

ENUMERATIONS OF FINITE TOPOLOGIES ASSOCIATED WITH A FINITE GRAPH

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ABSTRACT. The number of topologies and non-homeomorphic topologies on a fixed finite set are now known up to $n = 18$, $n = 16$ but still no complete formula yet [11]. There are one to one correspondence among topologies, preorder and digraphs. In this article, we enumerate topologies and non-homeomorphic topologies whose underlying graph is a given finite graph.

1. INTRODUCTION

Topology on a finite set, *finite topology* is often used to demonstrate interesting phenomena and counterexamples to plausible sounding conjectures. Finite topology also plays a key role in the theory of image analysis [5,6], the structures of molecular [8, 9], geometries of finite sets [10] and digital topology. One of research on finite topology is to enumerate the number of topologies and non-homeomorphic topologies on a fixed finite set, we denoted it by $\bar{\tau}(n)$ and $\bar{h}(n)$, respectively, where n is the cardinality of the finite set. Due to many different works, these numbers are now known up to $n = 18$, $n = 16$ but still no complete formula yet [11].

It is not difficult to relate a topology on a finite set with a preorder on the finite set. It can be briefly explained as follows. Let X be a finite set. A relation R on X is called a *preorder* if it is reflexive and transitive. For a topology \mathcal{T} on X , we define a relation $R(\mathcal{T})$ on X by a rule that $(x, y) \in R(\mathcal{T})$ if and only if every open set containing x also containing y , that is, $x \in \overline{\{y\}}^{\mathcal{T}}$, where $\overline{A}^{\mathcal{T}}$ is the closure of A with respect to the topological \mathcal{T} . Then $R(\mathcal{T})$ is a preorder on X . We call it *the preorder associated with the topology \mathcal{T}* . Conversely, for a preorder R on X , let $\mathcal{T}(R) = \{U \subset X : \forall x \in U, R(x) \subset U\}$, where $R(x) = \{y : (x, y) \in R\}$. Then $\mathcal{T}(R)$ is a topology on X and call it *the topology corresponding to the preorder R* . Notice that $\{R(x); x \in X\}$ is a base for the topology $\mathcal{T}(R)$.

Ever since the pioneer work of Evans, Harary and Lynn [3], counting such topologies can be done by counting digraphs as follows. Let G be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. We use $|X|$ for the cardinality of a set X . For a preorder R on X , let $D(R)$ be the direct graph whose vertex set is X and arc set is $R \setminus \Delta(X) = \{(x, y) : x \neq y \text{ and } (x, y) \in R\}$. For a digraph D , *the underlying graph of D* is the graph whose vertex set is equal to that of $V(D)$ and edge set is $\{\{x, y\} : xy \text{ or } yx \text{ is an arc of } D\}$. Notice that there are $3^{|E(G)|}$ directed graphs (or digraphs) whose underlying graph is a given graph G . For a topology \mathcal{T} on X , we say the underlying graph of \mathcal{T} is that of the digraph corresponding to the preorder

associated with \mathcal{T} . Notice that the digraph $D(R)$ corresponding to a preorder R is transitive, i.e., for any pair of distinct vertices a and c if ab and bc are arcs of $D(R)$, then ac is also an arc of $D(R)$. Example 2.1 can help to understand these relations if the reader is not familiar with graph theory. For terms in graph theory, we refer to [4].

There have been a few results on finite topology, preorder and digraphs. Evans *et.al* found a relation between labeled topologies on n points and the labeled transitive digraphs with n points [3]. Marijuan found a few useful properties on digraphs and topologies and relations between finite acyclic transitive digraphs and T_0 topologies [7].

In the present article, we mainly focus on graphs and unlabeled(non-homeomorphic) topologies and we enumerate topologies whose underlying graph is a given finite graph G , in other word, we enumerate the number of topologies and non-homeomorphic topologies with respect to the underlying graphs instead of the number of vertices of graphs. For a given graph G , let $\text{Top}(G)$ be the set of topologies having G as its underlying graph. Similarly, let $\bar{\tau}(G) = |\text{Top}(G)|$ and let $\hbar(G)$ be the number of non-homeomorphic topologies whose underlying graph is G . Notice that $\bar{\tau}(G)$ is equal to the number of transitive digraphs whose underlying graph is G .

