TANGENT POWER SUMS AND THEIR APPLICATIONS

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Abstract. For integer m, p, we study tangent power sum $\sum_{k=1}^{m} \tan^{2p} \frac{\pi k}{2m+1}$. We give recurrent, asymptotical and explicit formulas for these polynomials and indicate their connections with Newman's digit sums in base $2m$.

1. Introduction

Everywhere below we suppose that $n \geq 1$ is an odd number and p is a positive integer. In the present paper we study tangent power sum of the form

$$\sigma(n,p) = \sum_{k=1}^{\frac{n-1}{2}} \tan^{2p} \frac{\pi k}{n}. \tag{1}$$

In 2002, Chen [1] found formulas for $\sigma(n,p)$ in case $p \leq 5$ as polynomials in n. In 2007-2008, Shevelev [12] and Hassan [4] independently proved the following statements:

Theorem 1. *For every p, $\sigma(n,p)$ is integer and multiple of n.*

Theorem 2. *For a fixed p, $\sigma(n,p)$ is a polynomial in n of degree $2p$ with the leading term*

$$\frac{2^{2p-1}(2^{2p}-1)}{(2p)!} |B_{2p}| n^{2p}, \tag{2}$$

where B_{2p} is Bernoulli number.

Hassan [4] proved these results (see his Theorem 4.3 and formula 4.19), using a sampling theorem associated with the second-order discrete eigenvalue problem.

Shevelev [12] (see his Remark 2 and Remark 1) used some elementary arguments including the best-known Littlewood expression for the power sum of elementary polynomials in a determinant form [5].

In this paper we give another proof of these two theorems. Besides, we find several other representations and identities involving $\sigma(n,p)$ and numerical results for them. We give applications of $\sigma(n,p)$ in digit theory (Section 5). In the conclusive Section 7, using the digit interpretation and a combinatorial idea, we found an explicit expression for $\sigma(n,p)$ (Theorem 7).

2. Proof of Theorem 1

Denote $\omega = e^{\frac{2\pi i}{n}}$. Note that

$$\tan\frac{\pi k}{n} = i\frac{1-\omega^k}{1+\omega^k} = -i\frac{1-\omega^{-k}}{1+\omega^{-k}}, \quad \tan^2\frac{\pi k}{n} = \frac{1-\omega^{-k}}{1+\omega^k}\frac{1-\omega^k}{1+\omega^{-k}} \tag{3}$$

and, for the factors of $\tan^2\frac{\pi k}{n}$, we have

$$\frac{1-\omega^{-k}}{1+\omega^k} = \frac{(-\omega^k)^{n-1}-1}{(-\omega^k)-1} = \sum_{j=0}^{n-2}(-\omega^k)^j, \quad \frac{1-\omega^k}{1+\omega^{-k}} = \sum_{j=0}^{n-2}(-\omega^{-k})^j. \tag{4}$$

Since $\tan\frac{\pi k}{n} = -\tan\frac{\pi(n-k)}{n}$, then we have

$$2\sigma(n,p) = \sum_{k=1}^{n-1}\tan^{2p}\frac{\pi k}{n} \tag{5}$$

and, by (3)-(5),

$$2\sigma(n,p) = \sum_{k=1}^{n-1}\Big(\sum_{j=0}^{n-2}(-\omega^k)^j\Big)^p\Big(\sum_{j=0}^{n-2}(-\omega^{-k})^j\Big)^p =$$

$$\sum_{k=1}^{n-1}\Big(\prod_{l=0}^{p-1}\sum_{j=0}^{n-2}(-\omega^k)^j\prod_{l=0}^{p-1}\sum_{j=0}^{n-2}(-\omega^{-k})^j\Big) =$$

$$= \sum_{k=1}^{n-1}\Big(\prod_{t=0}^{2p-1}\sum_{j=0}^{n-2}(-\omega^{(-1)^t k})^j\Big). \tag{6}$$

