)$.*

Proof. Clearly, $D_n(x) \in \mathrm{RZ}(-\infty, 0)$ if and only if $d_n(x) \in \mathrm{RZ}(-1, 1)$. Since $d_2(x) = x^2$ and $d_3(x) = 3x^3 - 2x$, it suffices to consider the case $n \geq 4$.

Note that the polynomials $a_n(x)$ and $b_n(x)$ have the following expressions:

$$a_n(x) = \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} (-1)^k p(n, n - 2k + 1) x^{n-2k+1},$$

$$b_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k q(n, n-2k) x^{n-2k}.$$

Using Lemma 3.1, we write

$$a_{2n-1}(x) = (2n-1)! \prod_{i=1}^{n} (x - s_i)(x + s_i),$$

$$a_{2n}(x) = (2n)! x \prod_{i=1}^{n} (x - a_i)(x + a_i),$$

$$b_{2n}(x) = (2n)! 4^n \prod_{j=1}^{n} (x - r_j)(x + r_j),$$

and

$$b_{2n+1}(x) = (2n+1)! 2^{2n+1} x \prod_{j=1}^{n} (x - b_j)(x + b_j),$$

where

$$1 = s_1 > r_1 > s_2 > r_2 > \cdots > r_{n-1} > s_n > r_n > 0 \tag{3.2}$$

and

$$1 = a_1 > b_1 > a_2 > b_2 > \cdots b_{n-1} > a_n > b_n > 0. \tag{3.3}$$

Using (2.7) and (2.8), the inequalities (3.2) and (3.3) for zeros can be easily proved by induction on n. We omit the proof of this for brevity.

By (2.5), we get

$$d_{2n}(x) = \frac{b_{2n}(x)}{4^n} - n a_{2n-1}(x).$$

Let $F(x) = \prod_{i=1}^{n} (x - s_i)$ and $f(x) = \prod_{j=1}^{n} (x - r_j)$. Then

$$d_{2n}(x) = (2n-1)! (-1)^n n \{2 f(x) f(-x) - F(x) F(-x)\}.$$

Note that $\operatorname{sgn} d_{2n}(s_{j+1}) = (-1)^j$ and $\operatorname{sgn} d_{2n}(r_j) = (-1)^{j+1}$, where $1 \le j \le n-1$. . Therefore, $d_{2n}(x)$ has precisely one zero in each of $2n-2$ intervals (s_{j+1}, r_j) and $(-r_j, -s_{j+1})$ Note that $\operatorname{sgn} d_{2n}(r_n) = (-1)^{n-1}$ and $\operatorname{sgn} d_{2n}(-r_n) = (-1)^{n+1}$. It follows from (1.2) that $\operatorname{sgn} d_{2n}(0) = (-1)^n$. Therefore, $d_{2n}(x)$ has precisely one zero in each of the intervals $(-r_n, 0)$ and $(0, r_n)$. Thus $d_{2n}(x) \in \mathrm{RZ}(-1, 1)$.

Along the same lines, by (2.5), we get

$$d_{2n+1}(x) = \frac{b_{2n+1}(x)}{2^{2n+1}} - \frac{1}{2}(2n+1) a_{2n}(x).$$

Let $G(x) = \prod_{i=1}^{n} (x - a_i)$ and $g(x) = \prod_{j=1}^{n} (x - b_j)$. Then

$$d_{2n+1}(x) = (2n+1)! (-1)^n x \{g(x) g(-x) - \frac{1}{2} G(x) G(-x)\}.$$

Note that $\operatorname{sgn} d_{2n+1}(a_{j+1}) = (-1)^j$ and $\operatorname{sgn} d_{2n+1}(b_j) = (-1)^{j+1}$, where $1 \le j \le n-1$. Therefore, $d_{2n+1}(x)$ has precisely one zero in each of $2n-2$ intervals (a_{j+1}, b_j) and $(-b_j, -a_{j+1})$. Note that $\operatorname{sgn} d_{2n+1}(b_n) = (-1)^{n+1}$ and $\operatorname{sgn} d_{2n+1}(-b_n) = (-1)^n$. It follows from (2.9) that

$$\operatorname{sgn} \lim_{x \to 0} \frac{d_{2n+1}(x)}{x} = (-1)^n.$$

Hence

$$\operatorname{sgn} \lim_{x\to 0^-} d_{2n+1}(x) = (-1)^{n+1} \quad and \quad \operatorname{sgn} \lim_{x\to 0^+} d_{2n+1}(x) = (-1)^{n}.$$

