ON AN IDEAL OF MULTISYMMETRIC POLYNOMIALS ASSOCIATED WITH PERFECT CUBOIDS.

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Abstract. A perfect Euler cuboid is a rectangular parallelepiped with integer edges, with integer face diagonals, and with integer space diagonal as well. Finding such parallelepipeds or proving their non-existence is an old unsolved mathematical problem. Algebraically the problem is described by a system of Diophantine equations. Symmetry approach to the cuboid problem is based on the natural S_3 symmetry of its Diophantine equations. Factorizing these equations with respect to their S_3 symmetry, one gets some certain ideal within the ring of multisymmetric polynomials. In the present paper this ideal is completely calculated and presented through its basis.

1. Introduction.

The search for perfect cuboids extends from now back to the year of 1719 (see [1–39]), though one needs only to solve a very small system of Diophantine equations with respect to seven integer variables x_1, x_2, x_3, d_1, d_2, d_3, and L:

$$\begin{aligned} &(x_1)^2+(x_2)^2-(d_3)^2=0, && (d_3)^2+(x_3)^2-L^2=0,\\ &(x_2)^2+(x_3)^2-(d_1)^2=0, && (d_1)^2+(x_1)^2-L^2=0,\\ &(x_3)^2+(x_1)^2-(d_2)^2=0, && (d_2)^2+(x_2)^2-L^2=0. \end{aligned} \tag{1.1}$$

Here x_1, x_2, x_3 are the edges of a cuboid and d_1, d_2, d_3 are its face diagonals, while L is its space diagonal. Actually the number of the equations (1.1) can be reduced from six to four since the equations of the right column in (1.1) are equivalent to one equation $(x_1)^2+(x_2)^2+(x_3)^2=L^2$.

Recently in [40] the equations (1.1) were reduced to a single Diophantine equation with respect to four especially introduced parameters a, b, c, and u. On the base of this equation in [41] three cuboid conjectures were formulated. These conjectures were studied in [42–44]. However, they are not yet proved.

In [45] another approach to the equations (1.1) was tested. It is based on the intrinsic S_3 symmetry of the equations (1.1). Indeed, if $\sigma\in S_3$, then we can write

$$\sigma(x_i)=x_{\sigma i},\qquad \sigma(d_i)=d_{\sigma i},\qquad \sigma(L)=L. \tag{1.2}$$

2000 *Mathematics Subject Classification.* 11D41, 11D72, 13A50, 13F20, 13P10.

Each transformation $\sigma \in S_3$ permutes the equations (1.1), but the system in whole remains unchanged. Factor equations are produced from (1.1) by introducing new variables which are unchanged under the transformations (1.2). In [45] such variables were defined as values of elementary multisymmetric polynomials:

$$\begin{aligned} e_{[1,0]} &= x_1 + x_2 + x_3, \\ e_{[2,0]} &= x_1\,x_2 + x_2\,x_3 + x_3\,x_1, \\ e_{[3,0]} &= x_1\,x_2\,x_3, \end{aligned} \tag{1.3}$$

$$\begin{aligned} e_{[0,1]} &= d_1 + d_2 + d_3, \\ e_{[0,2]} &= d_1\,d_2 + d_2\,d_3 + d_3\,d_1, \\ e_{[0,3]} &= d_1\,d_2\,d_3, \end{aligned} \tag{1.4}$$

$$\begin{aligned} e_{[2,1]} &= x_1\,x_2\,d_3 + x_2\,x_3\,d_1 + x_3\,x_1\,d_2, \\ e_{[1,1]} &= x_1\,d_2 + d_1\,x_2 + x_2\,d_3 + d_2\,x_3 + x_3\,d_1 + d_3\,x_1, \\ e_{[1,2]} &= x_1\,d_2\,d_3 + x_2\,d_3\,d_1 + x_3\,d_1\,d_2. \end{aligned} \tag{1.5}$$

The polynomials (1.3) coincide with regular elementary symmetric polynomials in x_1, x_2, x_3 (see [46]). The polynomials (1.4) coincide with regular elementary symmetric polynomials in d_1, d_2, d_3. As for the polynomials (1.5), they are actually multisymmetric, i. e. they depend on double set of variables.

General multisymmetric polynomials, which are also known as vector symmetric polynomials, diagonally symmetric polynomials, McMahon polynomials etc, were initially studied in [47–53] (see also later publications [54–67]).

A general multisymmetric polynomial in our case is defined as an element of the ring $\mathbb{Q}[x_1, x_2, x_2, d_1, d_2, d_3, L]$ invariant with respect to the transformations (1.2). The variables x_1, x_2, x_3 and d_1, d_2, d_3 are usually arranged into a matrix:

$$M = \begin{Vmatrix} x_1 & x_2 & x_3 \\ d_1 & d_2 & d_3 \end{Vmatrix}. \tag{1.6}$$

Due to (1.2) the group S_3 act upon the matrix (1.6) by permuting its columns. The polynomials from $\mathbb{Q}[x_1, x_2, x_2, d_1, d_2, d_3, L]$ invariant with respect this action of S_3 constitute a ring[1]. We denote this ring through $\operatorname{Sym}\mathbb{Q}[M, L]$.

Let's denote through p_1, p_2, p_3, p_4, p_5, p_6 the left hand sides of the cuboid equations (1.1). Then we have the following six polynomials:

$$\begin{aligned} p_1 &= (x_1)^2 + (x_2)^2 - (d_3)^2, & p_4 &= (d_3)^2 + (x_3)^2 - L^2, \\ p_2 &= (x_2)^2 + (x_3)^2 - (d_1)^2, & p_5 &= (d_1)^2 + (x_1)^2 - L^2, \\ p_3 &= (x_3)^2 + (x_1)^2 - (d_2)^2, & p_6 &= (d_2)^2 + (x_2)^2 - L^2. \end{aligned} \tag{1.7}$$

The polynomials (1.7) generate an ideal in the ring $\mathbb{Q}[x_1, x_2, x_2, d_1, d_2, d_3, L]$:

$$I = \bigl\langle p_1, p_2, p_3, p_4, p_5, p_6 \bigr\rangle. \tag{1.8}$$

[1] Actually both rings $\mathbb{Q}[x_1, x_2, x_2, d_1, d_2, d_3, L]$ and $\operatorname{Sym}\mathbb{Q}[M, L]$ are algebras over the field of rational numbers $\mathbb{Q}$.

The intersection of the ideal (1.8) with the subring $\operatorname{Sym}\mathbb{Q}[M,L]$ of the polynomial ring $\mathbb{Q}[x_1,x_2,x_2,d_1,d_2,d_3,L]$ is an ideal in $\operatorname{Sym}\mathbb{Q}[M,L]$:

$$I_{\text{sym}} = I \cap \operatorname{Sym}\mathbb{Q}[M,L]. \tag{1.9}$$

Definition 1.1. A polynomial equation $p(x_1,x_2,x_2,d_1,d_2,d_3,L)=0$ with the polynomial $p \in I_{\text{sym}}$ is called a *factor equation* of the cuboid equations (1.1) with respect to their S_3 symmetry.

The main goal of this paper is to describe the ideal (1.9) in the ring of multisymmetric polynomials by calculating a finite basis of this ideal.

