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Orthogonal multiplet bases in $\mathrm{SU}(N_c)$ color space

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Abstract: We develop a general recipe for constructing orthogonal bases for the calculation of color structures appearing in QCD for any number of partons and arbitrary N_c. The bases are constructed using hermitian gluon projectors onto irreducible subspaces invariant under $\mathrm{SU}(N_c)$. Thus, each basis vector is associated with an irreducible representation of $\mathrm{SU}(N_c)$. The resulting multiplet bases are not only orthogonal, but also minimal for finite N_c. As a consequence, for calculations involving many colored particles, the number of basis vectors is reduced significantly compared to standard approaches employing overcomplete bases. We exemplify the method by constructing multiplet bases for all processes involving a total of 6 external colored partons.

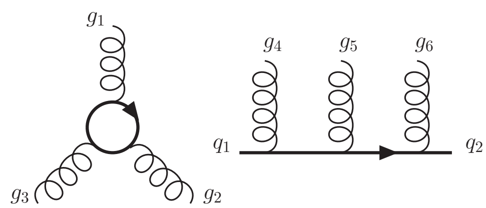
Contents

1. Introduction

With the start of the Large Hadron Collider follows an increased demand for accurately calculated processes in perturbative quantum chromodynamics (QCD), as the higher energies open up for events with more colored partons. A major challenge for these calculations is the complication brought about by the non-abelian gauge structure in QCD.

Several methods have been developed for treating the color structure in special cases [1–5]. The most general, and probably most widely used approach for exact calculations employs a decomposition of the color space into open and closed quark-lines [6–14], i.e. linear combinations of terms like

$$\mathrm{tr}[t^{g_1}t^{g_2}t^{g_3}]\;(t^{g_4}t^{g_5}t^{g_6})^{q_1}_{\;q_2} =$$



$$, \qquad (1.1)$$

where the involved partons may be combined to form any structures allowed in QCD. Here t^{g_j}, $g_j = 1, \ldots, N_c^2 - 1$ denotes a generator in the fundamental representation of $\mathrm{SU}(N_c)$, $N_c = 3$ for QCD, and $q_{1,2} = 1, ..., N_c$. We refer to this type of basis as a *trace basis*. Any amplitude, at tree level and beyond, can be decomposed in this way, and for fixed order calculations only a small fraction of all possible products of open and closed quark-lines have non-vanishing amplitudes. For calculations involving many partons, approximative Monte Carlo techniques [15–20] and, for higher efficiency, the color-flow basis [16], may be employed.

Two drawbacks of the trace basis are that it is non-orthogonal and, in general, overcomplete, i.e. it is not a proper basis but just a spanning set. In an alternative approach the state spaces of incoming and outgoing partons are decomposed into multiplets, i.e. into irreducible subspaces invariant under SU(3), or, more generally, $\mathrm{SU}(N_c)$. It is then possible to construct (minimal) orthogonal bases for color spaces. We refer to this kind of basis as a *multiplet basis*. These bases have the potential to significantly speed up QCD color calculations. However, to the best of our knowledge, multiplet bases have so far only been employed for processes with up to five colored partons [3–5, 21–23], typically in the context of resummation. One reason is that, in general, the construction of these bases is far from obvious. This is the problem we want to shed light on in this article.

Our main result is a general recipe for constructing orthogonal multiplet bases for QCD processes with an arbitrary number of quarks and gluons, to arbitrary order in perturbation theory and for arbitrary N_c. We explicitly demonstrate the method by constructing bases for all processes with six colored partons.

This article is organized as follows: In the remainder of the introduction we discuss the notion of color space (section 1.1), review the trace basis approach (section 1.2) and set the stage for our method with an example in section 1.3. Thereafter we address the construction of projection operators (section 2) and basis vectors (section 3) in the quarks-only case. We discuss the importance of hermitian projection operators, and define quark

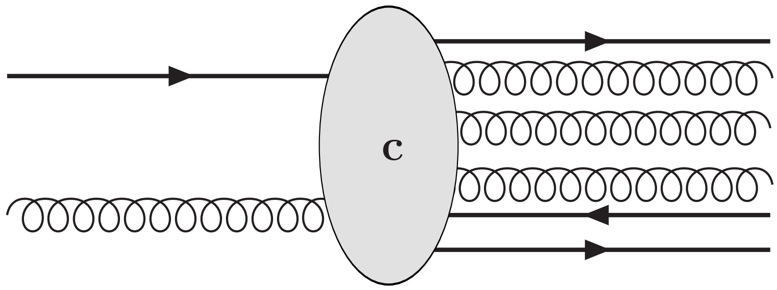


Figure 1: A color structure for a process with $n_q = 2$ and $N_g = 4$.

projectors which we use in the subsequent construction of gluon projectors. In section 4 we address the considerably more involved task of constructing projection operators for an arbitrary number of gluons. Starting from these projectors, we outline in section 5 how to build orthogonal bases for processes involving only gluons. After having addressed the construction in the gluon-only case, we find in section 6 that the extension to processes involving both quarks and gluons is straightforward. We conclude with some remarks in section 7.

1.1 Color space

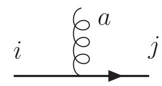
Consider a process with a certain number of incoming and outgoing quarks, anti-quarks and gluons. We denote by n_q the number of outgoing quarks plus the number of incoming anti-quarks, and by N_g the number gluons (incoming plus outgoing). Due to the QCD Feynman rules the number of incoming quarks plus the number of outgoing anti-quarks also has to equal n_q. Focusing on the color degrees of freedom, i.e. ignoring spin and momentum, quark states are elements of $V = \mathbb{C}^{N_c}$ and transform under the fundamental or defining representation of $\mathrm{SU}(N_c)$, anti-quarks states are elements of the dual space $\overline{V} \cong \mathbb{C}^{N_c}$ and transform in the complex conjugate of the fundamental representation, whereas gluons transform in the adjoint representation, i.e. gluon states are elements of a real $N_c^2 - 1$-dimensional vector space which we complexify to $A \cong \mathbb{C}^{N_c^2-1}$. Thus, with a QCD amplitude is associated a tensor $\mathbf{c} \in (V \otimes \overline{V})^{\otimes n_q} \otimes A^{\otimes N_g}$, its *color structure*.

Let us briefly remark on some conventions. We refer to $\mathrm{SU}(N_c)$-invariant irreducible subspaces as *multiplets*. A multiplet carries an irreducible representation of $\mathrm{SU}(N_c)$. As there is a unique irreducible representation associated with each multiplet, we often use the two terms interchangeably, e.g. we refer to the trivial representation as singlet or to the adjoint representation of SU(3) as octet.

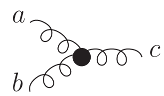
Eventually we will use Cvitanović's birdtrack notation [8, 24] representing a tensor $\mathbf{c} \in (V \otimes \overline{V})^{\otimes n_q} \otimes A^{\otimes N_g}$ as a blob with $2n_q + N_g$ legs, where straight lines with outward pointing arrows correspond to outgoing quarks (or incoming anti-quarks), straight lines with inward pointing arrows correspond to incoming quarks (or outgoing anti-quarks) and curly lines correspond to gluons[1], see figure 1. Inside the blob the lines can be connected

[1] Cvitanović [8, 24] represents gluons by thin instead of curly lines.

directly or via any number of

quark-gluon vertices, 

$= (t^a)^i{}_j$ and

triple-gluon vertices, 

$= \mathrm{i} f_{abc}\,,$ (1.2)

where $i, j = 1, \ldots, N_c,\ a, b, c = 1, \ldots, N_c^2 - 1$ and the indices in the triple gluon vertex are to be read *anti-clockwise*. Here t^a denotes a generator of $\mathrm{SU}(N_c)$ in the fundamental representation and f_{abc} are the $\mathrm{SU}(N_c)$ structure constants. We do not include the four gluon vertex in this list since its color part can be built from linear combinations of (one-gluon) contracted products of two triple-gluon vertices.

Since QCD-processes conserve color we are only interested in color structures that are overall singlets, i.e. invariant tensors, see appendix F. Thus, we define the *color space* as the color singlet subspace of $(V \otimes \overline{V})^{\otimes n_q} \otimes A^{\otimes N_g}$, i.e. the span of all tensors that transform under the trivial representation of $\mathrm{SU}(N_c)$. For instance, consider a process with two incoming and two outgoing quarks, $qq \to qq$ for short. The color space for this process is spanned by the singlets in $\overline{V} \otimes \overline{V} \otimes V \otimes V$ and as such has dimension 2, with a possible basis being given by the linear operators $\mathbf{P}_s,\ \mathbf{P}_a : V \otimes V \to V \otimes V$ projecting onto the symmetric (sextet) and anti-symmetric (anti-triplet) tensors in $V \otimes V$, respectively.

We are only interested in color summed (averaged) cross sections, which depend on the norm squared of the color structure,

$$\|\mathbf{c}\|^2 = \langle \mathbf{c} | \mathbf{c} \rangle\,, \tag{1.3}$$

where the scalar product is given by summing over all external color indices, i.e.

$$\langle \mathbf{c}_1 | \mathbf{c}_2 \rangle = \sum_{a_1, a_2, \ldots} \mathbf{c}_1^{*a_1\, a_2 \ldots}\, \mathbf{c}_2^{a_1\, a_2 \ldots} \tag{1.4}$$

with $a_i = 1, \ldots, N_c$ if parton i is a quark or anti-quark and $a_i = 1, \ldots, N_c^2 - 1$ if parton i is a gluon. For N_g even, i.e. $N_g = 2n_g$, color structures $\mathbf{c} \in (V \otimes \overline{V})^{\otimes n_q} \otimes A^{\otimes 2n_g}$ can be viewed as linear operators $\mathbf{c} : V^{\otimes n_q} \otimes A^{\otimes n_g} \to V^{\otimes n_q} \otimes A^{\otimes n_g}$ and the scalar product (1.4) reads

$$\langle \mathbf{c}_1 | \mathbf{c}_2 \rangle = \mathrm{tr}(\mathbf{c}_1^\dagger \mathbf{c}_2)\,. \tag{1.5}$$

Out of these operators the hermitian projectors,

$$\mathbf{P}_{i_1 \ldots i_{n_q}, o_1 \ldots o_{n_q}} = (\mathbf{P}^\dagger)_{i_1 \ldots i_{n_q}, o_1 \ldots o_{n_q}} = (\mathbf{P}_{o_1 \ldots o_{n_q}, i_1 \ldots i_{n_q}})^* \tag{1.6}$$

onto $\mathrm{SU}(N_c)$-invariant subspaces of $V^{\otimes n_q} \otimes A^{\otimes n_g}$ play a special role. They are examples of color singlets in $(V \otimes \overline{V})^{\otimes n_q} \otimes A^{\otimes 2n_g}$. If we chose them to be mutually transversal,

$$\mathbf{P}_j \mathbf{P}_k = 0 \quad \forall\, j \neq k\,, \tag{1.7}$$

i.e. the image of each projector is contained in the kernel of all the others, then hermiticity implies that they project onto mutually orthogonal subspaces, and that the projectors are themselves orthogonal with respect to the scalar product (1.5),

$$\langle \mathbf{P}_j | \mathbf{P}_k \rangle = \mathrm{tr}(\mathbf{P}_j^\dagger \mathbf{P}_k) = \mathrm{tr}(\mathbf{P}_j \mathbf{P}_k) = 0 \quad \forall\, j \neq k\,. \tag{1.8}$$

These projectors can therefore be used in the construction of orthogonal bases. Denoting by d_j the dimension of the image of $\mathbf{P}_j$ we also find

$$\|\mathbf{P}_j\|^2 = \mathrm{tr}(\mathbf{P}_j^\dagger \mathbf{P}_j) = \mathrm{tr}(\mathbf{P}_j^2) = \mathrm{tr}(\mathbf{P}_j) = d_j\,, \tag{1.9}$$

and thus $\mathbf{P}_j/\sqrt{d_j}$ is normalized with respect to the scalar product (1.4).

In the example of $qq \to qq$ above, the color space was spanned by these projectors alone. In general this is not the case. If the same multiplet appears several times in the decomposition of $V^{\otimes n_q} \otimes A^{\otimes n_g}$, then also operators describing transitions from one instance of a multipet to a different instance of the same multiplet constitute linearly independent vectors in color space.

Hermitian projectors onto SU(N_c)-invariant irreducible subspaces of $A^{\otimes n_g}$ will be our starting point for the construction of orthogonal bases of the color space within $A^{\otimes 2n_g}$. Then we will see that these projectors also enable the construction of orthogonal bases for the color space for $A^{\otimes n_g} \to A^{\otimes (n_g+1)}$, i.e. for the color singlet space within $A^{\otimes(2n_g+1)}$. Finally, when there are external quarks, we take advantage of $V \otimes \overline{V} = \bullet \oplus A$, where $\bullet$ denotes the singlet, i.e. a subspace transforming under the trivial representation. This implies

$$(V \otimes \overline{V})^{\otimes n_q} = (\bullet \oplus A)^{\otimes n_q} = \bigoplus_{\nu=0}^{n_q} \binom{n_q}{\nu} A^{\otimes \nu}\,, \tag{1.10}$$

where $A^{\otimes 0} = \bullet$, and thus, we are able to construct orthogonal bases for the color spaces within $(V \otimes \overline{V})^{\otimes n_q} \otimes A^{\otimes 2n_g}$ or $(V \otimes \overline{V})^{\otimes n_q} \otimes A^{\otimes(2n_g+1)}$, as soon as we have constructed the projectors for $A^{\otimes \nu}$ $\forall\, \nu = 1, \ldots, n_q + n_g$.

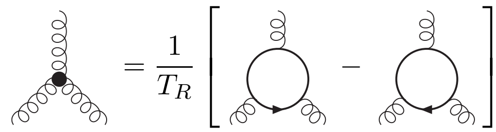
1.2 Trace bases

For tree level processes involving only gluons, the most popular way to keep track of the color structure is probably to use a basis consisting of traces over SU(3) generators [6–12, 14]. A general amplitude $\mathcal{A}$ can then be written as

$$\mathcal{A} = \sum_{\sigma \in S_{N_g-1}} \mathcal{A}_\sigma\, \mathrm{tr}[t^1 t^{\sigma(2)} \cdots t^{\sigma(N_g)}], \tag{1.11}$$

where σ denotes a permutation, i.e. $\mathcal{A}$ is a sum over (color) scalar subamplitudes $\mathcal{A}_\sigma$ (also referred to as color ordered, dual or partial amplitudes) multiplying the color structures given by the traces. Note that fixing the position of the first generator does not impose any restriction due to the cyclicity of the trace. For tree-level gluon-only processes there are thus $(N_g - 1)!$ basis vectors. In diagrammatic notation these traces are quark loops with N_g gluon lines attached. That every tree-level gluon amplitude can be written in this way can be seen as follows [8]. Consider any tree level diagram and first rewrite any four-gluon vertex in terms of three gluon vertices. Then replace the triple gluon vertices using

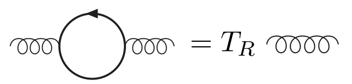
$$\mathrm{i}f_{abc} = \frac{1}{T_R}\left[\,\mathrm{tr}(t^a t^b t^c) - \mathrm{tr}(t^b t^a t^c)\right] \quad \Leftrightarrow$$



(1.12)

where the arbitrary normalization constant T_R is defined by

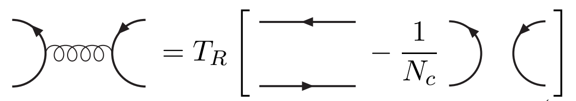
$$\mathrm{tr}[t^a t^b] = T_R\, \delta_{ab} \quad \Leftrightarrow$$



(1.13)

Finally, remove every internal gluon propagator using the Fierz-type identity

$$(t^a)^i_j (t^a)^k_l = T_R\left[\delta^i_l \delta^k_j - \frac{1}{N_c}\delta^i_j \delta^k_l\right] \quad \Leftrightarrow$$



(1.14)

Noting that the color suppressed terms drop out, see e.g. [24, sec. 9.14], the final result is a sum of traces of the form given in eq. (1.11). At loop level it is necessary to also incorporate basis vectors which are products of traces. In general, considering processes to order N_{loop}, it is necessary to include states which are direct products of up to N_{loop} different traces. As $\mathrm{tr}[t^a] = 0$, the basis vectors for calculations to arbitrary order in the coupling constant have at most $N_g/2$ traces since each trace has to contain at least two generators. Considering all ways of partitioning N_g gluons into traces does thus always give a basis which can be used to any order in perturbation theory. This basis is complete for $N_g \le N_c$, but it is overcomplete for $N_g > N_c$ [7,8]. Moreover, it is not orthogonal. This is a significant drawback due to the rapid growth of the number of basis vectors with the number of external gluons (partons in general). Counting the number of basis vectors can be reduced to the problem of mapping N_g units to N_g units without mapping a single one to itself (no generator can stand alone inside a trace). There are thus

$$\mathrm{subfactorial}(N_g) = N_g! \sum_{\nu=0}^{N_g} \frac{(-1)^\nu}{\nu!} \approx N_g!/e \tag{1.15}$$

basis vectors, giving rise to $\approx (N_g!/e)^2$ terms when calculating scalar products.

For processes involving quarks the basis may be constructed similarly, by starting with connecting all n_q quark ends to the n_q anti-quark ends, and then attaching the gluons in all possible ways to these open quark lines. Again, at loop level, new color structures have to be considered. For calculations up to N_{loop} we, in general, also have to include color structures which, in addition to the n_q open quark lines, also have up to N_{loop} closed quarks lines, i.e. traces of subsets of generators. Again the basis vectors will be non-orthogonal, and the number of basis vectors will grow roughly like a factorial. The exact number of basis vectors for a total of N_g gluons and n_q $q\bar{q}$-pairs can be found using the recursion relation

$$N_{\mathrm{vec}}[n_q, N_g] = N_{\mathrm{vec}}[n_q, N_g - 1](N_g - 1 + n_q) + N_{\mathrm{vec}}[n_q, N_g - 2](N_g - 1)\,, \tag{1.16}$$

with

$$N_{\text{vec}}[n_q, 0] = n_q! \ , \quad N_{\text{vec}}[n_q, 1] = n_q n_q! \ , \tag{1.17}$$

or, alternatively, by using an exponential generating function [8, 25]. The first term in eq. (1.16) comes from attaching the new gluon line to any of the existing (open or closed) quark lines, whereas the last term comes from basis vectors in which the generator for the new gluon stands inside the same trace as *one* of the $N_g - 1$ other gluons.

For special cases the number of degrees of freedom for the sub-amplitudes have been seen to reduce significantly, and powerful recursion relations have been derived. Especially, this is the case for tree-level pure Yang-Mills theory, as in eq. (1.11), [10, 26–33]. While these strategies may significantly reduce the computational effort in the situations they are tailored for, we here pursue a general approach. We aim for minimal orthogonal bases, which can be used for any number and kind of partons, and to any order in perturbation theory. We demonstrate that such bases can be constructed using hermitian transversal projectors onto different irreducible representations. The resulting bases are orthogonal, and can easily be chosen minimal for any finite N_c, such as $N_c = 3$.

1.3 Illustration: $gg \to gg$

Our method will be based on first constructing hermitian projectors which decompose $A^{\otimes n_g}$ into irreducible subspaces invariant under $\mathrm{SU}(N_c)$. We will then show how these can be used for constructing complete orthogonal bases, for processes involving up to $2n_g + 1$ gluons, and processes where a subset of the gluons has been replaced by $q\overline{q}$-pairs. Let us sketch this procedure for $n_g = 2$.

The $\mathrm{SU}(N_c)$ irreducible representations involved in the decomposition of $A^{\otimes n_g}$ can, e.g., be obtained, by multiplying the corresponding Young diagrams [34],

$$\begin{array}{ccccccccccccccccc}
(N_c\text{-}1,1) & & (N_c\text{-}1,1) & & (N_c) & & (N_c\text{-}1,1) & & (N_c\text{-}1,1) & & (N_c\text{-}2,1,1) & & (N_c\text{-}1,N_c\text{-}1,2) & & (N_c\text{-}1,N_c\text{-}1,1,1) & & (N_c\text{-}2,2) \\
\text{(2,1)} & \otimes & \text{(2,1)} & = & \bullet & \oplus & \text{(2,1)} & \oplus & \text{(2,1)} & \oplus & \text{(3)} & \oplus & \text{(3,3)} & \oplus & \text{(4,2)} & \oplus & \circ \ . \\
8 & & 8 & & 1 & & 8 & & 8 & & 10 & & \overline{10} & & 27 & & 0
\end{array} \tag{1.18}$$

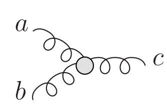
Here and in the following we represent irreducible representations in several ways: On the first line we uniquely specify the multiplets in terms of the lengths of the *columns* of the corresponding Young diagrams. On the second line we specialize to $N_c = 3$ displaying actual Young diagrams. There we denote by $\circ$ any irreducible representation that does not exist for $N_c = 3$ but only for sufficiently large N_c. Also recall that we denote by $\bullet$ the trivial rep, i.e. $\bullet = $ (single column of 3 boxes) for $N_c = 3$. Finally, on the third line we give the dimensions of the respective SU(3) multiplets.