The outline of this article is as follows. First, we will provide precise definitions and some general formulae in Section 2. In Section 3, we study how $\bar{\tau}(G)$, $\hbar(G)$ are related with graph operations. In Section 4, we find $\bar{\tau}(G)$, $\hbar(G)$ for a few graphs. A brief conclusion and some research problems are presented in Section 5.

2. GENERAL FORMULAE

For a finite set X , let $\text{Top}(X)$ be the set of all topologies on X . Two topologies \mathcal{T}_1 on a set X and \mathcal{T}_2 on a set Y are equivalent if the two topological spaces (X, \mathcal{T}_1) and (Y, \mathcal{T}_2) are homeomorphic. For a natural number n , let N_n be the set $\{1, 2, \dots, n\}$. For convenience, let $\bar{\tau}(n) = |\text{Top}(N_n)|$ and let $\hbar(n)$ be the number of non-homeomorphic topologies on N_n . It is clear that $\bar{\tau}(1) = 1$ and $\hbar(1) = 1$.

Example 2.1. Let $X = \{a, b\}$ and let $\mathcal{T} = \{\emptyset, \{a\}, X\}$. Then \mathcal{T} is a topology on X . The preorder $R(\mathcal{T})$ associated with \mathcal{T} is $\{(a, a), (b, b), (b, a)\}$ and the underlying graph of \mathcal{T} is the complete graph K_2 on two vertices a and b . Let $R = \Delta(X) = \{(a, a), (b, b)\}$. Then R is a preorder on X . The topology $\mathcal{T}(R)$ associated with R is the discrete topology on X and the underlying graph of $\mathcal{T}(R)$ is the null graph \mathcal{N}_2 on two vertices a and b .

The following example demonstrates the topologies whose underlying graph is the complete graph K_2 with 2 vertices $\{a, b\}$.

Example 2.2. Let $X = \{a, b\}$. Then $\mathcal{T}_1 = \{\emptyset, X\}$, $\mathcal{T}_2 = \{\emptyset, \{a\}, X\}$, $\mathcal{T}_3 = \{\emptyset, \{b\}, X\}$ and $\mathcal{T}_4 = \{\emptyset, \{a\}, \{b\}, X\}$ are the list of all topologies on X . So, $\bar{\tau}(2) = 4$. Since (X, \mathcal{T}_2) and (X, \mathcal{T}_3) are homeomorphic, $\hbar(2) = 3$. It is clear that $\text{Top}(K_2) = \{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3\}$. This implies that $\bar{\tau}(K_2) = 3$ and $\hbar(K_2) = 2$.

Some properties of finite topological spaces can be described by graph theoretical terminologies. The following lemma should be previously known, however, we could not find. Thus, we provide it in our language.

Lemma 2.3. *Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be finite topological spaces. Then we have*

- (a) *a function $f : (X, \mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y)$ is continuous if and only if $f : (X, R(\mathcal{T}_X)) \rightarrow (Y, R(\mathcal{T}_Y))$ preserves the relation, that is, f is a graph homomorphism between the two digraphs $D(R(\mathcal{T}_X))$ and $D(R(\mathcal{T}_Y))$,*
- (b) *the number of components of the topological space (X, \mathcal{T}_X) is equal to that of the underlying graph of \mathcal{T}_X . In particular, (X, \mathcal{T}_X) is connected if and only if the underlying graph of \mathcal{T}_X is connected.*

It comes from Lemma 2.3 (a) that $\bar{h}(G)$ is equal to the number of isomorphism classes of transitive digraphs whose underlying graph is G .