Furthermore, we note that

$$(n-1)^t \equiv (-1)^t \pmod{n}. \tag{7}$$

Indeed, it is evident for odd t. If t is even and $t = 2^h s$ with odd s, then

$$(n-1)^t - (-1)^t = ((n-1)^s)^{2^h} - ((-1)^s)^{2^h} =$$

$$((n-1)^s - (-1)^s)((n-1)^s + (-1)^s)((n-1)^{2s} +$$

$$(-1)^{2s})\cdot \ldots \cdot ((n-1)^{2^{h-1}s} + (-1)^{2^{h-1}s}),$$

and, since $(n-1)^s + 1 \equiv 0 \pmod{n}$, we are done. Using (7), we can write (6) in the form (we sum from $k=0$, adding the zero summand)

$$2\sigma(n,p) = \sum_{k=0}^{n-1}\prod_{t=0}^{2p-1}(1 - \omega^{k(n-1)^t} + \omega^{2k(n-1)^t} - \ldots - \omega^{(n-2)k(n-1)^t}). \tag{8}$$

Considering $0, 1, 2, \ldots, n-2$ as digits in the base $n-1$, after the multiplication of factors of the product in (8) we obtain summands of the form $(-1)^{s(r)}\omega^{kr}$, $r = 0, \ldots, (n-1)^{2p}-1$, where $s(r)$ is the digit sum of r in the base $n-1$. Thus we have

$$2\sigma(n,p) = \sum_{k=0}^{n-1}\sum_{r=0}^{(n-1)^{2p}-1}(-1)^{s(r)}\omega^{kr} = \sum_{r=0}^{(n-1)^{2p}-1}(-1)^{s(r)}\sum_{k=0}^{n-1}(\omega^k)^r. \tag{9}$$

However,

$$\sum_{k=0}^{n-1}(\omega^k)^r = \begin{cases} n, \ if\ r \equiv 0 \pmod n \\ 0, \ otherwise. \end{cases}$$

Therefore, by (9),

$$2\sigma(n,p) = n \sum_{r=0,\ n|r}^{(n-1)^{2p}-1} (-1)^{s(r)} \tag{10}$$

and, consequently, $2\sigma(n,p)$ is integer multiple of n. It is left to show that the right hand side of (10) is even. It is sufficient to show that the sum contains even number of summands. The number of summands is

$$1 + \lfloor \frac{(n-1)^{2p}}{n} \rfloor = 1 + \frac{(n-1)^{2p}-1}{n} =$$

$$1 + \sum_{l=0}^{2p-1}(-1)^l \binom{2p}{l} n^{2p-1-l} \equiv 1 + \sum_{l=0}^{2p-1}(-1)^l \binom{2p}{l} \pmod 2 =$$

$$1 - (-1)^{2p}\binom{2p}{2p} = 0.$$

This completes proof of the theorem. □

3. Proof of Theorem 2

As is well known,

$$\sin n\alpha = \sum_{i=0}^{\frac{n-1}{2}}(-1)^i \binom{n}{2i+1} \cos^{n-(2i+1)}\alpha \sin^{2i+1}\alpha,$$

or

$$\sin n\alpha = \tan\alpha \cos^n\alpha \sum_{i=0}^{\frac{n-1}{2}}(-1)^i \binom{n}{2i+1} \tan^{2i}\alpha.$$

Put here $\alpha = \frac{k\pi}{n},\ k = 1, 2, ..., \frac{n-1}{2}$. Since $\tan\alpha \neq 0,\ \cos\alpha \neq 0$, then

$$0 = \sum_{i=0}^{\frac{n-1}{2}}(-1)^i \binom{n}{2i+1} \tan^{2i}\alpha =$$

$$(-1)^{\frac{n-1}{2}}(\tan^{n-1}\alpha - \binom{n}{n-2}\tan^{n-3}\alpha + ... -$$

$$(-1)^{\frac{n-1}{2}}\binom{n}{3}\tan^2\alpha + (-1)^{\frac{n-1}{2}}\binom{n}{1}).$$

This means that the equation

$$\lambda^{\frac{n-1}{2}} - \binom{n}{2}\lambda^{\frac{n-3}{2}} + \binom{n}{4}\lambda^{\frac{n-5}{2}} - ... + (-1)^{\frac{n-1}{2}}\binom{n}{n-1} = 0 \tag{11}$$

has $\frac{n-1}{2}$ roots: $\lambda_k = \tan^2\frac{k\pi}{n},\ k = 1, 2, ..., \frac{n-1}{2}$. Note that (11) is the characteristic equation for the following difference equation