Hermitian projectors corresponding to eq. (1.18) have been given in several places. The earliest reference known to us is [35], where they are given for $N_c = 3$. A derivation for arbitrary N_c in terms of birdtracks is given by Cvitanović in [36, sec. 6.D & tab. 6.3], see also [24, sec. 9.12 & tab. 9.4]. Cvitanović employs characteristic equations for invariant matrices in order to construct the projectors. Our approach described in section 4, which is inspired by [24], will avoid factorizing characteristic equations but instead provide an algorithm for directly writing down the projectors. A slightly different diagrammatic derivation, also for arbitrary N_c, is given by Dokshitzer and Marchesini [3]. Our construction for a certain class of projectors in section 4.4 is a generalization of their method. For the moment we list the two gluon projectors without derivation,

$$
\begin{aligned}
\mathbf{P}^{1}_{g_1\,g_2\,g_3\,g_4} &= \frac{1}{N_c^2-1}\delta_{g_1\,g_2}\delta_{g_3\,g_4}\,, \\
\mathbf{P}^{8s}_{g_1\,g_2\,g_3\,g_4} &= \frac{N_c}{2T_R(N_c^2-4)}d_{g_1 g_2 i_1}d_{i_1 g_3 g_4}\,, \\
\mathbf{P}^{8a}_{g_1\,g_2\,g_3\,g_4} &= \frac{-1}{2N_cT_R}\mathrm{i}f_{g_1\,g_2\,i_1}\mathrm{i}f_{i_1\,g_3\,g_4}\,, \\
\mathbf{P}^{10}_{g_1\,g_2\,g_3\,g_4} &= \frac{1}{4}(\delta_{g_1\,i_1}\delta_{g_2\,i_2}-\delta_{g_1\,i_2}\delta_{g_2\,i_1})\left[\delta_{i_1\,g_3}\delta_{i_2\,g_4}+\frac{1}{T_R^2}\,\mathrm{tr}(t^{i_1}t^{g_4}t^{i_2}t^{g_3})\right]-\frac{1}{2}\,\mathbf{P}^{8a}_{g_1\,g_2\,g_3\,g_4}\,, \\
\mathbf{P}^{\overline{10}}_{g_1\,g_2\,g_3\,g_4} &= \frac{1}{4}(\delta_{g_1\,i_1}\delta_{g_2\,i_2}-\delta_{g_1\,i_2}\delta_{g_2\,i_1})\left[\delta_{i_1\,g_3}\delta_{i_2\,g_4}-\frac{1}{T_R^2}\,\mathrm{tr}(t^{i_1}t^{g_4}t^{i_2}t^{g_3})\right]-\frac{1}{2}\,\mathbf{P}^{8a}_{g_1\,g_2\,g_3\,g_4}\,, \\
\mathbf{P}^{27}_{g_1\,g_2\,g_3\,g_4} &= \frac{1}{4}(\delta_{g_1\,i_1}\delta_{g_2\,i_2}+\delta_{g_1\,i_2}\delta_{g_2\,i_1})\left[\delta_{i_1\,g_3}\delta_{i_2\,g_4}+\frac{1}{T_R^2}\,\mathrm{tr}(t^{i_1}t^{g_4}t^{i_2}t^{g_3})\right] \\
&\quad -\frac{N_c-2}{2N_c}\,\mathbf{P}^{8s}_{g_1\,g_2\,g_3\,g_4}-\frac{N_c-1}{2N_c}\,\mathbf{P}^{1}_{g_1\,g_2\,g_3\,g_4}\,, \\
\mathbf{P}^{0}_{g_1\,g_2\,g_3\,g_4} &= \frac{1}{4}(\delta_{g_1\,i_1}\delta_{g_2\,i_2}+\delta_{g_1\,i_2}\delta_{g_2\,i_1})\left[\delta_{i_1\,g_3}\delta_{i_2\,g_4}-\frac{1}{T_R^2}\,\mathrm{tr}(t^{i_1}t^{g_4}t^{i_2}t^{g_3})\right] \\
&\quad -\frac{N_c+2}{2N_c}\,\mathbf{P}^{8s}_{g_1\,g_2\,g_3\,g_4}-\frac{N_c+1}{2N_c}\,\mathbf{P}^{1}_{g_1\,g_2\,g_3\,g_4}\,,
\end{aligned}
\tag{1.19}
$$

where we have introduced the totally symmetric tensor

$$
d_{abc} := \frac{1}{T_R}\left[\,\mathrm{tr}(t^a t^b t^c)+\mathrm{tr}(t^b t^a t^c)\right] = \tag{1.20}
$$



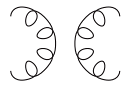
In eq. (1.19), and often in the following, we label projection operators by the dimensions of SU(3) multiplets, although our construction is for arbitrary N_c, and for $N_c \neq 3$ the dimensions differ. If a multiplet appears several times we add some additional label, as for the octets above. Since $\mathbf{P}^0$ vanishes for $N_c = 3$, we also have $\mathbf{P}^{27} = \mathbf{P}^{27} + \mathbf{P}^0$ in this case, which allows to write $\mathbf{P}^{27}$ in a simpler form,

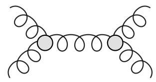
$$
\mathbf{P}^{27}_{g_1\,g_2\,g_3\,g_4} \underset{N_c=3}{=} \frac{1}{2}(\delta_{g_1\,g_3}\delta_{g_2\,g_4}+\delta_{g_1\,g_4}\delta_{g_2\,g_3})-\mathbf{P}^{8s}_{g_1\,g_2\,g_3\,g_4}-\mathbf{P}^{1}_{g_1\,g_2\,g_3\,g_4}\,. \tag{1.21}
$$

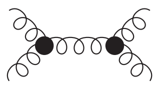
This is the way in which $\mathbf{P}^{27}$ is given in [35]. As gluons transform in a real representation, for processes involving only gluons, the decuplet projectors occur only in the real combination [5, 14, 22, 35, 37]

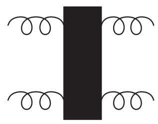
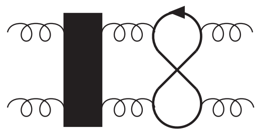
$$
\left(\mathbf{P}^{10}+\mathbf{P}^{\overline{10}}\right)_{g_1\,g_2\,g_3\,g_4} = \frac{1}{2}(\delta_{g_1\,g_3}\delta_{g_2\,g_4}-\delta_{g_1\,g_4}\delta_{g_2\,g_3})-\mathbf{P}^{8a}_{g_1\,g_2\,g_3\,g_4}\,. \tag{1.22}
$$

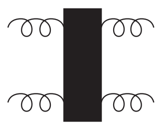
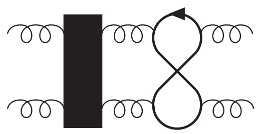
However, for processes involving quarks $\mathbf{P}^{10}$ and $\mathbf{P}^{\overline{10}}$ can appear independently. In birdtrack notation [8, 24] eq. (1.19) reads

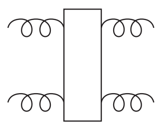
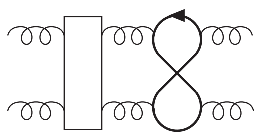
$\mathbf{P}^{1} = \frac{1}{N_c^2-1}$ 

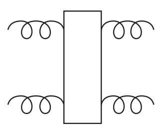
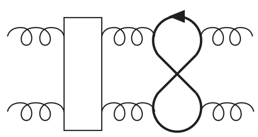
$\mathbf{P}^{8s} = \frac{N_c}{2T_R(N_c^2-4)}$ 

$\mathbf{P}^{8a} = \frac{1}{2N_cT_R}$ 

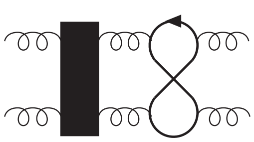
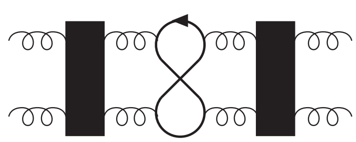
$\mathbf{P}^{10} = \frac{1}{2}$  $+ \frac{1}{2T_R^2}$  $- \frac{1}{2}\mathbf{P}^{8a}$ (1.23)

$\mathbf{P}^{\overline{10}} = \frac{1}{2}$  $- \frac{1}{2T_R^2}$  $- \frac{1}{2}\mathbf{P}^{8a}$

$\mathbf{P}^{27} = \frac{1}{2}$  $+ \frac{1}{2T_R^2}$  $- \frac{N_c-2}{2N_c}\mathbf{P}^{8s} - \frac{N_c-1}{2N_c}\mathbf{P}^{1}$

$\mathbf{P}^{0} = \frac{1}{2}$  $- \frac{1}{2T_R^2}$  $- \frac{N_c+2}{2N_c}\mathbf{P}^{8s} - \frac{N_c+1}{2N_c}\mathbf{P}^{1}.$

The black and white bars denote anti-symmetrization and symmetrization, respectively, see appendix A. One can easily verify that

 $=$  , (1.24)

and similarly for the symmetrized expression, making the hermiticity of these projectors manifest.

From these projection operators orthogonal bases have been constructed for processes involving up to five gluons [5]. In general, knowing the projection operators for up to n_g gluons it is possible to construct orthogonal bases for QCD processes involving up to $2n_g+1$ gluons (where we assume for the moment that there are no quarks). The basis for $2n_g+1$ gluons can be constructed by considering, e.g., $n_g \to n_g+1$. The incoming gluons may then be projected onto a multiplet M using the projection operators for n_g gluons. If the incoming n_g gluons are in M the outgoing n_g+1 gluons must, due to color conservation, be in the same multiplet, see appendix F. However, the multiplet M may appear more than once in $A^{\otimes n_g}$ or $A^{\otimes(n_g+1)}$ or both. For example, there are six 27-plets in $A^{\otimes 3}$, and one in

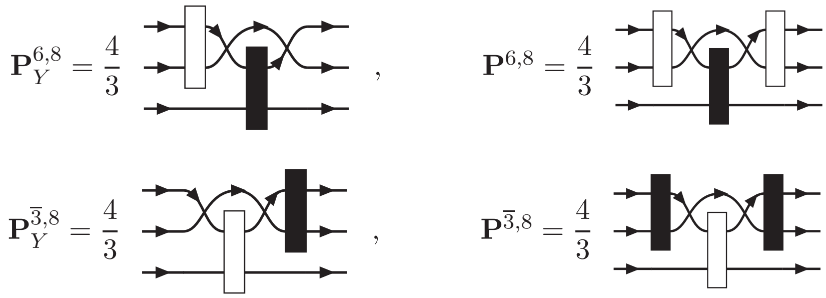


Figure 2: The standard Young projection operators $\mathbf{P}_Y^{6,8}$ and $\mathbf{P}_Y^{\overline{3},8}$ compared to their hermitian versions $\mathbf{P}^{6,8}$ and $\mathbf{P}^{\overline{3},8}$ from eq. (2.2). Clearly $\mathbf{P}^{6,8\dagger}\mathbf{P}^{\overline{3},8} = \mathbf{P}^{6,8}\mathbf{P}^{\overline{3},8} = 0$. However, as can be seen from the symmetries, $\mathbf{P}_Y^{6,8\dagger}\mathbf{P}_Y^{\overline{3},8} \neq 0$.

$A^{\otimes 2}$. For $2g \to 3g$ there are thus one (from the incoming side) $\times$ six (from the outgoing side) possibilities for the gluons to be in matching 27-plets. The 27-plets corresponding to the case that two of the gluons in the outgoing $A^{\otimes 3}$ are in a decuplet and an anti-decuplet, do, however, only appear in combination.

2. Hermitian quark projectors

In this section we discuss projection operators for n_q quarks. Later, in section 3 we use the hermitian versions of these projection operators in order to construct an orthogonal basis of the color space for n_q $q\overline{q}$ pairs.

A standard method for constructing projection operators onto irreducible subspaces invariant under $\mathrm{SU}(N_c)$ is to symmetrize and anti-symmetrize according to the corresponding Young tableaux, and – in the case of five or more quarks – successively project out already constructed projectors for Young tableaux of equal shape, see e.g. [38, sec. 5.4]. In this way, a complete set of projection operators can be constructed for any number of quarks. These projection operators are, however, *not* hermitian, see figure 2, which implies that they are not suited for constructing an orthogonal basis of the color space for n_q $q\overline{q}$ pairs, as we cannot use eq. (1.8).

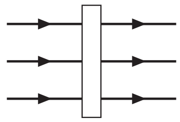
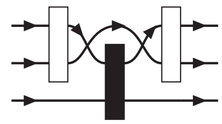
Diagrammatically speaking, these operators have been constructed such that products of distinct projectors vanish when contracting the *outgoing* indices of the first projector with the *incoming* indices of the second one; however, when calculating scalar products in the color space of n_q $q\overline{q}$-pairs, the outgoing indices of the first vector are contracted with the *outgoing* indices of the second vector, cf. eq. (1.4). Therefore, standard Young projection operators are not orthogonal in the sense of eq. (1.4). By utilizing hermitian projection operators this problem can be circumvented.

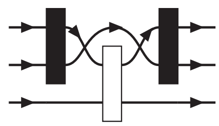
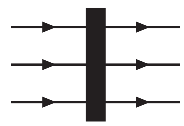
Hermitian Young projectors for three quarks were given in [9]. In [24] a general method for constructing hermitian Young projectors is developed. This method is based on solving certain characteristic equations. An alternative approach for directly writing down hermitian Young projectors will be presented elsewhere [39].

The projectors can be expressed in terms of symmetrization and anti-symmetrization operators, cf. eq. (A.8). Here and in the following we label projection operators by the multiplets built up successively when multiplying the partons, i.e.

$$\mathbf{P}^{M_2,M_3} \tag{2.1}$$

denotes a projector onto states where parton 1 and parton 2 are in a multiplet M_2, and together with parton 3 form a multiplet M_3. In this notation we have

$\mathbf{P}^{6,10} =$  , $\mathbf{P}^{6,8} = \frac{4}{3}$  ,

$\mathbf{P}^{\overline{3},8} = \frac{4}{3}$  , $\mathbf{P}^{\overline{3},1} =$  . (2.2)

In index notation, and written out as sums over permutations, these projection operators read

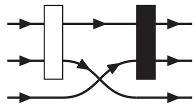
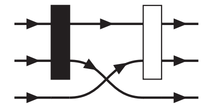
$$\begin{aligned}
\mathbf{P}^{6,10}{}_{q_1\,q_2\,q_3\,q_4\,q_5\,q_6} &= \frac{1}{6}\big(\delta^{q_1}_{q_4}\delta^{q_2}_{q_5}\delta^{q_3}_{q_6} + \delta^{q_1}_{q_4}\delta^{q_2}_{q_6}\delta^{q_3}_{q_5} + \delta^{q_1}_{q_5}\delta^{q_2}_{q_4}\delta^{q_3}_{q_6} \\
&\qquad + \delta^{q_1}_{q_5}\delta^{q_2}_{q_6}\delta^{q_3}_{q_4} + \delta^{q_1}_{q_6}\delta^{q_2}_{q_4}\delta^{q_3}_{q_5} + \delta^{q_1}_{q_6}\delta^{q_2}_{q_5}\delta^{q_3}_{q_4}\big)\,, \\
\mathbf{P}^{6,8}{}_{q_1\,q_2\,q_3\,q_5\,q_4\,q_6} &= \frac{1}{6}\big(2\delta^{q_1}_{q_5}\delta^{q_2}_{q_4}\delta^{q_3}_{q_6} - \delta^{q_1}_{q_5}\delta^{q_2}_{q_6}\delta^{q_3}_{q_4} + 2\delta^{q_1}_{q_4}\delta^{q_2}_{q_5}\delta^{q_3}_{q_6} \\
&\qquad - \delta^{q_1}_{q_4}\delta^{q_2}_{q_6}\delta^{q_3}_{q_5} - \delta^{q_1}_{q_6}\delta^{q_2}_{q_5}\delta^{q_3}_{q_4} - \delta^{q_1}_{q_6}\delta^{q_2}_{q_4}\delta^{q_3}_{q_5}\big)\,, \\
\mathbf{P}^{\overline{3},8}{}_{q_1\,q_2\,q_3\,q_5\,q_4\,q_6} &= \frac{1}{6}\big(-2\delta^{q_1}_{q_5}\delta^{q_2}_{q_4}\delta^{q_3}_{q_6} - \delta^{q_1}_{q_5}\delta^{q_2}_{q_6}\delta^{q_3}_{q_4} + 2\delta^{q_1}_{q_4}\delta^{q_2}_{q_5}\delta^{q_3}_{q_6} \\
&\qquad + \delta^{q_1}_{q_4}\delta^{q_2}_{q_6}\delta^{q_3}_{q_5} + \delta^{q_1}_{q_6}\delta^{q_2}_{q_5}\delta^{q_3}_{q_4} - \delta^{q_1}_{q_6}\delta^{q_2}_{q_4}\delta^{q_3}_{q_5}\big)\,, \\
\mathbf{P}^{\overline{3},1}{}_{q_1\,q_2\,q_3\,q_4\,q_5\,q_6} &= \frac{1}{6}\big(\delta^{q_1}_{q_4}\delta^{q_2}_{q_5}\delta^{q_3}_{q_6} - \delta^{q_1}_{q_4}\delta^{q_2}_{q_6}\delta^{q_3}_{q_5} - \delta^{q_1}_{q_5}\delta^{q_2}_{q_4}\delta^{q_3}_{q_6} \\
&\qquad + \delta^{q_1}_{q_5}\delta^{q_2}_{q_6}\delta^{q_3}_{q_4} + \delta^{q_1}_{q_6}\delta^{q_2}_{q_4}\delta^{q_3}_{q_5} - \delta^{q_1}_{q_6}\delta^{q_2}_{q_5}\delta^{q_3}_{q_4}\big)\,.
\end{aligned} \tag{2.3}$$

As these projectors are hermitian, i.e. their birdtrack diagrams (2.2) are invariant under reflection about a vertical line and simultaneous inversion of all arrows, cf. appendix A, they are not only mutually transversal, cf. eq. (1.7), but also orthogonal with respect to the scalar product (1.4).

3. Quark bases from hermitian quark projectors

When viewed as vectors in the color space for $(V \otimes \overline{V})^{\otimes 3}$ the projectors in eq. (2.2) do not span the full space, since operators describing transitions from one instance of a multiplet to any other instance of that multiplet also transform as singlets under $\mathrm{SU}(N_c)$. A basis of the color space for three $q\overline{q}$ pairs thus contains four different vectors derived from the octets. Normalized orthogonal basis vectors can be chosen as follows, [9, Fig. 21],

$$\mathbf{V}^{6,10;6,10} = \sqrt{\frac{6}{N_c(N_c^2+3N_c+2)}}\,\mathbf{P}^{6,10}\,, \qquad \mathbf{V}^{6,8;6,8} = \sqrt{\frac{3}{N_c(N_c^2-1)}}\,\mathbf{P}^{6,8}\,,$$

$$\mathbf{V}^{6,8;\overline{3},8} = \frac{2}{\sqrt{N_c(N_c^2-1)}}$$  $$,\qquad \mathbf{V}^{\overline{3},8;6,8} = \frac{2}{\sqrt{N_c(N_c^2-1)}}$$  $$,$$

$$\mathbf{V}^{\overline{3},8;\overline{3},8} = \sqrt{\frac{3}{N_c(N_c^2-1)}}\,\mathbf{P}^{\overline{3},8}\,, \qquad \mathbf{V}^{\overline{3},1;\overline{3},1} = \sqrt{\frac{6}{N_c(N_c^2-3N_c+2)}}\,\mathbf{P}^{\overline{3},1}\,, \tag{3.1}$$

where each basis vector is denoted by the construction history on the incoming and outgoing side in sequence, and the normalization is consistent with eq. (1.9). In index notation the two vectors describing transitions between the octets can be written as

$$\begin{aligned}
\mathbf{V}^{6,8;\overline{3},8}_{q_1\,q_2\,q_3\,q_4\,q_5\,q_6} &= \frac{1}{2\sqrt{N_c(N_c^2-1)}}\left(\delta^{q_1}_{q_4}\delta^{q_2}_{q_6}\delta^{q_3}_{q_5} - \delta^{q_1}_{q_5}\delta^{q_2}_{q_6}\delta^{q_3}_{q_4} + \delta^{q_1}_{q_6}\delta^{q_2}_{q_4}\delta^{q_3}_{q_5} - \delta^{q_1}_{q_6}\delta^{q_2}_{q_5}\delta^{q_3}_{q_4}\right),\\
\mathbf{V}^{\overline{3},8;6,8}_{q_1\,q_2\,q_3\,q_4\,q_5\,q_6} &= \frac{1}{2\sqrt{N_c(N_c^2-1)}}\left(\delta^{q_1}_{q_4}\delta^{q_2}_{q_6}\delta^{q_3}_{q_5} + \delta^{q_1}_{q_5}\delta^{q_2}_{q_6}\delta^{q_3}_{q_4} - \delta^{q_1}_{q_6}\delta^{q_2}_{q_4}\delta^{q_3}_{q_5} - \delta^{q_1}_{q_6}\delta^{q_2}_{q_5}\delta^{q_3}_{q_4}\right).
\end{aligned} \tag{3.2}$$

These two basis vectors can be constructed as follows. In order to find a vector describing a transition from $\overline{3}, 8$ to $6, 8$ write down the birdtrack expression for $\mathbf{P}^{6,8}$ on the left and that for $\mathbf{P}^{\overline{3},8}$ on the right; now one has to find a non-vanishing way for connecting these diagrams. To this end, note that there is a symmetrizer to the very left in $\mathbf{P}^{6,8}$ and an anti-symmetrizer to the very right in $\mathbf{P}^{\overline{3},8}$. If both lines leaving the white bar enter the black bar then the whole expression vanishes, i.e. one of the lines leaving the white bar has to be connected to the third line on the right. One such choice is displayed in the diagram for $\mathbf{V}^{6,8;\overline{3},8}$ above. Any other non-vanishing choice yields the same vector up to a factor. Finally, the result has to be normalized using the scalar product (1.5). The remaining vector $\mathbf{V}^{\overline{3},8;6,8}$ can either be constructed in the same way, or by taking the hermitian conjugate of $\mathbf{V}^{6,8;\overline{3},8}$.

Knowing the hermitian projection operators it is possible to similarly construct the orthogonal basis vectors for processes involving more $q\overline{q}$ pairs. The orthogonality can be seen by noting that contracting the incoming or the outgoing indices gives 0. We also note that there are six basis vectors, in agreement with what is obtained from the $n_q!$ ways of connecting quark and anti-quark lines in eq. (1.17). In this case there are thus equally many vectors for $N_c = 3$ as for $N_c \to \infty$.

4. Hermitian gluon projectors

In this section we outline a general algorithm for constructing hermitian projectors for all multiplets appearing in $A^{\otimes n_g}$ for arbitrary n_g. The construction is recursive, i.e. the projectors for the decomposition of $A^{\otimes \nu}$ with $\nu \leq n_g - 1$ along with their properties are used when constructing the projectors onto multiplets within $A^{\otimes n_g}$. As an illustrating example we treat the case $n_g = 3$ along with the outline of the general construction.

For our algorithm it is important to keep track of for which n a given multiplet M appears for the first time in the sequence $A^{\otimes n}$, $n = 0, 1, 2, \ldots$. We denote this number by $n_f(M)$ and call it that multiplet's *first occurrence*. For instance the singlet has $n_f(\bullet) = 0$

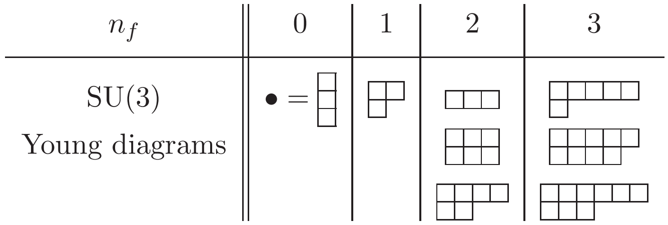
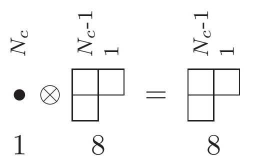


Table 1: Examples of SU(3) Young diagrams sorted according to their first occurrence n_f.

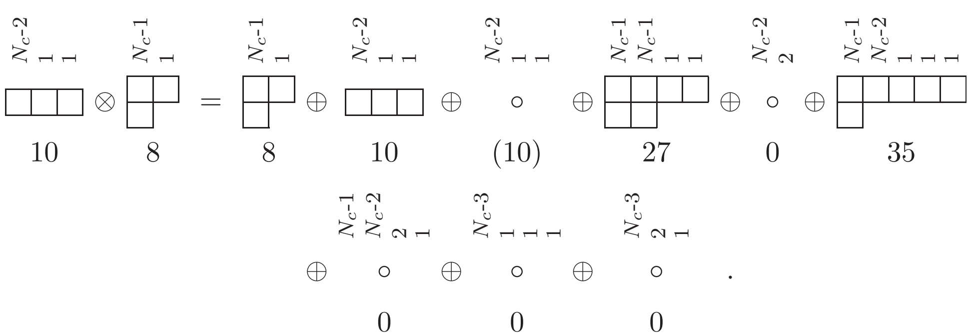
and the adjoint representation has first occurrence one. Some more examples, labeled by SU(3) Young diagrams, are listed in table 1.