Let \mathcal{T}_1 and \mathcal{T}_2 be two topologies on N_n . If they are the same, then their underlying graphs are also the same. So, two topologies having distinct underlying graphs can not be the same. For a graph G , let $\text{Aut}(G)$ be the group of graph automorphisms of G . Then there are $\frac{|V(G)|!}{|\text{Aut}(G)|}$ graphs on $V(G)$ that are isomorphic to G . The following theorem comes from this observation.

Theorem 2.4. *Let n be a natural number. Then we have*

$$\bar{\tau}(n) = \sum_G \frac{n!}{|\text{Aut}(G)|} \bar{\tau}(G) \text{ and } \bar{h}(n) = \sum_G \bar{h}(G),$$

where G runs over all representatives of isomorphism classes of graphs of n vertices and $\text{Aut}(G)$ is the group of all graph automorphisms of G .

In order to complete the computation $\bar{\tau}(n)$ and $\bar{h}(n)$, we need to compute $\bar{\tau}(G)$ and $\bar{h}(G)$ for a given graph G .

For a graph G , let $\mathcal{TD}(G)$ be the set of transitive digraphs whose underlying graph is G . Then $\text{Aut}(G)$ acts on the set $\mathcal{TD}(G)$ and $\bar{h}(G) = |\mathcal{TD}(G)/\text{Aut}(G)|$ by Lemma 2.3 (a). Now, the following lemma comes from the Burnside lemma.

Lemma 2.5. *Let G be a connected graph and let $\mathcal{TD}(G)$ be the set of transitive digraph whose underlying graph is G . Then*

$$\bar{h}(G) = \frac{1}{|\text{Aut}(G)|} \sum_{\sigma \in \text{Aut}(G)} |\text{Fix}_\sigma|,$$

where $\text{Fix}_\sigma = \{D \in \mathcal{TD}(G) : \sigma(D) = D\}$.

It is easy to show that every vertex induced subgraph of a transitive digraph is transitive. From this, we can have the following lemma.

Lemma 2.6. *For any graph G , $\bar{\tau}(G) = 0$ if and only if there exists a vertex induced subgraph H of G such that $\bar{\tau}(H) = 0$.*

A graph G is said to be *triangle free* if G does not contain any triangles. Let G be a triangle free graph having at most 3 vertices. Then every vertex of a transitive digraph having G as its underlying graph is a source or a sink. Since every source is adjacent to a sink and vice versa, $\bar{\tau}(G) \neq 0$ if and only if G is bipartite. To complete the computation $\bar{\tau}(G)$ for a triangle free graph G , we need to compute $\bar{\tau}(G)$ and

$\bar{h}(G)$ for a connected bipartite graph G . If G is a connected bipartite graph having 2 vertices, then G must be K_2 . It is observed in Example 2.2 that $\bar{\tau}(K_2) = 3$ and $\bar{h}(K_2) = 2$. Let G be a connected bipartite graph having at least 3 vertices and let X_1 and X_2 be a vertex bipartition of G . Let D be a transitive digraph whose underlying graph is G . Then every vertex in X_1 is a source and every vertex in X_2 is a sink or vice versa. It implies that the number of transitive digraphs having G as its underlying graph is 2. Moreover, the two digraphs are isomorphic if and only if G is reflexible. We summarize this discussion as follows.

Theorem 2.7. *For a triangle free graph G , $\bar{\tau}(G) \neq 0$ if and only if G is bipartite. Moreover, for a connected bipartite graph G having at least two vertices, we have*

$$\bar{\tau}(G) = \begin{cases} 3 & \text{if } G = K_2 \\ 2 & \text{if } G \neq K_2 \end{cases} \quad \text{and} \quad \bar{h}(G) = \begin{cases} 1 & \text{if } G \neq K_2 \text{ and } G \text{ is reflexible} \\ 2 & \text{otherwise} \end{cases}$$

3. TOPOLOGIES AND GRAPH OPERATIONS

In this section, we will compute the number $\bar{\tau}(G \odot H)$ and $\bar{h}(G \odot H)$ when $\bar{\tau}(G)$, $\bar{\tau}(H)$, $\bar{h}(G)$, and $\bar{h}(H)$ are known, where \odot is the product, disjoint union, or amalgamation of graphs.