$$y(p) = \binom{n}{2} y(p-1) - \binom{n}{4} y(p-2) + ... -$$

(12) $$(-1)^{\frac{n-1}{2}} \binom{n}{n-1} y(p - \frac{n-1}{2})$$

which, consequently, has a private solution

$$y(p) = \sum_{k=1}^{\frac{n-1}{2}} (\tan^2 \frac{k\pi}{n})^p = \sigma(n,p).$$

Now, using Newton's formulas for equation (11),

$$\sigma(n,1) = \binom{n}{2},$$

$$\sigma(n,2) = \binom{n}{2}\sigma(n,1) - 2\binom{n}{4},$$

(13) $$\sigma(n,3) = \binom{n}{2}\sigma(n,2) - \binom{n}{4}\sigma(n,1) + 3\binom{n}{6}, \ \textit{etc.}$$

we conclude that $\sigma(n,p)$ is a polynomial in n of degree $2p$. Note that, by induction, all these polynomials are integer-valued and thus we have another independent proof of Theorem 1. To find the leading terms of these polynomials, we make some transformations of (1). Put $\frac{n-1}{2} = m$. Changing in (1) the order of summands ($l = m - k$) and noting that

$$\frac{(m-l)\pi}{2m+1} + \frac{(2l+1)\pi}{4m+2} = \frac{\pi}{2},$$

we have

(14) $$\sigma(n,p) = \sum_{l=0}^{m-1} \cot^{2p} \frac{(2l+1)\pi}{4m+2}.$$

Further we have

$$\sigma(n,p) = \sum_{0 \le l \le \sqrt{m}} \cot^{2p} \frac{(2l+1)\pi}{4m+2} +$$

(15) $$\sum_{\sqrt{m} < l \le m-1} \cot^{2p} \frac{(2l+1)\pi}{4m+2} = \Sigma_1 + \Sigma_2.$$

Let $p > 1$. Let us estimate the second sum Σ_2. The convexity of $\sin x$ on $[0, \frac{\pi}{2}]$ gives the inequality $\sin x \ge \frac{2}{\pi} x$. Therefore, for summands in the second sum, we have

$$\cot^{2p} \frac{(2l+1)\pi}{4m+2} < \sin^{-2p} \frac{(2l+1)\pi}{4m+2} <$$
$$(\frac{2m+1}{2l+1})^{2p} < (\frac{2m+1}{2\sqrt{m}+1})^{2p} < m^p.$$

This means that $\Sigma_2 < m^{p+1} < m^{2p}$ and not influences on the leading term. Now note that, evidently,

$$\frac{(2l+1)\pi}{4m+2} cot \frac{(2l+1)\pi}{4m+2} \to 1$$

uniformly over $l \leq \sqrt{m}$. Thus

$$\Sigma_1 = \sum_{0 \leq l \leq \sqrt{m}} (\frac{(4m+2)}{(2l+1)\pi})^{2p} + \alpha(m) =$$
$$(\frac{(4m+2)}{\pi})^{2p} \sum_{0 \leq l \leq \sqrt{m}} \frac{1}{(2l+1)^{2p}} + \alpha(m),$$

where $\alpha(m) \leq \varepsilon\sqrt{m}$. Thus the coefficient of the leading term of the polynomial $\sigma(n,p)$ is

$$\lim_{m\to\infty} \frac{\Sigma_1}{n^{2p}} = (\frac{2}{\pi})^{2p} \sum_{l=0}^{\infty} \frac{1}{(2l+1)^{2p}} =$$
$$(\frac{2}{\pi})^{2p} (\zeta(2p) - \sum_{l=1}^{\infty} \frac{1}{(2l)^{2p}}) =$$
$$(\frac{2}{\pi})^{2p} (\zeta(2p) - \frac{1}{2^{2p}} \zeta(2p)) = \frac{2^p(2^{2p}-1)}{\pi^{2p}} \zeta(2p).$$

It is left to note that, by very known formula, $\zeta(2p) = \frac{|B_{2p}| 2^{2p-1} \pi^{2p}}{(2p)!}$, we find that the leading coefficient is defined by formula (2). □