Therefore, $d_{2n+1}(x)$ has precisely one zero in each of the intervals $(-b_n, 0)$ and $(0, b_n)$. Moreover, $d_{2n+1}(x)$ has a simple zero $x = 0$. Thus $d_{2n+1}(x) \in \mathrm{RZ}(-1, 1)$.

In conclusion, we define

$$d_{2n}(x) = \frac{(2n)!}{2} \prod_{i=1}^{n} (x - c_i)(x + c_i)$$

and

$$d_{2n+1}(x) = \frac{(2n+1)!}{2} x \prod_{i=1}^{n} (x - d_i)(x + d_i),$$

where $c_1 > c_2 > \cdots > c_{n-1} > c_n$ and $d_1 > d_2 > \cdots > d_{n-1} > d_n$. Then

$$r_1 > c_1 > s_2 > r_2 > c_2 > s_3 > \cdots > r_{n-1} > c_{n-1} > s_n > r_n > c_n > 0 \tag{3.4}$$

and

$$b_1 > d_1 > a_2 > b_2 > d_2 > a_3 > \cdots > b_{n-1} > d_{n-1} > a_n > b_n > d_n > 0. \tag{3.5}$$

This completes the proof. □

We say that the polynomials $f_1(x), \ldots, f_k(x)$ are *compatible* if for all nonnegative real numbers $c_1, c_2, \ldots, c_k$, we have $\sum_{i=1}^{k} c_i f_i(x) \in \mathrm{RZ}$. Let $f(x), g(x) \in \mathrm{RZ}$. A *common interleaver* for $f(x)$ and $g(x)$ is a polynomial that interleaves $f(x)$ and $g(x)$ simultaneously. Denote by $n_f(x)$ the number of real zeros of a polynomial $f(x)$ that lie in the interval $[x, \infty)$ (counted with their multiplicities). Chudnovsky and Seymour [7] established the following two lemmas.

Lemma 3.3 ([7, 3.5])**.** *Let $f(x), g(x) \in \mathrm{RZ}$. Then $f(x)$ and $g(x)$ have a common interleaver if and only if $|n_f(x) - n_g(x)| \le 1$ for all $x \in \mathbb{R}$.*

Lemma 3.4 ([7, 3.6])**.** *Let $f_1(x), f_2(x), \ldots, f_k(x)$ be polynomials with positive leading coefficients and all zeros real. Then following three statements are equivalent:*

- (a) *$f_1(x), f_2(x), \ldots, f_k(x)$ are pairwise compatible,*
- (b) *for all s, t such that $1 \le s < t \le k$, the polynomials f_s, f_t have a common interleaver,*
- (c) *$f_1(x), f_2(x), \ldots, f_k(x)$ are compatible.*

By (3.4) and (3.5), we obtain

$$|n_{a_{n-1}}(x) - n_{b_n}(x)| \le 1, \quad |n_{a_{n-1}}(x) - n_{d_n}(x)| \le 1$$

and

$$|n_{d_n}(x) - n_{b_n}(x)| \le 1$$

for all $x \in \mathbb{R}$. Combining Lemma 3.3 and Lemma 3.4, we get the following result.

Theorem 3.5. *For $n \ge 2$, the polynomials $a_{n-1}(x), b_n(x)$ and $d_n(x)$ are compatible. Equivalently, the polynomials $A_{n-1}(x), B_n(x)$ and $D_n(x)$ are compatible.*

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