The following lemma gives a computation formula for the graph that can be expressed by a disjoint union of some connected graphs.

Lemma 3.1. *For a natural number ℓ , let G_1, \dots, G_ℓ be pairwise nonisomorphic ℓ connected graphs and let n_1, \dots, n_ℓ be ℓ natural numbers. Let $G = n_1 G_1 \cup n_2 G_2 \cup \dots \cup n_\ell G_\ell$. Then*

$$\bar{\tau}(G) = \prod_{i=1}^{\ell} \bar{\tau}(G_i)^{n_i} \quad \text{and} \quad \bar{h}(G) = \prod_{i=1}^{\ell} \bar{h}(n_i G_i) = \prod_{i=1}^{\ell} \binom{\bar{h}(G_i) + n_i - 1}{n_i},$$

where mH stands for the disjoint union of m copies of H .

Proof. Let H_1 and H_2 be two graphs having disjoint vertex sets. Then $\bar{\tau}(H_1 \cup H_2) = \bar{\tau}(H_1)\bar{\tau}(H_2)$. It implies that $\bar{\tau}(G) = \prod_{i=1}^{\ell} \bar{\tau}(G_i)^{n_i}$. To prove the second statement, we firstly show that $\bar{h}(nH) = \binom{\bar{h}(H) + n - 1}{n}$. Let us identify a topology $\mathcal{T} \in \text{Top}(nH)$ with a finite sequence $(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_n)$ of length n with $\mathcal{T}_i \in \text{Top}(H)$ for each $i = 1, 2, \dots, n$. Then two topologies $(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_n)$ and $(\mathcal{T}'_1, \mathcal{T}'_2, \dots, \mathcal{T}'_n)$ are equivalent if and only if there exists a bijection $\sigma : N_n \rightarrow N_n$ such that \mathcal{T}_i and $\mathcal{T}'_{\sigma(i)}$ are equivalent for each $i = 1, 2, \dots, n$. It implies that the number $\bar{h}(nH)$ is equal to the number of selections with repetitions of n objects chosen from $\bar{h}(H)$ types of objects, i.e., $\bar{h}(nH) = \binom{\bar{h}(H) + n - 1}{n}$. For given two nonisomorphic graphs H_1 and H_2 , it is not hard to show that $\bar{h}(H_1 \cup H_2) = \bar{h}(H_1)\bar{h}(H_2)$. It completes the proof. \square

For two graphs G and H , the Cartesian product $G \square H$ is a graph such that $V(G \square H) = V(G) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if $(u_1 = u_2 \text{ and } v_1 v_2 \in E(H))$ or $(u_1 u_2 \in V(G) \text{ and } v_1 = v_2)$. We aim to compute $\bar{\tau}(G \square H)$ and $\bar{h}(G \square H)$.

Lemma 3.2. *For any odd number $n \geq 3$, $\bar{\tau}(K_2 \square C_n) = 0$.*

Proof. If n is greater than or equal to 5, then it is not hard to show that there is no transitive digraph whose underlying graph is C_n . Since $K_2 \square C_n$ has an induced subgraph isomorphic to C_n , $\bar{\tau}(K_2 \square C_n) = 0$ by Lemma 2.6.

Let $n = 3$ and let D be a digraph whose underlying graph is $K_2 \square C_3$. Let $V(K_2) = \{u_1, u_2\}$ and $V(C_3) = \{v_1, v_2, v_3\}$. One can check that there are $u_i \in V(K_2)$ and $v_j, v_k \in V(C_3)$ such that both $((u_i, v_j), (u_{3-i}, v_j))$ and $((u_{3-i}, v_j), (u_{3-i}, v_k))$ are directed edges in D . Since $((u_i, v_j)$ and $(u_{3-i}, v_k))$ are not adjacent in $K_2 \square C_3$, D is not transitive. Hence $\bar{\tau}(K_2 \square C_3) = 0$. \square

Note that for two graphs G and H , $G \square H$ is bipartite if and only if both G and H are bipartite. Furthermore, for bipartite graph $G \square H$, one can check that $G \square H$ is reflexible if and only if either G or H is reflexible.