In order to make sure that projectors onto all invariant subspaces are constructed we first decompose $A^{\otimes n_g}$ into multiplets, $A^{\otimes n_g} = \bigoplus_j M_j$, by multiplying Young diagrams. The $n_g = 2$ decomposition has already been preformed in eq. (1.18). For $n_g = 3$ we have to multiply the r.h.s. of eq. (1.18) term by term with another gluon. Multiplication of the singlet trivially yields an octet,



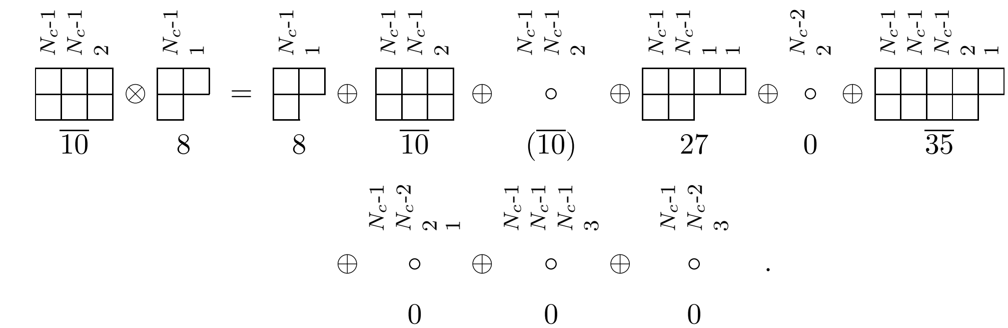
(4.1)

The product of two octets is already displayed in eq. (1.18). When multiplying the decuplet with an octet we have



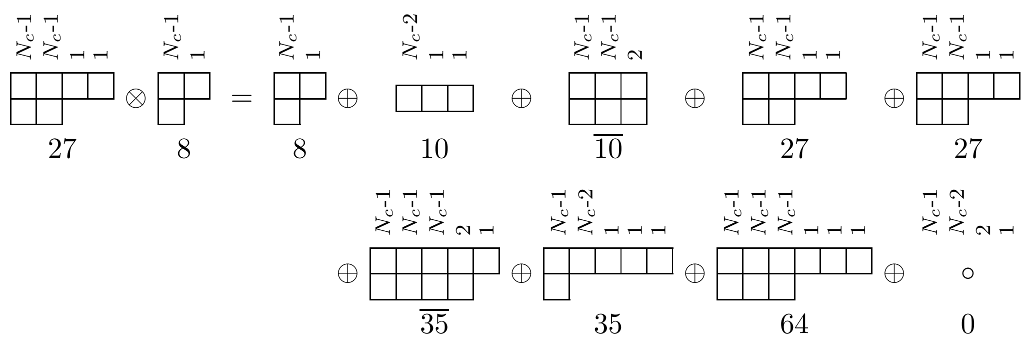
(4.2)

As above, we in general denote multiplets that do not appear for $N_c = 3$, but only for sufficiently large N_c, by $\circ$. While the second $(N_c - 2, 1, 1)$-multiplet, has a Young tableaux shape which is admissible for SU(3), it can be seen from Young tableaux multiplication that it cannot appear. For such multiplets – which are forbidden only by the construction – we display the corresponding SU(3)-dimension in brackets. Similarly, for the anti-decuplet we get

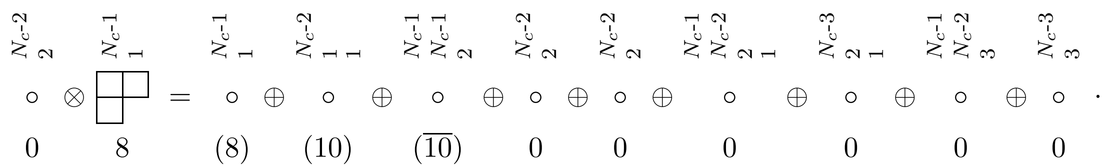


(4.3)

Finally, for the products with the remaining two multiplets, 0 and 27, we obtain



(4.4)



(4.5)

Here the first three multiplets on the r.h.s. of the last equation, $(N_c-1,1)$, $(N_c-2,1,1)$ and $(N_c-1,N_c-1,2)$, would be allowed Young diagrams for $N_c=3$. However, we denote them by $\circ$ and set the dimensions in brackets since they were obtained by multiplication of $(N_c-2,2)$, a multiplet that does not exist for $N_c=3$.

Looking at these decompositions of tensor products one can make two observations:

1. A multiplet $M' \subseteq M \otimes A$ always has first occurrence
$$n_f(M') = n_f(M)-1\,,\ n_f(M) \text{ or } n_f(M)+1\,. \tag{4.6}$$
In particular, there are no singlets in eqs. (4.2 – 4.5).

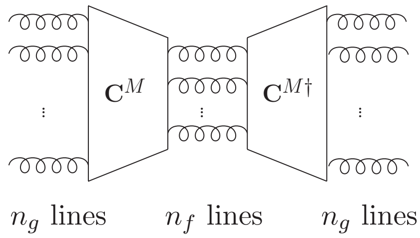
2. The only multiplet which can show up several times in $M \otimes A$ is M itself, all other multiplets appear at most once. In fact, M can appear up to N_c-1 times.

Both observations are true in general and we prove them in appendix B.

Below we outline the construction of the corresponding projectors $\mathbf{P}^{M_j}$ having the following properties:

(i) $\mathbf{P}^{M_j}\mathbf{P}^{M_k} = \delta_{jk}\mathbf{P}^{M_j}$. We call this property *transversality*, cf. eq. (1.7).

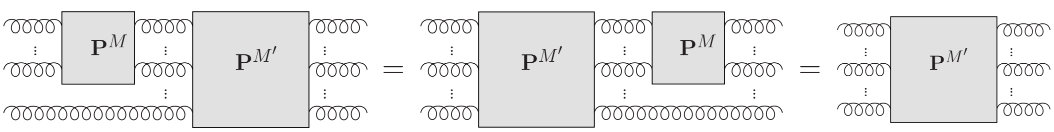
(ii) $\mathbf{P}^M = \mathbf{C}^M \mathbf{C}^{M\dagger}$ with $\mathbf{C}^M : A^{\otimes n_f} \to A^{\otimes n_g}$, where n_f is the first occurrence of M. Choosing suitable bases in $A^{\otimes n_g}$ and $A^{\otimes n_f}$ the matrix elements of $\mathbf{C}^M$ are Clebsch-Gordan coefficients. In birdtrack notation this means that there is always an intermediate section with n_f gluon lines in the middle of the diagram for each projector,

$\mathbf{P}^M =$  . (4.7)

(iii) A projector $\mathbf{P}^{M'}$ onto a multiplet $M' \subseteq A^{\otimes n_g}$ appearing in the decomposition of $M \otimes A^{\otimes(n_g-\nu)}$, i.e. $M \otimes A^{\otimes(n_g-\nu)} = M' \oplus \ldots$, satisfies

$$(\mathbf{P}^M \otimes \mathbb{1}_{A^{\otimes(n_g-\nu)}})\mathbf{P}^{M'} = \mathbf{P}^{M'}(\mathbf{P}^M \otimes \mathbb{1}_{A^{\otimes(n_g-\nu)}}) = \mathbf{P}^{M'}\,, \tag{4.8}$$

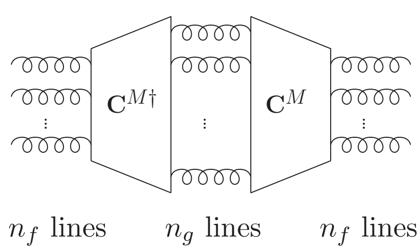
where $\mathbb{1}_{A^{\otimes(n_g-\nu)}} : A^{\otimes(n_g-\nu)} \to A^{\otimes(n_g-\nu)}$ denotes the identity operator. In terms of birdtracks this is written

 , (4.9)

i.e. the first ν gluons are in multiplet M and together with the remaining gluons they form an overall multiplet M'.

The hermiticity of $\mathbf{P}^M$ is obvious from (ii). Also note its birdtrack manifestation in eq. (4.7): The diagram is invariant under simultaneous mirroring about a vertical line through the n_f gluon lines and reversing all arrows (which may appear inside $\mathbf{C}^M$), cf. appendix A. Together with transversality (i) hermiticity ensures that the projectors project onto mutually orthogonal subspaces and are themselves mutually orthogonal with respect to the scalar product (1.5), cf. the discussion in section 1.1.

From (ii) one can infer that by multiplying the Clebsch-Gordan matrices in reverse order we obtain

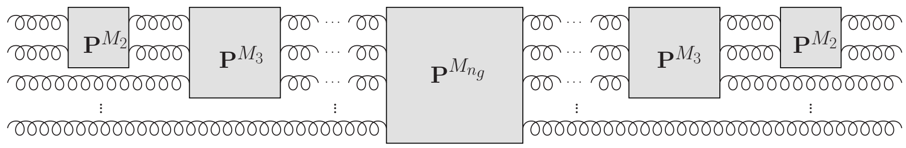
$\mathbf{C}^{M\dagger}\mathbf{C}^M =$  $= \mathbf{P}^{M_f}$, (4.10)

where $M_f \subset A^{\otimes n_f}$ carries the same irreducible representation as M. For $M \neq M'$ we have

$$\mathbf{C}^{M\dagger}\mathbf{C}^{M'} = 0\,. \tag{4.11}$$

We include proofs of eqs. (4.10) and (4.11) in appendix C.

According to property (iii) a projector $A^{\otimes n_g} \to A^{\otimes n_g}$ not only projects onto a definite multiplet, but also ensures that the first ν gluons are in multiplet M_ν, $\nu = 2, \ldots, n_g$. We refer to the sequence $M_2, M_3, \ldots, M_{n_g}$ as the projector's *construction history*. It is convenient to label projectors by their construction histories,

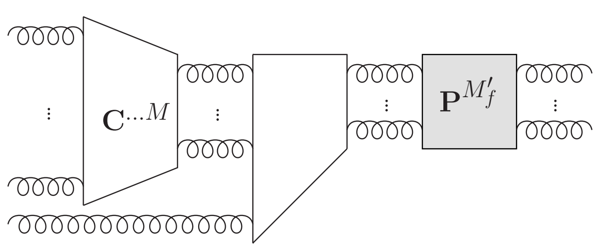
$\mathbf{P}^{M_2,M_3,\ldots,M_{n_g}} =$  . (4.12)

We prove our algorithm by induction in n_g. We can start from either $n_g = 0$ or $n_g = 1$, for which all properties are satisfied trivially. However, it is instructive to revisit the $n_g = 2$ projectors, which were given in eqs. (1.19) and (1.23), and verify that they also satisfy the properties (i)–(iii). The only property which may not be immediately obvious is (ii) for the $n_f = 2$ projectors $\mathbf{P}^{10}, \mathbf{P}^{\overline{10}}, \mathbf{P}^{27}$ and $\mathbf{P}^{0}$. Note, however, that for $n_g = n_f$ property (ii) is satisfied trivially with $\mathbf{C}^M = \mathbf{P}^M = \mathbf{C}^{M\dagger}$.

Below we outline the recursive construction of projectors for the decomposition of $A^{\otimes n_g}$ from the projectors for the decompositions of $A^{\otimes \nu}$, $\nu \le n_g - 1$. Making sure that the properties (i)–(iii) are retained by this algorithm will establish the induction step. In order to keep track of which projectors have to be constructed in step n_g we proceed as follows. For each multiplet $M \subseteq A^{\otimes(n_g-1)}$ we decompose $M \otimes A$ by multiplying the corresponding Young diagrams, as done in eq. (1.18) and eqs. (4.2–4.5) above.

Multiplets $M' \subseteq A^{\otimes n_g}$ with first occurrence $n_f(M') = n_g$ we refer to as *new multiplets*. For all other multiplets $M' \subseteq A^{\otimes n_g}$ we have $n_f(M') < n_g$ and, correspondingly, they are referred to as *old multiplets*. Multiplets $M' \subseteq M \otimes A$ with $n_f(M') = n_f(M) - 1$ or $n_f(M') = n_f(M)$ are necessarily old multiplets. Multiplets with $n_f(M') = n_f(M) + 1$ can be either old or new depending on whether M was old or new within $A^{\otimes(n_g-1)}$. Our general strategy for obtaining all projectors, is to first construct all projectors onto old multiplets, and then to use these projectors in the subsequent construction of projectors onto new multiplets.

Projectors $\mathbf{P}^{\cdots M,M'}$ onto old multiplets $M' \subseteq M \otimes A$ can always be constructed as follows. Consider the corresponding Clebsch-Gordan matrix $\mathbf{C}^{\cdots M,M'}$. In order to satisfy property (iii), there has to be a $\mathbf{C}^{\cdots M} \otimes \mathbb{1}_A$ at the left end, whereas property (ii) requires a $\mathbf{P}^{M'_f}$ at the right end,

$\mathbf{C}^{\cdots M,M'} =$  . (4.13)

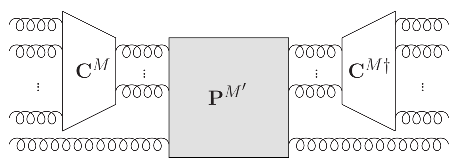
In the middle one has to connect the $n_f(M'_f)$ gluon lines on the right to the $n_f(M) + 1$

gluon lines on the left in such a way that the whole expression does not vanish. Then, after appropriate normalization, $\mathbf{P}^{\ldots M,M'} = \mathbf{C}^{\ldots M,M'}\mathbf{C}^{\ldots M,M'\dagger}$ is the desired projector.

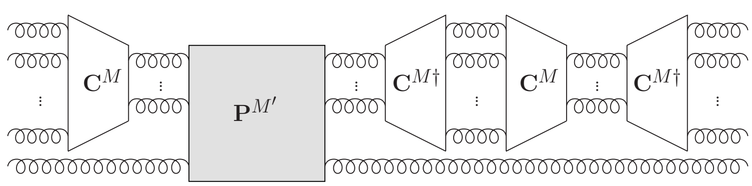
In principle, a non-vanishing connection can always be found be splitting all gluon lines entering the trapezoid into $q\bar{q}$-pairs, and then considering all ways of attaching the quark- and anti-quark ends. At least one such connection has to be non-zero. As this procedure may be tedious, in particular for many gluons, we provide more explicit recipes for the construction of projectors onto old multiplets in sections 4.1–4.3. These recipes cover most of the frequently occurring cases. In particular, they directly yield the full set of 3-gluon projectors onto old multiplets. Projectors onto new multiplets require an independent construction which we develop in section 4.4.

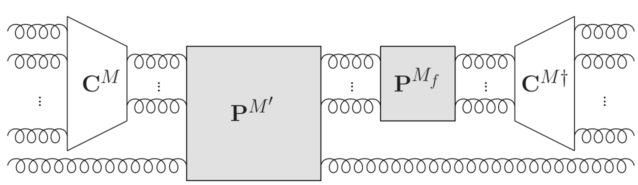
4.1 Starting from an old multiplet, $n_f(M) < n_g - 1$

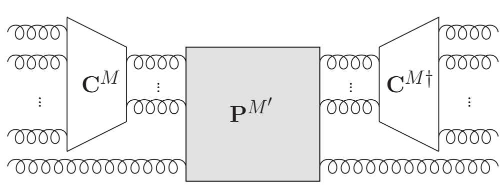
We begin with $M' \subseteq M \otimes A$ where the starting multiplet for $n_g - 1$ gluons was old, i.e. $n_f(M) < n_g - 1$. In this case we write down $\mathbf{P}^{\ldots,M}$ as depicted in property (ii), add a gluon line below and draw a projector $\mathbf{P}^{M'}$ over the $n_f(M) + 1$ gluon lines in the middle,

$\mathbf{P}^{\ldots,M,M'} =$  . (4.14)

The projector $\mathbf{P}^{M'}$ has already been constructed in an earlier step according to the induction hypothesis. The operator $\mathbf{P}^{\ldots,M,M'}$ satisfies (ii) since $\mathbf{P}^{M'}$ does. To make it explicit we could insert another copy of $\mathbf{P}^{M'}$ in the middle of the r.h.s. above. $\mathbf{P}^{\ldots,M,M'}$ also satisfies (iii) since, $\mathbf{P}^{\ldots,M}$ does and since

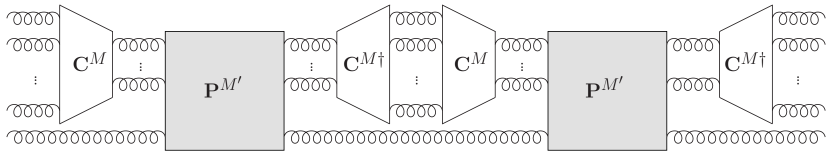
$\mathbf{P}^{\ldots,M,M'}\,(\mathbf{P}^{\ldots,M} \otimes \mathbb{1}) =$ 

$=$ 

$=$  $= \mathbf{P}^{\ldots,M,M'}$. (4.15)

Here the second line holds since $\mathbf{P}^{\ldots,M}$ fulfills (ii) which implies eq. (4.10). The first occurrence projector $\mathbf{P}^{M_f}$ appearing in this way, however, also has to appear in the construction history of $\mathbf{P}^{M'}$. Since $\mathbf{P}^{M'}$ in turn satisfies (iii) we can readily omit $\mathbf{P}^{M_f}$ from the equation. By the same arguments we establish that $\mathbf{P}^{\ldots,M,M'}$ actually is a projector, i.e. that

$(\mathbf{P}^{\ldots,M,M'})^2 = \mathbf{P}^{\ldots,M,M'}$. Writing $(\mathbf{P}^{\ldots,M,M'})^2$ in terms of birdtracks

$$\left(\mathbf{P}^{\ldots,M,M'}\right)^2 =$$  , (4.16)

the $\mathbf{C}^{M\,\dagger}\mathbf{C}^{M}$ again produces a $\mathbf{P}^{M_f}$ in the middle which can be absorbed into one of the $\mathbf{P}^{M'}$, and, since the latter is a projector, we have $(\mathbf{P}^{M'})^2 = \mathbf{P}^{M'}$ and thus get back $\mathbf{P}^{\ldots,M,M'}$.

The transversality property (i) for projectors constructed from different starting multiplets M is obvious due to (iii). For two different projectors constructed in the above described way, starting from the same multiplet M, transversality follows by repeating the last calculation with the second $\mathbf{P}^{M'}$ replaced by $\mathbf{P}^{M''}$ which then cancel.

The $n_g = 3$ projectors which can be constructed in this way are $\mathbf{P}^{1,8}$, $\mathbf{P}^{8s,M'}$ and $\mathbf{P}^{8a,M'}$ with $M' = 1,\ 8s,\ 8a,\ 10,\ \overline{10},\ 27,\ 0$. As an example we note that $\mathbf{P}^{8s,27}$ may be written

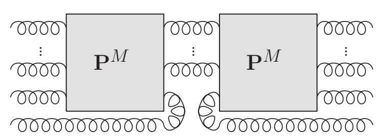
$$\mathbf{P}^{8s,27}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} = \frac{1}{T_R}\frac{N_c}{2(N_c^2-4)} d_{g_1\,g_2\,i_1}\mathbf{P}^{27}_{i_1\,g_3\,i_2\,g_6} d_{i_2\,g_4\,g_5} \tag{4.17}$$

where the normalization derives from $\mathbf{P}^{8s}$ in eq. (1.19). The other projectors are stated in appendix D.

4.2 Starting from a new multiplet and going back to one with lower n_f

When constructing multiplets $M' \subseteq M \otimes A$ where M was new in the step before, i.e. $n_f(M) = n_g - 1$, we have to distinguish the three cases $n_f(M') = n_f(M) - 1, n_f(M)$ and $n_f(M) + 1$.

We first treat the case $n_f(M') = n_f(M) - 1 = n_g - 2$. If M' also appeared in the construction history immediately before M, then we find

$$\mathbf{P}^{\ldots,M,M'} = \frac{\dim M'}{\dim M}$$  . (4.18)

Clearly, $\mathbf{P}^{\ldots,M,M'}$ satisfies properties (i), (ii) and (iii). Due to the construction history the $n_g - 2$ lines in the middle carry the desired irreducible representation. In appendix E we prove that $\mathbf{P}^{\ldots,M,M'}$ as given in eq. (4.18) is a projector, and calculate the normalization. For the three gluon case, the construction in eq. (4.18) always works as only the octet can precede the first occurrence two multiplet M. The projectors which can be constructed in this way are $\mathbf{P}^{10,8}$, $\mathbf{P}^{\overline{10},8}$, $\mathbf{P}^{27,8}$ and $\mathbf{P}^{0,8}$. As an example we note that $\mathbf{P}^{27,8}$ can be written

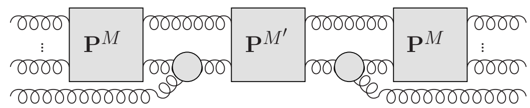
$$\mathbf{P}^{27,8}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} = \frac{4(N_c+1)}{N_c^2(N_c+3)}\mathbf{P}^{27}_{g_1\,g_2\,i_1\,g_3}\mathbf{P}^{27}_{i_1\,g_6\,g_4\,g_5} \tag{4.19}$$

where the prefactor is the ratio of the general N_c dimensions of the "octet" and the "27"-plet.

For more than three gluons it may happen that M' differs from the multiplet preceding M in the construction history. The above method would then give 0. In this case we resort to eq. (4.13) in order to find the corresponding projectors.

4.3 Starting from a new multiplet and going to one with same n_f

We now turn to the case $n_f(M') = n_f(M) = n_g - 1$. If the multiplicity of M' in $M \otimes A$ is one, we construct the corresponding projector by attaching a gluon to one of the internal gluons,

$$\mathbf{P}^{\dots,M,M'} = \frac{\dim(M')}{B(M,M')}$$  . (4.20)

where the big gray circles can be any of $\mathrm{i}f$ or d (same on both sides of $\mathbf{P}^{M'}$), and B is the normalization factor from eq. (E.6). The projection operators $\mathbf{P}^{27,10}$, $\mathbf{P}^{27,\overline{10}}$, $\mathbf{P}^{0,10}$, $\mathbf{P}^{0,\overline{10}}$, $\mathbf{P}^{10,27}$, $\mathbf{P}^{\overline{10},27}$, $\mathbf{P}^{10,0}$, and $\mathbf{P}^{\overline{10},0}$ are constructed as indicated above.

As an example we note that, after calculating the normalization, the projector $\mathbf{P}^{27,10}$ may be written

$$\mathbf{P}^{27,10}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} = \frac{1}{T_R}\frac{2(N_c+2)}{N_c(N_c+3)}\mathbf{P}^{27}_{g_1\,g_2\,i_1\,i_2} d_{i_2\,g_3\,i_3} \mathbf{P}^{10}_{i_1\,i_3\,i_4\,i_6} d_{i_6\,g_6\,i_5} \mathbf{P}^{27}_{i_4\,i_5\,g_4\,g_5}. \tag{4.21}$$

For $n_g > 3$ it can happen that the above construction does not work. This is the case if there is no instance of M' in $m \otimes A$, where m is the multiplet preceding M in the construction history. In this case we refer to the general strategy from eq. (4.13).

For the multiplet M appearing in many instances in $M \otimes A$, it has to be guaranteed that projectors corresponding to all instances are constructed and that the operators are hermitian and transversal. We start by defining two projectors of the form (4.20), $\mathbf{P}^{\dots,M,Md}$, for which both big gray circles represent d and $\mathbf{P}^{\dots,M,Mf}$, where both circles represent $\mathrm{i}f$. For $n_g = 3$ and $M = 0$ or $M = 27$ these two projectors are transversal and we are done. We note in passing that the two-gluon octet-projectors are also constructed in this way, cf. eq. (1.23).

