

# YET ANOTHER TABLE OF INTEGRALS

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ABSTRACT. This collection of sums and integrals has been harvested from the mathematical and physical literature in unstructured ways. Its main use is backtracking the original sources whenever an integral of the reader's application resembles one of the items in the collection.

## INTRODUCTION

Dealing with the analysis of real numbers in the physical sciences shows a strange attraction towards integrals. Closed-form integration beats numerical integration, and often adaptive series expansion helps to crumble cumbersome integral kernels to digestable pieces.

The current table started as a incoherent list of bookmarks pointing to “interesting” formulas that complement or correct the Gradstein-Rhysik tables [91], see <http://www.mathtable.com/gr/>. As such it does not replicate the original sources in full but is to be merely regarded as an aid to find places at which certain forms and classes of integrals or sums have been targeted.

The notation is generally not harmonized. Stirling numbers appear in bracketed and indexed notations, and at least two different meanings of harmonic numbers  $H$  with lower and upper indices are met.

There is only one hint of use: The list of references appears *prior* to each formula.

### 0.1. Finite Series. [181, 176]

$$(0.1) \quad \sum_{j=1}^n j^k = \frac{1}{n} \left( \rho(n, i) + \sum_{i=1}^n \sigma_{k+1}(i) \right) = \frac{B_{k+1}(n+1) - B_{k+1}}{k+1}$$

where  $\rho(n, k) \equiv \sum_{d=1}^n d^k (n \bmod d)$  is a sum over  $n \bmod d$  multiplied, then summed, over  $d^k$ , and  $\sigma_k(n) = \sum_{d|n} d^k$ .

[55, 67]

$$(0.2) \quad \sum_{k=1}^{n-1} k^r = \sum_{k=0}^r \frac{B_k}{k!} \frac{r!}{(r-k+1)!} n^{r-k+1}.$$

[182]

$$(0.3) \quad \sum_{k=0}^n k^m = \sum_{j=0}^m \left\{ \begin{matrix} m \\ j \end{matrix} \right\} \binom{n+1}{j+1} j!.$$

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By telescoping the last term in this formula becomes

$$(0.4) \quad n^m = \sum_{j=0}^m \left\{ \begin{matrix} m \\ j \end{matrix} \right\} \binom{n}{j} j!.$$

[186]

$$(0.5) \quad \sum_{0 \leq j \leq n/m} \binom{n-jm}{k} = P(n+m, k, m) - P(r, k, m)$$

for  $n \geq 0$ ,  $k \geq 1$ ,  $m \geq 1$  where

$$(0.6) \quad P(x, k, m) \equiv \frac{1}{m} \sum_{j=1}^{k+1} \binom{x}{j} A(m, k+1-j)$$

with g.f.

$$(0.7) \quad \frac{mx}{(1+x)^m - 1} = \sum_{j=0}^{\infty} A(m, j) x^j.$$

[166]

$$(0.8) \quad \sum_{k=0}^n \binom{r+k}{k} = \binom{r+n+1}{n}, \quad n = 0, 1, 2, \dots$$

[182]

$$(0.9) \quad \sum_k \binom{n}{2k} k = n2^{n-3}, \quad n \geq 2.$$

[182]

$$(0.10) \quad \sum_k \binom{n}{2k+1} k = (n-2)2^{n-3}, \quad n \geq 2.$$

[182]

$$(0.11) \quad \sum_k \binom{n}{2k} k^m = n(n-m-1)^{\overline{m-1}} 2^{n-2m-1}, \quad n \geq m+1,$$

[182]

$$(0.12) \quad \sum_k \binom{n}{2k+1} k^m = n(n-m-1)^{\overline{m}} 2^{n-2m-1}, \quad n \geq m+1,$$

where  $k^{\overline{m}}$  is the falling factorial  $k(k-1)(k-2)\cdots(k-m+1)$ .

[182]

$$(0.13) \quad \sum_{k=0}^n \binom{n}{k} k^m = \sum_{j=0}^m \left\{ \begin{matrix} m \\ j \end{matrix} \right\} \binom{n}{j} j! 2^{n-j}.$$

[182]

$$(0.14) \quad \sum_{k=0}^n \binom{n}{k} (-1)^k k^m = \left\{ \begin{matrix} m \\ n \end{matrix} \right\} (-1)^n n!.$$

[182]

$$(0.15) \quad \sum_{k=0}^n \binom{n}{2k} k^m = n \sum_{j=1}^{\min(m, n-1)} \left\{ \begin{matrix} m \\ j \end{matrix} \right\} \binom{n-j-1}{j-1} (j-1)! 2^{n-2j-1}.$$