4. Several numerical results

Since, by (1), $\sigma(1,p) = 0$, then $\sigma(n,p) \equiv 0 \pmod{n(n-1)}$. Put

$$\sigma^*(n,p) = 2\sigma(n,p)/(n(n-1)).$$

By formulas (13), the first polynomials $\{\sigma^*(n,p)\}$ are

$$\sigma^*(n,1) = 1,$$
$$\sigma^*(n,2) = \frac{n^2+n}{3} - 1,$$
$$\sigma^*(n,3) = \frac{2(n^2+n)(n^2-4)}{15} + 1,$$
$$\sigma^*(n,4) = \frac{(n^2+n)(17n^4 - 95n^2 + 213)}{315} - 1,$$
$$\sigma^*(n,5) = \frac{2(n^2+n)(n^2-4)(31n^4 - 100n^2 + 279)}{2835} + 1, \ etc.$$

As well known (cf. Problem 85 in [8]), the integer-valued polynomials have integer coefficients in the binomial basis $\{\binom{n}{k}\}$. The first integer-valued polynomials $\{\sigma(n,p)\}$ represented in binomial basis have the form

$$\sigma(n,1) = \binom{n}{2},$$

$$\sigma(n,2)=\binom{n}{2}+6\binom{n}{3}+4\binom{n}{4},$$
$$\sigma(n,3)=\binom{n}{2}+24\binom{n}{3}+96\binom{n}{4}+120\binom{n}{5}+48\binom{n}{6},$$
$$\sigma(n,4)=\binom{n}{2}+78\binom{n}{3}+836\binom{n}{4}+3080\binom{n}{5}+5040\binom{n}{6}+3808\binom{n}{7}+1088\binom{n}{8},$$

etc.

Note that the recursion (12) presupposes a fixed n. In general, by (12), we have

$$\sigma(n,p)=\binom{n}{2}\sigma(n,p-1)-\binom{n}{4}\sigma(n,p-2)+...-$$

(16) $$(-1)^{\frac{n-1}{2}}\binom{n}{n-1}\sigma(n,p-\frac{n-1}{2}),\ p\geq\frac{n-1}{2}.$$

Since from (1) $\sigma(n,0)=\frac{n-1}{2},\ n=3,5,...,$ then, calculating other initials by (13), we have the recursions:

$$\sigma(3,p)=3\sigma(3,p-1),\ p\geq 1,\ \sigma(3,0)=1;$$
$$\sigma(5,p)=10\sigma(5,p-1)-5\sigma(5,p-2),\ p\geq 2,\ \sigma(5,0)=2,\ \sigma(5,1)=10;$$
$$\sigma(7,p)=21\sigma(7,p-1)-35\sigma(7,p-2)+7\sigma(7,p-3),\ p\geq 3,$$
$$\sigma(7,0)=3,\ \sigma(7,1)=21,\ \sigma(7,2)=371;$$
$$\sigma(9,p)=36\sigma(9,p-1)-126\sigma(9,p-2)+84\sigma(9,p-3)-9\sigma(9,p-4),\ p\geq 4,$$
$$\sigma(9,0)=4,\ \sigma(9,1)=36,\ \sigma(9,2)=1044,\ \sigma(9,3)=33300;\ etc.$$

Thus

(17) $$\sigma(3,p)=3^p,$$

and a few terms of the other sequences $\{\sigma(n,p)\}$ are

$n=5)$ $2,10,90,850,8050,76250,722250,6841250,64801250,$
$613806250,5814056250,...;$

$n=7)$ $3,21,371,7077,135779,2606261,50028755,960335173,$
$18434276035,353858266965,6792546291251,...;$

$n=9)$ $4,36,1044,33300,1070244,34420356,1107069876,$
$35607151476,1145248326468,36835122753252,...;$

$n=11)$ $5,55,2365,113311,5476405,264893255,12813875437,$
$619859803695,29985188632421,1450508002869079,\ldots.$

5. Applications to digit theory

For $x \in \mathbb{N}$ and odd $n \geq 3$, denote by $S_n(x)$ the sum

$$S_n(x) = \sum_{0 \leq r < x:\ r \equiv 0 \pmod{n}} (-1)^{s_{n-1}(r)}, \tag{18}$$

where $s_{n-1}(r)$ is the digit sum of r in base $n-1$.
Note that, in particular, $S_3(x)$ equals the difference between the numbers of multiples of 3 with even and odd binary digit sums (or multiples of 3 from sequences A001969 and A000069 in [14]) in interval $[0, x)$.