For the decuplets, i.e. for $M = 10$ and $M = \overline{10}$, within $A^{\otimes 3}$ the situation is slightly more complicated. It turns out that

$$\mathbf{P}^{\dots,M,Mf} \neq \mathbf{P}^{\dots,M,Md} \qquad \text{but} \qquad \mathbf{P}^{\dots,M,Mf}\,\mathbf{P}^{\dots,M,Md} \neq 0\,, \tag{4.22}$$

i.e. the projectors are different but not transversal. In this case we keep one of them, $\mathbf{P}^{\dots,M,Mf}$, for our final list. Then we construct another transversal operator by projecting $\mathbf{P}^{\dots,M,Md}$ onto the orthogonal complement of the image of $\mathbf{P}^{\dots,M,Mf}$,

$$\mathbf{T}^{\dots,M,Mfd} := \left(\mathbb{1}_{A^{\otimes 3}} - \mathbf{P}^{\dots,M,Mf}\right)\mathbf{P}^{\dots,M,Md}\left(\mathbb{1}_{A^{\otimes 3}} - \mathbf{P}^{\dots,M,Mf}\right). \tag{4.23}$$

The resulting tensor is proportional to the desired projector, $\mathbf{P}^{\dots,M,Mfd} = \alpha\mathbf{T}^{\dots,M,Mfd}$, and the normalization is determined by taking the trace and solving for α. This yields the second projector

$$\mathbf{P}^{\dots,M,Mfd} = \frac{\dim M}{\operatorname{tr}\mathbf{T}^{\dots,M,Mfd}}\,\mathbf{T}^{\dots,M,Mfd}\,, \tag{4.24}$$

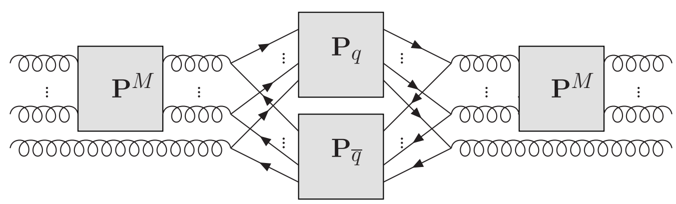
which is transversal to $\mathbf{P}^{\dots,M,Mf}$. In appendix E we show that $\mathbf{P}^{\dots,M,Mfd}$ can be written in the form of eq. (4.20) where the big gray circles represent a linear combination of d and if, and in appendix D we give the corresponding projection operators in this form.

For $N_c > 3$ and $n_g > 3$, there may be more than two instances of M in $M \otimes A$, allowing for the definition of more than two projectors, which – if not already transversal – can be made so by recursively projecting onto orthogonal complements as above. In this case the original set of projectors can always be found by applying the method outlined in eq. (4.13).

4.4 Projection operators onto new multiplets

In order to be able to reach a new multiplet, i.e. one with first occurrence $n_f(M') = n_g$, we have to start with a multiplet M which was new for $n_g - 1$ gluons, i.e. $n_f(M) = n_g - 1$.

For constructing the projection operators we split the gluons into $q\bar{q}$ pairs, such that symmetrization and anti-symmetrization can be done in the quark and anti-quark indices separately. We start by constructing tensors

$\mathbf{T}^{\dots M,M'} =$  , (4.25)

where $\mathbf{P}_q$ projects onto a multiplet $M_q \subset V^{\otimes n_g}$ and $\mathbf{P}_{\overline{q}}$ onto $\overline{M_{\overline{q}}} \subset \overline{V}^{\otimes n_g}$, (in the notation of appendix B). The motivation for the definition $\mathbf{T}^{\dots M,M'}$ is best understood by reading the expression from the center outwards. In the middle section we project onto $M_q \otimes \overline{M_{\overline{q}}}$, which contains at most one (and for sufficiently large N_c exactly one) new multiplet, as we show in appendix B.

The projector onto $M_q \otimes \overline{M_{\overline{q}}}$ is sandwiched between generators projecting $(V \otimes \overline{V})^{\otimes n_g} \to A^{\otimes n_g}$. This removes projections onto some (not all) old multiplets, but leaves the projection onto the new multiplet untouched, as the difference between $(V \otimes \overline{V})^{\otimes n_g} = (A \oplus \bullet)^{\otimes n_g}$ and $A^{\otimes n_g}$ is $\bigoplus_{\nu=0}^{n_q-1} \binom{n_q}{\nu} A^{\otimes \nu}$, cf. eq. (1.10), i.e. contains only multiplets of lower first occurrence.

Finally we sandwich between $\mathbf{P}^M \otimes \mathbb{1}_A$, thus projecting onto $M \otimes A$ and making sure that property (iii), i.e. the construction history, is satisfied. Since there is only one new multiplet within $M_q \otimes \overline{M_{\overline{q}}}$ and only one within $M \otimes A$, and since $A^{n_g} \subseteq (V \otimes \overline{V})^{n_g}$ we can always choose the quark and anti-quark multiplets in such a way that the resulting $\mathbf{T}^{\dots M,M'}$ is non-zero and contains a part which is proportional to the projector onto M' which we want to construct. The details of how to choose these quark and anti-quark projectors are discussed towards the end of this section.

The tensor $\mathbf{T}^{\dots M,M'}$, seen as linear operator from $M \otimes A$ to $M \otimes A$ contains a part mapping the new multiplet M' to itself. However, there are also pieces mapping other multiplets $m \subseteq M \otimes A$ to (sometimes a different instance of) m. These pieces have to be projected out. To this end we define the hermitian operator

$$\mathbf{Q} = \mathbb{1}_{A^{\otimes n_g}} - \sum_{\substack{m \subseteq A^{\otimes n_g} \\ n_f(m) < n_g}} \mathbf{P}^m \,, \tag{4.26}$$

which projects onto the new multiplets within $A^{\otimes n_g}$, and sandwich $\mathbf{T}^{\dots M,M'}$ between it,

$$\widetilde{\mathbf{T}}^{\dots M,M'} := \mathbf{Q}\,\mathbf{T}^{\dots M,M'}\,\mathbf{Q}\,. \tag{4.27}$$

Note that for practical purposes in eq. (4.26) it is sufficient to sum over $m \subseteq M \otimes A$ with $n_f(m) < n_g$, as all other terms vanish in eq. (4.27). Using

$$\mathbf{T}^{\dots,M,M'} = \sum_{m,m' \in M \otimes A} t_{m,m'} \mathbf{C}^m \mathbf{C}^{m'\dagger} \,, \tag{4.28}$$

and rewriting all projection operators in terms of Clebsch-Gordan matrices, it is not hard to prove that eq. (4.27) can be further simplified to

$$\widetilde{\mathbf{T}}^{\dots,M,M'} = \mathbf{T}^{\dots,M,M'} - \sum_{\substack{m \subseteq M \otimes A \\ n_f(m) < n_g}} \mathbf{P}^m\,\mathbf{T}^{\dots,M,M'}\,. \tag{4.29}$$

This is the way in which the three gluon projection operators are constructed. The desired projector is proportional to $\widetilde{\mathbf{T}}^{\dots M,M'}$ and the normalization is found by taking the trace in $\mathbf{P}^{\dots M,M'} = \alpha \widetilde{\mathbf{T}}^{\dots M,M'}$, yielding

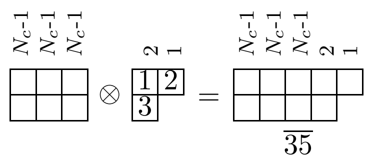
$$\mathbf{P}^{\dots M,M'} = \frac{\dim M'}{\mathrm{tr}\left(\widetilde{\mathbf{T}}^{\dots M,M'}\right)} \widetilde{\mathbf{T}}^{\dots M,M'}\,. \tag{4.30}$$

It remains to show that the resulting projector is hermitian. Using hermitian quark projectors, see section 2, the hermiticity of $\mathbf{P}^{\dots M,M'}$ is obvious from inserting eq. (4.25) into eq. (4.27).

Interestingly $\mathbf{P}^{\dots M,M'}$ is also hermitian when using conventional, i.e. non-hermitian, Young projectors. It follows from Schur's lemma, see appendix F, that invariant projectors onto multiplets can only be non-hermitian if the multiplet is not unique within the space from which one projects. Within $A^{\otimes 3}$, e.g., the 64-plet is unique and thus an invariant projector onto it is automatically hermitian, whereas the 35-plet appears within both, $10 \otimes 8$, and $27 \otimes 8$, see eqs. (4.2) and (4.4). In the latter case, of a multiplet which is not unique within $A^{\otimes n_g}$, it can (and actually does) happen that, when constructing the projector onto a particular instance of $M' = 35$, the central part of eq. (4.25) – without the $\mathbf{P}^M$-projectors – contains terms projecting onto an instance of 35 having a different construction history *and* terms which map one of the instances to the other. Of these only the latter terms can be non-hermitian. However, since M' is unique within $M \otimes A$ these terms are removed by the $\mathbf{P}^M$-projectors in eq. (4.25), and, after removing the lower first occurrence parts in eq. (4.27), the resulting $\widetilde{\mathbf{T}}^{\dots M,M'}$ is hermitian – irrespectively of whether the P_q and $P_{\overline{q}}$ are hermitian or not.

Finally, we note that property (ii) follows trivially from the hermiticity using $\mathbf{C}^{\cdots M,M'} = \mathbf{C}^{\dagger \cdots M,M'} = \mathbf{P}^{\cdots M,M'}$, and that the construction history property is manifest. The projectors thus fulfill all of (i)-(iii).

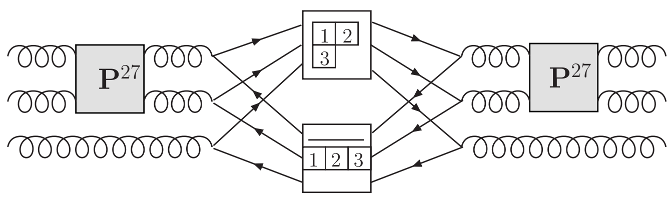
As an example consider the projector onto $\overline{35} \subset A^{\otimes 3}$ coming from $27 \subset A \otimes A$. From the Young tableau $\begin{smallmatrix}1&2\\3&\end{smallmatrix}$ for the quarks, and $\overline{\begin{smallmatrix}1&2&3\end{smallmatrix}}$ for the anti-quarks we get



$$\oplus \text{ old multiplets}\,. \tag{4.31}$$

As $\begin{smallmatrix}1&2\\3&\end{smallmatrix}$ is symmetric in the first two indices, and since the 27-plet is contained in $\begin{smallmatrix}1&2\end{smallmatrix} \otimes \overline{\begin{smallmatrix}1&2\end{smallmatrix}}$, these q- and $\overline{q}$-diagrams are chosen such that the projection onto the 27-plet in the first two indices is non-vanishing,

$$\mathbf{T}^{27,\overline{35}} =$$



$$\neq 0\,. \tag{4.32}$$

In this way, by considering the symmetry of the quarks and the anti-quarks in the $n_g - 1$ projector, and using the n_g quark and anti-quark projectors where the n_g^{th} quark and anti-quark are added to their respective Young diagrams, in such a way that the Young diagrams have the right shapes to guarantee $M' \subseteq M_q \otimes \overline{M_{\overline{q}}}$, it is always possible to find suitable q- and $\overline{q}$-diagrams. This allows for a unique construction of all instances of new multiplets, i.e. each new multiplet within $A^{\otimes n_g}$ has a corresponding new multiplet within $(V \otimes \overline{V})^{n_g}$, defined in this way.

For the three gluon case the projection operators are written down in appendix D. It has been checked that these projection operators sum to unity, i.e.

$$\sum_{I \in A^{\otimes 3}} \mathbf{P}^{I}_{g_1\, g_2\, g_3\, g_4\, g_5\, g_6} = \delta_{g_1\, g_3} \delta_{g_2\, g_4} \delta_{g_3\, g_6}\,, \tag{4.33}$$

and that they are hermitian

$$\mathbf{P}_{g_1\, g_2\, g_3\, g_4\, g_5\, g_6} = \mathbf{P}^{*}_{g_4\, g_5\, g_6\, g_1\, g_2\, g_3}\,. \tag{4.34}$$

As a consequence of the hermiticity it follows that the real projection operators, and the real linear combinations are symmetric, i.e.

$$\mathbf{P}_{g_1\, g_2\, g_3\, g_4\, g_5\, g_6} = \mathbf{P}_{\,g_4\, g_5\, g_6\, g_1\, g_2\, g_3} \quad \text{for real combinations.} \tag{4.35}$$

4.5 Further remarks

We conclude this section with some general remarks. First we note that the fact that the multiplets can be uniquely constructed from the quark and anti-quark projection operators

Case	Projectors $N_c = 3$	Projectors $N_c = \infty$	Vectors $N_c = 3$	Vectors $N_c = \infty$
2g $\to$ 2g	6	7	8	9
3g $\to$ 3g	29	51	145	265
4g $\to$ 4g	166	513	3 598	14 833
5g $\to$ 5g	1 002	6 345	107 160	1 334 961

Table 2: Number of projectors and basis vectors for $n_g \to n_g$ gluons (*without* imposing projection operators and vectors to appear in self-conjugate combinations). In the $N_c \to \infty$ limit the number of vectors for a total of N_g, incoming plus outgoing, gluons is given by subfactorial(N_g) as in eq. (1.15). For $N_c = 3$ the number of projectors are found by counting all irreducible representations occurring in $A^{\otimes n_g}$, and the number of vectors are obtained by considering all possible transitions between multiplets of same type on the incoming and outgoing side.

also introduces an alternative N_c-independent description of all multiplets. They can be labeled using first the lengths of all quark columns, and then the lengths of all anti-quark columns, e.g. 12 and 111 in the above case of $\overline{35}$. We use this notation for the new multiplets arising for three gluons, which have dimension 0 for SU(3). As an example we note that $\mathbf{P}^{27,\overline{35}}$ alternatively could have been written $\mathbf{P}^{c11c11,c21c111}$. Using this notation also translates straightforwardly to the representations of Young diagrams used in eqs. (4.1)-(4.5). The length of the first column is N_c minus the length of the last anti-quark column, the length of the second column is N_c minus the length of the second last anti-quark column, etc. After this follow columns with lengths given by the quark columns. The notation also immediately reveals the first occurrence of a multiplet; it is simply the sum of the quark-column lengths (which equals the sum of the anti-quark column lengths – before conjugating). To distinguish the different *instances* of the new multiplets we note that they could alternatively have been labeled using Young tableaux with quark and anti-quark numbers filled in as in eq. (4.32).

In this context we also remark that the fact that N_c is small in QCD leads to a significant reduction of the number of projection operators in the n_g gluon space, partly since many new projection operators vanish for small N_c, and partly because projectors may be forbidden by construction, such as $\mathbf{P}^{0,8}$. Similarly, there is a reduction in the number of basis vectors in the space of $2n_g$ gluons. In fact, as is argued in appendix G, the number of basis vectors grows only exponentially, as opposed to factorially, c.f. eq. (1.15), for finite N_c. As can be seen in table 2, for more than a few gluons, the reduction in the number of projectors and in the dimension of the vector space is significant.

We finally remark that as the gluon transforms under a real representation, only projection operators and basis vectors which are invariant under charge conjugation may appear for purely gluonic processes [5, 14, 22, 37]. Non-invariant projectors may only appear together with their charge conjugated versions. Thus, for example $\mathbf{P}^{10,35}$ may only occur together with $\mathbf{P}^{\overline{10},\overline{35}}$. This reduces the number of projection operators for $n_g = 3$ gluons from 51 to 36 in the $N_c \to \infty$ limit, and from 29 to 21 for $N_c = 3$.

5. Multiplet based gluon bases

In order to construct the vectors in the color space for $A^{\otimes 6}$ we may group the gluons as $g_1\, g_2\, g_3 \to g_4\, g_5\, g_6$. We can (naively) combine any instance of an incoming multiplet with any (other) instance of the same outgoing multiplet as there is no conserved quantum number which forbids the transition from one instance of a multiplet to another instance of the same multiplet. However, as long as only gluons are involved, all vectors have to be invariant under charge conjugation. The construction of the basis for the six gluon space, is discussed in detail in appendix H.

Alternatively we could have grouped the gluons as $g_1 g_2 \to g_3 g_4 g_5 g_6$ in which case only multiplets with first occurrence up to two could have appeared on the left hand side, and therefore also on the right hand side. On the incoming side we then have the multiplets as enumerated in eq. (1.18). For keeping track of the multiplets on the outgoing side we would need to find all the multiplets with first occurrence up to two, when Young multiplying four gluons.

The multiplets with first occurrence up to two, arising when multiplying a first occurrence two multiplet with an octet can be read off from eqs. (4.1)-(4.5). However, we also have multiplets with first occurrence two, when multiplying the three gluon multiplets with first occurrence three with a gluon. To enumerate these multiplets we need to Young multiply the first occurrence three states in eqs. (4.1)-(4.5) with a gluon. After having performed this task we can, however, construct the basis vectors using projection operators with first occurrence up to three.

As the first occurrence cannot change by more then one unit when multiplying with a gluon, it is in general the case that we never need projection operators with first occurrence larger than n_g when treating up to $2\, n_g + 1$ gluons. This is true independently of how the gluons are grouped. In particular the $n_g = 3$ projection operators are also sufficient to construct orthonormal bases for up to seven gluons.

6. General construction of multiplet bases

To treat the general case involving both quarks and gluons we note that for each quark (outgoing quark or incoming anti-quark) there is an anti-quark (incoming quark or outgoing anti-quark). We can therefore always start the process of sub-grouping by paring up each quark with an anti-quark. Each $q\overline{q}$ pair can either be in a singlet, reducing the basis construction to the corresponding problem without this $q\overline{q}$ pair, or in an octet. In the latter case the basis construction is equivalent to the construction where the $q\overline{q}$ pair is traded for a gluon. Below we exemplify the basis construction by constructing orthogonal bases for processes involving six colored partons. Since, for each quark there is a corresponding anti-quark, we may have three $q\overline{q}$-pairs, two $q\overline{q}$-pairs and 2 gluons, one $q\overline{q}$-pair and four gluons or six gluons, as treated above. Mathematica .m-files containing the bases constructed in this way are attached electronically.

SU(3) dimension/notation	1	8
Multiplet, general notation	c0c0	c1c1
In $q_1\overline{q}_2$	$(12)^1$	$(12)^8$
Out $q_3\overline{q}_4 q_5\overline{q}_6$	$((34)^1(56)^1)^1$ $((34)^8(56)^8)^1$	$((34)^1(56)^8)^8$, $((34)^8(56)^1)^8$ $((34)^8(56)^8)^{8s}$, $((34)^8(56)^8)^{8a}$

Table 3: The multiplets appearing in the construction of the bases corresponding to $q_1\overline{q}_2 \to q_3\overline{q}_4 q_5\overline{q}_6$. On the incoming side $q_1\overline{q}_2$ may be in a singlet or in an octet. Due to color conservation the outgoing multiplet must be the same. However, in this case, there are may ways to build up the singlet or octet. To have an overall singlet the quarks q_3 and $\overline{q}_4$ may separately form a singlet if q_5 and $\overline{q}_6$ do too. Alternatively q_3 and $\overline{q}_4$ may be in an octet which then combines with another octet from q_5 and $\overline{q}_6$ to from a total singlet. For the octets we have a total of four options. All in all, the multiplets may be enumerated as above. The vector space has thus 1×2 dimensions for the singlets and 1×4 dimensions from the octets.

6.1 Example: Three $q\overline{q}$ pairs

This case is dealt with extensively in section 3 where a basis is constructed using the hermitian quark projection operators for $q_1q_2q_3 \to q_4q_5q_6$. In this section we note that we can equally well construct the basis using the gluon projection operators and grouping the partons as $q_1\overline{q}_2 \to q_3\overline{q}_4 q_5\overline{q}_6$. The multiplets on the ingoing and outgoing side may then be constructed as in table 3, by first grouping $q\overline{q}$-pairs to form octets or singlets. Clearly the dimension of the basis must still be 6 as in section 3. Having enumerated all the basis vectors as in table 3, we may write down the basis using the (somewhat redundant) notation $\mathbf{V}^{M_{12};M_{34},M_{56},M_{3456}}$,

$$
\begin{aligned}
\mathbf{V}^{1;1,1,1}_{q_1\,q_2\,q_3\,q_4\,q_5\,q_6} &= \frac{1}{\sqrt{N_c^3}}\delta^{q_1}_{q_2}\delta^{q_4}_{q_3}\delta^{q_6}_{q_5}\,,\\
\mathbf{V}^{1;8,8,1}_{q_1\,q_2\,q_3\,q_4\,q_5\,q_6} &= \frac{1}{T_R}\frac{1}{\sqrt{N_c(N_c^2-1)}}\delta^{q_1}_{q_2}(t^{i_1})^{q_4}_{q_3}(t^{i_1})^{q_6}_{q_5}\,,\\
\mathbf{V}^{8;1,8,8}_{q_1\,q_2\,q_3\,q_4\,q_5\,q_6} &= \frac{1}{T_R}\frac{1}{\sqrt{N_c(N_c^2-1)}}(t^{i_1})^{q_1}_{q_2}\delta^{q_4}_{q_3}(t^{i_1})^{q_6}_{q_5}\,,\\
\mathbf{V}^{8;8,1,8}_{q_1\,q_2\,q_3\,q_4\,q_5\,q_6} &= \frac{1}{T_R}\frac{1}{\sqrt{N_c(N_c^2-1)}}(t^{i_1})^{q_1}_{q_2}(t^{i_1})^{q_4}_{q_3}\delta^{q_6}_{q_5}\,,\\
\mathbf{V}^{8;8,8,8s}_{q_1\,q_2\,q_3\,q_4\,q_5\,q_6} &= \frac{1}{T_R^2}\sqrt{\frac{N_c}{2(N_c^4-5N_c^2+4)}}(t^{i_1})^{q_1}_{q_2}d_{i_1\,i_2\,i_3}(t^{i_2})^{q_4}_{q_3}(t^{i_3})^{q_6}_{q_5}\,,\\
\mathbf{V}^{8;8,8,8a}_{q_1\,q_2\,q_3\,q_4\,q_5\,q_6} &= \frac{1}{T_R^2}\frac{1}{\sqrt{2N_c(N_c^2-1)}}(t^{i_1})^{q_1}_{q_2}\mathrm{i}f_{i_1\,i_2\,i_3}(t^{i_2})^{q_4}_{q_3}(t^{i_3})^{q_6}_{q_5}\,,
\end{aligned}
\tag{6.1}
$$

where the normalization has been fixed to get an orthonormal basis. We note that in this case we did not have to make use of the two or three gluon projection operators, which is immediately clear from the grouping of the partons, since on the left hand side we cannot get a multiplet with first occurrence larger than one. The basis in eq. (6.1) is related to that in eq. (3.1) by an orthogonal transformation.

SU(3) dim	1	8	10	$\overline{10}$	27	0
Multiplet	c0c0	c1c1	c11c2	c2c11	c11c11	c2c2
In $q_1\overline{q}_2g_3$	$((12)^8 3)^1$	$((12)^1 3)^8$ $((12)^8 3)^{8s}$ $((12)^8 3)^{8a}$	$((12)^8 3)^{10}$	$((12)^8 3)^{\overline{10}}$	$((12)^8 3)^{27}$	$((12)^8 3)^0$

Table 4: Table describing the multiplets used in the construction of the bases corresponding to $q_1\overline{q}_2g_3 \to q_4\overline{q}_5g_6$. As the incoming and outgoing particle content is the same, the possible multiplets for the outgoing partons are identical to those for the incoming. For each basis vector, any instance of a multiplet M on the incoming side can be combined with any instance of M on the outgoing side. For $q_1\overline{q}_2g_3 \to q_4\overline{q}_5g_6$ we thus have $1+3^2+1+1+1+1 = 14$ basis vectors, reducing to 13 for SU(3).

6.2 Example: Two $q\overline{q}$ pairs and two gluons

In order to construct a basis for processes involving two $q\overline{q}$-pairs and two gluons we may for example group the partons as $q_1\overline{q}_2g_3 \to q_4\overline{q}_5g_6$ and use Young multiplication to arrive at the multiplet possibilities in table 4. (Alternatively we could use $g_3\,g_6 \to q_1\overline{q}_2q_4\overline{q}_5$ in which case we would not have any projection operators.)

The $q_1\overline{q}_2$, can be either in a singlet or in an octet. If they are in a singlet, the singlet is combined with the gluon g_3 to an overall octet. On the other hand, if $q_1\overline{q}_2$ are in an octet, when combined with g_3, the overall multiplet may be any of $1, 8^s, 8^a, 10, \overline{10}, 27, 0$. On the outgoing side, the same method of subgrouping can be applied. From this it is immediately clear that we need gluon projectors with first occurrence up to two, but not higher.