2. The substitution homomorphism.

Let $\mathbb{Q}[E,L]=\mathbb{Q}[E_{10},E_{20},E_{30},E_{01},E_{02},E_{03},E_{21},E_{11},E_{12},L]$ be a polynomial ring with ten independent variables. If $q \in \mathbb{Q}[E,L]$, then substituting the elementary multisymmetric polynomials (1.3), (1.4), and (1.5) for E_{10}, E_{20}, E_{30}, E_{01}, E_{02}, E_{03}, E_{21}, E_{11}, E_{12} into the arguments of q, we get a polynomial $p \in \operatorname{Sym}\mathbb{Q}[M,L]$. This means that we have a mapping

$$\varphi\colon\ \mathbb{Q}[E,L] \longrightarrow \operatorname{Sym}\mathbb{Q}[M,L]. \tag{2.1}$$

It is easy to see that the mapping (2.1) is a ring homomorphism. Such a homomorphism is called a *substitution homomorphism*.

Theorem 2.1. *The elementary multisymmetric polynomials* (1.3)*,* (1.4)*, and* (1.5) *generate the ring of all multisymmetric polynomials, i. e. each multisymmetric polynomial $p \in \operatorname{Sym}\mathbb{Q}[M,L]$ can be expressed as a polynomial with rational coefficients through these elementary multisymmetric polynomials.*

The theorem 2.1 is known as the fundamental theorem for elementary multisymmetric polynomials. Its proof can be found in [53]. The theorem 2.1 means that the mapping (2.1) is surjective. Unfortunately the elementary multisymmetric polynomials (1.3), (1.4), and (1.5) are not algebraically independent over $\mathbb{Q}$. For this reason the homomorphism (2.1) is not bijective. It has a nonzero kernel:

$$\operatorname{Ker}\varphi = K \neq \{0\}. \tag{2.2}$$

The kernel (2.2) is an ideal of the ring $\mathbb{Q}[E,L]$. According to Hilbert's basis theorem (see [68] and [69]) each ideal of the $\mathbb{Q}[E,L]$ is finitely generated. This means that

$$K = \bigl\langle q_1,\,\ldots\,,\,q_n\bigr\rangle. \tag{2.3}$$

At present time I know seven polynomials belonging to the ideal (2.3). They are found by means of direct calculations. Here is the first of these seven polynomials:

$$\begin{aligned} q_1 &= 4\,E_{01}\,E_{02}\,E_{20} - E_{02}\,E_{10}^2\,E_{01} - E_{01}^3\,E_{20} + {}\\ &+ E_{10}\,E_{11}\,E_{01}^2 - E_{11}^2\,E_{01} - 2\,E_{10}\,E_{01}\,E_{12} + 3\,E_{03}\,E_{10}^2 - {}\\ &- 9\,E_{03}\,E_{20} - 3\,E_{21}\,E_{02} + E_{21}\,E_{01}^2 + 3\,E_{11}\,E_{12}, \end{aligned} \tag{2.4}$$

The other six polynomials are given by the following formulas:

$$\begin{gathered}q_2=4\,E_{10}\,E_{20}\,E_{02}-E_{20}\,E_{01}^2\,E_{10}-E_{10}^3\,E_{02}+\\+E_{01}\,E_{11}\,E_{10}^2-E_{11}^2\,E_{10}-2\,E_{01}\,E_{10}\,E_{21}+3\,E_{30}\,E_{01}^2-\\-9\,E_{30}\,E_{02}-3\,E_{12}\,E_{20}+E_{12}\,E_{10}^2+3\,E_{11}\,E_{21},\end{gathered}\tag{2.5}$$

$$\begin{gathered}q_3=9\,E_{21}\,E_{12}-E_{01}^2\,E_{10}\,E_{21}-6\,E_{10}\,E_{11}\,E_{12}-6\,E_{01}\,E_{12}\,E_{20}+\\+5\,E_{01}\,E_{10}^2\,E_{12}-3\,E_{11}^3+7\,E_{10}\,E_{11}^2\,E_{01}+12\,E_{11}\,E_{20}\,E_{02}-\\-3\,E_{01}^2\,E_{11}\,E_{20}-3\,E_{02}\,E_{10}^2\,E_{11}-4\,E_{01}^2\,E_{10}^2\,E_{11}-81\,E_{03}\,E_{30}+\\+18\,E_{01}\,E_{02}\,E_{30}-3\,E_{01}^3\,E_{30}+36\,E_{20}\,E_{10}\,E_{03}-9\,E_{03}\,E_{10}^3-\\-16\,E_{01}\,E_{02}\,E_{20}\,E_{10}+4\,E_{01}^3\,E_{10}\,E_{20}+4\,E_{01}\,E_{10}^3\,E_{02},\end{gathered}\tag{2.6}$$

$$\begin{gathered}q_4=3\,E_{01}\,E_{21}^2-2\,E_{01}^2\,E_{21}\,E_{20}-9\,E_{01}\,E_{12}\,E_{30}+\\+E_{10}\,E_{12}\,E_{01}\,E_{20}-E_{11}^2\,E_{20}\,E_{01}+3\,E_{01}^2\,E_{30}\,E_{11}+E_{11}\,E_{20}\,E_{01}^2\,E_{10}-\\-3\,E_{01}\,E_{30}\,E_{02}\,E_{10}+4\,E_{01}\,E_{20}^2\,E_{02}-E_{01}^3\,E_{20}^2-E_{01}\,E_{20}\,E_{10}^2\,E_{02},\end{gathered}\tag{2.7}$$

$$\begin{gathered}q_5=-27\,E_{10}\,E_{21}\,E_{03}+E_{10}\,E_{01}^3\,E_{21}+9\,E_{10}\,E_{12}^2-E_{11}^2\,E_{10}\,E_{01}^2-\\-6\,E_{02}\,E_{12}\,E_{10}^2-2\,E_{01}^2\,E_{12}\,E_{10}^2-3\,E_{02}\,E_{11}^2\,E_{10}-E_{01}^2\,E_{10}^3\,E_{02}+\\+9\,E_{11}\,E_{03}\,E_{10}^2+3\,E_{01}\,E_{02}\,E_{10}^2\,E_{11}+E_{01}^3\,E_{11}\,E_{10}^2-3\,E_{10}^3\,E_{02}^2+\\+3\,E_{10}^3\,E_{01}\,E_{03}+12\,E_{10}\,E_{20}\,E_{02}^2+E_{02}\,E_{20}\,E_{01}^2\,E_{10}-\\-E_{01}^4\,E_{20}\,E_{10}-18\,E_{10}\,E_{01}\,E_{03}\,E_{20}+3\,E_{11}\,E_{01}\,E_{10}\,E_{12},\end{gathered}\tag{2.8}$$

$$\begin{gathered}q_6=-27\,E_{03}\,E_{21}+E_{21}\,E_{01}^3+9\,E_{12}^2+3\,E_{12}\,E_{01}\,E_{11}-\\-2\,E_{01}^2\,E_{10}\,E_{12}-3\,E_{02}\,E_{11}^2-E_{01}^2\,E_{11}^2+9\,E_{03}\,E_{11}\,E_{10}-\\-3\,E_{10}^2\,E_{02}^2+3\,E_{01}\,E_{02}\,E_{11}\,E_{10}+E_{01}^3\,E_{11}\,E_{10}-\\-18\,E_{20}\,E_{01}\,E_{03}+3\,E_{03}\,E_{01}\,E_{10}^2-6\,E_{02}\,E_{10}\,E_{12}-\\-E_{01}^4\,E_{20}+12\,E_{02}^2\,E_{20}+E_{01}^2\,E_{02}\,E_{20}-E_{01}^2\,E_{10}^2\,E_{02},\end{gathered}\tag{2.9}$$