Leo Moser (cf. [7], Introduction) conjectured that always

$$S_3(x) > 0. \tag{19}$$

Newman [7] proved this conjecture. Moreover, he obtained the inequalities

$$\frac{1}{20} < S_3(x)x^{-\lambda} < 5, \tag{20}$$

where

$$\lambda = \frac{\ln 3}{\ln 4} = 0.792481... . \tag{21}$$

In connection with these remarkable Newman results, the qualitative result (19) we call a weak Newman phenomenon (or Moser-Newman phenomenon), while an estimating result of the form (20) we call a strong Newman phenomenon.

In 1983, Coquet [2] studied a very complicated continuous and nowhere differentiable fractal function $F(x)$ with period 1 for which

$$S_3(3x) = x^{\lambda} F\left(\frac{\ln x}{\ln 4}\right) + \frac{\eta(x)}{3}, \tag{22}$$

where

$$\eta(x) = \begin{cases} 0, & if\ x\ is\ even, \\ (-1)^{s_2(3x-1)}, & if\ x\ is\ odd. \end{cases} \tag{23}$$

He obtained that

$$\limsup_{x \to \infty,\, x \in \mathbb{N}} S_3(3x)x^{-\lambda} = \frac{55}{3}\left(\frac{3}{65}\right)^{\lambda} = 1.601958421\ldots , \tag{24}$$

$$\liminf_{x\to\infty,\ x\in\mathbb{N}} S_3(3x)x^{-\lambda} = \frac{2\sqrt{3}}{3} = 1.154700538\ldots\ . \tag{25}$$

In 2007, Shevelev [11] gave an elementary proof of Coquet's formulas (24)-(25) and his sharp estimates in the form

$$\frac{2\sqrt{3}}{3}x^{\lambda} \le S_3(3x) \le \frac{55}{3}\left(\frac{3}{65}\right)^{\lambda} x^{\lambda},\quad x\in\mathbb{N}. \tag{26}$$

Besides, Shevelev showed that the sequence $\{(-1)^{s_2(n)}(S_3(n)-3S_3(\lfloor n/4\rfloor))\}$, is periodic with period 24 taking the values $-2,-1,0,1,2$. This gives a simple recursion for $S_3(n)$. In 2008, Drmota and Stoll [3] proved a generalized weak Newman phenomenon, showing that (19) is valid for sum (18) for every $n\ge 3$, at least beginning with $x\ge x_0(n)$. Our proof of Theorem 1 allows to consider a strong form of this generalization, but yet only in "full" intervals in even base $n-1$ of the form $[0,(n-1)^{2p})$ (see also preprint of Shevelev [12]).

Theorem 3. *For $x_{n,p}=(n-1)^{2p},\ p\ge 1,$ we have*

$$S_n(x_{n,p}) \sim \frac{2}{n}x_{n,p}^{\lambda}\ \ \sigma(n,p)\sim x_{n,p}^{\lambda}\ \ (p\to\infty), \tag{27}$$

where

$$\lambda=\lambda_n=\frac{\ln\cot(\frac{\pi}{2n})}{\ln(n-1)}. \tag{28}$$

Proof. According to (10) and (18), we have

$$S_n(x_{n,p}) = \frac{2}{n}\sigma(n,p),\ p\ge 1. \tag{29}$$

Thus, choosing the maximal exponent in (1) as $p\to\infty$, we find

$$S_n(x_{n,p}) \sim \frac{2}{n}\tan^{2p}\frac{(n-1)\pi}{2n} =$$
$$\frac{2}{n}\cot^{2p}\frac{\pi}{2n} = \exp(\ln\frac{2}{n} + 2p\ln\cot\frac{\pi}{2n}) =$$
$$\exp(\ln\frac{2}{n} + 2p\lambda\ln(n-1)) = \exp(\ln\frac{2}{n}+\ln x_{n,p}^{\lambda}) = \frac{2}{n}x_{n,p}^{\lambda}. \tag{30}$$

□

In particular, in the cases of $n=3,5,7,9,11$ we have $\lambda_3=\frac{\ln 3}{\ln 4}=0.79248125...$, $\lambda_5=0.81092244...$, $\lambda_7=0.82452046...$, $\lambda_9=0.83455828...$, $\lambda_{11}=0.84230667...$ respectively.