To construct the projectors, the states with corresponding construction history on the incoming and outgoing side have to be joined, giving

$$
\begin{aligned}
\mathbf{P}^{8,1}_{q_1\,q_2\,g_3\,q_4\,q_5\,g_6} &= \frac{1}{T_R}(t^{i_1})^{q_1}_{q_2}\,\mathbf{P}^{1}_{g_3\,i_1\,g_6\,i_2}(t^{i_2})^{q_5}_{q_4}\,,\\
\mathbf{P}^{1,8}_{q_1\,q_2\,g_3\,q_4\,q_5\,g_6} &= \frac{1}{N_c}\delta^{q_1}_{q_2}\;\delta_{g_3\,g_6}\;\delta^{q_5}_{q_4}\,,\\
\mathbf{P}^{8,8s}_{q_1\,q_2\,g_3\,q_4\,q_5\,g_6} &= \frac{1}{T_R}(t^{i_1})^{q_1}_{q_2}\,\mathbf{P}^{8s}_{g_3\,i_1\,g_6\,i_2}(t^{i_2})^{q_5}_{q_4}\,,\\
\mathbf{P}^{8,8a}_{q_1\,q_2\,g_3\,q_4\,q_5\,g_6} &= \frac{1}{T_R}(t^{i_1})^{q_1}_{q_2}\,\mathbf{P}^{8a}_{g_3\,i_1\,g_6\,i_2}(t^{i_2})^{q_5}_{q_4}\,,\\
\mathbf{P}^{8,10}_{q_1\,q_2\,g_3\,q_4\,q_5\,g_6} &= \frac{1}{T_R}(t^{i_1})^{q_1}_{q_2}\,\mathbf{P}^{10}_{g_3\,i_1\,g_6\,i_2}(t^{i_2})^{q_5}_{q_4}\,,\\
\mathbf{P}^{8,\overline{10}}_{q_1\,q_2\,g_3\,q_4\,q_5\,g_6} &= \frac{1}{T_R}(t^{i_1})^{q_1}_{q_2}\,\mathbf{P}^{\overline{10}}_{g_3\,i_1\,g_6\,i_2}(t^{i_2})^{q_5}_{q_4}\,,\\
\mathbf{P}^{8,27}_{q_1\,q_2\,g_3\,q_4\,q_5\,g_6} &= \frac{1}{T_R}(t^{i_1})^{q_1}_{q_2}\,\mathbf{P}^{27}_{g_3\,i_1\,g_6\,i_2}(t^{i_2})^{q_5}_{q_4}\,,\\
\mathbf{P}^{8,0}_{q_1\,q_2\,g_3\,q_4\,q_5\,g_6} &= \frac{1}{T_R}(t^{i_1})^{q_1}_{q_2}\,\mathbf{P}^{0}_{g_3\,i_1\,g_6\,i_2}(t^{i_2})^{q_5}_{q_4}\,,
\end{aligned}
\tag{6.2}
$$

where the factor $1/T_R$ in the norm (when present) compensates for the factor T_R coming from contraction of quarks to form gluon projectors when squaring the above projectors.

SU(3) dim	1	8	10	$\overline{10}$	27	0
Multiplet	c0c0	c1c1	c11c2	c2c11	c11c11	c2c2
In $q_1\overline{q}_2g_3$	$((12)^8 3)^1$	$((12)^1 3)^8$	$((12)^8 3)^{10}$	$((12)^8 3)^{\overline{10}}$	$((12)^8 3)^{27}$	$((12)^8 3)^0$
		$((12)^8 3)^{8s}$				
		$((12)^8 3)^{8a}$				
Out $g_4g_5g_6$		$((45)^1 6)^8$				
	$((45)^{8s} 6)^1$	$((45)^{8s} 6)^{8s/a}$	$((45)^{8s} 6)^{10}$	$((45)^{8s} 6)^{\overline{10}}$	$((45)^{8s} 6)^{27}$	$((45)^{8s} 6)^0$
	$((45)^{8a} 6)^1$	$((45)^{8a} 6)^{8s/a}$	$((45)^{8a} 6)^{10}$	$((45)^{8a} 6)^{\overline{10}}$	$((45)^{8a} 6)^{27}$	$((45)^{8a} 6)^0$
		$((45)^{10} 6)^8$	$((45)^{10} 6)^{10}$	$((45)^{\overline{10}} 6)^{\overline{10}}$	$((45)^{10} 6)^{27}$	$((45)^{10} 6)^0$
		$((45)^{\overline{10}} 6)^8$	$((45)^{10} 6)^{10}$	$((45)^{\overline{10}} 6)^{\overline{10}}$	$((45)^{\overline{10}} 6)^{27}$	$((45)^{\overline{10}} 6)^0$
		$((45)^{27} 6)^8$	$((45)^{27} 6)^{10}$	$((45)^{27} 6)^{\overline{10}}$	$((45)^{27} 6)^{27}$	$((45)^0 6)^0$
		$((45)^0 6)^8$	$((45)^0 6)^{10}$	$((45)^0 6)^{\overline{10}}$	$((45)^{27} 6)^{27}$	$((45)^0 6)^0$

Table 5: Table describing the multiplets used in the construction of the bases corresponding to $q_1\overline{q}_2g_3 \to g_4g_5g_6$. In each case basis vectors corresponding to any instance of a multiplet M on the incoming side can be combined with any instance of the same outgoing multiplet. We thus have a $2 + 3 \times 9 + 6 + 6 + 6 + 6 = 53$ dimensional vector space, reducing to $2 + 3 \times 8 + 4 + 4 + 6 + 0 = 40$ for SU(3).

The basis vectors in the six parton space are given by allowing all instances of a given multiplet to go into any instance of the same multiplet, as enumerated in table 4. The vector space is thus 14 dimensional, reducing to 13 for SU(3). Their explicit forms are given in appendix I.

6.3 Example: One $q\overline{q}$ pair and four gluons

To find an orthonormal basis for processes involving one $q\overline{q}$ pair and four gluons, we again utilize the method of sub-grouping, sorting the partons as $q_1\,\overline{q}_2\,g_3 \to g_4\,g_5\,g_6$, and finding the multiplets listed in table 5.

We note that we could as well have sorted the partons (for example) as $q_1\,\overline{q}_2 \to g_3\,g_4\,g_5\,g_6$, in which case we would have had to perform the four gluon Young tableaux multiplication on the right hand side.

With the chosen treatment the incoming side is treated precisely as in the case of $q_1\overline{q}_2g_3 \to q_4\overline{q}_5g_6$ above. Due to color conservation, only states with first occurrence up to two appearer on the outgoing side, meaning that we again only need the gluon projection operators with first occurrence up to two.

If $q_1\overline{q}_2$ are in an octet, the basis construction is similar to the case of $2g \to 3g$ [5], with the exception that also non-self conjugate states may appear. This leads to a doubling of the number of basis vectors for the part of the sub-space where $q_1\overline{q}_2$ are in an octet.

As the types of partons on the incoming and outgoing side are not the same, there are no projection operators, but there are 53 orthogonal basis vectors out of which 13 vanish for SU(3). Due to the size of the basis, we do not display the basis vectors but only attach them electronically to this publication. A future publication of the Mathematica package used in the construction of the projection operators and basis vectors is planned.

7. Conclusions and outlook

In this paper we outline a general algorithm for constructing orthogonal (normalized) multiplet bases for color summed calculations in QCD, for any number of quarks and gluons, any N_c, and to any order in perturbation theory.

This is accomplished by first constructing gluon projection operators projecting onto irreducible representations. We outline, how to construct these projection operators recursively for an arbitrary number of gluons, and illustrate the method by constructing a complete set of orthogonal projection operators for three gluons.

A key idea for the construction is the splitting of gluons into $q\overline{q}$-pairs, and the subsequent usage of hermitian (anti-) quark projection operators, as illustrated in eq. (4.25). We find that, in the $N_c \to \infty$ limit, there is a one to one correspondence between the quark and anti-quark symmetries, and the gluon projection operators, cf. appendix B, whereas for small N_c many projection operators vanish. As a consequence of this uniqueness, we note that the Young tableaux corresponding to different N_c stand in a one to one – or one to zero – relation to each other. We also remark that choosing explicit indices we are able to calculate the Clebsch-Gordan matrices $\mathbf{C}^M$ in an N_c independent manner.

As an illustration we explicitly construct three gluon projection operators projecting onto mutually orthogonal subspaces, and the corresponding six gluon basis. Note, however, that the three gluon projectors can also be used for constructing orthogonal bases for up to seven gluons, or up to $n_g + n_q = 7$ gluons and $q\overline{q}$-pairs in general.

Using this type of gluon projection operators, we note that we can easily construct complete sets of basis vectors for an arbitrary number of $q\overline{q}$-pairs and gluons. The bases constructed in this way have the advantage of being orthogonal. As the number of basis vectors (for $N_c \to \infty$) scales roughly as a factorial in the number of gluons and $q\overline{q}$-pairs, cf. eq. (1.15) and eq. (1.16), this is a very strong advantage for processes involving many partons. These bases can also trivially be made minimal for the N_c under consideration, by just omitting the vanishing basis vectors. For many partons, this leads to a significant reduction in the number of terms that have to be treated. For example, for a total of 10 gluons, there are about one million basis vectors for $N_c \to \infty$, which in the standard, non-orthogonal, trace-type basis would give rise to an unmanageable $\approx 10^{12}$ elements to calculate while squaring an amplitude. On the other hand, for the $N_c = 3$, in the minimal multiplet basis, we instead have to treat about 10^5 terms. In fact we prove in appendix G that the number of basis vectors scales at most as an exponential, rather than a factorial, in the number of gluons plus $q\overline{q}$-pairs.

The usage of these orthogonal SU(3) minimal bases therefore has the potential to speed up exact multi-parton calculations significantly. However, it should be remarked that in order to facilitate this, additional theoretical progress is advantageous. For the standard trace-type bases powerful recursion relations in the number of external particles can be employed for special cases [10, 26–33] and it remains to be explored how these would manifest themselves in a multiplet type basis.

In general (and especially for classes of processes where no efficient recursion relations can be found), one would want efficient algorithms for sorting Feynman diagrams in the multiplet basis. If this can be done at tree level, and the effect of gluon exchange could be found, the color structure of higher order calculations could probably be dealt with efficiently. Although the effect of gluon emission is simple in many situations, and although the soft anomalous dimension matrices [3–5, 21, 22] describing the effect of gluon exchange on the various basis vectors, have been found to be relatively sparse, a complete systematic treatment is still pending.

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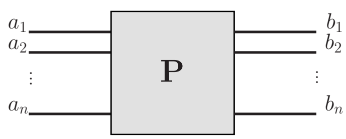
A. Some birdtrack conventions

We introduced Cvitanović's birdtrack notation [8, 24] in sections 1.1 and 1.3. Here we collect some properties and conventions.

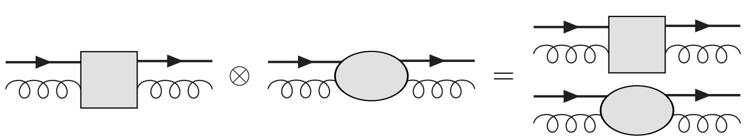
When translating the birdtrack diagram for a projector to index notation we subgroup the indices on the l.h.s. and on the r.h.s., respectively, i.e.



$$= \mathbf{P}_{a_1\, a_2\, \ldots\, a_n\, b_1\, b_2\, \ldots\, b_n}\ , \tag{A.1}$$

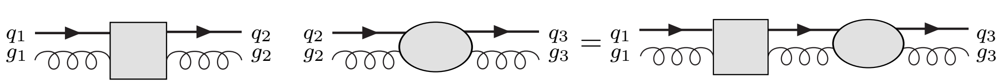
where all lines could be gluon , quark  or anti-quark  lines. With this convention we deviate from [24, sec. 4.1], where all indices are read off in anti-clockwise order, in which case the r.h.s. of eq. (A.1) would read $\mathbf{P}_{a_1\, \ldots\, a_n\, b_n\, \ldots\, b_1}$. However, for the structures constants f_{abc} and the totally symmetric tensor d_{abc} we do adopt the convention of assigning indices in an anti-clockwise order, see eqs. (1.2) and (1.20).

Tensor products are taken by writing the birdtrack diagrams one below the other,



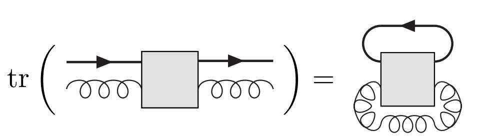
(A.2)

and index contractions are achieved by joining lines,



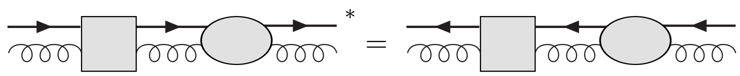
(A.3)

In particular this implies that traces are taken by joining left and right legs,



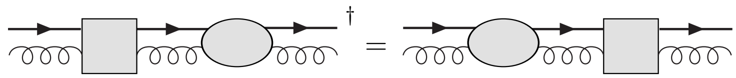
(A.4)

The complex conjugate of a diagram is obtained by reversing all arrows



$$= \quad , \tag{A.5}$$

and hermitian conjugation amounts to mirror the diagram across a vertical line *and* to reverse all arrows,



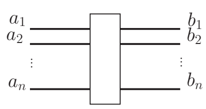
$$= \quad . \tag{A.6}$$

The latter is to be compared to

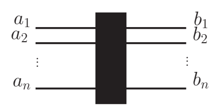
$$(\mathbf{P}^\dagger)_{a_1\, a_2\, \ldots\, a_n\, b_1\, b_2\, \ldots\, b_n} = (\mathbf{P}_{b_1\, b_2\, \ldots\, b_n\, a_1\, a_2\, \ldots\, a_n})^* \tag{A.7}$$

in index notation.

We frequently have to symmetrize or anti-symmetrize over a set of indices which in birdtrack notation is indicated by a white or a black bar, respectively,



$$= \mathbf{S}_{a_1\, a_2\, \ldots\, a_n\, b_1\, b_2\, \ldots\, b_n} \, ,$$



$$= \mathbf{A}_{a_1\, a_2\, \ldots\, a_n\, b_1\, b_2\, \ldots\, b_n} \, . \tag{A.8}$$

Written out as sums over permutations these read

$$\begin{aligned} \mathbf{S}_{a_1\, a_2\, \ldots\, a_n\, b_1\, b_2\, \ldots\, b_n} &= \frac{1}{n!} \sum_{\sigma \in S_n} \delta_{a_1 b_{\sigma(1)}}\, \delta_{a_2 b_{\sigma(2)}} \cdots \delta_{a_n b_{\sigma(n)}} \, , \\ \mathbf{A}_{a_1\, a_2\, \ldots\, a_n\, b_1\, b_2\, \ldots\, b_n} &= \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sign}(\sigma)\, \delta_{a_1 b_{\sigma(1)}}\, \delta_{a_2 b_{\sigma(2)}} \cdots \delta_{a_n b_{\sigma(n)}} \, . \end{aligned} \tag{A.9}$$

B. First occurrence in Young multiplication

In this appendix we analyze the first occurrence of multiplets in terms of Young diagrams. In particular we show that there is at most one new multiplet, having $n_f = n$, within $\overline{M_{\overline{q}}} \otimes M_q$ with $M_{\overline{q}}, M_q \subseteq V^{\otimes n}$. We also derive a rule for determining the first occurrence $n_f(M)$ of a given multiplet M, and show that $n_f(M') - n_f(M) \in \{-1, 0, 1\}$ for $M' \subseteq M \otimes A$. We assume that the reader is familiar with the labeling of irreducible representation of $\mathrm{SU}(N_c)$ by Young diagrams as well as the rules for conjugating and multiplying Young diagrams, see e.g. [34, secs. 7.12 & 10] and [24, sec. 9.8].

The first occurrence $n_f(M)$ of a multiplet M, which is contained in $A^{\otimes n}$ for some n, was defined in section 4. We now define the first occurrence $\widetilde{n}_f(M)$ within the sequence $(V \otimes \overline{V})^{\otimes n}$ in the same way, as the smallest $n \geq 0$ for which $M \subseteq (V \otimes \overline{V})^{\otimes n}$. Let us show that

$$n_f(M) = \widetilde{n}_f(M) \tag{B.1}$$

for any M which appears in either one of the sequences. On the one hand, note that due to $A^{\otimes n} \subseteq (V \otimes \overline{V})^{\otimes n}$ any $M \subseteq A^{\otimes n}$ is also contained in $(V \otimes \overline{V})^{\otimes n}$, and thus $\widetilde{n}_f(M) \leq n_f(M)$.

On the other hand, due to eq. (1.10) any $M \subseteq (V \otimes \overline{V})^{\otimes n}$ also has to be contained in $A^{\otimes \nu}$ for some $\nu \leq n$, and therefore $\widetilde{n}_f(M) \geq n_f(M)$. Together these two inequalities establish eq. (B.1) and, consequently, we no longer distinguish between n_f and $\widetilde{n}_f$.

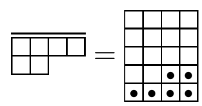
In order to analyze first occurrences we exploit that $(V \otimes \overline{V})^{\otimes n}$ is isomorphic to $\overline{V}^{\otimes n} \otimes V^{\otimes n}$, i.e. we have

$$A^{\otimes n} \subseteq (V \otimes \overline{V})^{\otimes n} \cong \overline{V}^{\otimes n} \otimes V^{\otimes n} \,. \tag{B.2}$$

In terms of Young diagrams this means that for every instance of a multiplet $M \subseteq A^{\otimes n}$ there exist n-box Young diagrams corresponding to multiplets M_q and $M_{\overline{q}}$ such that[2]

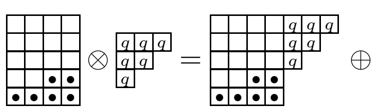
$$M \subseteq \overline{M_{\overline{q}}} \otimes M_q \,. \tag{B.3}$$

In the following we refer the Young diagrams for M_q and $\overline{M_{\overline{q}}}$ as quark diagram and anti-quark diagram, respectively, or q- and $\overline{q}$-diagram for short. Recall that the $\overline{q}$-diagram is obtained from the diagram for $M_{\overline{q}}$ by first supplementing the latter with additional boxes at the bottom until all columns have length N_c, then rotating the resulting rectangular diagram by 180°, and finally removing the original boxes. For instance, for $N_c = 5$ we have

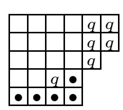

(B.4)

where ⊡ denotes the *absence* of a box, i.e. these are the boxes which were removed in the last step. We may thus view the $\overline{q}$-diagram as a rectangular diagram with N_c rows with n boxes cut out.

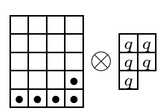
Now consider $\overline{M_{\overline{q}}} \otimes M_q$ with N_c such that the rightmost column of the $\overline{q}$-diagram has as many boxes as the leftmost column of the q-diagram. In this case we obtain


diagrams in which at least one quark box occupies a cut out space . (B.5)

However, all diagrams in which a quark box occupies a cut out space, such as


, (B.6)

are already contained in $\overline{V}^{\otimes(n-1)} \otimes V^{\otimes(n-1)}$, in the above case within

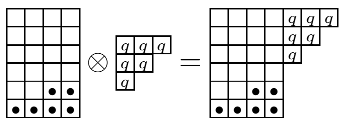

. (B.7)

Thus, the first occurrence of these diagrams is less than n, here $n - 1$. The first diagram on the r.h.s. of eq. (B.5), however, cannot be obtained for smaller n, as there would not be enough boxes available. Hence, its first occurrence is $n_f = n$, i.e. it corresponds to a new

[2]By "n-box diagram" we only mean that $M_q, M_{\overline{q}} \subseteq V^{\otimes n}$, i.e. if the diagrams have one or more columns with N_c boxes it does not matter for our discussion whether we keep these columns or omit them – as one is allowed to for SU(N_c) Young diagrams.

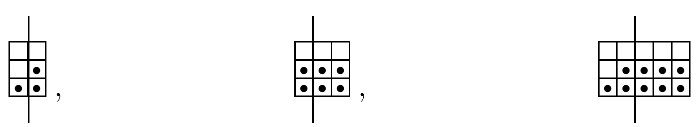
multiplet in the terminology of section 4. We also note that all Young diagrams of this kind, i.e. with n cut out spaces ⊡ from the $\overline{q}$-diagram and n boxes $\boxed{q}$ from the q-diagram, appear within $(V \otimes \overline{V})^{\otimes n}$, and thus also within $A^{\otimes n}$. Therefore, we can use the shapes of the quark and anti-quark diagrams for determining the first occurrence n_f.

For larger N_c, such that the rightmost column of the $\overline{q}$-diagram is longer than the leftmost column of the q-diagram, the structure of eq. (B.5) remains the same, e.g., for $N_c = 6$,

 $\oplus$ diagrams in which at least one quark box occupies a cut out space , (B.8)

as all other ways of appending all quark boxes to the $\overline{q}$-diagram without occupying a cut out space are forbidden by the rules for Young multiplication. Therefore, also in this case we obtain exactly one multiplet with first occurrence $n_f = n$. On the other hand, for smaller N_c, such that the rightmost column of the $\overline{q}$-diagram is shorter than the leftmost column of the q-diagram, it is impossible to append all quark boxes to the $\overline{q}$-diagram without occupying a cut out space. In this case we thus obtain only diagrams with $n_f < n$, i.e. diagrams corresponding to old multiplets. We conclude that *within* $\overline{M_{\overline{q}}} \otimes M_q$*, with* $M_{\overline{q}}, M_q \subseteq V^{\otimes n}$*, there is always at most one new multiplet*, and for sufficiently large N_c there is exactly one new multiplet. Our construction of projectors onto new multiplets builds on this. Another consequence is that new multiplets can be labeled uniquely by the q- and $\overline{q}$-diagrams. We introduce a corresponding notation in section 4.5.

The above discussion also provides us with a graphical rule for determining the first occurrence $n_f(M)$ of a given multiplet M. First, draw the Young diagram corresponding to M and mark empty spaces at the bottom of it by ⊡ until all columns have length N_c. Then draw a vertical line through the diagram such that the number of cut out spaces ⊡ to the left of the line is the same as the number of boxes □ to the right of it. This number is the first occurrence of M. As examples we consider the octet, the decuplet and the 35-plet for SU(3),

 (B.9)

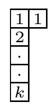
and obtain their first occurrences $1, 2$ and 3, respectively.

We now turn to the question of how the first occurrence changes under multiplication with an additional gluon. More precisely, for $M' \subseteq M \otimes A$ we want to know how $n_f(M)$ and $n_f(M')$ are related. By definition we can obtain $n_f(M') = n_f(M) + 1$ but not higher, we will see that this also is in agreement with the first occurrence counting above. Concerning the lowest possible value of $n_f(M')$ one may heuristically argue as follows. The additional gluon can form a singlet with one of the other gluons, leaving the remaining gluons in a multiplet with first occurrence $n_f(M) - 1$. (If they were in a state with lower first occurrence, they could not have built up a state with first occurrence $n_f(M)$ before the multiplication with the additional gluon.) Overall we thus expect

$$n_f(M) - 1 \le n_f(M') \le n_f(M) + 1 \quad \forall\ M' \subseteq M \otimes A\,. \tag{B.10}$$

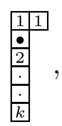
In order to prove this we carry out the Young multiplication with the adjoint representation and use the rule for determining the first occurrence, which we have derived above.