$$\begin{gathered}q_7=3\,E_{21}^2-2\,E_{20}\,E_{01}\,E_{21}-9\,E_{30}\,E_{12}+E_{10}\,E_{12}\,E_{20}-\\-E_{20}\,E_{11}^2+3\,E_{30}\,E_{11}\,E_{01}+E_{10}\,E_{20}\,E_{11}\,E_{01}-\\-3\,E_{02}\,E_{10}\,E_{30}+4\,E_{20}^2\,E_{02}-E_{01}^2\,E_{20}^2-E_{10}^2\,E_{20}\,E_{02}.\end{gathered}\tag{2.10}$$

Theorem 2.2. *Seven polynomials* (2.4), (2.5), (2.6), (2.7), (2.8), (2.9), (2.10) *constitute a basis for the ideal K being the kernel of the homomorphism* (2.1).

Proving the theorem 2.1 is an algorithmically solvable problem. For this purpose the Gröbner bases technique should be applied to the ring

$$\mathbb{Q}[x_1,x_2,x_3,d_1,d_2,d_3,E_{10},E_{20},E_{30},E_{01},E_{02},E_{03},E_{21},E_{11},E_{12},L].\tag{2.11}$$

Gröbner bases are associated with monomial orderings (see [69] or [70]). The lexicographic ordering (**lex**) is the most simple one. It is defined through some ordering of variables. In the case of the ring (2.11) one should choose the ordering

$$\begin{gathered} x_1 > x_2 > x_3 > d_1 > d_2 > d_3 > E_{21} > E_{12} > E_{11} > \\ > E_{30} > E_{03} > E_{20} > E_{02} > E_{10} > E_{01} > L. \end{gathered} \tag{2.12}$$

Due to the lexicographic ordering based on (2.12) each polynomial r of the ring (2.11) gains its leading term $\mathrm{LT}(r)$ with respect to this **lex**-ordering.

Definition 2.1. For each ideal I of a polynomial ring the ideal $\mathrm{LT}(I)$ is generated by leading terms of all polynomials of this ideal.

Definition 2.2. A basis $r_1, \ldots, r_s$ of an ideal I is called a *Gröbner basis* if the leading terms $\mathrm{LT}(r_1), \ldots, \mathrm{LT}(r_s)$ generate the ideal $\mathrm{LT}(I)$.

An algorithm for computing Gröbner bases was first published by Bruno Buchberger in 1965 in his PhD thesis [71]. Wolfgang Gröbner was Buchberger's thesis adviser. Similar algorithms were developed for local rings by Heisuke Hironaka in 1964 (see [72] and [73]) and for free Lie algebras by A. I. Shirshov in 1962 (see [74]).

The ring (2.11) comprises both of the rings $\mathbb{Q}[M, L]$ and $\mathbb{Q}[E, L]$. For this reason one can consider the following nine polynomials in this ring:

$$\begin{aligned} &r_1 = E_{10} - e_{[1,0]}, && r_2 = E_{20} - e_{[2,0]}, && r_3 = E_{30} - e_{[3,0]}, \\ &r_4 = E_{01} - e_{[0,1]}, && r_5 = E_{02} - e_{[0,2]}, && r_6 = E_{03} - e_{[0,3]}, \\ &r_7 = E_{21} - e_{[2,1]}, && r_8 = E_{11} - e_{[1,1]}, && r_9 = E_{12} - e_{[1,2]}. \end{aligned} \tag{2.13}$$

The polynomials (2.13) are constructed with the use of the elementary multisymmetric polynomials (1.3), (1.4), and (1.5). They generate the ideal

$$K_0 = \langle r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9 \rangle \tag{2.14}$$

of the ring (2.11). The kernel of the homomorphism (2.1) in (2.2) coincides with the 6-th elimination ideal for the ideal (2.14) with respect to the ordering (2.12):

$$K = \operatorname{Ker} \varphi = K_6 = K_0 \cap \mathbb{Q}[E, L]. \tag{2.15}$$

Definition 2.3. Let I be an ideal in the polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$. Then the intersection of the ideal I with the subring $\mathbb{Q}[x_{k+1}, \ldots, x_n] \subset \mathbb{Q}[x_1, \ldots, x_n]$ is called the *k-th elimination ideal* of the ideal I:

$$I_k = I \cap \mathbb{Q}[x_{k+1}, \ldots, x_n]. \tag{2.16}$$

Theorem 2.3 (elimination theorem). *Let I be an ideal in the polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$ and let $G = \{g_1, \ldots, g_s\}$ be its Gröbner basis with respect to the* **lex**-*ordering with $x_1 > x_2 > \ldots > x_n$. Then for any $0 \leqslant k \leqslant n$ the intersection*

$$G_k = G \cap \mathbb{Q}[x_{k+1}, \ldots, x_n] \tag{2.17}$$

is a Gröbner basis for the k-th elimination ideal I_k.

The definition 2.3 and the formula (2.16) explain the formula (2.15), while the theorem 2.3 along with the formula (2.17) yields an algorithm for calculating a basis for the ideal (2.2) and thus for proving the theorem 2.2. The proof of the theorem 2.3 can be found in [69].

The algorithm provided by the theorem 2.3 is already implemented in many packages for symbolic computations. For instance, the Maxima package (version 5.22.1) contains the Gröbner subpackage (revision 1.6) with the command

$$\texttt{poly\_elimination\_ideal(L,k,V)}, \tag{2.18}$$

where L is a list of polynomials, k is the integer number from (2.17), and V is a list of variables. Due to (2.12), (2.13), and (2.15) in my case I have $k = 6$ and

$$\begin{aligned}
&L = [r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9],\\
&V = [x_1, x_2, x_3, d_1, d_2, d_3, E_{21}, E_{12}, E_{11}, E_{30}, E_{03}, E_{20}, E_{02}, E_{10}, E_{01}, L].
\end{aligned}$$

After running the command (2.18) with the above parameters on a machine with dual core Prescott 2.8E Intel Pentium-4 processor and with 500 megabytes RAM on board I have got a Gröbner basis G_K of the ideal K consisting of 14 polynomials. Some of them are rather huge for to typeset them here. Using this Gröbner basis, I have verified that the polynomials (2.4), (2.5), (2.6), (2.7), (2.8), (2.9), and (2.10) do actually belong to the kernel of the homomorphism (2.1).

Conversely, the polynomials (2.4), (2.5), (2.6), (2.7), (2.8), (2.9), and (2.10) generate their own Gröbner basis G_Q. Using this second Gröbner basis G_Q, I have tested each polynomial of the first Gröbner basis G_K and have found that all of these polynomials belong to the ideal $Q = \langle q_1, q_2, q_3, q_4, q_5, q_6, q_7\rangle$ generated by the polynomials (2.4), (2.5), (2.6), (2.7), (2.8), (2.9), and (2.10). This result means that the ideals $K = \operatorname{Ker}\varphi$ and Q do coincide, i. e. I have got a computer aided proof of the theorem 2.2.