Show that

$$1-\frac{\ln\frac{\pi}{2}}{\ln(n-1)} \le \lambda_n \le 1-\frac{\ln\frac{\pi}{2}}{\ln(n-1)}+\frac{1}{(n-1)\ln(n-1)}. \tag{31}$$

Indeed, by the convexity of $\cos x$ on $[0, \frac{\pi}{2}]$, $\cos x \geq 1 - \frac{2}{\pi}x$, and, therefore, $\cos\frac{\pi}{2n} \geq 1 - \frac{1}{n}$. Using also that $\tan\frac{\pi}{2n} \geq \frac{\pi}{2n} \geq \sin\frac{\pi}{2n}$, we have

$$\frac{2}{\pi}(n-1) \leq \cot\frac{\pi}{2n} \leq \frac{2}{\pi}n$$

and, by (28),

$$1 - \frac{\ln\frac{\pi}{2}}{\ln(n-1)} \leq \lambda_n \leq 1 - \frac{\ln\frac{\pi}{2}}{\ln(n-1)} + \frac{\ln(1+\frac{1}{n-1})}{\ln(n-1)}$$

which yields (31), since, for $n \geq 3$, $\ln(1+\frac{1}{n-1}) < \frac{1}{n-1}$. Finally, let us show the monotonic increasing of λ_n. For function $f(x) = \frac{\ln\cot(\frac{\pi}{2x})}{\ln(x-1)}$, we have

$$\ln(x-1)f'(x) = \frac{\pi}{x^2\sin\frac{\pi}{x}} - \frac{f(x)}{x-1}. \tag{32}$$

As in (31), we also have

$$f(x) \leq 1 - \frac{\ln\frac{\pi}{2}}{\ln(x-1)} + \frac{1}{(x-1)\ln(x-1)}. \tag{33}$$

On the other hand, since $\sin\frac{\pi}{x} \leq \frac{\pi}{x}$, then

$$\frac{\pi(x-1)}{x^2\sin\frac{\pi}{x}} \geq 1 - \frac{1}{x},$$

and, by (32), in order to show that $f'(x) > 0$, it is sufficient to prove that $f(x) < 1 - \frac{1}{x}$, or, by (33), to show that

$$1 - \frac{\ln\frac{\pi}{2}}{\ln(x-1)} + \frac{1}{(x-1)\ln(x-1)} < 1 - \frac{1}{x},$$

or

$$\frac{\ln(x-1)}{x} + \frac{1}{x-1} < \ln\frac{\pi}{2}.$$

This inequality holds for $x \geq 7$, and since $\lambda_3 < \lambda_5 < \lambda_7$, then the monotonicity of λ_n follows. Thus we have the monotonic strengthening of the strong form of Newman-like phenomenon for the base $n-1$ in the considered intervals.

6. An identity

Since (29) was proved for $x_{n,p} = (n-1)^{2p}$, $p \geq 1$, then, by (16), for $S_n(x_{n,p})$ in the case $p \geq \frac{n+1}{2}$, we have the relations

$$\sum_{k=0}^{\frac{n-1}{2}}(-1)^k\binom{n}{2k}\sigma(n, p-k) =$$

$$\sum_{k=0}^{\frac{n-1}{2}}(-1)^k\binom{n}{2k}S_n((n-1)^{2p-2k}) = 0.$$

In case $p = \frac{n-1}{2}$ the latter relation does not hold. Let us show that in this case we have the identity

$$\sum_{k=0}^{\frac{n-1}{2}}(-1)^k\binom{n}{2k}S_n((n-1)^{n-2k-1}) = (-1)^n,$$

or, putting $n-2k-1=2j$, the identity

$$\sum_{j=0}^{\frac{n-1}{2}}(-1)^j\binom{n}{2j+1}S_n((n-1)^{2j}) = 1. \tag{34}$$

Indeed, in case $j=0$, we, evidently, have $S_n(1)=1$, while, formally, by (29), for $p=0$, we obtain "$S_n(1) = \frac{2}{n}\sigma(n,\ 0) = \frac{2}{n}\frac{n-1}{2} = \frac{n-1}{n}$", i.e., the error is $-\frac{1}{n}$, and the error in the corresponding sum is $n(-\frac{1}{n}) = -1$. Therefore, in the latter formula, instead of 0, we have 1. Note that (34) one can rewrite also in the form

$$\sum_{j=1}^{\frac{n-1}{2}}(-1)^{j-1}\binom{n}{2j+1}\sigma(n,j) = \binom{n}{2}.$$

7. Explicit combinatorial representation

In its turn, the representation (29) allows to get an explicit combinatorial representation for $\sigma(n,p)$. We need three lemmas.