[182]

$$(0.16) \quad \sum_{k=0}^n \binom{n}{2k+1} k^m = \sum_{j=1}^{\min(m, n-1)} \left\{ \begin{matrix} m \\ j \end{matrix} \right\} \binom{n-j-1}{j-1} j! 2^{n-2j-1}.$$

[107]

$$(0.17) \quad \sum_{0 \leq k \leq N, k \neq K} \binom{N}{k} (-1)^k \frac{1}{(k-K)^m} = \binom{N}{K} (-1)^{K+1} \frac{1}{m!} Y_m(\dots, (i-1)! [H_{N-K}^{(i)} + (-1)^i H_K^{(i)}], \dots)$$

where  $0 \leq K \leq N$ ,  $Y_m(\dots, x_i, \dots)$  are the Bell polynomials and  $H_r^{(i)} = \sum_{j=1}^r j^{-i}$  Harmonic numbers of the  $i$ -th order.

[107]

$$(0.18) \quad \sum_{0 \leq k \leq N, k \neq K} \binom{N}{k} (-1)^k \frac{1}{k-K} = \binom{N}{K} (-1)^{K+1} (H_{N-K} - H_K)$$

where  $0 \leq K \leq N$ ,  $H_r^{(i)} = \sum_{j=1}^r j^{-i}$  Harmonic numbers of the  $i$ -th order.

[107]

$$(0.19) \quad \sum_{0 \leq k \leq N} \binom{N}{k} (-1)^k \frac{1}{k^m} = -\frac{1}{m!} Y_m(\dots, (i-1)! H_N^{(i)}, \dots)$$

where  $H_r^{(i)} = \sum_{j=1}^r j^{-i}$  Harmonic numbers of the  $i$ -th order.

[107]

$$(0.20) \quad \sum_{0 \leq k \leq N} \binom{N}{k} (-1)^k \frac{1}{(k-\xi)^m} = \Gamma(-\xi) \frac{\Gamma(N+1)}{\Gamma(N+1-\xi)} \frac{1}{(m-1)!} Y_{m-1}(\dots, (i-1)! \zeta_N(i, -\xi), \dots),$$

where  $\zeta_N(i, -\zeta) = \sum_{j=0}^N (j-\xi)^{-j}$ .

[161, §4.3]

$$(0.21) \quad a_n = \sum_{k=0}^n \binom{p+k}{k} b_{n-k} \Leftrightarrow b_n = \sum_{k=0}^n (-1)^k \binom{p+k}{k} a_{n-k}.$$

[161, §4.3]

$$(0.22) \quad a_n = \sum_{k=0}^n \binom{p+k}{k} b_{n-2k} \Leftrightarrow b_n = \sum_{k=0}^n (-1)^k \binom{p+k}{k} a_{n-2k}.$$

[161, §4.3]

$$(0.23) \quad a_n = \sum_{k=0}^n \binom{2k}{k} b_{n-k} \Leftrightarrow b_n = \sum_{k=0}^n \frac{1}{1-2k} \binom{2k}{k} a_{n-k}.$$

[161, §4.3]

$$(0.24) \quad a_n = \sum_{k=0}^n \frac{1}{k+1} \binom{2k}{k} b_{n-k} \Leftrightarrow b_n = a_n - \sum_{k=1}^n \frac{1}{k} \binom{2k}{k} a_{n-k}.$$

[156]

$$(0.25) \quad x_n = a_n + 2^{1-n} \sum_{k=0}^n \binom{n}{k} x_k \Leftrightarrow x_n = x_0 + \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{k\bar{x}_1 + \hat{a}_k - \bar{x}_0}{1 - 2^{1-k}}$$

where  $\bar{x}_0 \equiv a_0 + x_0$ ,  $\bar{x}_1 = a_1 + x_0$  and

$$(0.26) \quad \hat{a}_n = \sum_{k=0}^n \binom{n}{k} (-1)^k a_k.$$

[156]

$$(0.27) \quad x_{n+1} = a_{n+1} + 2^{1-n} \sum_{k=0}^n \binom{n}{k} x_k \Leftrightarrow x_n = - \sum_{k=0}^n (-1)^k \binom{n}{k} \hat{x}_{k-2}$$

where  $\bar{x}_0 = 0$ ,

$$(0.28) \quad \hat{x}_n = Q_n \sum_{i=1}^{n+1} (\hat{a}_i - \hat{a}_{i+1} - a_1) / Q_{i-1}, \quad Q_n \equiv \prod_{k=1}^n (1 - 2^{-k}).$$