3. The fine structure of the ideal I_{sym}.

The ideal I producing I_{sym} in (1.9) is generated by six polynomials (1.7) in (1.8). Actually, the number of generating polynomials of the ideal I can be reduced from six to four. Indeed, we can write

$$I = \langle p_0, p_1, p_2, p_3\rangle, \tag{3.1}$$

where p_0 is a symmetric polynomial given by the formula

$$p_0 = (x_1)^2 + (x_2)^2 + (x_3)^2 - L^2. \tag{3.2}$$

Due to the relationship (3.1) each polynomial $p \in I_{\text{sym}}$ is written as

$$p = \alpha_0\, p_0 + \sum_{i=1}^{3} \alpha_i\, p_i, \tag{3.3}$$

where $\alpha_i \in \mathbb{Q}[M, L]$. Since p is a multisymmetric polynomial, it should be invariant with respect to the symmetrization operator S defined by the formula

$$S(p) = \sum_{\sigma\in S_3} \frac{\sigma^{-1}(p)}{6}. \tag{3.4}$$

The invariance of p with respect to the operator (3.4) is written as $p = S(p)$. Therefore, applying S to (3.3), we derive the formula

$$p = S(\alpha_0\, p_0) + \sum_{i=1}^{3} S(\alpha_i\, p_i). \tag{3.5}$$

Now let's recall the formulas (1.2). Applying them to the polynomials (1.7) and (3.2), we derive the analogous formulas

$$\sigma(p_i) = p_{\sigma i}, \qquad\qquad \sigma(p_0) = p_0 \tag{3.6}$$

for p_1, p_2, p_3, and p_0. Relying on (3.6) we introduce the following notations:

$$\tilde{\alpha}_0 = S(\alpha_0), \qquad\qquad \tilde{\alpha}_i = \sum_{\sigma\in S_3} \frac{\sigma^{-1}(\alpha_{\sigma i})}{6} \tag{3.7}$$

Using (3.7), we can transform the formula (3.5) as follows:

$$p = \tilde{\alpha}_0\, p_0 + \sum_{i=1}^{3} \tilde{\alpha}_i\, p_i. \tag{3.8}$$

The formula (3.8) is analogous to the formula (3.3). However, unlike the original coefficients α_1, α_2, α_3, and α_0 in (3.3), the coefficients (3.7) obey the relationships

$$\sigma(\tilde{\alpha}_i) = \tilde{\alpha}_{\sigma i}, \qquad\qquad \sigma(\tilde{\alpha}_0) = \tilde{\alpha}_0. \tag{3.9}$$

The formulas (3.8) and (3.9) mean that we have proved the following lemma.

Lemma 3.1. *Each polynomial $p \in I_{\text{sym}} = I \cap \operatorname{Sym}\mathbb{Q}[M, L]$ is given by the formula (3.3) with the coefficients $\alpha_i \in \mathbb{Q}[M, L]$ obeying the relationships*

$$\sigma(\alpha_i) = \alpha_{\sigma i}, \qquad\qquad \sigma(\alpha_0) = \alpha_0. \tag{3.10}$$

The formulas (3.10) in the lemma 3.1 are important since, applying them back to the formula (3.5) and taking into account (3.6), we derive

$$p = \alpha_0\, p_0 + 3\, S(\alpha_1\, p_1). \tag{3.11}$$

Note that $\alpha_1 \in \mathbb{Q}[M, L]$ in (3.11) is a polynomial, i. e. it is a sum of monomials:

$$\alpha_1 = \sum_{\substack{i,j,k\\ m,n,r,s}} \theta_{ijkmnrs}\; x_1^i\, x_2^j\, x_3^k\, d_1^m\, d_2^n\, d_3^r\, L^s. \tag{3.12}$$

Substituting (3.12) into the formula (3.11), we easily derive the following lemma.

Lemma 3.2. *The ideal $I_{\text{sym}} = I \cap \operatorname{Sym}\mathbb{Q}[M, L]$ of the ring $\operatorname{Sym}\mathbb{Q}[M, L]$ is generated by the polynomial p_0 and by various polynomials of the form*

$$S(p_1\, x_1^i\, x_2^j\, x_3^k\, d_1^m\, d_2^n\, d_3^r\, L^s). \tag{3.13}$$

Note that the factor L^s is invariant with respect to the operator S. It can be split out from the polynomial (3.13). Similarly, if $\mu = \min(i, j, k) > 0$ and/or $\nu = \min(m, n, r) > 0$, we can split out the invariant factors $(x_1\, x_2\, x_3)^\mu$ and/or $(d_1\, d_2\, d_3)^\nu$. As a result we modify the lemma 3.2 as follows.

Lemma 3.3. *The ideal* $I_{\mathrm{sym}} = I \cap \mathrm{Sym}\mathbb{Q}[M, L]$ *of the ring* $\mathrm{Sym}\mathbb{Q}[M, L]$ *is generated by the polynomial* p_0 *and by various polynomials of the form*

$$S(p_1\, x_1^i\, x_2^j\, x_3^k\, d_1^m\, d_2^n\, d_3^r),$$

where at least one of the nonnegative numbers i, j, k *is zero and at least one of the nonnegative numbers* m, n, r *is zero.*

The lemma 3.3 yields a basis for the ideal I_{sym}. However this basis is not finite. Getting a finite basis of the ideal I_{sym} is a little bit more tricky.

4. Partially multisymmetric polynomials.

Let's consider the formulas (3.10). The polynomial α_0 in (3.10) is multisymmetric, i. e. it is invariant with respect to the transformations (1.2) for all $\sigma \in S_3$. As for the polynomials α_1, α_2, and α_3 in (3.10), they are partially multisymmetric. The formulas (3.10) for these polynomials yield

$$\begin{aligned}
&\sigma(\alpha_1) = \alpha_1 \quad \text{if and only if} \quad \sigma 1 = 1,\\
&\sigma(\alpha_2) = \alpha_2 \quad \text{if and only if} \quad \sigma 2 = 2,\\
&\sigma(\alpha_3) = \alpha_3 \quad \text{if and only if} \quad \sigma 3 = 3.
\end{aligned} \tag{4.1}$$

The formulas (4.1) mean that the polynomials α_1, α_2, and α_3 are S_2 invariant, but they are invariant with respect to different subgroups of the group S_3 isomorphic to the group S_2. In order to describe such partially multisymmetric polynomials we split out the following three matrices from the matrix (1.6):

$$M_1 = \begin{Vmatrix} x_2 & x_3 \\ d_2 & d_3 \end{Vmatrix}, \qquad M_2 = \begin{Vmatrix} x_1 & x_3 \\ d_1 & d_3 \end{Vmatrix}, \qquad M_3 = \begin{Vmatrix} x_1 & x_2 \\ d_1 & d_2 \end{Vmatrix}. \tag{4.2}$$