Theorem 3.3. *For any nontrivial graphs G and H ,*

$$\bar{\tau}(G \square H) = \begin{cases} 0 & \text{if } G \text{ or } H \text{ is not bipartite,} \\ 2 & \text{otherwise} \end{cases}$$

and

$$\bar{h}(C_n) = \begin{cases} 0 & \text{if } G \text{ or } H \text{ is not bipartite,} \\ 1 & \text{if (both } G \text{ and } H \text{ are bipartite) and (either } G \text{ or } H \text{ is reflexible),} \\ 2 & \text{otherwise.} \end{cases}$$

Proof. Assume that G or H is not bipartite. Then $G \square H$ is not bipartite and $G \square H$ contains an induced subgraph isomorphic to $K_2 \square C_n$ for some odd n . By Lemmas 2.6 and Lemma 3.2, we have $\bar{\tau}(G \square H) = 0$.

Suppose G and H are bipartite. Now $G \square H$ is bipartite and hence $\bar{\tau}(G \square H) = 2$ by Theorem 2.7. Furthermore, $\bar{h}(G \square H)$ is 1 if both G and H are reflexible; 2 otherwise. \square

Let v be a cut vertex of a graph G . Then v is a sink or a source in every transitive digraph having G as its underlying graph. Let G and H be two graphs. For two vertices $u \in V(G)$ and $v \in V(H)$, a graph $G *_{u=v} H$ is obtained from G and H by identifying the vertices u and v . We call it *the amalgamation of G and H along the vertices u and v* . We note that u (or v) is a cut vertex of $G *_{u=v} H$. For a graph G and a vertex $u \in V(G)$, let $\bar{\tau}_{si}(G, u)$ ($\bar{\tau}_{so}(G, u)$, respectively) be the number of topologies having G as its underlying graph and v as a sink(source, respectively) in the digraphs corresponding them. Similarly, we define $\bar{h}_{si}(G, u)$ and $\bar{h}_{so}(G, u)$. For any transitive digraph D , if we change direction of every directed edge in D , then the resulting digraph is also a transitive digraph. Moreover the correspondence is bijective and hence $\bar{\tau}_{si}(G, u) = \bar{\tau}_{so}(G, u)$ and $\bar{h}_{si}(G, u) = \bar{h}_{so}(G, u)$. Note that for any preorder R , the digraph obtained by changing direction of every edge in $D(R)$ corresponds to the topology composed of all closed sets in $\mathcal{T}(R)$.

Now the following lemma comes from the fact that v is a cut vertex of $G *_{u=v} H$.

Lemma 3.4. *Let G and H be two graphs.*

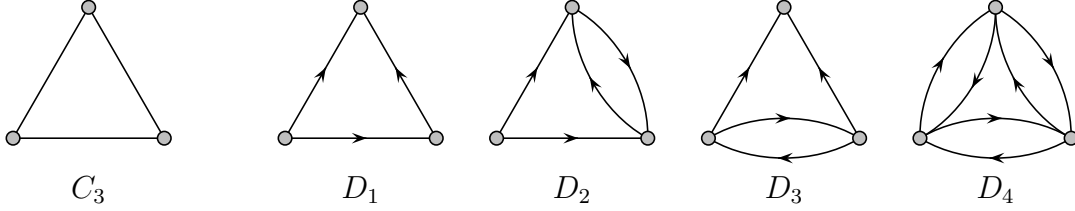


FIGURE 1. C_3 and four nonisomorphic transitive digraphs whose underlying graph is C_3 .