Recall that the Young diagram for the adjoint representation has two columns, the first one with $N_c - 1$ boxes and the second column with one box. For the following discussion it is convenient to define $k = N_c - 1$. In order to calculate $M \otimes A$ we have to append the labeled boxes of



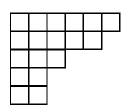
(B.11)

in all allowed ways to the Young diagram for M. We label the way in which the boxes are appended by a pattern like



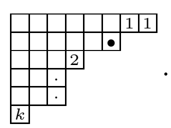
(B.12)

which stands for appending the boxes $\boxed{1}\boxed{1}$ to row 1, nothing to row 2, box $\boxed{2}$ to row 3, etc., and box $\boxed{k}$ to row N_c. For instance, if the initial multiplet was



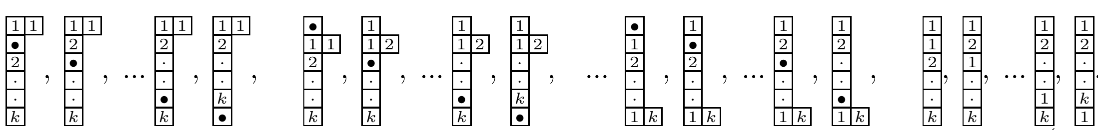
(B.13)

then the pattern (B.12) is a shorthand notation for the resulting Young diagram



(B.14)

In this way we can enumerate all multiplets which could possibly arise after Young multiplication with the adjoint representation by



(B.15)

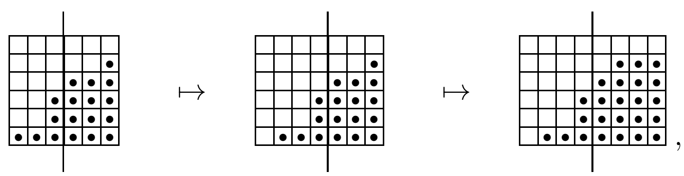
For a given initial multiplet many of these are typically forbidden, but the list is always exhaustive.

We first consider the last patterns, all of the same shape, with N_c unit length rows. They give rise to different instances of the initial multiplet. There are thus up to $N_c - 1$ instances of the multiplet M within $M \otimes A$, coming from the different placements of the second $\boxed{1}$. In these cases the first occurrence of the resulting multiplet M' is, trivially, equal to the first occurrence of the initial multiplet M.

All other patterns in (B.15) have different shapes, and, accordingly, the corresponding resulting Young diagrams (if they are allowed) also have different shapes. Thus, all of these patterns correspond to unique multiplets in $M \otimes A$, which carry non-equivalent irreducible representations of $\mathrm{SU}(N_c)$. It remains to determine their first occurrence.

Assume that we have calculated the first occurrence of the initial multiplet M according to the rules, described before the examples (B.9). In particular we have placed the vertical line such that the number of boxes $\square$ to the right, and the number of cut out spaces $\boxdot$ to the left of it are the same. Keep this line in place while appending the boxes according to the pattern. Now each box which we append to the Young diagram for M either increases the number of boxes $\square$ to the right of the vertical line by one or decreases the number of cut out spaces $\boxdot$ to the left of it by one. Thus, the difference between the former and the latter number always increases by N_c. In order to determine the first occurrence of the resulting multiplet M' we thus have to compensate this increase by moving the vertical line one column to the right, as by this, in each row we either decrease the number of boxes $\square$ to the right of it by one or increase the number of cut out spaces $\boxdot$ to the left of it by one.

Consider now again the patterns in (B.15) which append two boxes to one row and nothing to one of the other rows. Think of applying these according to the following two-step procedure. First append one box to each row and move the vertical line one box to the right as justified in the preceding paragraph. Then move one box from the row in which the pattern has the $\boxdot$ to the row in which the pattern has two boxes. We illustrate this for the example of eqs. (B.12)–(B.14),


(B.16)

where in the second step the box is moved from row 2 to row 1. Note once more that it does not matter whether we keep or omit the first column of the resulting Young diagram.

The first occurrence of the resulting diagram now depends only on whether the box, which is moved in the last step, crosses the vertical line, and, if it does, in which direction. If it does not cross the line, then the first occurrence does not change (but the shape of the diagram does). This is the case in the example (B.16), and consequently the first occurrence remains 5. However, the first occurrence changes whenever the moving box crosses the vertical line. If the box crosses the line from left to right then $n_f(M') = n_f(M) + 1$, e.g. if it is moved from row 3, 4, 5 or 6 to row 1 or 2 in the example. If the box crosses the line from right to left, then $n_f(M') = n_f(M) - 1$, e.g. if it is moved from row 1 or 2 to row 4, 5 or 6 in the example. As only one box moves, the first occurrence can change by at most 1. From this reasoning we also see immediately from the Young diagram shapes that there cannot be any "27"-plet within "0"$\otimes$"8"$\subset A^{\otimes 3}$.

We summarize the results of what can happen under multiplication with one additional gluon, i.e. for $M' \subseteq M \otimes A$:

(i) If $M' = M$ then it can appear up to $N_c - 1$ times in $M \otimes A$.

(ii) All other $M' \subseteq M \otimes A$ appear only once. In particular all new multiplets within $M \otimes A$ appear with multiplicity one.

(iii) The first occurrence of M' differs from that of M by at most 1, i.e. $n_f(M') \in \{n_f(M)-1, n_f(M), n_f(M)+1\}$.

The fact that n_f can change by at most one unit has a well-known manifestation for SU(2) (spin), namely that when multiplying a spin-j state with a spin-1 state then the resulting state's total spin is either $j-1$, j or $j+1$.

C. Properties of the Clebsch-Gordan matrices $\mathbf{C}^M$

In this appendix we prove some properties of the Clebsch-Gordan matrices $\mathbf{C}^M : A^{\otimes n_f} \to A^{\otimes n_g}$ and projection operators introduced in section 4.

We first consider eq. (4.10) which states the following. If $\mathbf{P}^M = \mathbf{C}^M \mathbf{C}^{M\dagger}$ projects onto a multiplet $M \subseteq A^{\otimes n_g}$ then $\mathbf{P}^{M_f} = \mathbf{C}^{M\dagger}\mathbf{C}^M$ is also projector, projecting onto $M_f \subseteq A^{\otimes n_f}$, a first occurrence of the multiplet M. Notice first that by cyclic permutation under the trace

$$d := \dim M = \operatorname{tr} \mathbf{P}^M = \operatorname{tr} \mathbf{P}^{M_f} = \dim M_f\,, \tag{C.1}$$

i.e. if $\mathbf{P}^{M_f}$ is a projector, then it projects onto a subspace with the same dimension as M. It remains to show that (a) $\mathbf{P}^{M_f}$ actually is a projector and that (b) M_f carries an irreducible representation of SU(N_c) equivalent to that carried by M.

(a) Recall that both $\mathbf{C}^M\mathbf{C}^{M\dagger}$ and $\mathbf{C}^{M\dagger}\mathbf{C}^M$ are diagonalizable since they are manifestly hermitian. All their non-zero eigenvalues are identical, including their multiplicities, for the following reason: If $\lambda \neq 0$ is an eigenvalue of $\mathbf{C}^M\mathbf{C}^{M\dagger}$ then there exists a non-zero $v \in A^{\otimes n_g}$ such that $\mathbf{C}^M\mathbf{C}^{M\dagger} v = \lambda v$. Defining $u \in A^{\otimes n_f}$ by $u = \mathbf{C}^{M\dagger} v$ the eigenvalue equation can be rewritten as $\mathbf{C}^M u = \lambda v$ and by multiplication with $\mathbf{C}^{M\dagger}$ on the left we obtain

$$\mathbf{C}^{M\dagger}\mathbf{C}^M u = \mathbf{C}^{M\dagger}\lambda v = \lambda u\,, \tag{C.2}$$

i.e. λ is also an eigenvalue of $\mathbf{C}^{M\dagger}\mathbf{C}^M$ with eigenvector u. As $\mathbf{P}^M = \mathbf{C}^M\mathbf{C}^{M\dagger}$ is a projector its only non-zero eigenvalue is 1 and by the above argument this is also true for $\mathbf{P}^{M_f} = \mathbf{C}^{M\dagger}\mathbf{C}^M$, i.e. $\mathbf{P}^{M_f}$ is also a projector.

(b) In order to analyze whether M_f is invariant under SU(N_c) it is necessary to distinguish carefully between a multiplet M and the irreducible representation which it carries. We denote by Ad the adjoint representation of SU(N_c), in particular, for every group element $G \in \mathrm{SU}(N_c)$, Ad(G) is a linear operator $A \to A$. The space $A^{\otimes n_g}$ carries $\mathrm{Ad}^{\otimes n_g}$, the n_g-fold tensor product of the adjoint representation. By Γ_M we denote the irreducible representation carried by $M \subseteq A^{\otimes n_g}$.

Now we choose bases $\{v_j\}$ for $A^{\otimes n_g}$ and $\{u_j\}$ for $A^{\otimes n_f}$ such that $v_1, \ldots, v_d$ span M and $u_1, \ldots, u_d$ span M_f. In this basis $\mathbf{P}^M$ reads

$$\mathbf{P}^M = \begin{pmatrix} \mathbb{1}_d & 0 \\ 0 & 0 \end{pmatrix} \tag{C.3}$$

where $\mathbb{1}_d$ denotes the $d \times d$ unit matrix. Similarly, $\mathbf{P}^{M_f} = \left(\begin{smallmatrix} \mathbb{1}_d & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Recall, however, that in general the zero blocks have different sizes for $\mathbf{P}^M$ and $\mathbf{P}^{M_f}$.

Next we choose a vector $v \in M$ and define $u := \mathbf{C}^{M\dagger} v$. As a consequence of (a) we have $u \in M_f$ and $v = \mathbf{C}^M u$. We know how v transforms under $\mathrm{SU}(N_c)$,

$$\mathrm{Ad}^{\otimes n_g}(G)\, v = \begin{pmatrix} \Gamma_M(G) & 0 \\ 0 & 0 \end{pmatrix} v \;=: \tilde{v} \in \; M \qquad \forall\, G \in \mathrm{SU}(N_c)\,. \tag{C.4}$$

Again we have $\tilde{u} := \mathbf{C}^{M\dagger}\, \tilde{v} \in M_f$, i.e. the transformation $\mathrm{Ad}^{\otimes n_g}(G)$ leads to a transformation $\Gamma_{M_f}(G) : M_f \to M_f$ defined by

$$\tilde{u} =: \begin{pmatrix} \Gamma_{M_f}(G) & 0 \\ 0 & 0 \end{pmatrix} u = \mathbf{C}^{M\dagger} \begin{pmatrix} \Gamma_M(G) & 0 \\ 0 & 0 \end{pmatrix} \mathbf{C}^M\, u. \tag{C.5}$$

It remains to show that Γ_{M_f} is a representation of $\mathrm{SU}(N_c)$ and that it is equivalent to Γ_M. Denoting the upper left $d \times d$ block of $\mathbf{C}^M$ by U, i.e.

$$\mathbf{C}^M = \begin{pmatrix} U & \cdots \\ \vdots & \ddots \end{pmatrix}\,, \tag{C.6}$$

we observe that

$$\Gamma_{M_f}(G) = U^\dagger\, \Gamma_M(G)\, U\,. \tag{C.7}$$

Finally, we show that U is unitary which concludes the proof. To this end notice that Γ_M maps the identity $\mathbb{1}_{N_c} \in \mathrm{SU}(N_c)$ to the $d \times d$ unit matrix, i.e. $\Gamma_M(\mathbb{1}_{N_c}) = \mathbb{1}_d$, since Γ_M is a representation. From eq. (C.5) we infer

$$\begin{aligned} \begin{pmatrix} \Gamma_{M_f}(\mathbb{1}_{N_c}) & 0 \\ 0 & 0 \end{pmatrix} u = \mathbf{C}^{M\dagger} \begin{pmatrix} \mathbb{1}_d & 0 \\ 0 & 0 \end{pmatrix} \mathbf{C}^M\, u \underset{(\mathrm{C.3})}{=} \mathbf{C}^{M\dagger}\mathbf{P}^M\mathbf{C}^M\, u = \mathbf{C}^{M\dagger}\mathbf{C}^M\mathbf{C}^{M\dagger}\mathbf{C}^M\, u \\ = \left(\mathbf{P}^{M_f}\right)^2 u = \begin{pmatrix} \mathbb{1}_d & 0 \\ 0 & 0 \end{pmatrix} u\,, \end{aligned} \tag{C.8}$$

i.e. $\mathbb{1}_d \underset{(\mathrm{C.8})}{=} \Gamma_{M_f}(\mathbb{1}_{N_c}) \underset{(\mathrm{C.7})}{=} U^\dagger\, \Gamma_M(\mathbb{1}_{N_c})\, U = U^\dagger U$.

Having shown eq. (4.10) we turn to eq. (4.11) which is proved easily using $\mathbf{C}^{M\dagger}\mathbf{C}^{M'} = 0 \;\Leftrightarrow\; \|\mathbf{C}^{M\dagger}\mathbf{C}^{M'}\| = 0$ and the norm (1.5):

$$\|\mathbf{C}^{M\dagger}\mathbf{C}^{M'}\|^2 = \mathrm{tr}(\mathbf{C}^{M'\dagger}\mathbf{C}^M\mathbf{C}^{M\dagger}\mathbf{C}^{M'}) = \mathrm{tr}(\mathbf{P}^M\mathbf{P}^{M'}) = 0 \quad \text{for } M \neq M'\,. \tag{C.9}$$

When constructing basis vectors from projectors in appendix H we are also interested in the norm of $\mathbf{V} := \mathbf{C}^M\mathbf{C}^{M'\dagger}$,

$$\|\mathbf{V}\|^2 = \mathrm{tr}(\mathbf{C}^{M'}\mathbf{C}^{M\dagger}\mathbf{C}^M\mathbf{C}^{M'\dagger}) = \mathrm{tr}(\mathbf{P}^{M_f}\mathbf{P}^{M'_f}) = \delta_{M_f M'_f}\, \mathrm{tr}(\mathbf{P}^{M_f}) = \delta_{M_f M'_f}\, \dim M_f, \tag{C.10}$$

where $\delta_{M_f M'_f}$ indicates that the instances of the first occurrence multiplets must be the same.

D. Projectors for $ggg \to ggg$ in $\mathrm{SU}(N_c)$

For large enough N_c ($N_c \geq 6$) there are 51 projection operators projecting onto irreducible subspaces. They are stated in eq. (D.1). Note, however, that for $N_c = 3$, this number is reduced to 29, as many projectors correspond to multiplets whose Young diagrams are not admissible (as, for example, column 2 is longer than column 1), or were constructed using a non-admissible intermediate state (as is the case for $\mathbf{P}^{0,8}$).

In addition, as the gluon transforms under a real representation, for processes with only gluons, the projection operators can only occur in real (i.e. symmetric) combinations, further reducing the number of physically independent projection operators to 21. The full set of 51 projection operators reads

$$
\begin{aligned}
\mathbf{P}^{8s,1}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= \frac{1}{T_R}\frac{N_c}{2\,(N_c^4-5N_c^2+4)}d_{g_1\,g_2\,g_3}d_{g_4\,g_5\,g_6}\\
\mathbf{P}^{8a,1}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= \frac{1}{T_R}\frac{-1}{2N_c(N_c^2-1)}\mathrm{i}f_{g_1\,g_2\,g_3}\mathrm{i}f_{g_4\,g_5\,g_6}\\
\mathbf{P}^{1,8}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= \frac{1}{N_c^2-1}\delta_{g_1\,g_2}\delta_{g_4\,g_5}\delta_{g_3\,g_6}\\
\mathbf{P}^{8s,8s}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= \frac{1}{T_R^2}\frac{N_c^2}{4\,(N_c^2-4)^2}d_{g_1\,g_2\,i_1}d_{i_1\,g_3\,i_2}d_{g_4\,g_5\,i_3}d_{i_3\,g_6\,i_2}\\
\mathbf{P}^{8s,8a}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= \frac{1}{T_R^2}\frac{-1}{4(N_c^2-4)}d_{g_1\,g_2\,i_1}\mathrm{i}f_{i_1\,g_3\,i_2}d_{g_4\,g_5\,i_3}\mathrm{i}f_{i_3\,g_6\,i_2}\\
\mathbf{P}^{8a,8s}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= \frac{1}{T_R^2}\frac{-1}{4(N_c^2-4)}\mathrm{i}f_{g_1\,g_2\,i_1}d_{i_1\,g_3\,i_2}\mathrm{i}f_{g_4\,g_5\,i_3}d_{i_3\,g_6\,i_2}\\
\mathbf{P}^{8a,8a}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= \frac{1}{T_R^2}\frac{1}{4N_c^2}\mathrm{i}f_{g_1\,g_2\,i_1}\mathrm{i}f_{i_1\,g_3\,i_2}\mathrm{i}f_{g_4\,g_5\,i_3}\mathrm{i}f_{i_3\,g_6\,i_2}\\
\mathbf{P}^{10,8}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= \frac{4}{N_c^2-4}\mathbf{P}^{10}_{g_1\,g_2\,i_1\,g_3}\mathbf{P}^{10}_{i_1\,g_6\,g_4\,g_5}\\
\mathbf{P}^{\overline{10},8}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= \frac{4}{N_c^2-4}\mathbf{P}^{\overline{10}}_{g_1\,g_2\,i_1\,g_3}\mathbf{P}^{\overline{10}}_{i_1\,g_6\,g_4\,g_5}\\
\mathbf{P}^{27,8}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= \frac{4(N_c+1)}{N_c^2(N_c+3)}\mathbf{P}^{27}_{g_1\,g_2\,i_1\,g_3}\mathbf{P}^{27}_{i_1\,g_6\,g_4\,g_5}\\
\mathbf{P}^{0,8}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= \frac{4(N_c-1)}{(N_c-3)N_c^2}\mathbf{P}^{0}_{g_1\,g_2\,i_1\,g_3}\mathbf{P}^{0}_{i_1\,g_6\,g_4\,g_5}\\
\mathbf{P}^{8s,10}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= \frac{1}{T_R}\frac{N_c}{2\,(N_c^2-4)}d_{g_1\,g_2\,i_1}\mathbf{P}^{10}_{i_1\,g_3\,i_2\,g_6}d_{i_2\,g_4\,g_5}\\
\mathbf{P}^{8a,10}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= -\frac{1}{T_R}\frac{1}{2N_c}\mathrm{i}f_{g_1\,g_2\,i_1}\mathbf{P}^{10}_{i_1\,g_3\,i_2\,g_6}\mathrm{i}f_{i_2\,g_4\,g_5}\\
\mathbf{P}^{10,10f}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= \mathbf{C}^{10f}_{g_1\,g_2\,g_3;\,i_1\,i_2}\mathbf{C}^{10f\,\dagger}_{i_1\,i_2;\,g_4\,g_5\,g_6}\\
\mathbf{P}^{10,10fd}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= \mathbf{C}^{10fd}_{g_1\,g_2\,g_3;\,i_1\,i_2}\mathbf{C}^{10fd\,\dagger}_{i_1\,i_2;\,g_4\,g_5\,g_6}\\
\mathbf{P}^{27,10}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= \frac{1}{T_R}\frac{2(N_c+2)}{N_c(N_c+3)}\mathbf{P}^{27}_{g_1\,g_2\,i_1\,i_2}d_{i_2\,g_3\,i_3}\mathbf{P}^{10}_{i_1\,i_3\,i_4\,i_6}d_{i_6\,g_6\,i_5}\mathbf{P}^{27}_{i_4\,i_5\,g_4\,g_5}\\
\mathbf{P}^{0,10}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= \frac{1}{T_R}\frac{2(N_c-2)}{N_c(N_c-3)}\mathbf{P}^{0}_{g_1\,g_2\,i_1\,i_2}d_{i_2\,g_3\,i_3}\mathbf{P}^{10}_{i_1\,i_3\,i_4\,i_6}d_{i_6\,g_6\,i_5}\mathbf{P}^{0}_{i_4\,i_5\,g_4\,g_5}
\end{aligned}
$$

$$\mathbf{P}^{8s,\overline{10}}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} = \frac{1}{T_R}\frac{N_c}{2\,(N_c^2-4)}d_{g_1\,g_2\,i_1}\mathbf{P}^{\overline{10}}_{i_1\,g_3\,i_2\,g_6}d_{i_2\,g_4\,g_5}$$

$$\mathbf{P}^{8a,\overline{10}}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} = -\frac{1}{T_R}\frac{1}{2N_c}\mathrm{i}f_{g_1\,g_2\,i_1}\mathbf{P}^{\overline{10}}_{i_1\,g_3\,i_2\,g_6}\mathrm{i}f_{i_2\,g_4\,g_5}$$

$$\mathbf{P}^{\overline{10},\overline{10}f}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} = \mathbf{C}^{\overline{10}f}_{g_1\,g_2\,g_3;\,i_1\,i_2}\mathbf{C}^{\overline{10}f\,\dagger}_{i_1\,i_2;\,g_4\,g_5\,g_6}$$

$$\mathbf{P}^{\overline{10},\overline{10}fd}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} = \mathbf{C}^{\overline{10}fd}_{g_1\,g_2\,g_3;\,i_1\,i_2}\mathbf{C}^{\overline{10}fd\,\dagger}_{i_1\,i_2;\,g_4\,g_5\,g_6}$$

$$\mathbf{P}^{27,\overline{10}}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} = \frac{1}{T_R}\frac{2(N_c+2)}{N_c(N_c+3)}\mathbf{P}^{27}_{g_1\,g_2\,i_1\,i_2}d_{i_2\,g_3\,i_3}\mathbf{P}^{\overline{10}}_{i_1\,i_3\,i_4\,i_6}d_{i_6\,g_6\,i_5}\mathbf{P}^{27}_{i_4\,i_5\,g_4\,g_5}$$

$$\mathbf{P}^{0,\overline{10}}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} = \frac{1}{T_R}\frac{2(N_c-2)}{N_c(N_c-3)}\mathbf{P}^{0}_{g_1\,g_2\,i_1\,i_2}d_{i_2\,g_3\,i_3}\mathbf{P}^{\overline{10}}_{i_1\,i_3\,i_4\,i_6}d_{i_6\,g_6\,i_5}\mathbf{P}^{0}_{i_4\,i_5\,g_4\,g_5}$$

$$\mathbf{P}^{8s,27}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} = \frac{1}{T_R}\frac{N_c}{2(N_c^2-4)}d_{g_1\,g_2\,i_1}\mathbf{P}^{27}_{i_1\,g_3\,i_2\,g_6}d_{i_2\,g_4\,g_5}$$

$$\mathbf{P}^{8a,27}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} = \frac{1}{T_R}\frac{-1}{2N_c}\mathrm{i}f_{g_1\,g_2\,i_1}\mathbf{P}^{27}_{i_1\,g_3\,i_2\,g_6}\mathrm{i}f_{i_2\,g_4\,g_5}$$

$$\mathbf{P}^{10,27}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} = \frac{1}{T_R}\frac{2N_c}{N_c^2-N_c-2}\mathbf{P}^{10}_{g_1\,g_2\,i_1\,i_2}d_{i_2\,g_3\,i_3}\mathbf{P}^{27}_{i_1\,i_3\,i_4\,i_6}d_{i_6\,g_6\,i_5}\mathbf{P}^{10}_{i_4\,i_5\,g_4\,g_5}$$

$$\mathbf{P}^{\overline{10},27}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} = \frac{1}{T_R}\frac{2N_c}{N_c^2-N_c-2}\mathbf{P}^{\overline{10}}_{g_1\,g_2\,i_1\,i_2}d_{i_2\,g_3\,i_3}\mathbf{P}^{27}_{i_1\,i_3\,i_4\,i_6}d_{i_6\,g_6\,i_5}\mathbf{P}^{\overline{10}}_{i_4\,i_5\,g_4\,g_5}$$

$$\mathbf{P}^{27,27d}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} = \frac{1}{T_R}\frac{N_c(N_c+2)}{N_c^3+3N_c^2-6N_c-8}\mathbf{P}^{27}_{g_1\,g_2\,i_1\,i_2}d_{i_2\,g_3\,i_3}\mathbf{P}^{27}_{i_1\,i_3\,i_4\,i_6}d_{i_6\,g_6\,i_5}\mathbf{P}^{27}_{i_4\,i_5\,g_4\,g_5}$$