Like the matrix (1.6), the matrices (4.2) can be used for producing elementary multisymmetric polynomials. Here are these polynomials:

$$\begin{aligned}
&f_{[1,0]}[1] = x_2 + x_3, && f_{[2,0]}[1] = x_2\, x_3,\\
&f_{[0,1]}[1] = d_2 + d_3, && f_{[0,1]}[1] = d_2\, d_3,\\
&f_{[1,0]}[2] = x_3 + x_1, && f_{[2,0]}[2] = x_3\, x_1,\\
&f_{[0,1]}[2] = d_3 + d_1, && f_{[0,1]}[2] = d_3\, d_1,\\
&f_{[1,0]}[3] = x_1 + x_2, && f_{[2,0]}[3] = x_1\, x_2,\\
&f_{[0,1]}[3] = d_1 + d_2, && f_{[0,1]}[3] = d_1\, d_2.
\end{aligned} \tag{4.3}$$

Apart from (4.3) there are three other elementary multisymmetric polynomials:

$$\begin{aligned}
&f_{[1,1]}[1] = x_2\, d_3 + x_3\, d_2,\\
&f_{[1,1]}[2] = x_3\, d_1 + x_1\, d_3,\\
&f_{[1,1]}[3] = x_1\, d_2 + x_2\, d_1.
\end{aligned} \tag{4.4}$$

The polynomials in (4.3) and (4.4) are subdivided into three groups depending on which matrix (4.2) is used for their production.

Like α_1, α_2, and α_3, the polynomials (4.3) and (4.4) are partially multisymmetric. They obey the following relationships very similar to (3.10):

$$
\begin{aligned}
&\sigma(f_{[1,0]}[i]) = f_{[1,0]}[\sigma i], &&\sigma(f_{[2,0]}[i]) = f_{[2,0]}[\sigma i],\\
&\sigma(f_{[0,1]}[i]) = f_{[0,1]}[\sigma i], &&\sigma(f_{[0,2]}[i]) = f_{[0,2]}[\sigma i],\\
&\sigma(f_{[1,1]}[i]) = f_{[1,1]}[\sigma i].
\end{aligned}
\tag{4.5}
$$

The polynomials (4.3) and (4.4) obey a theorem similar to the theorem 2.1.

Theorem 4.1. *The elementary multisymmetric polynomials $f_{[1,0]}$, $f_{[2,0]}$, $f_{[0,1]}$, $f_{[0,2]}$, $f_{[1,1]}$ generate the ring of all S_2 multisymmetric polynomials, i. e. each S_2 multisymmetric polynomial can be expressed as a polynomial with rational coefficients through these elementary multisymmetric polynomials.*

The theorem 4.1 is an S_2 version of the fundamental theorem on elementary multisymmetric polynomials which is formulated for the general case of S_n multisymmetric polynomials (see [53]). Applying this theorem to α_1, α_2, and α_3, we get

$$
\alpha_i = q_i(x_i, d_i, f_{[1,0]}[i], f_{[2,0]}[i], f_{[0,1]}[i], f_{[0,2]}[i], f_{[1,1]}[i], L),
\tag{4.6}
$$

where q_i is some polynomial of eight independent variables. The polynomials α_1, α_2, and α_3 are not independent. They are related to each other by means of the formulas (3.10). Therefore, applying (1.2), (3.10), and (4.5) to (4.6), we conclude that the polynomials q_i in (4.6) can be chosen so that they do coincide, i. e.

$$
q_1 = q_2 = q_3 = q(x, d, f_{[1,0]}, f_{[2,0]}, f_{[0,1]}, f_{[0,2]}, f_{[1,1]}, L)
\tag{4.7}
$$

Applying (4.7) to (4.6), we write (4.6) as follows:

$$
\alpha_i = q(x_i, d_i, f_{[1,0]}[i], f_{[2,0]}[i], f_{[0,1]}[i], f_{[0,2]}[i], f_{[1,1]}[i], L),
\tag{4.8}
$$

Now let's return back to the formulas (3.5) and (3.11). Applying (4.8) to (3.11), we get the following expression for p:

$$
p = \alpha_0\, p_0 + 3\, S(q\, p_1).
\tag{4.9}
$$

Here $q = q(x_1, d_1, f_{[1,0]}[1], f_{[2,0]}[1], f_{[0,1]}[1], f_{[0,2]}[1], f_{[1,1]}[1], L)$ and S is the symmetrization operator (3.4). The formula (4.9) applies to any polynomial $p \in I_{\text{sym}}$.

5. The module structure of the ideal I_{sym}.

Each ideal is a module over that ring for which it is an ideal. When applied to the ideal I_{sym}, this fact means that

$$
p \in I_{\text{sym}} \ \text{ implies } \ \alpha\, p \in I_{\text{sym}} \ \text{ for any } \ \alpha \in \operatorname{Sym}\mathbb{Q}[M, L].
\tag{5.1}
$$

Relying on (5.1), let us consider the the product $\alpha\, p$ for a polynomial p given by the formula (4.9). As a result we obtain the formula

$$
\alpha p = \alpha\, \alpha_0\, p_0 + 3\, \alpha\, S(q\, p_1).
\tag{5.2}
$$

Note that α in (5.2) is a multisymmetric polynomial. Therefore it goes through the symmetrization operator S as a scalar factor. This yields

$$\alpha p=\alpha\,\alpha_0\,p_0+3\,S(\alpha\,q\,p_1). \tag{5.3}$$

Comparing the formulas (5.3) and (4.9), we conclude that the multiplication by α in I_{sym} is equivalent to the transformation

$$\alpha_0\mapsto\alpha\,\alpha_0, \qquad\qquad q\mapsto\alpha\,q. \tag{5.4}$$

The polynomial α in the formulas (5.4) is expressed the through elementary multisymmetric polynomials (1.3), (1.4), and (1.5):

$$\alpha=\alpha(e_{[1,0]},e_{[2,0]},e_{[3,0]},e_{[0,1]},e_{[0,2]},e_{[0,3]},e_{[2,1]},e_{[1,1]},e_{[1,2]},L), \tag{5.5}$$

while the polynomial q in (5.4) is given by the formula

$$q=q(x_1,d_1,f_{[1,0]}[1],f_{[2,0]}[1],f_{[0,1]}[1],f_{[0,2]}[1],f_{[1,1]}[1],L) \tag{5.6}$$

Formally, the polynomials α and q depend on different sets of variables, though due to (1.3), (1.4), (1.5), (4.3), and (4.4) both sets reduce to x_1, x_2, x_3, d_1, d_2, d_3, and L. Our next goal is to study the mutual relations of arguments in (5.5) and (5.6). By means of direct calculations we derive the formulas

$$\begin{aligned} &f_{[1,0]}[1]=e_{[1,0]}-x_1, && f_{[2,0]}[1]=e_{[2,0]}-x_1\,e_{[1,0]}+x_1^2,\\ &f_{[0,1]}[1]=e_{[0,1]}-x_1, && f_{[0,2]}[1]=e_{[0,2]}-d_1\,e_{[0,1]}+d_1^2. \end{aligned} \tag{5.7}$$