- (a) If v is a cut vertex of G , then v is sink or source in every transitive digraph having G as its underlying graph and hence $\bar{\tau}(G) = 2\bar{\tau}_{si}(G, v)$. Moreover, if v is the unique cut vertex, then $\bar{h}(G) = 2\bar{h}_{si}(G, v)$.
- (b) If $u \in V(G)$ and $v \in V(H)$, then $\bar{\tau}(G *_u H) = 2\bar{\tau}_{si}(G, u)\bar{\tau}_{si}(H, v)$. Moreover, both either G and H have no cut vertices, then

$$\bar{h}(G *_u H) = \begin{cases} (\bar{h}_{si}(G, u) + 1) \bar{h}_{si}(G, u) & \text{if } u \simeq_f v, \\ 2\bar{h}_{si}(G, u) \bar{h}_{si}(H, v) & \text{otherwise,} \end{cases}$$

where $u \simeq_f v$ means that there exists a graph isomorphism $f : G \rightarrow H$ such that $f(u) = v$.

From Theorem 2.4 and Lemma 3.1, we can see that the computation of $\bar{\tau}(n)$ and $\bar{h}(n)$ can be completed if we can compute $\bar{\tau}(G)$ and $\bar{h}(G)$ for any connected graph with n vertices. In the next section, we compute $\bar{\tau}(G)$ and $\bar{h}(G)$ for some special classes of connected graphs.

4. TOPOLOGIES HAVING A FIXED UNDERLYING GRAPH

In this section, we compute $\bar{\tau}(G)$ and $\bar{h}(G)$ when G is a cycle, a wheel or a complete graph.

First, we aim to compute $\bar{\tau}(C_n)$ and $\bar{h}(C_n)$ for a natural number $n \geq 3$. Notice that C_3 is the complete graph K_3 on three vertices. There are 4 nonisomorphic transitive digraphs whose underlying graph is K_3 as illustrated in Figure 1 and the following is the list of all representatives of them; $D_1 = \{1|2|3\}$, $D_2 = \{1|2, 3\}$, $D_3 = \{1, 2|3\}$, $D_4 = \{1, 2, 3\}$, where $D_3 = \{1, 2|3\}$ stands for the digraph with vertex set $\{1, 2, 3\}$ and arc set $\{12, 21, 13, 23\}$. Hence $\bar{h}(C_3) = 4$. For convenience, let α_i be the number of digraphs that are isomorphic to D_i , Then $\alpha_1 = 4! = 6$, $\alpha_2 = \alpha_3 = \frac{3!}{2!} = 3$, $\alpha_4 = \frac{3!}{3!} = 1$. Hence, $\bar{\tau}(C_3) = \sum_{i=1}^4 \alpha_i = 6 + 6 + 1 = 13$. Since there is no transitive digraphs whose underlying graph is C_n when n is odd greater than 3.

Now, the following corollary comes from Lemma 2.6, Theorem 2.7 and the fact that every cycle of even length is reflexible.

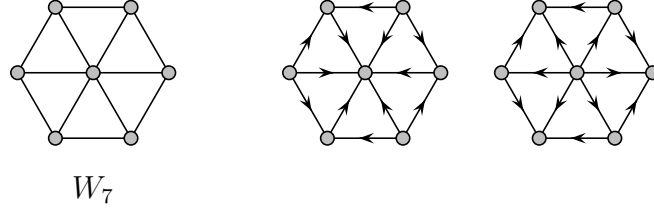


FIGURE 2. W_7 and two nonisomorphic transitive digraphs whose underlying graph is W_7 .

Corollary 4.1. *For a natural number $n \geq 3$,*

$$\bar{\tau}(C_n) = \begin{cases} 13 & \text{if } n = 3, \\ 0 & \text{if } n \text{ is odd and } n \geq 5, \\ 2 & \text{otherwise,} \end{cases}$$

and

$$\bar{h}(C_n) = \begin{cases} 4 & \text{if } n = 3, \\ 0 & \text{if } n \text{ is odd and } n \geq 5, \\ 1 & \text{otherwise.} \end{cases}$$