$$\mathbf{P}^{27,27f}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} = \frac{1}{T_R}\frac{1}{N_c+1}\mathbf{P}^{27}_{g_1\,g_2\,i_1\,i_2}\mathrm{i}f_{i_2\,g_3\,i_3}\mathbf{P}^{27}_{i_1\,i_3\,i_4\,i_6}\mathrm{i}f_{i_6\,g_6\,i_5}\mathbf{P}^{27}_{i_4\,i_5\,g_4\,g_5}$$

$$\mathbf{P}^{8s,0}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} = \frac{1}{T_R}\frac{N_c}{2\,(N_c^2-4)}d_{g_1\,g_2\,i_1}\mathbf{P}^{0}_{i_1\,g_3\,i_2\,g_6}d_{i_2\,g_4\,g_5}$$

$$\mathbf{P}^{8a,0}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} = -\frac{1}{T_R}\frac{1}{2N_c}\mathrm{i}f_{g_1\,g_2\,i_1}\mathbf{P}^{0}_{i_1\,g_3\,i_2\,g_6}\mathrm{i}f_{i_2\,g_4\,g_5}$$

$$\mathbf{P}^{10,0}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} = \frac{1}{T_R}\frac{2N_c}{N_c^2+N_c-2}\mathbf{P}^{10}_{g_1\,g_2\,i_1\,i_2}\mathrm{i}f_{i_2\,g_3\,i_3}\mathbf{P}^{0}_{i_1\,i_3\,i_4\,i_6}\mathrm{i}f_{i_6\,g_6\,i_5}\mathbf{P}^{10}_{i_4\,i_5\,g_4\,g_5}$$

$$\mathbf{P}^{\overline{10},0}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} = \frac{1}{T_R}\frac{2N_c}{N_c^2+N_c-2}\mathbf{P}^{\overline{10}}_{g_1\,g_2\,i_1\,i_2}\mathrm{i}f_{i_2\,g_3\,i_3}\mathbf{P}^{0}_{i_1\,i_3\,i_4\,i_6}\mathrm{i}f_{i_6\,g_6\,i_5}\mathbf{P}^{\overline{10}}_{i_4\,i_5\,g_4\,g_5}$$

$$\mathbf{P}^{0,0d}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} = \frac{1}{T_R}\frac{(N_c-2)N_c}{N_c^3-3N_c^2-6N_c+8}\mathbf{P}^{0}_{g_1\,g_2\,i_1\,i_2}d_{i_2\,g_3\,i_3}\mathbf{P}^{0}_{i_1\,i_3\,i_4\,i_6}d_{i_6\,g_6\,i_5}\mathbf{P}^{0}_{i_4\,i_5\,g_4\,g_5}$$

$$\mathbf{P}^{0,0f}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} = \frac{1}{T_R}\frac{1}{N_c-1}\mathbf{P}^{0}_{g_1\,g_2\,i_1\,i_2}\mathrm{i}f_{i_2\,g_3\,i_3}\mathbf{P}^{0}_{i_1\,i_3\,i_4\,i_6}\mathrm{i}f_{i_6\,g_6\,i_5}\mathbf{P}^{0}_{i_4\,i_5\,g_4\,g_5}$$

$$\begin{aligned}\mathbf{P}^{27,64=c111c111}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= \frac{1}{T_R^3}\mathbf{T}^{27,64}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} - \frac{2N_c^2}{9(N_c+1)(N_c+2)}\mathbf{P}^{27,8}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6}\\ &- \frac{4(N_c^2-N_c-2)}{9N_c\,(N_c+2)}\mathbf{P}^{27,27d}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6}\end{aligned}$$

$$\begin{aligned}\mathbf{P}^{10,35=c111c21}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= \frac{1}{T_R^3}\frac{4}{3}\mathbf{T}^{10,35}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} - \frac{N_c-2}{3N_c}\mathbf{P}^{10,8}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} - \frac{N_c+3}{18N_c}\mathbf{P}^{10,10f}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6}\\ &+ \frac{\sqrt{N_c^2-9}}{6N_c}\mathbf{C}^{10f}_{g_1\,g_2\,g_3;\,i_1\,i_2}\mathbf{C}^{10fd\,\dagger}_{i_1\,i_2;\,g_4\,g_5\,g_6} + \frac{\sqrt{N_c^2-9}}{6N_c}\mathbf{C}^{10fd}_{g_1\,g_2\,g_3;\,i_1\,i_2}\mathbf{C}^{10f\,\dagger}_{i_1\,i_2;\,g_4\,g_5\,g_6}\end{aligned}$$

$$
\begin{aligned}
& - \frac{N_c-3}{2N_c}\mathbf{P}^{10,10fd}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} - \frac{N_c-2}{3N_c}\mathbf{P}^{10,27}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} \\
\mathbf{P}^{27,35=c111c21}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= \frac{1}{T_R^3}\frac{4}{3}\mathbf{T}^{27,35}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} - \frac{N_c(N_c+3)}{9(N_c+1)(N_c+2)}\mathbf{P}^{27,8}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} \\
& - \frac{N_c}{3(N_c+2)}\mathbf{P}^{27,10}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} - \frac{N_c^2+2N_c-8}{18N_c(N_c+2)}\mathbf{P}^{27,27d}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} \\
& - \frac{1}{T_R}\frac{1}{6(N_c+1)}\mathbf{P}^{27}_{g_1\,g_2\,i_1\,i_2} d_{i_2\,g_3\,i_3}\mathbf{P}^{27}_{i_1\,i_3\,i_4\,i_6}\mathrm{i}f_{i_6\,g_6\,i_5}\mathbf{P}^{27}_{i_4\,i_5\,g_4\,g_5} \\
& - \frac{1}{T_R}\frac{1}{6(N_c+1)}\mathbf{P}^{27}_{g_1\,g_2\,i_1\,i_2}\mathrm{i}f_{i_2\,g_3\,i_3}\mathbf{P}^{27}_{i_1\,i_3\,i_4\,i_6} d_{i_6\,g_6\,i_5}\mathbf{P}^{27}_{i_4\,i_5\,g_4\,g_5} - \frac{1}{2}\mathbf{P}^{27,27f}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} \\
\mathbf{P}^{\overline{10},\overline{35}=c21c111}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= \frac{1}{T_R^3}\frac{4}{3}\mathbf{T}^{\overline{10},\overline{35}}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} - \frac{N_c-2}{3N_c}\mathbf{P}^{\overline{10},8}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} - \frac{N_c+3}{18N_c}\mathbf{P}^{\overline{10},\overline{10}f}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} \\
& - \frac{\sqrt{N_c^2-9}}{6N_c}\mathbf{C}^{\overline{10}f}_{g_1\,g_2\,g_3;\,i_1\,i_2}\mathbf{C}^{\overline{10}fd\,\dagger}_{i_1\,i_2;\,g_4\,g_5\,g_6} - \frac{\sqrt{N_c^2-9}}{6N_c}\mathbf{C}^{\overline{10}fd}_{g_1\,g_2\,g_3;\,i_1\,i_2}\mathbf{C}^{\overline{10}f\,\dagger}_{i_1\,i_2;\,g_4\,g_5\,g_6} \\
& - \frac{N_c-3}{2N_c}\mathbf{P}^{\overline{10},\overline{10}fd}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} - \frac{N_c-2}{3N_c}\mathbf{P}^{\overline{10},27}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} \\
\mathbf{P}^{27,\overline{35}=c21c111}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= \frac{1}{T_R^3}\frac{4}{3}\mathbf{T}^{27,\overline{35}}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} - \frac{N_c(N_c+3)}{9(N_c+1)(N_c+2)}\mathbf{P}^{27,8}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} \\
& - \frac{N_c}{3(N_c+2)}\mathbf{P}^{27,\overline{10}}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} - \frac{N_c^2+2N_c-8}{18N_c(N_c+2)}\mathbf{P}^{27,27d}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} \\
& + \frac{1}{T_R}\frac{1}{6(N_c+1)}\mathbf{P}^{27}_{g_1\,g_2\,i_1\,i_2} d_{i_2\,g_3\,i_3}\mathbf{P}^{27}_{i_1\,i_3\,i_4\,i_6}\mathrm{i}f_{i_6\,g_6\,i_5}\mathbf{P}^{27}_{i_4\,i_5\,g_4\,g_5} \\
& + \frac{1}{T_R}\frac{1}{6(N_c+1)}\mathbf{P}^{27}_{g_1\,g_2\,i_1\,i_2}\mathrm{i}f_{i_2\,g_3\,i_3}\mathbf{P}^{27}_{i_1\,i_3\,i_4\,i_6} d_{i_6\,g_6\,i_5}\mathbf{P}^{27}_{i_4\,i_5\,g_4\,g_5} - \frac{1}{2}\mathbf{P}^{27,27f}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} \\
\mathbf{P}^{10,c21c21}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= \frac{1}{T_R^3}\frac{16}{9}\mathbf{T}^{10,c21c21}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} - \frac{1}{3}\mathbf{P}^{10,8}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} - \frac{4}{9}\mathbf{P}^{10,10f}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} \\
& - \frac{2\,(N_c+1)}{3N_c}\mathbf{P}^{10,27}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} - \frac{2\,(N_c-1)}{3N_c}\mathbf{P}^{10,0}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} \\
\mathbf{P}^{\overline{10},c21c21}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= \frac{1}{T_R^3}\frac{16}{9}\mathbf{T}^{\overline{10},c21c21}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} - \frac{1}{3}\mathbf{P}^{\overline{10},8}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} - \frac{4}{9}\mathbf{P}^{\overline{10},\overline{10}f}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} \\
& - \frac{2\,(N_c+1)}{3N_c}\mathbf{P}^{\overline{10},27}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} - \frac{2\,(N_c-1)}{3N_c}\mathbf{P}^{\overline{10},0}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} \\
\mathbf{P}^{27,c21c21}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= \frac{1}{T_R^3}\frac{16}{9}\mathbf{T}^{27,c21c21}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} - \frac{(N_c+3)(5N_c+6)}{9(N_c+1)(N_c+2)}\mathbf{P}^{27,8}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} \\
& - \frac{2(N_c+3)}{3(N_c+2)}\mathbf{P}^{27,10}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} - \frac{2(N_c+3)}{3(N_c+2)}\mathbf{P}^{27,\overline{10}}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} \\
& - \frac{4(N_c+1)(N_c+4)}{9N_c(N_c+2)}\mathbf{P}^{27,27d}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} \\
\mathbf{P}^{0,c21c21}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= \frac{1}{T_R^3}\frac{16}{9}\mathbf{T}^{0,c21c21}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} + \frac{(6-5N_c)(N_c-3)}{9(N_c-2)(N_c-1)}\mathbf{P}^{0,8}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} \\
& - \frac{2(N_c-3)}{3(N_c-2)}\mathbf{P}^{0,10}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} - \frac{2(N_c-3)}{3(N_c-2)}\mathbf{P}^{0,\overline{10}}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6}
\end{aligned}
$$

$$
\begin{aligned}
& - \frac{4(N_c-4)(N_c-1)}{9(N_c-2)N_c}\mathbf{P}^{0,0d}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} \\
\mathbf{P}^{10,c111c3}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= \frac{1}{T_R^3}\mathbf{T}^{10,c111c3}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} - \frac{4}{9}\mathbf{P}^{10,10f}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} \\
\mathbf{P}^{\overline{10},c3c111}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= \frac{1}{T_R^3}\mathbf{T}^{\overline{10},c3c111}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} - \frac{4}{9}\mathbf{P}^{\overline{10},\overline{10}f}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} \\
\mathbf{P}^{0,c21c3}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= \frac{1}{T_R^3}\frac{4}{3}\mathbf{T}^{0,c21c3}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} - \frac{(N_c-3)N_c}{9(N_c-2)(N_c-1)}\mathbf{P}^{0,8}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} \\
& - \frac{N_c}{3(N_c-2)}\mathbf{P}^{0,10}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} - \frac{N_c^2-2N_c-8}{18(N_c-2)N_c}\mathbf{P}^{0,0d}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} \\
& + \frac{1}{T_R}\frac{1}{6(N_c-1)}\mathbf{P}^{0}_{g_1\,g_2\,i_1\,i_2} d_{i_2\,g_3\,i_3}\mathbf{P}^{0}_{i_1\,i_3\,i_4\,i_6}\mathrm{i}f_{i_6\,g_6\,i_5}\mathbf{P}^{0}_{i_4\,i_5\,g_4\,g_5} \\
& + \frac{1}{T_R}\frac{1}{6(N_c-1)}\mathbf{P}^{0}_{g_1\,g_2\,i_1\,i_2}\mathrm{i}f_{i_2\,g_3\,i_3}\mathbf{P}^{0}_{i_1\,i_3\,i_4\,i_6} d_{i_6\,g_6\,i_5}\mathbf{P}^{0}_{i_4\,i_5\,g_4\,g_5} - \frac{1}{2}\mathbf{P}^{0,0f}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} \\
\mathbf{P}^{10,c21c3}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= \frac{1}{T_R^3}\frac{4}{3}\mathbf{T}^{10,c21c3}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} - \frac{N_c+2}{3N_c}\mathbf{P}^{10,8}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} \\
& - \frac{N_c-3}{18N_c}\mathbf{P}^{10,10f}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} - \frac{\sqrt{N_c^2-9}}{6N_c}\mathbf{C}^{10f}_{g_1\,g_2\,g_3;\,i_1\,i_2}\mathbf{C}^{10fd\,\dagger}_{i_1\,i_2;\,g_4\,g_5\,g_6} \\
& - \frac{\sqrt{N_c^2-9}}{6N_c}\mathbf{C}^{10fd}_{g_1\,g_2\,g_3;\,i_1\,i_2}\mathbf{C}^{10f\,\dagger}_{i_1\,i_2;\,g_4\,g_5\,g_6} - \frac{N_c+3}{2N_c}\mathbf{P}^{10,10fd}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} - \frac{N_c+2}{3N_c}\mathbf{P}^{10,0}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} \\
\mathbf{P}^{0,c3c21}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= \frac{1}{T_R^3}\frac{4}{3}\mathbf{T}^{0,c3c21}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} - \frac{(N_c-3)N_c}{9(N_c-2)(N_c-1)}\mathbf{P}^{0,8}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} \\
& - \frac{N_c}{3(N_c-2)}\mathbf{P}^{0,\overline{10}}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} - \frac{\left(N_c^2-2N_c-8\right)}{18(N_c-2)N_c}\mathbf{P}^{0,0d}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} \\
& - \frac{1}{T_R}\frac{1}{6(N_c-1)}\mathbf{P}^{0}_{g_1\,g_2\,i_1\,i_2} d_{i_2\,g_3\,i_3}\mathbf{P}^{0}_{i_1\,i_3\,i_4\,i_6}\mathrm{i}f_{i_6\,g_6\,i_5}\mathbf{P}^{0}_{i_4\,i_5\,g_4\,g_5} \\
& - \frac{1}{T_R}\frac{1}{6(N_c-1)}\mathbf{P}^{0}_{g_1\,g_2\,i_1\,i_2}\mathrm{i}f_{i_2\,g_3\,i_3}\mathbf{P}^{0}_{i_1\,i_3\,i_4\,i_6} d_{i_6\,g_6\,i_5}\mathbf{P}^{0}_{i_4\,i_5\,g_4\,g_5} - \frac{1}{2}\mathbf{P}^{0,0f}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} \\
\mathbf{P}^{\overline{10},c3c21}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= \frac{1}{T_R^3}\frac{4}{3}\mathbf{T}^{\overline{10},c3c21}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} - \frac{N_c+2}{3N_c}\mathbf{P}^{\overline{10},8}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} \\
& - \frac{N_c-3}{18N_c}\mathbf{P}^{\overline{10},\overline{10}f}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} + \frac{\sqrt{N_c^2-9}}{6N_c}\mathbf{C}^{\overline{10}f}_{g_1\,g_2\,g_3;\,i_1\,i_2}\mathbf{C}^{\overline{10}fd\,\dagger}_{i_1\,i_2;\,g_4\,g_5\,g_6} \\
& + \frac{\sqrt{N_c^2-9}}{6N_c}\mathbf{C}^{\overline{10}fd}_{g_1\,g_2\,g_3;\,i_1\,i_2}\mathbf{C}^{\overline{10}f\,\dagger}_{i_1\,i_2;\,g_4\,g_5\,g_6} - \frac{N_c+3}{2N_c}\mathbf{P}^{\overline{10},\overline{10}fd}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} - \frac{N_c+2}{3N_c}\mathbf{P}^{\overline{10},0}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} \\
\mathbf{P}^{0,c3c3}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= \frac{1}{T_R^3}\mathbf{T}^{0,c3c3}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} - \frac{2N_c^2}{9(N_c-2)(N_c-1)}\mathbf{P}^{0,8}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} \\
& - \frac{4(N_c^2+N_c-2)}{9(N_c-2)N_c}\mathbf{P}^{0,0d}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6}
\end{aligned}
\tag{D.1}
$$

where the tensors $\mathbf{T}^{M_{12},M_{123}}$ are constructed as indicated in eq. (4.25), and

$$
\mathbf{C}^{10f\,\dagger}_{i_1\,i_2;\,g_4\,g_5\,g_6} = \frac{1}{\sqrt{N_c T_R}}\mathbf{P}^{10}_{i_1\,i_2\,i_3\,i_4}\mathrm{i}f_{i_4\,g_6\,i_5}\mathbf{P}^{10}_{i_3\,i_5\,g_4\,g_5}
$$

$$\mathbf{C}^{\overline{10}f\,\dagger}_{i_1\,i_2;\,g_4\,g_5\,g_6} = \frac{1}{\sqrt{N_c T_R}}\mathbf{P}^{\overline{10}}_{i_1\,i_2\,i_3\,i_4}\mathrm{i}f_{i_4\,g_6\,i_5}\mathbf{P}^{\overline{10}}_{i_3\,i_5\,g_4\,g_5}$$

$$\mathbf{C}^{10fd\,\dagger}_{i_1\,i_2;\,g_4\,g_5\,g_6} = \frac{\sqrt{N_c}}{\sqrt{T_R(N_c^2-9)}}\mathbf{P}^{10}_{i_1\,i_2\,i_3\,i_4}(d_{i_4\,g_6\,i_5} - \frac{1}{N_c}\mathrm{i}f_{i_4\,g_6\,i_5})\mathbf{P}^{10}_{i_3\,i_5\,g_4\,g_5}$$

$$\mathbf{C}^{\overline{10}fd\,\dagger}_{i_1\,i_2;\,g_4\,g_5\,g_6} = \frac{\sqrt{N_c}}{\sqrt{T_R(N_c^2-9)}}\mathbf{P}^{\overline{10}}_{i_1\,i_2\,i_3\,i_4}(d_{i_4\,g_6\,i_5} + \frac{1}{N_c}\mathrm{i}f_{i_4\,g_6\,i_5})\mathbf{P}^{\overline{10}}_{i_3\,i_5\,g_4\,g_5}\,. \quad \text{(D.2)}$$

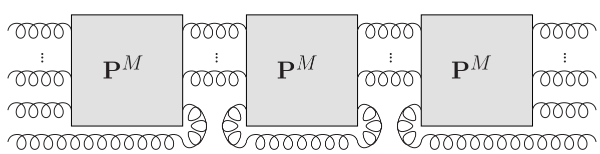
To understand the presence of T_R in eq. (D.1) we note that expressed in terms of pure traces, such as for example $\mathrm{tr}[t^{g_2}t^{g_5}t^{g_6}t^{g_4}t^{g_3}t^{g_1}]$, the T_R factor would enter as $1/T_R^3$, to compensate for the contraction of three gluons (giving T_R^3) when squaring the projection operator. However, every internal index gives rise to an extra factor T_R when contracted, and the $\mathrm{i}f_{abc}$ and d_{abc} come with a factor $1/T_R$, as can be seen from eqs. (1.12) and (1.20). Similarly, the two gluon projection operators contain a compensating factor $1/T_R^2$. When all this is accounted for the T_R appears as in eq. (D.1) and eq. (D.2).

E. Properties of some projection operators

In this appendix we collect and prove some properties of the gluon projection operators in section 4.

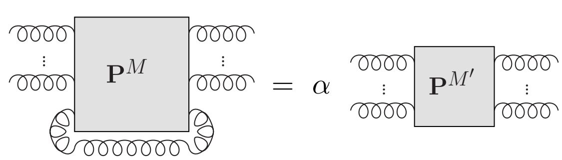
First we prove eq. (4.18). We explicitly verify that $\mathbf{P}^{M'}$ is a projector by squaring,

$$\left(\mathbf{P}^{\dots,M',M,M'}\right)^2 = \left(\frac{\dim M'}{\dim M}\right)^2$$



(E.1)

Due to property (ii) the term in the middle has to be proportional to $\mathbf{P}^{M'} : A^{\otimes(n_g-2)} \to A^{\otimes(n_g-2)}$, i.e.

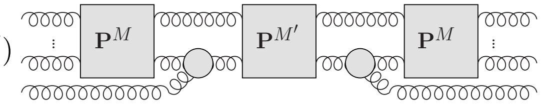


(E.2)

The constant can be found by taking the trace, $\dim M = \alpha \dim M'$, and thus $\alpha = \dim M/\dim M'$. Substituting into eq. (E.1) we obtain $\left(\mathbf{P}^{\dots,M,M'}\right)^2 = \mathbf{P}^{\dots,M,M'}$ as desired.

We now turn to eq. (4.20). Letting

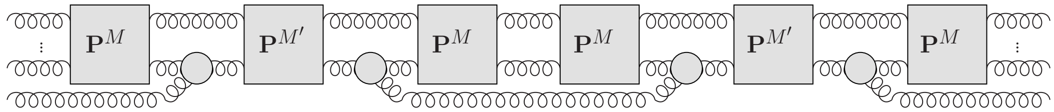
$$\mathbf{P}^{\dots,M,M'} = \mathcal{N}(M,M')$$



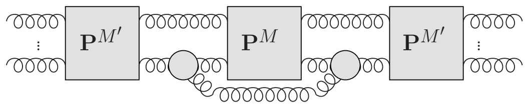
(E.3)

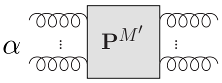
where $\mathcal{N}(M,M')$ denotes a normalization factor, we have

$$(\mathbf{P}^{\dots,M,M'})^2 = \mathcal{N}^2(M,M')\,\times \quad \text{(E.4)}$$

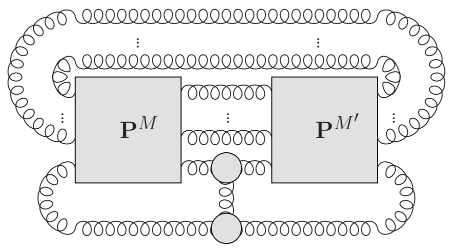
×  .

After employing $(\mathbf{P}^M)^2 = \mathbf{P}^M$ we observe that the middle part can be written as



$= \alpha$  (E.5)

for some α. We find α and thus $\mathcal{N}(M, M')$ by taking the trace

$B(M, M') =$  $= \alpha \dim(M')$, (E.6)

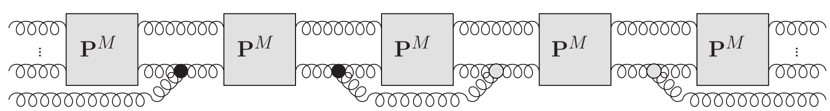
i.e. $\mathcal{N}(M, M') = \dim(M')/B(M, M')$.