The polynomial $f_{[1,1]}[1]$ is reexpressed by the formula

$$f_{[1,1]}[1]=e_{[1,1]}-d_1\,e_{[1,0]}-x_1\,e_{[0,1]}+2\,d_1\,x_1. \tag{5.8}$$

In addition to (5.7) and (5.8), there are the following four equations:

$$\begin{aligned} &x_1^3=x_1^2\,e_{[1,0]}-x_1\,e_{[2,0]}+e_{[3,0]},\\ &d_1^3=d_1^2\,e_{[0,1]}-d_1\,e_{[0,2]}+e_{[0,3]}, \end{aligned} \tag{5.9}$$

$$\begin{aligned} &d_1\,x_1^2=\frac{2\,d_1\,x_1}{3}\,e_{[1,0]}+\frac{x_1^2}{3}\,e_{[0,1]}-\frac{x_1}{3}\,e_{[1,1]}-\frac{d_1}{3}\,e_{[2,0]}+\frac{1}{3}\,e_{[2,1]},\\ &x_1\,d_1^2=\frac{2\,x_1\,d_1}{3}\,e_{[0,1]}+\frac{d_1^2}{3}\,e_{[1,0]}-\frac{d_1}{3}\,e_{[1,1]}-\frac{x_1}{3}\,e_{[0,2]}+\frac{1}{3}\,e_{[1,2]}. \end{aligned} \tag{5.10}$$

The equations (5.9) and (5.10) are easily derived by means of direct calculations with the use of the formulas (1.3), (1.4), and (1.5).

Let's substitute (5.7) and (5.8) into the arguments of the polynomial (5.6). As a result the polynomial q is expressed in the form

$$q=\tilde{q}(x_1,d_1,e_{[1,0]},e_{[2,0]},e_{[3,0]},e_{[0,1]},e_{[0,2]},e_{[0,3]},e_{[2,1]},e_{[1,1]},e_{[1,2]},L), \tag{5.11}$$

where $\tilde{q}$ is some arbitrary polynomial of twelve variables. The first formula (5.9) expresses x_1^3 through x_1^2 and x_1. Similarly, the second formula (5.9) expresses d_1^3 through d_1^2 and d_1. Therefore, without loss of generality we can assume that the order of the polynomial $\tilde{q}$ in x_1 and in d_1 is not higher than 2, i.e. the variables x_1 and d_1 enter this polynomial through the following monomials:

$$x_1^2\,d_1^2,\qquad x_1^2\,d_1,\qquad x_1\,d_1^2,\qquad x_1\,d_1,\qquad x_1^2,\qquad d_1^2,\qquad x_1,\qquad d_1,\qquad 1. \tag{5.12}$$

Due to the equations (5.10) we can exclude the monomials $x_1^2\,d_1$ and $x_1\,d_1^2$ from the above list (5.12) and write the formula (5.11) as

$$q = Q_{22}\,x_1^2\,d_1^2 + Q_{11}\,x_1\,d_1 + Q_{20}\,x_1^2 + Q_{02}\,d_1^2 + Q_{10}\,x_1 + Q_{01}\,d_1 + Q_{00}. \tag{5.13}$$

The coefficients Q_{ij} in (5.13) are produced by polynomials of ten variables:

$$Q_{ij} = Q_{ij}(e_{[1,0]}, e_{[2,0]}, e_{[3,0]}, e_{[0,1]}, e_{[0,2]}, e_{[0,3]}, e_{[2,1]}, e_{[1,1]}, e_{[1,2]}, L). \tag{5.14}$$

The values of the expressions (5.5) and (5.14) are regular multisymmetric polynomials from the ring $\operatorname{Sym}\mathbb{Q}[M, L]$, while the values of the expression (5.11) constitute a module over this ring. Due to (5.13) this module is finitely generated.

6. A basis of the ideal I_{sym}.

Now we can substitute the formula (5.13) with the coefficients (5.14) into the formula (4.9). As a result we can write (4.9) as

$$\begin{aligned} p = \alpha_0\,p_0 + Q_{22}\,S(3\,x_1^2\,d_1^2\,p_1) + Q_{11}\,S(3\,x_1\,d_1\,p_1) + Q_{20}\,S(3\,x_1^2\,p_1) +\\ + Q_{02}\,S(3\,d_1^2\,p_1) + Q_{10}\,S(3\,x_1\,p_1) + Q_{01}\,S(3\,d_1\,p_1) + Q_{00}\,S(3\,p_1), \end{aligned} \tag{6.1}$$

where p is an arbitrary polynomial from the ideal I_{sym} and α_0, Q_{22}, Q_{11}, Q_{10}, Q_{01}, Q_{00} are arbitrary polynomials from the ring $\operatorname{Sym}\mathbb{Q}[M, L]$. The formula (6.1) proves the following theorem, which is the main result of the present paper.

Theorem 6.1. *The ideal I_{sym} in the ring $\operatorname{Sym}\mathbb{Q}[M, L]$ defined by the left hand sides of the cuboid equations (1.1) through the formulas (1.7), (1.8), (1.9) is finitely generated. Eight multisymmetric polynomials*

$$\begin{aligned} &\tilde{p}_1 = p_0, && \tilde{p}_2 = S(3\,p_1),\\ &\tilde{p}_3 = S(3\,d_1\,p_1), && \tilde{p}_4 = S(3\,x_1\,p_1),\\ &\tilde{p}_5 = S(3\,x_1\,d_1\,p_1), && \tilde{p}_6 = S(3\,x_1^2\,p_1),\\ &\tilde{p}_7 = S(3\,d_1^2\,p_1), && \tilde{p}_8 = S(3\,x_1^2\,d_1^2\,p_1) \end{aligned} \tag{6.2}$$

belong to the ideal I_{sym} and constitute a basis of this ideal.

The polynomial $\tilde{p}_1 = p_0$ from the first formula (6.2) is already known in an explicit form. It is given by the formula (3.2). The polynomial p_1 used in the other formulas (6.2) is also known in an explicit form (see (1.7)). Now, applying the formula (3.4) for S, we can explicitly calculate the polynomials $\tilde{p}_2$, $\tilde{p}_3$, $\tilde{p}_4$, $\tilde{p}_5$, $\tilde{p}_6$, $\tilde{p}_7$, and $\tilde{p}_8$. Here is the formula for the polynomial $\tilde{p}_2$:

$$\tilde{p}_2 = (x_2^2 + x_3^2 - d_1^2) + (x_3^2 + x_1^2 - d_2^2) + (x_1^2 + x_2^2 - d_3^2). \tag{6.3}$$

The explicit formulas for $\tilde p_3$, $\tilde p_4$, $\tilde p_5$, $\tilde p_6$, $\tilde p_7$, and $\tilde p_8$ are listed just below:

$$\tilde p_3 = d_1\,(x_2^2+x_3^2-d_1^2)+d_2\,(x_3^2+x_1^2-d_2^2)+d_3\,(x_1^2+x_2^2-d_3^2), \tag{6.4}$$

$$\tilde p_4 = x_1\,(x_2^2+x_3^2-d_1^2)+x_2\,(x_3^2+x_1^2-d_2^2)+x_3\,(x_1^2+x_2^2-d_3^2), \tag{6.5}$$

$$\begin{aligned}\tilde p_5 = x_1\,d_1\,(x_2^2+x_3^2-d_1^2)+x_2\,d_2\,(x_3^2+x_1^2-d_2^2)\,+\\ +\,x_3\,d_3\,(x_1^2+x_2^2-d_3^2),\end{aligned} \tag{6.6}$$

$$\tilde p_6 = x_1^2\,(x_2^2+x_3^2-d_1^2)+x_2^2\,(x_3^2+x_1^2-d_2^2)+x_3^2\,(x_1^2+x_2^2-d_3^2), \tag{6.7}$$

$$\tilde p_7 = d_1^2\,(x_2^2+x_3^2-d_1^2)+d_2^2\,(x_3^2+x_1^2-d_2^2)+d_3^2\,(x_1^2+x_2^2-d_3^2), \tag{6.8}$$

$$\begin{aligned}\tilde p_8 = x_1^2\,d_1^2\,(x_2^2+x_3^2-d_1^2)+x_2^2\,d_2^2\,(x_3^2+x_1^2-d_2^2)\,+\\ +\,x_3^2\,d_3^2\,(x_1^2+x_2^2-d_3^2).\end{aligned} \tag{6.9}$$