Next, we will compute $\bar{\tau}(G)$ when G is the wheel graph. For a natural number $n \geq 4$, the wheel W_n is a graph with n vertices which contains a cycle C_{n-1} as an induced subgraph, and every vertex in the cycle is adjacent to one other vertex. Note that W_4 is the complete graph K_4 on 4 vertices. A wheel graph W_7 and 4 nonisomorphic transitive digraphs whose underlying graph is W_7 is given in Figure 2. There are 8 nonisomorphic transitive digraphs whose underlying graph is K_4 and the following is the list of all representatives of them; $D_1 = \{1|2|3|4\}$, $D_2 = \{1|2|3, 4\}$, $D_3 = \{1|2, 3|4\}$, $D_4 = \{1, 2|3|4\}$, $D_5 = \{1|2, 3, 4\}$, $D_6 = \{1, 2, 3|4\}$, $D_7 = \{1, 2|3, 4\}$, and $D_8 = \{1, 2, 3, 4\}$. For convenience, let β_i be the number of digraphs that are isomorphic to D_i , Then $\beta_1 = 24$, $\beta_2 = \beta_3 = \beta_4 = 12$, $\beta_5 = \beta_6 = 4$, $\beta_7 = 6$, and $\beta_8 = 1$. Hence $\bar{\tau}(K_4) = \sum_{i=1}^8 \beta_i = 24 + 36 + 8 + 6 + 1 = 75$. Now, the following comes from Lemma 2.6 and a simple computation.

Theorem 4.2. *For a natural number $n \geq 4$, let W_n be the wheel graph. Then we have*

$$\bar{\tau}(W_n) = \begin{cases} 4 & \text{if } n \text{ is odd and } n \geq 7, \\ 0 & \text{if } n \text{ is even and } n \geq 6, \\ 75 & \text{if } n = 4, \\ 8 & \text{if } n = 5, \end{cases}$$

and

$$\bar{h}(W_n) = \begin{cases} 2 & n \text{ is odd and } n \geq 7, \\ 0 & n \text{ is even and } n \geq 6, \\ 8 & \text{if } n = 4, \\ 4 & \text{if } n = 5. \end{cases}$$

Finally, we will compute $\bar{\tau}(K_n)$ and $\bar{h}(K_n)$ for the complete graph K_n on n vertices.

Theorem 4.3. *For a natural number n , we have*

$$\bar{\tau}(K_n) = \sum_{k=1}^n \text{Surj}(n, k) = \sum_{k=1}^n S(n, k)k! \quad \text{and} \quad \bar{h}(K_n) = 2^{n-1},$$

where $\text{Surj}(n, k)$ is the number of surjections from N_n to N_k and $S(n, k)$ is the number of ways of partitions of N_n into exactly k nonempty parts which is known as a Stirling number the second kind.

Proof. Let R be a preorder on N_n whose underlying graph is the complete graph K_n . Then $(i, j) \in R$ or $(j, i) \in R$ for any two distinct elements i and j in N_n . We define another relation $E(R)$ on N_n by (x, y) in $E(R)$ if and only if both (x, y) and (y, x) are in R . Then $E(R)$ is an equivalence relation on N_n . Let f_R be a relation on $N_n/E(R)$ defined by $([x], [y]) \in \tilde{R}$ if and only if $(x, y) \in R$ for some $x \in [x]$ and $y \in [y]$. By transitivity, if $(x, y) \in R$ for some $x \in [x]$ and $y \in [y]$ then $(x', y') \in R$ for all $x' \in [x]$ and $y' \in [y]$. Now \tilde{R} is a total order. For convenience, let $N_n/E(R) = \{[i_1], [i_2], \dots, [i_k]\}$ and $([i_s], [i_t]) \in \tilde{R}$ if and only if $s \leq t$. We define $f_R : N_n \rightarrow N_k$ by $f_R(i) = s$ if $(i, i_s) \in E(R)$. Then f_R is a surjection. Conversely, for a given surjection $f : N_n \rightarrow N_k$ we define a relation R_f on N_n by $(i, j) \in R_f$ if and only if $f(i) \leq f(j)$. Then R_f is a preorder on N_n whose underlying graph is K_n . Since $R_{f_R} = R$ and $f_{R_f} = f$, the correspondence is one-to-one and hence $\bar{\tau}(K_n) = \sum_{k=1}^n \text{Surj}(n, k)$. Since $\text{Surj}(n, k) = S(n, k)k!$, $\bar{\tau}(K_n) = \sum_{k=1}^n \text{Surj}(n, k) = \sum_{k=1}^n S(n, k)k!$.