Next we show that the projector (4.24) can also be written in the form (4.20). Expanding the products in eq. (4.23) we have

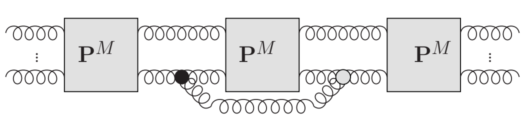
$$\begin{aligned}\mathbf{T}^{\dots,M,Mfd} = \mathbf{P}^{\dots,M,Md} - \mathbf{P}^{\dots,M,Mf}\mathbf{P}^{\dots,M,Md} - \mathbf{P}^{\dots,M,Md}\mathbf{P}^{\dots,M,Mf} \\ + \mathbf{P}^{\dots,M,Mf}\mathbf{P}^{\dots,M,Md}\mathbf{P}^{\dots,M,Mf} .\end{aligned} \tag{E.7}$$

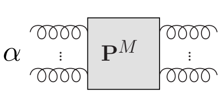
The second term on the r.h.s. reads

$$\mathbf{P}^{\dots,M,Mf}\mathbf{P}^{\dots,M,Md} = \frac{(\dim M)^2}{B_f(M)\, B_d(M)} \times$$

×  . (E.8)

Here and in the following we denote by $B_f(M)$ the bubble diagram (E.6) with $M' = M$ and where both gray circles are chosen as if, and by $B_d(M)$ the same diagram with both gray circles being d. The middle part in eq. (E.8) is proportional to $\mathbf{P}^{\dots,M}$,

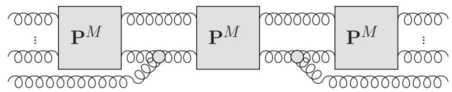


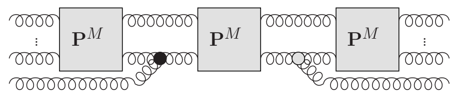
$= \alpha$  . (E.9)

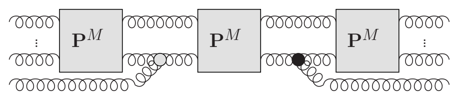
We find α by taking the trace,

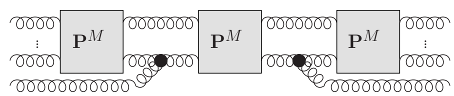
$$\alpha = \frac{C(M)}{\dim M} , \tag{E.10}$$

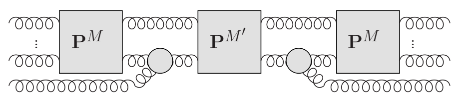
where $C(M)$ denotes the bubble diagram (E.6) with $M' = M$ and where one gray circle represents if and the other one equals d. Note that α is real. By similar calculations we also find the other terms,

$$\mathbf{T}^{\dots,M,Mfd} = \frac{\dim M}{B_d(M)} \Bigg($$ 

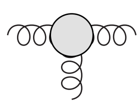
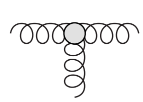
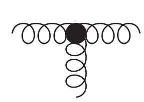
$$- \frac{C(M)}{B_f(M)}$$ 

$$- \frac{C(M)}{B_f(M)}$$  (E.11)

$$+ \left(\frac{C(M)}{B_f(M)}\right)^2$$  $$\Bigg)$$

$$= \frac{\dim M}{B_d(M)}$$  ,

where on the last line the big gray circles are given by

 $=$  $- \frac{C(M)}{B_f(M)}$  . (E.12)

Upon normalization we have cast $\mathbf{P}^{\dots,M,Mfd}$ in the form (4.20) with $M' = M$ and where all big gray circles, also those inside $B(M, M)$, are given by eq. (E.12).

F. Invariant tensors, Schur's lemma and color conservation

We briefly summarize properties of invariant tensors and their relation to the multiplet version of color conservation which follows from Schur's Lemma.

Let $v_j \in V$, $j = 1, \dots, n_q$, $w_k \in \overline{V}$, $k = 1, \dots, n_{\overline{q}}$, and $u_l \in A$, $l = 1, \dots, N_g$. Under $G \in \mathrm{SU}(N_c)$ the v_j transform in the defining representation, $v_j \mapsto Gv_j$, the w_k transform in the complex conjugate of the defining representation, $w_k \mapsto G^* w_k$ and the u_l transform in the adjoint representation, $u_l \mapsto \mathrm{Ad}(G)u_l$. A tensor $\mathbf{T} \in V^{\otimes n_q} \otimes \overline{V}^{\otimes n_{\overline{q}}} \otimes A^{\otimes N_g}$ is called *invariant* if

$$\begin{aligned} &\left\langle \mathbf{T} \,\middle|\, v_1 \otimes \dots \otimes v_{n_q} \otimes w_1 \otimes \dots \otimes w_{n_{\overline{q}}} \otimes u_1 \otimes \dots \otimes u_{N_g} \right\rangle \\ &= \left\langle \mathbf{T} \,\middle|\, Gv_1 \otimes \dots \otimes Gv_{n_q} \otimes G^* w_1 \otimes \dots \otimes G^* w_{n_{\overline{q}}} \otimes \mathrm{Ad}(G)u_1 \otimes \dots \otimes \mathrm{Ad}(G)u_{N_g} \right\rangle \end{aligned} \tag{F.1}$$

$\forall\, G \in \mathrm{SU}(N_c)$ and $\forall\, v_j, w_k, u_l$, where we use the scalar product of section 1.1. Another way to express invariance is to say that $\mathbf{T}$ is a (color) singlet, i.e. that it transforms under the trivial representation of $\mathrm{SU}(N_c)$. Viewing $\mathbf{T}$ as a linear map, e.g., as $\mathbf{T} : V^{\otimes n_{\overline{q}}} \otimes A^{\otimes n_g} \to V^{\otimes n_q} \otimes A^{\otimes (N_g - n_g)}$, the invariance condition can be rewritten as

$$\begin{aligned} &\left\langle \mathbf{T}(w_1^* \otimes \dots \otimes w_{n_{\overline{q}}}^* \otimes u_1 \otimes \dots \otimes u_{n_g}) \,\middle|\, v_1 \otimes \dots \otimes v_{n_q} \otimes u_{n_g+1} \otimes \dots \otimes u_{N_g} \right\rangle \\ &= \left\langle \mathbf{T}\,\Gamma^a(G)(w_1^* \otimes \dots \otimes w_{n_{\overline{q}}}^* \otimes u_1 \otimes \dots \otimes u_{n_g}) \,\middle|\, \Gamma^b(G)(v_1 \otimes \dots \otimes v_{n_q} \otimes u_{n_g+1} \otimes \dots \otimes u_{N_g}) \right\rangle \end{aligned} \tag{F.2}$$

where Γ^a and Γ^b are product representations of $\mathrm{SU}(N_c)$,

$$\Gamma^a(G) = G^{\otimes n_{\overline{q}}} \otimes \mathrm{Ad}(G)^{\otimes n_g}\,, \qquad \Gamma^b(G) = G^{\otimes n_q} \otimes \mathrm{Ad}(G)^{\otimes (N_g - n_g)}\,. \tag{F.3}$$

Using unitarity of $\Gamma^b(G)$ we can further rewrite the invariance condition as,

$$\langle \mathbf{T}(\cdots) \mid \cdots \rangle = \langle \Gamma^b(G)^\dagger\, \mathbf{T}\, \Gamma^a(G)(\cdots) \mid \cdots \rangle = \langle \Gamma^b(G)^{-1}\, \mathbf{T}\, \Gamma^a(G)(\cdots) \mid \cdots \rangle\,, \tag{F.4}$$

where the dots stand for the tensor products of eq. (F.2). Since the condition has to hold for any v_j, w_k, u_l an invariant tensor $\mathbf{T}$ commutes with the representations Γ^a and Γ^b in the sense that

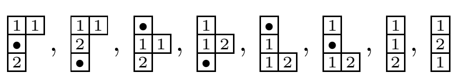
$$\mathbf{T}\, \Gamma^a(G) = \Gamma^b(G)\, \mathbf{T} \quad \forall\; G \in \mathrm{SU}(N_c)\,. \tag{F.5}$$

In the case of irreducible representations Γ^a and Γ^b, Schur's lemma, see e.g. [34, sec. 3-14], states that $\mathbf{T}$ has to be a scalar multiple of the identity if the two representations are equivalent and that $\mathbf{T}$ vanishes if they are not equivalent. In general, the representations (F.4) are not irreducible, but their carrier spaces $V^{\otimes n_{\overline{q}}} \otimes A^{\otimes n_g}$ and $V^{\otimes n_q} \otimes A^{\otimes (N_g - n_g)}$ can be decomposed into a direct sum of multiplets, $M_1 \oplus M_2 \oplus \ldots$. Introducing bases, such that the first $\dim(M_1)$ vectors span M_1, the next $\dim(M_2)$ span M_2 etc., the invariant tensor $\mathbf{T}$ gets a block structure. Due to Schur's lemma all blocks which would map elements of a multiplet M to a multiplet M' carrying a non-equivalent irreducible representation vanish identically. For instance, $\mathbf{T}$ can map an octet state to any other octet state – and there can be several octets in the decompositions of $V^{\otimes n_{\overline{q}}} \otimes A^{\otimes n_g}$ and $V^{\otimes n_q} \otimes A^{\otimes (N_g - n_g)}$ – but it can never map an octet state to a decuplet state. In the context of QCD we refer to this property as color conservation.

Examples for $\mathrm{SU}(N_c)$-invariant tensors are $\delta_{q_1}^{q_2}$ (quark or anti-quark lines), $\delta_{g_1 g_2}$ (gluon lines), the three-gluon vertices $\mathrm{i} f_{g_1 g_2 g_3}$ and $d_{g_1 g_2 g_3}$ or the generators $(t^g)_{q_1}^{q_2}$ of the defining representation, see e.g. [7, 8, 40]. Tensor products and contractions of invariant tensors are again invariant tensors, i.e. *all* tensors appearing in our constructions of projectors and basis vectors are invariant tensors.

G. Exponential scaling of number of basis vectors

We here show that for finite N_c the number of projection operators required for $n_g \to n_g$ gluons grows at most exponentially in n_g. To see this, we note that starting in any multiplet M we have, for finite N_c, according to eq. (B.15), at most a finite number of new multiplets $N_{\max}$. For $N_c = 3$, the possibilities may, in accordance with eq. (B.15), be enumerated as



For $N_c = 3$ we thus have $N_{\max} = 8$, and the number of projection operators increases at most by a factor of 8 for each new gluon. In fact, it grows much slower for a small number of gluons, as then most placements above are forbidden. However, we observe that it approaches this increase for many gluons. In general we have $N_{\max} = (N_c + 1)(N_c - 1)$ as there are N_c ways of placing the only 2-row, and, for each option, $(N_c - 1)$ ways of

SU(3) dim	64	35	$\overline{35}$	0	
Multiplet	c111c111	c111c21	c21c111	c21c21	
Out $g_4 g_5 g_6$	$((45)^{27}6)^{64}$	$((45)^{10}6)^{35}$ $((45)^{27}6)^{35}$	$((45)^{\overline{10}}6)^{\overline{35}}$ $((45)^{27}6)^{\overline{35}}$	$((45)^{10}6)^{c21c21}$ $((45)^{\overline{10}}6)^{c21c21}$ $((45)^{27}6)^{c21c21}$ $((45)^{0}6)^{c21c21}$	
SU(3) dim	0	0	0	0	0
Multiplet	c111c3	c3c111	c21c3	c3c21	c3c3
Out $g_4 g_5 g_6$	$((45)^{10}6)^{c111c3}$	$((45)^{\overline{10}}6)^{c3c111}$	$((45)^{10}6)^{c21c3}$ $((45)^{0}6)^{c21c3}$	$((45)^{\overline{10}}6)^{c3c21}$ $((45)^{0}6)^{c3c21}$	$((45)^{0}6)^{c3c3}$

Table 6: The new multiplets for $3g \to 3g$, appearing in addition to those listed in table 5. Putting together the information here and in table 5 we note that there are $2^2+9^2+6^2+6^2+6^2+6^2+1+2^2+2^2+4^2+1+1+2^2+2^2+1 = 265$ bases vectors, reducing to $2^2+8^2+4^2+4^2+6^2+1+2^2+2^2 = 145$ for SU(3).

placing the $\boxdot$. Finally, when there is no $\boxdot$ (rightmost cases above), there are $N_c - 1$ ways of placing the second 1. As we always, irrespective of the shape of the starting multiplet, have a finite $N_{\max}$ the number of projection operators grows at most as $N_{\max}^{n_g}$.

Having this upper bound for the growth of the number of projection operators we note that the number of basis vectors in the $2n_g$ space grows at most as $N_{\max}^{2n_g}$. (This would correspond to allowing transitions between all multiplets.) Again, in reality, it grows much slower for a small number of gluons.

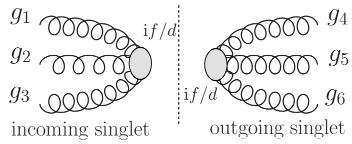
Finally, we note that for processes involving quarks, each $q\overline{q}$-pair is either in an octet, in which case the above counting for the gluon case carries over, or in a singlet, corresponding to the case of one less gluon above. We thus find that also in the general case of both, quarks and gluons, the number of projectors and basis vectors grows (at most) exponentially.

H. Orthogonal basis for six gluons in SU(N_c)

In this appendix we discuss in detail how to construct a basis for the six gluon color space. As always we utilize the method of subgrouping. In order to construct all basis vectors we have to combine every instance of a multiplet M on the incoming side with every instance of M on the outgoing side. One way of splitting the gluons is to consider $g_1\, g_2\, g_3 \to g_4\, g_5\, g_6$. Then, clearly, the possible multiplets are the same on the incoming and outgoing side. The multiplets with first occurrence up to two are in fact already listed in table 5. In addition to these multiplets there are also new multiplets, as listed in table 6. In total this gives rise to 265 basis vectors, in agreement with subfactorial(6), eq. (1.15). For $N_c = 3$ this number is reduced to 145.

To construct the corresponding basis we note that we may divide the basis vectors into four different categories, according to their first occurrence zero, one, two and three. For the singlets we effectively have zero gluons passing from the incoming to the outgoing side. They are thus of the form

$$\mathbf{C}^{8a/s,1}_{g_1\,g_2\,g_3}\mathbf{C}^{8a/s,1\,\dagger}_{g_4\,g_5\,g_6} \propto$$

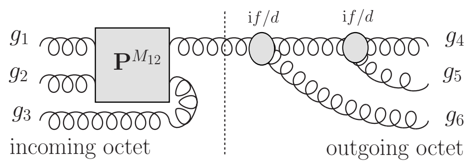


(H.1)

where – as opposed to in the projector case, c.f. eq. (4.7) – the color structure on the incoming side in general differs from that on the outgoing side, giving rise to a total of four singlets.

Similarly, in order to construct the basis vectors corresponding to multiplets with first occurrence one (octets), we note that they may be drawn having one gluon passing from the incoming to the outgoing side, for example

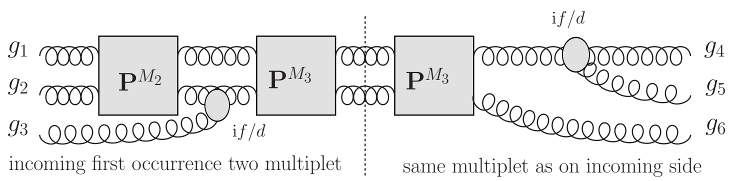
$$\mathbf{C}^{M_2,8}_{g_1\,g_2\,g_3;\,i_1}\mathbf{C}^{8a/s,8a/s\,\dagger}_{i_1;g_4\,g_5\,g_6} \propto$$



. (H.2)

Moreover, the first occurrence two projectors can be treated in the same way, e.g

$$\mathbf{C}^{M_2,M_3}_{g_1\,g_2\,g_3;\,i_1\,i_2}\mathbf{C}^{8a/s,M_3\,\dagger}_{i_1\,i_2;\,g_4\,g_5\,g_6} \propto$$



. (H.3)

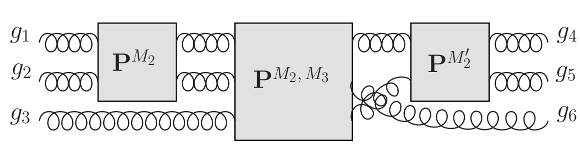
For the first occurrence zero, one, and two case, the normalization of the basis vectors is always given by eq. (C.10).

For multiplets with first occurrence three and higher we encounter a new situation. Even for n_f gluons there are several instances of the same multiplet corresponding to different construction histories. Therefore the basis vectors are in general *not* proportional to

$$\mathbf{C}^{M_2,M_3}_{g_1\,g_2\,g_3;\,i_1\,i_2\,i_3}\mathbf{C}^{M_2',M_3\,\dagger}_{i_1\,i_2\,i_3;\,g_4\,g_5\,g_6} \tag{H.4}$$

for all first occurrence 3 projectors. They can only be constructed in this way if $M_2 = M_2'$, in which case the six gluon vector is proportional to the corresponding $ggg \to ggg$ projector, and the normalization is given by eq. (1.9). If $M_2 \neq M_2'$ the above color structure vanishes, due to the construction history property (iii) in section 4. In order to find the corresponding vector we instead permute the gluon lines before contracting the projectors

$$\mathbf{P}^{M_2,M_3}_{g_1\,g_2\,g_3\,i_1\,g_6\,i_2}\mathbf{P}^{M_2'}_{i_1\,i_2\,g_4\,g_5} \propto$$



. (H.5)

The electronically attached six gluon vectors corresponding to first occurrence three multiplets are constructed in this way. Having more than three gluons, there are yet more possibilities of having transitions between different instances of the same multiplets on the incoming and outgoing side. In this case we remark that having all projectors at hand, we can always construct the basis by pursuing the same strategy as in eq. (4.13).

I. Multiplet basis for two $q\overline{q}$ pairs and two gluons

Here the multiplet basis for $q_1\overline{q}_2g_3 \to q_4\overline{q}_5g_6$, discussed in section 6.2 is given:

$$\begin{aligned}
\mathbf{V}^{8,1;8,1}_{q_1\,q_2\,g_3\,q_4\,q_5\,g_6} &= \frac{1}{T_R}\frac{1}{N_c^2-1}(t^{g_3})^{q_1}_{\;q_2}\,(t^{g_6})^{q_5}_{\;q_4}\\
\mathbf{V}^{1,8;1,8}_{q_1\,q_2\,g_3\,q_4\,q_5\,g_6} &= \frac{1}{N_c\sqrt{N_c^2-1}}\delta^{q_1}_{\;q_2}\,\delta_{g_3\,g_6}\,\delta^{q_5}_{\;q_4}\\
\mathbf{V}^{1,8;8,8s}_{q_1\,q_2\,g_3\,q_4\,q_5\,g_6} &= \frac{1}{T_R}\frac{1}{\sqrt{2(N_c^4-5N_c^2+4)}}\delta^{q_1}_{\;q_2}\,d_{g_3\,g_6\,i_1}\,(t^{i_1})^{q_5}_{\;q_4}\\
\mathbf{V}^{1,8;8,8a}_{q_1\,q_2\,g_3\,q_4\,q_5\,g_6} &= \frac{1}{T_R}\frac{1}{N_c\sqrt{2(N_c^2-1)}}\delta^{q_1}_{\;q_2}\,\mathrm{i}f_{g_3\,g_6\,i_1}\,(t^{i_1})^{q_5}_{\;q_4}\\
\mathbf{V}^{8,8s;1,8}_{q_1\,q_2\,g_3\,q_4\,q_5\,g_6} &= \frac{1}{T_R}\frac{1}{\sqrt{2(N_c^4-5N_c^2+4)}}(t^{i_1})^{q_1}_{\;q_2}\,d_{i_1\,g_3\,g_6}\,\delta^{q_5}_{\;q_4}\\
\mathbf{V}^{8,8s;8,8s}_{q_1\,q_2\,g_3\,q_4\,q_5\,g_6} &= \frac{1}{T_R^2}\frac{N}{2(N_c^2-4)\sqrt{N_c^2-1}}(t^{i_1})^{q_1}_{\;q_2}\,d_{i_1\,g_3\,i_2}\,d_{i_2\,g_6\,i_3}\,(t^{i_3})^{q_5}_{\;q_4}\\
\mathbf{V}^{8,8s;8,8a}_{q_1\,q_2\,g_3\,q_4\,q_5\,g_6} &= \frac{1}{T_R^2}\frac{1}{2\sqrt{N_c^4-5N_c^2+4}}(t^{i_1})^{q_1}_{\;q_2}\,d_{i_1\,g_3\,i_2}\,\mathrm{i}f_{i_2\,g_6\,i_3}\,(t^{i_3})^{q_5}_{\;q_4}\\
\mathbf{V}^{8,8a;1,8}_{q_1\,q_2\,g_3\,q_4\,q_5\,g_6} &= \frac{1}{T_R}\frac{1}{N_c\sqrt{2(N_c^2-1)}}(t^{i_1})^{q_1}_{\;q_2}\,\mathrm{i}f_{i_1\,g_3\,g_6}\,\delta^{q_5}_{\;q_4}\\
\mathbf{V}^{8,8a;8,8s}_{q_1\,q_2\,g_3\,q_4\,q_5\,g_6} &= \frac{1}{T_R^2}\frac{1}{2\sqrt{N_c^4-5N_c^2+4}}(t^{i_1})^{q_1}_{\;q_2}\,\mathrm{i}f_{i_1\,g_3\,i_2}\,d_{i_2\,g_6\,i_3}\,(t^{i_3})^{q_5}_{\;q_4}\\
\mathbf{V}^{8,8a;8,8a}_{q_1\,q_2\,g_3\,q_4\,q_5\,g_6} &= \frac{1}{T_R^2}\frac{1}{2N_c\sqrt{N_c^2-1}}(t^{i_1})^{q_1}_{\;q_2}\,\mathrm{i}f_{i_1\,g_3\,i_2}\,\mathrm{i}f_{i_2\,g_6\,i_3}\,(t^{i_3})^{q_5}_{\;q_4}\\
\mathbf{V}^{8,10;8,10}_{q_1\,q_2\,g_3\,q_4\,q_5\,g_6} &= \frac{1}{T_R}\frac{2}{\sqrt{N_c^4-5N_c^2+4}}(t^{i_1})^{q_1}_{\;q_2}\,\mathbf{P}^{10}_{g_3\,i_1 g_6\,i_2}(t^{i_2})^{q_5}_{\;q_4}\\
\mathbf{V}^{8,\overline{10};8,\overline{10}}_{q_1\,q_2\,g_3\,q_4\,q_5\,g_6} &= \frac{1}{T_R}\frac{2}{\sqrt{N_c^4-5N_c^2+4}}(t^{i_1})^{q_1}_{\;q_2}\,\mathbf{P}^{\overline{10}}_{g_3\,i_1\,g_6\,i_2}(t^{i_2})^{q_5}_{\;q_4}\\
\mathbf{V}^{8,27;8,27}_{q_1\,q_2\,g_3\,q_4\,q_5\,g_6} &= \frac{1}{T_R}\frac{2}{N_c\sqrt{N_c^2+2N_c-3}}(t^{i_1})^{q_1}_{\;q_2}\,\mathbf{P}^{27}_{g_3\,i_1\,g_6\,i_2}(t^{i_2})^{q_5}_{\;q_4}\\
\mathbf{V}^{8,0;8,0}_{q_1\,q_2\,g_3\,q_4\,q_5\,g_6} &= \frac{1}{T_R}\frac{2}{N_c\sqrt{N_c^2-2N_c-3}}(t^{i_1})^{q_1}_{\;q_2}\,\mathbf{P}^{0}_{g_3\,i_1\,g_6\,i_2}(t^{i_2})^{q_5}_{\;q_4}.
\end{aligned} \tag{I.1}$$

The normalization is fixed such that the norm square is 1. Note that for the basis vectors with the same construction history on the incoming and outgoing side, the vectors are proportional to the corresponding projector, and the normalization is given by eq. (1.9).

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