Using the formulas (3.2), (6.3), (6.4), (6.5), (6.6), (6.7), (6.8), and (6.9), now we can write the S_3 factor equations for the cuboid equations (1.1). For this purpose it is convenient to use the polynomials p_1, p_2, and p_3 from (1.7):

$$\begin{aligned}
&x_1^2+x_2^2+x_3^2-L^2=0, && p_1+p_2+p_3=0,\\
&d_1\,p_1+d_2\,p_2+d_3\,p_3=0, && x_1\,p_1+x_2\,p_2+x_3\,p_3=0,\\
&d_1^2\,p_1+d_2^2\,p_2+d_3^2\,p_3=0, && x_1^2\,p_1+x_2^2\,p_2+x_3^2\,p_3=0,\\
&x_1\,d_1\,p_1+x_2\,d_2\,p_2\,+ && x_1^2\,d_1^2\,p_1+x_2^2\,d_2^2\,p_2\,+\\
&\qquad+\,x_3\,d_3\,p_3=0, && \qquad+\,x_3^3\,d_3^2\,p_3=0.
\end{aligned} \tag{6.10}$$

Since the polynomials $\tilde p_1$, $\tilde p_2$, $\tilde p_3$, $\tilde p_4$, $\tilde p_5$, $\tilde p_6$, $\tilde p_7$, $\tilde p_8$ constitute a basis of the ideal I_{sym}, the equations (6.10) compose a complete set of S_3 factor equations.

7. Comparison with the previously obtained factor equations.

In the previous paper [45] eight factor equations were already derived. But they were written in so-called E-form, i. e. in terms of the values of the elementary multisymmetric polynomials (1.3), (1.4), and (1.5). In order to compare the previously obtained equations from [45] with the equations (6.10) we need to convert them into xd-form by means of the mapping φ from (2.1).

Let's consider the first of the previously obtained factor equations. In its E-form this equation is written as follows (see (4.3) in [45]):

$$E_{10}^2-2\,E_{20}-L^2=0. \tag{7.1}$$

In order to apply φ to (7.1) we should substitute $E_{10}=e_{[1,0]}$, $E_{20}=e_{[2,0]}$ and then use the formulas (1.3). As a result we get the equation coinciding with the first equation in the left column of (6.10).

The second of the previously obtained factor equations is the equation (4.6) in [45]. In its E-form this equation is written as follows:

$$E_{01}^2-2\,E_{02}-2\,L^2=0. \tag{7.2}$$

Upon applying the mapping φ to (7.2) we get the equation

$$2\,(x_1^2+x_2^2+x_3^2-L^2)-(p_1+p_2+p_3)=0, \tag{7.3}$$

which is derived from the first equations in the left and right columns of (6.10).

Let's proceed to the third of the previously obtained factor equations. This is the equation (4.12) in [45]. In its E-form this equation is written as follows:

$$2\,E_{12}+6\,E_{30}-2\,E_{01}\,E_{11}+E_{10}\,E_{01}^2+3\,E_{10}\,L^2-E_{10}^3=0. \tag{7.4}$$

Upon converting to the xd-form the equation (7.4) looks like

$$\begin{gathered}e_{[1,0]}\,((p_1+p_2+p_3)-3\,(x_1^2+x_2^2+x_3^2-L^2))-\\-\,2\,(x_1\,p_1+x_2\,p_2+x_3\,p_3)=0.\end{gathered} \tag{7.5}$$

It is easy to see that (7.5) can be derived from the first and the second equations in the right column of (6.10) and from the first equation in the left column of (6.10).

The fourth of the previously obtained factor equations is the equation (4.19) in [45]. In its E-form this equation is written as follows:

$$2\,E_{21}+6\,E_{03}-2\,E_{10}\,E_{11}+E_{01}\,E_{10}^2+5\,E_{01}\,L^2-E_{01}^3=0. \tag{7.6}$$

Upon applying the mapping φ to (7.6) we get the equation

$$\begin{gathered}e_{[0,1]}\,(3\,(p_1+p_2+p_3)-5\,(x_1^2+x_2^2+x_3^2-L^2))-\\-\,2\,(d_1\,p_1+d_2\,p_2+d_3\,p_3)=0.\end{gathered} \tag{7.7}$$

The equation (7.7) can be derived from the first and the second equations in the left column of (6.10) and from the first equation in the right column of (6.10).

The fifth of the previously obtained factor equations is more complicated. It is given by the formula (5.5) in [45]. Here is its E-form:

$$\begin{gathered}8\,E_{10}\,E_{12}-8\,E_{01}\,E_{21}-8\,E_{11}^2+4\,E_{01}^2\,E_{10}^2-\\-\,E_{01}^4-3\,E_{10}^4+10\,E_{10}^2\,L^2+4\,E_{01}^2\,L^2+L^4=0.\end{gathered} \tag{7.8}$$

Upon converting to the xd-form the equation (7.8) looks like

$$\begin{gathered}18\,(x_1^2\,p_1+x_2^2\,p_2+x_3^2\,p_3)+6\,(x_1\,d_1\,p_1+x_2\,d_2\,p_2+x_3\,d_3\,p_3)-\\-\,8\,e_{[0,1]}\,(d_1\,p_1+d_2\,p_2+d_3\,p_3)-24\,e_{[1,0]}\,(x_1\,p_1+x_2\,p_2+x_3\,p_3)+\\+\,(8\,e_{[2,0]}+3\,e_{[0,1]}^2+4\,e_{[1,0]}^2+6\,e_{[0,2]})\,(p_1+p_2+p_3)+\\+\,(2\,e_{[2,0]}-4\,e_{[0,1]}^2-11\,e_{[1,0]}^2-L^2)\,(x_1^2+x_2^2+x_3^2-L^2)=0.\end{gathered} \tag{7.9}$$

Like (7.5) and (7.7), the equation (7.9) is a linear combination of the equations (6.10) with coefficients in $\operatorname{Sym}\mathbb{Q}[M,L]$, i. e. it can be derived from (6.10).