Lemma 4. ([10], *p.* 215) *The number of compositions $C(m,n,s)$ of m with n positive parts not exceeding s is given by formula*

$$C(m,n,s) = \sum_{j=0}^{\min(n,\lfloor\frac{m-n}{s}\rfloor)}(-1)^j\binom{n}{j}\binom{m-sj-1}{n-1}. \tag{35}$$

Since, evidently, $C(m,n,1) = \delta_{m,n}$, then, as a corollary, we have the identity

$$\sum_{j=0}^{\min(n,m-n)}(-1)^j\binom{n}{j}\binom{m-j-1}{n-1} = \delta_{m,n}. \tag{36}$$

Lemma 5. *The number of compositions $C_0(m,n,s)$ of m with n nonnegative parts not exceeding s is given by formula*

$$C_0(m,n,s) = \begin{cases} C(m+n,n,s+1),\ if\ m\geq n\geq 1,\ s\geq 2, \\ \sum_{\nu=1}^{m}C(m,\nu,s)\binom{n}{n-\nu},\ if\ 1\leq m<n,\ s\geq 2, \\ 1,\ if\ m=0,\ n\geq 1,\ s\geq 0, \\ 0,\ if\ m>n\geq 1,\ s=1, \\ \binom{n}{m},\ if\ 1\leq m\leq n,\ s=1. \end{cases} \tag{37}$$

Proof. Let firstly $s \geq 2,\ m \geq n \geq 1$. If to diminish on 1 every part of a composition of $m+n$ with n positive parts not exceeding $s+1$, then we obtain a composition of m with n nonnegative parts not exceeding s, such that zero parts allowed. Let, further, $s \geq 2,\ 1 \leq m < n$. Consider $C(m, \nu, s)$ compositions of m with $\nu \leq m$ parts. To obtain n parts, consider $n-\nu$ zero parts, which we choose in $\binom{n}{n-\nu}$ ways. The summing over $1 \leq \nu \leq m$ gives the required result. Other cases are evident. □

Let now $(n-1)^h \leq N < (n-1)^{h+1},\ n \geq 3$. Consider the representation of N in the base $n-1$:

$$N = g_h(n-1)^h + ... + g_1(n-1) + g_0,$$

where $g_i = g_i(N),\ i = 0, ..., h$, are digits of N, $\ 0 \leq g_i \leq n-2$. Let

$$s^e(N) = \sum_{i \text{ is even}} g_i,\ s^o(N) = \sum_{i \text{ is odd}} g_i.$$

Lemma 6. *N is multiple of n if and only if $s^o(N) \equiv s^e(N) \pmod{n}$.*

Proof. The lemma follows from the evident relation $(n-1)^i \equiv (-1)^i \pmod{n}$, $i \geq 0$. □

Now we obtain a combinatorial explicit formula for $\sigma(n, p)$.

Theorem 7. *For $n \geq 3,\ p \geq 1,$ we have*

$$\sigma(n,p) = \frac{n}{2} \sum_{j=0}^{(n-2)p} ((C_0(j,p,n-2))^2 +$$

$$2 \sum_{k=1}^{\lfloor \frac{(n-2)p-j}{n} \rfloor} (-1)^k C_0(j,p,n-2) C_0(j+nk,p,n-2)), \tag{38}$$

where $C_0(m,n,s)$ is defined by formula (37).

Proof. Consider all nonnegative integers $N's$ not exceeding $(n-1)^{2p}-1$, which have $2p$ digits $g_i(N)$ in base $n-1$ (the first 0's allowed). Let the sum of digits of N on even p positions be j, while on odd p positions such sum be $j+kn$ with a positive integer k. Then, by Lemma 6, such $N's$ are multiple of n. Since in the base $n-1$ the digits not exceed $n-2$, then the number of ways to choose such $N's$, for $k=0$, is $(C_0(j,p,n-2))^2$. In the case $k \geq 1$, we should also consider the symmetric case when on odd p positions the sum of digits of N be j, while on even p positions such sum be $j+kn$ with a positive integer k. This, for $k \geq 1$, gives $2C_0(j,p,n-2)C_0(j+kn,p,n-2)$ required numbers $N's$. Furthermore, since n is odd, then, if k is odd, then