[106]

$$(0.29) \quad u_n = Au_{n-1} - Bu_{n-2}, \quad v_n = Av_{n-1} - Bv_{n-2},$$

(0.30)

$$\rightsquigarrow u_{n+1} = \prod_{j=1}^n \left( A - 2i\sqrt{-B} \cos \frac{\pi j}{n+1} \right) = (i\sqrt{-B})^n \frac{\sin \left( (n+1) \cos^{-1} \frac{-iA}{2\sqrt{-B}} \right)}{\sin \left( \cos^{-1} \frac{-iA}{2\sqrt{-B}} \right)},$$

(0.31)

$$\rightsquigarrow v_n = \prod_{k=1}^n \left( A - 2i\sqrt{-B} \cos \frac{\pi(k-1/2)}{n} \right) = 2(i\sqrt{-B})^n \cos \left( n \cos^{-1} \frac{-iA}{2\sqrt{-B}} \right),$$

where  $u_0 = 0$ ,  $u_1 = 1$ ,  $v_0 = 2$ ,  $v_1 = A$

[201]

$$(0.32) \quad \sum_{x=1}^{p-1} F(x) + \sum_{x=1}^{p-1} \left( \frac{x}{p} \right) F(x) = \sum_{x=1}^{p-1} F(x^2),$$

if  $p$  is an odd prime and  $F(x) = F(x+p)$ , where  $\left( \frac{x}{p} \right)$  is the Legendre Symbol.

[201]

$$(0.33) \quad \sum_{x=1}^{p-1} \left( \frac{x}{p} \right) e(k(x+\bar{x})) = \left( \frac{k}{p} \right) i^{(p-1)^2/4} p^{1/2} (e(2k) + e(-2k)).$$

where  $\bar{x}$  is the unique solution to  $x\bar{x} \equiv 1 \pmod{p}$ , and  $e(t) = e^{2\pi it/p}$ .

[48]

$$(0.34) \quad \sum_{k=0}^m \binom{m}{k} \frac{(n+k)!}{(n+k+s)!} a_{n+k+s} = \sum_{k=0}^n \binom{n}{k} \frac{(-1)^{n-k} (m+k)!}{(m+k+s)!} b_{m+k+s} \\ + \sum_{j=0}^{s-1} \sum_{i=0}^{s-1-j} \binom{s-1-j}{i} \binom{s-1}{j} \frac{(-1)^{n+1+i} a_j}{(s-1)! (m+n+1+i) \binom{m+n+i}{n}},$$

where  $b_n \equiv \sum_{k=0}^n \binom{n}{k} a_k$ .

[48]

$$(0.35) \quad \sum_{k=0}^m \binom{m}{k} \binom{n+k}{s} a_{n+k+s} = \sum_{k=0}^n \binom{n}{k} \binom{m+k}{s} (-1)^{n-k} b_{m+k+s},$$

where  $b_n \equiv \sum_{k=0}^n \binom{n}{k} a_k$ .

[48]

$$(0.36) \quad \sum_{k=0}^m \frac{\binom{m}{k}}{\binom{n+k+s}{s}} x^{m-k} A_{n+k+s}(y) = \sum_{k=0}^n \frac{\binom{n}{k}}{\binom{m+k+s}{s}} (-1)^{n+m+s} x^{n-k} A_{m+k+s}^*(z) \\ + \sum_{j=0}^{s-1} \sum_{i=0}^{s-1-j} \binom{s-1-j}{i} \binom{s-1}{j} \frac{(-1)^{n+1+i} x^{m+n+s-j} s A_j(y)}{(m+n+1+i) \binom{m+n+i}{n}},$$

and

$$(0.37) \quad \sum_{k=0}^m \binom{m}{k} \binom{n+k}{s} x^{m-k} A_{n+k+s}(y) = \sum_{k=0}^n \binom{n}{k} \binom{m+k}{s} (-1)^{n+m+s} x^{n-k} A_{m+k+s}^*(z)$$

where  $A_n(x) \equiv \sum_{k=0}^n \binom{n}{k} (-1)^k a_k x^{n-k}$  and  $A_n^*(x) \equiv \sum_{k=0}^n \binom{n}{k} (-1)^k a_k^* x^{n-k}$  and  $a_n^* \equiv \sum_{k=0}^n \binom{n}{k} (-1)^k a_k$  and  $x + y + z = 1$ .