The next step is to consider the sixth of the previously obtained factor equations. It is given by the formula (5.10) in [45]. Here is its E-form:

$$\begin{gathered}-8\,E_{10}\,E_{12}+8\,E_{01}\,E_{21}-8\,E_{11}^2+4\,E_{01}^2\,E_{10}^2-\\-\,E_{10}^4-3\,E_{01}^4+20\,E_{01}^2\,L^2-2\,E_{10}^2\,L^2-5\,L^4=0.\end{gathered} \tag{7.10}$$

The equation is similar to (7.8) and is equally complicated as the equation (7.8) since it is of the same order with respect to its variables. Upon converting to the

xd-form the equation (7.10) is written as follows:

$$
\begin{aligned}
&6\,(x_1^2\,p_1+x_2^2\,p_2+x_3^2\,p_3)+18\,(x_1\,d_1\,p_1+x_2\,d_2\,p_2+x_3\,d_3\,p_3)-\\
&-24\,e_{[0,1]}\,(d_1\,p_1+d_2\,p_2+d_3\,p_3)-8\,e_{[1,0]}\,(x_1\,p_1+x_2\,p_2+x_3\,p_3)+\\
&+(8\,e_{[2,0]}+9\,e_{[0,1]}^2-4\,e_{[1,0]}^2+18\,e_{[0,2]})\,(p_1+p_2+p_3)-\\
&-(10\,e_{[2,0]}20\,e_{[0,1]}^2-7\,e_{[1,0]}^2-5\,L^2)\,(x_1^2+x_2^2+x_3^2-L^2)=0.
\end{aligned}
\tag{7.11}
$$

Like (7.9), the equation (7.11) is a linear combination of the equations (6.10) with coefficients in $\operatorname{Sym}\mathbb{Q}[M,L]$, i. e. it can be derived from (6.10).

Let's proceed to the seventh of the previously obtained factor equations. This is the equation (5.17) in [45]. In its E-form this equation is written as follows:

$$
\begin{aligned}
&4\,E_{11}\,E_{21}-2\,E_{11}\,E_{01}^3+6\,E_{12}\,E_{01}^2+2\,E_{12}\,E_{10}^2-\,E_{10}^3\,E_{01}^2+\\
&+E_{10}\,E_{01}^4-2\,E_{12}\,L^2-E_{10}\,E_{01}^2\,L^2+2\,E_{10}^3\,L^2-2\,E_{10}\,L^4=0.
\end{aligned}
\tag{7.12}
$$

Upon converting to the xd-form the equation (7.12) looks like

$$
\begin{aligned}
&-4\,e_{[1,0]}\,(d_1^2\,p_1+d_2^2\,p_2+d_3^2\,p_3)-8\,e_{[0,1]}\,(x_1\,d_1\,p_1+x_2\,d_2\,p_2+\\
&+x_3\,d_3\,p_3)-(4\,e_{[2,0]}+2\,e_{[0,2]}-2\,e_{[1,0]}^2-3\,e_{[0,1]}^2)\,(x_1\,p_1+x_2\,p_2+\\
&+x_3\,p_3)+8\,e_{[1,0]}\,e_{[0,1]}\,(d_1\,p_1+d_2\,p_2+d_3\,p_3)+(3\,e_{[3,0]}-\\
&-3\,e_{[1,1]}\,e_{[0,1]}+3\,e_{[2,0]}\,e_{[1,0]}-3\,e_{[0,2]}\,e_{[1,0]}-e_{[1,0]}^3-e_{[1,2]}+\\
&+L^2\,e_{[1,0]})\,(p_1+p_2+p_3)+(2\,e_{[1,2]}+2\,e_{[1,0]}^3-8\,e_{[2,0]}\,e_{[1,0]}+\\
&+2\,e_{[0,2]}\,e_{[1,0]}+2\,e_{[1,0]}\,L^2)\,(x_1^2+x_2^2+x_3^2-L^2)=0.
\end{aligned}
\tag{7.13}
$$

Again, looking at (7.13), we see that this equation is a linear combination of the equations (6.10) with coefficients in $\operatorname{Sym}\mathbb{Q}[M,L]$, i. e. it can be derived from (6.10).

The eighth of the previously obtained factor equations is similar to the seventh one. It is given by the formula (5.22) in [45]. Here is its E-form:

$$
\begin{aligned}
&4\,E_{11}\,E_{12}-2\,E_{11}\,E_{10}^3+6\,E_{21}\,E_{10}^2+2\,E_{21}\,E_{01}^2-E_{01}^3\,E_{10}^2+E_{01}\,E_{10}^4+\\
&+2\,E_{21}\,L^2-2\,E_{11}\,E_{10}\,L^2+2\,E_{01}\,E_{10}^2\,L^2+E_{01}^3\,L^2-3\,E_{01}\,L^4=0.
\end{aligned}
\tag{7.14}
$$

Converting the equation (7.14) to the xd-form, we obtain

$$
\begin{aligned}
&-4\,e_{[0,1]}\,(x_1^2\,p_1+x_2^2\,p_2+x_3^2\,p_3)-8\,e_{[1,0]}\,(x_1\,d_1\,p_1+x_2\,d_2\,p_2+\\
&+x_3\,d_3\,p_3)+8\,e_{[1,0]}\,e_{[0,1]}\,(x_1\,p_1+x_2\,p_2+x_3\,p_3)+(2\,e_{[1,0]}^2+\\
&+2\,L^2)\,(d_1\,p_1+d_2\,p_2+d_3\,p_3)-(4\,e_{[1,1]}\,e_{[1,0]}+4\,e_{[2,0]}\,e_{[0,1]}-\\
&-3\,e_{[0,1]}\,e_{[1,0]}^2+3\,L^2\,e_{[0,1]}-2\,e_{[2,1]})\,(p_1+p_2+p_3)-(4\,e_{[2,1]}+\\
&+6\,e_{[0,3]}-4\,e_{[2,0]}\,e_{[0,1]}-4\,e_{[1,1]}\,e_{[1,0]}-3\,L^2\,e_{[0,1]}+\\
&+5\,e_{[0,1]}\,e_{[1,0]}^2)\,(x_1^2+x_2^2+x_3^2-L^2)=0.
\end{aligned}
\tag{7.15}
$$

Like (7.13), the equation (7.15) is a linear combination of the equations (6.10) with coefficients in $\operatorname{Sym}\mathbb{Q}[M,L]$. This means that it can be derived from (6.10).

8. Concluding remarks.

The theorem 6.1 is the main result of this paper. It yields a basis for the ideal I_{sym} and a complete list (6.10) of the cuboid factor equations in xd-form. As we noted above the equation (7.1) is equivalent to the first equation (6.10). Looking attentively at (7.3), (7.5), (7.7), (7.9), (7.11), (7.13), and (7.15), we find that the first seven equations in (6.10) can be derived from the previously obtained eight factor equations (7.1), (7.2), (7.4), (7.6), (7.8), (7.10), (7.12), and (7.14). The last equation (6.10) is new. Upon converting to an E-form it can be added to the list of previously obtained factor equations. However, this is not enough for to complete the list. The matter is that in E-form a complete list should include kernel equations. Therefore the equations $q_1 = 0$, $q_2 = 0$, $q_3 = 0$, $q_4 = 0$, $q_5 = 0$, $q_6 = 0$, $q_7 = 0$ given by the kernel polynomials (2.4), (2.5), (2.6), (2.7), (2.8), (2.9), and (2.10) should be added.

9. Acknowledgments.

I am grateful to my colleague I. Yu. Cherdantsev who recommended me the book [69] for learning Gröbner bases and their applications.

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