For a proof of the second equation, let $f : N_n \rightarrow N_h$ and $g : N_n \rightarrow N_k$ be two surjections. Then $\mathcal{T}(R_f)$ and $\mathcal{T}(R_g)$ are equivalent if and only if $h = k$ and $|f^{-1}(i)| = |g^{-1}(i)|$ for all $i = 1, 2, \dots, h = k$ (by Lemma 2.3). For a k -tuple (n_1, n_2, \dots, n_k) of natural numbers such that $n = n_1 + n_2 + \dots + n_k$, we define a surjection $\varphi : N_n \rightarrow N_k$ such that $\varphi^{-1}(1) = \{1, 2, \dots, n_1\}$ and

$$\varphi^{-1}(i) = \left\{ \left(\sum_{t=1}^{i-1} n_t \right) + 1, \left(\sum_{t=1}^{i-1} n_t \right) + 2, \dots, \left(\sum_{t=1}^{i-1} n_t \right) + n_i - 1, \left(\sum_{t=1}^i n_t \right) \right\}$$

for each $i = 2, \dots, k$. Then the topology corresponding to φ is a representative of the equivalence class of topologies corresponding to all surjection f satisfying $|f^{-1}(i)| = n_i$ for all $i = 1, 2, \dots, k$. It is clear that two different k -tuples represent two different equivalence topologies. So, the number of equivalence classes of topologies corresponding to the set of all surjections from $N_n \rightarrow N_k$ is equal to the number of ways to choose $k - 1$ positions among $n - 1$ positions between the n numbers $1, 2, \dots, n$.

Hence we have $\bar{h}(K_n) = \sum_{k=1}^n \binom{n-1}{k-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} = 2^{n-1}$. \square

5. FURTHER REMARKS

We already know that $\bar{\tau}(1) = 1$, $\bar{h}(1) = 1$, $\bar{\tau}(2) = 4$ and $\bar{h}(2) = 3$. In order to compute $\bar{\tau}(3)$ and $\bar{h}(3)$, we list all representatives of isomorphism classes of graphs on 3 vertices as follows; the null graph \mathcal{N}_3 , H , the path P_2 of length 2, and the complete graph K_3 , where H is the disjoint union of K_2 and the null graph \mathcal{N}_1 . It

is not hard to show that $\bar{\tau}(\mathcal{N}_3) = 1$, $\bar{h}(\mathcal{N}_3) = 1$, $\bar{\tau}(H) = 3$, $\bar{h}(H) = 2$, $\bar{\tau}(P_2) = 2$, $\bar{h}(P_2) = 2$, $\bar{\tau}(K_3) = 13$, and $\bar{h}(K_3) = 4$. Now, It comes from Theorem 2.4 that $\bar{\tau}(3) = 1 + 3 \times 3 + 2 \times 3 + 13 = 29$ and $\bar{h}(3) = 1 + 2 + 2 + 4 = 9$. Similarly, we can see that $\bar{\tau}(4) = 355$ and $\bar{h}(4) = 33$.

At last, we conclude the article by proposing several further research problems as follows.

Problem 5.1. *Let G, H be graphs and K_n be a complete graph with n vertices.*

- (a) *Compute $\bar{\tau}(G)$ and $\bar{h}(G)$ for arbitrary graph G .*
- (b) *Compute $\bar{\tau}(G \odot H)$ and $\bar{h}(G \odot H)$ for any two distinct graphs and any binary operation \odot on the set of graphs.*
- (c) *Find an explicit numerical formula to compute $\bar{\tau}(K_n)$.*
- (d) *Find an explicit numerical formula to compute $\bar{\tau}(n)$ and $\bar{h}(n)$.*

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