[48]

$$(0.38) \quad a_{n,m} = \sum_{k=0}^n \binom{n}{k} \alpha^{n-k} \beta^k a_{0,m+k},$$

and

$$(0.39) \quad \sum_{k=0}^n \binom{n}{k} \alpha^{n-k} \beta^k a_{0,m+k} = \sum_{k=0}^m \binom{m}{k} (-\alpha)^{m-k} \beta^{-m} a_{n+k,0},$$

where  $a_{n,m} \equiv \alpha a_{n-1,m} + \beta a_{n-1,m+1}$  for  $n \geq 1, m \geq 0$ .

[92]

$$(0.40) \quad x_{nm} = \sum_{k=0}^m x_{mk} \binom{n+k}{2m} \rightsquigarrow x_{nm} = \sum_{k=0}^m \frac{2k+1}{m+k+1} \binom{n+k}{m+k} \binom{n-1-k}{m-k} x_{kk},$$

for  $m < n, n \geq 0, 0 \leq m \leq n$ .

[33]

$$(0.41) \quad F(x) \equiv \sum_{k \geq 1} f(x/k); \quad G(x) \equiv \sum_{k \geq 1} (-)^k f(x/k) \rightsquigarrow F(x) = 2^{-n} F(2^n x) + \sum_{k \geq 1} 2^{-k} G(2^k x).$$

[33]  
(0.42)

$$F(x) = r_1 F(m_1 x) + r_2 G(m_2 x) \rightsquigarrow r_2 \sum_{k=1}^n r_1^{k-1} G(m_1^{k-1} m_2 x) = F(x) - r_1^n F(m_1^n x).$$

[182]

$$(0.43) \quad \sum_{k=0}^n \binom{n}{k} H_k = 2^n \left( H_n - \sum_{k=1}^n \frac{1}{k 2^k} \right).$$

where  $H_k$  are the harmonic numbers.

[182]

$$(0.44) \quad \sum_k \binom{n}{2k} \frac{1}{k+1} = \frac{n 2^{n+1} + 2}{(n+1)(n+2)}.$$

[182]

$$(0.45) \quad \sum_k \binom{n}{2k+1} \frac{1}{k+1} = \frac{2^{n+1} - 2}{n+1}.$$

[164]

$$(0.46) \quad \sum_{j=0}^n \binom{m-a+b}{j} \binom{n+a-b}{n-j} \binom{a+j}{m+n} = \binom{a}{m} \binom{b}{n},$$

where  $n, m$  are integer and  $a, b$  real.

[164]

$$(0.47) \quad \sum_{m=0}^P (-)^m \binom{P}{m} \binom{a-m}{M} = \binom{a-P}{M-P},$$

where  $P, M$  are integer and  $a, b$  real.

[166]

$$(0.48) \quad \sum_k \binom{r}{k} \binom{s}{n+k} = \binom{r+s}{r+n}, \quad r = 0, 1, 2, \dots, \quad n = \dots - 2, -1, 0, 1, 2, 3, \dots$$

[164, 166]

$$(0.49) \quad \sum_{m=0}^n (-)^m \binom{n}{m} \binom{a+m}{p} = (-)^n \binom{a}{p-n},$$

where  $n, p$  are integer and  $a$  is real.

[166]

$$(0.50) \quad \sum_{k \geq 0} \binom{n+k}{m+2k} \binom{2k}{k} \frac{(-1)^k}{k+1} = \binom{n-1}{m-1}.$$

[166]

$$(0.51) \quad \sum_{\nu=0}^s (-)^{\nu} \binom{\beta}{\nu} \binom{\beta+s-\nu}{\beta} \frac{\alpha}{\alpha+s-\nu} = \frac{(\alpha-\beta)_s}{(\alpha+1)_s}.$$

[166]  
(0.52) 
$$\sum_{k=-l}^l (-)^k \binom{2l}{l+k} \binom{2m}{m+k} \binom{2n}{n+k} = \frac{(l+m+n)!(2l)!(2m)!(2n)!}{(l+m)!(m+n)!(n+l)!l!m!n!}, \quad l = \min(l, m, n).$$

[199]  
(0.53) 
$$\sum_k \sum_j \binom{n}{k} \binom{n+k}{k} \binom{k}{j}^3 = \sum_k \binom{n}{k}^2 \binom{n+k}{k}^2.$$

[199]  
(0.54) 
$$\sum_{i=0}^n \sum_{j=0}^n \binom{i+j}{i}^2 \binom{4n-2i-2j}{2n-2i} = (2n+1) \binom{2n}{n}.$$