$s_{n-1}(N)$ is odd, while, if k is even, then $s_{n-1}(N)$ is even. Thus the difference $S_n((n-1)^{2p})$ between n-multiple $N's$ with even and odd digit sums equals

$$S_n((n-1)^{2p}) = \sum_j ((C_0(j,p,n-2))^2 +$$

$$2\sum_k (-1)^k C_0(j,p,n-2)C_0(j+nk,p,n-2)).$$

Now to obtain (38), note that $0 \leq j \leq (n-2)p$, and, for $k \geq 1$, also $j + nk \leq (n-2)p$, such that $1 \leq k \leq \frac{(n-2)p-j}{n}$, and that, by (29), $\sigma(n,p) = \frac{n}{2}S_n((n-1)^{2p})$. □

Example 8. Let $n = 5, p = 2$. By Theorem 7, we have

$$\sigma(5,2) = 2.5\sum_{j=0}^{6}((C_0(j,2,3))^2 +$$

$$2\sum_{k=1}^{\lfloor\frac{6-j}{3}\rfloor}(-1)^k C_0(j,2,3)C_0(j+5k,2,3)). \tag{39}$$

We have

$$C_0(0,2,3) = 1, C_0(1,2,3) = 2, C_0(2,2,3) = 3,$$
$$C_0(3,2,3) = 4, C_0(4,2,3) = 3, C_0(5,2,3) = 2, C_0(6,2,3) = 1.$$

Thus

$$\sum_{j=0}^{6}((C_0(j,2,3))^2 = 44.$$

In the cases $j = 0, k = 1$ and $j = 1, k = 1$ we have

$$C_0(0,2,3)C_0(5,2,3) = 2,\ C_0(1,2,3)C_0(6,2,3) = 2.$$

Thus

$$2\sum_{j=0}^{6}\sum_{k=1}^{\lfloor\frac{6-j}{3}\rfloor}(-1)^k C_0(j,2,3)C_0(j+5k,2,3)) = -8$$

and, by (39), we have

$$\sigma(5,2) = 2.5(44-8) = 90.$$

On the other hand, by (1), we directly have

$$\sigma(5,2) = \sum_{k=1}^{2}\tan^4\frac{\pi k}{5} = 0.278640... + 89.721359... = 89.999999...$$

Example 9. In case $n = 3$, by Theorem 7 and formulas (17), (37), we have

$$3^p = \frac{3}{2}\sum_{j=0}^{p}((C_0(j,p,1))^2 +$$

$$2\sum_{k=1}^{\lfloor\frac{p-j}{3}\rfloor}(-1)^k C_0(j,p,1)C_0(j+3k,p,1)) =$$

$$\frac{3}{2}\sum_{j=0}^{p}(\binom{p}{j}^2+2\sum_{k=1}^{\lfloor\frac{p-j}{3}\rfloor}(-1)^k\binom{p}{j}\binom{p}{3k+j}.$$

Thus, using well known formula $\sum_{j=0}^{p}(\binom{p}{j}^2=\binom{2p}{p}$, we find the identity

$$\sum_{j=0}^{p}\sum_{k=1}^{\lfloor\frac{p-j}{3}\rfloor}(-1)^k\binom{p}{j}\binom{p}{3k+j}=3^{p-1}-\frac{1}{2}\binom{2p}{p},$$

or, changing the order of summing,

$$\sum_{k=1}^{\lfloor\frac{p}{3}\rfloor}(-1)^k\sum_{j=0}^{p-3k}\binom{p}{j}\binom{p}{3k+j}=3^{p-1}-\frac{1}{2}\binom{2p}{p}.$$

Since (cf.[9],p.8)

$$\sum_{j=0}^{p-3k}\binom{p}{j}\binom{p}{3k+j}=\binom{2p}{p+3k}, \tag{40}$$

then we obtain an identity

$$\sum_{k=1}^{\lfloor\frac{p}{3}\rfloor}(-1)^{k-1}\binom{2p}{p+3k}=\frac{1}{2}\binom{2p}{p}-3^{p-1},\ p\geq 1. \tag{41}$$

Note that firstly (41) was proved in a quite another way by Shevelev [13] (2007) and again proved by Merca [6] (2012).

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