[199]  
(0.55) 
$$\sum_{k_1} \sum_{k_2 \leq k_1} \sum_{k_3 \leq k_2} (k_1 - k_2)(k_1 - k_3)(k_2 - k_3) \binom{n}{k_1} \binom{n}{k_2} \binom{n}{k_3} = n^2(n-1)8^{n-2} \frac{(3/2)_{n-2}}{(3)_{n-2}}.$$

[199]  
(0.56) 
$$\sum_i \sum_j \binom{i+j}{i} \binom{n-i}{j} \binom{n-j}{n-i-j} = \sum_{k=0}^n \binom{2k}{k}.$$

[199]  
(0.57) 
$$\sum_{i=0}^n \sum_{j=0}^m \binom{i+j}{j}^2 \binom{m+m-i-j}{n-j}^2 = \frac{1}{2} \binom{2m+2n+2}{2n+1}.$$

[199]  
(0.58) 
$$\sum_{s=0}^k \sum_{b \geq 0} (-)^b \binom{s}{b} \binom{k-s}{2v-b} \binom{k-2v}{s-b} = \binom{k-v}{k-2v} 2^{k-2v}, \quad k \geq 2v.$$

[199]  
(0.59) 
$$\sum_{j,k} (-)^{j+k} \binom{j+k}{k+l} \binom{r}{j} \binom{n}{k} \binom{s+n-j-k}{m-j} = (-)^l \binom{n+r}{n+l} \binom{s-r}{m-n-l}.$$

[49]  
(0.60) 
$$\sum_{i=0}^n \sum_{j=0}^n \binom{i+j}{i} \binom{m-i+j}{j} \binom{n-j+i}{i} \binom{m+n-i-j}{m-i} = \frac{(m+n+1)!}{m!n!} \sum_k \frac{1}{2k+1} \binom{m}{k} \binom{n}{k}.$$

[49]  
(0.61) 
$$\sum_{i=0}^n \sum_{j=0}^n \binom{i+j}{i}^2 \binom{4n-2i-2j}{2n-2i} = (2n+1) \binom{2n}{n}^2.$$

[49]  
(0.62) 
$$\sum_{i=0}^{\lfloor m/2 \rfloor} \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{i+j}{i}^2 \binom{m+n-2i-2j}{n-2i} = \frac{\lfloor (m+n+1)/2 \rfloor! \lfloor (m+n+2)/2 \rfloor!}{\lfloor m/2 \rfloor! \lfloor (m+1)/2 \rfloor! \lfloor (n+1)/2 \rfloor!}.$$

[49]  
(0.63) 
$$\sum_i \sum_j \binom{n}{j} \binom{n+j}{j} \binom{j}{i}^2 \binom{2i}{i}^2 \binom{2i}{j-i} = \sum_k \binom{n}{k}^3 \binom{n+k}{k}^3.$$

[49]  
(0.64) 
$$\sum_j \sum_k (-1)^{j+k} \binom{j+k}{k+l} \binom{r}{j} \binom{n}{k} \binom{s+n-j-k}{m-j} = (-1)^l \binom{n+r}{n+l} \binom{s-r}{m-n-l}.$$

[49, 199]  
(0.65) 
$$\sum_r \sum_s (-1)^{n+r+s} \binom{n}{r} \binom{n}{s} \binom{n+s}{s} \binom{n+r}{r} \binom{2n-r-s}{n} = \sum_k \binom{n}{k}^4.$$

[172]  
(0.66) 
$$\binom{n}{i}^k = \binom{n+ik-i}{ik} + A_2 \binom{n+ik-i-1}{ik} + \cdots + A_{ik-i} \binom{n+1}{ik} + \binom{n}{ik},$$

$$\begin{aligned} S_i^k(n, 0) &\equiv \binom{n}{i}^k; \\ S_i^k(n, p) &\equiv S_i^k(1, p-1) + S_i^k(2, p-1) + \cdots + S_i^k(n, k-1) \\ &= \binom{n+ik-i+p}{ik+p} + A_2 \binom{n+ik-i+p-1}{ik+p} + \cdots + A_{ik-i} \binom{n+p+1}{ik+p} + \binom{n+p}{ik+p} \end{aligned}$$

where  $A_j = A_{ik-i-j+2}$  for  $j = 2, 3, \dots, ik-i$ ,

(0.67) 
$$A_j = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & \binom{i}{i}^k \\ \binom{ik+1}{ik} & 1 & 0 & \cdots & 0 & \binom{i+1}{i}^k \\ \binom{ik+2}{ik} & \binom{ik+1}{ik} & 1 & \cdots & 0 & \binom{i+2}{i}^k \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{ik+j-1}{ik} & \binom{ik+j-2}{ik} & \binom{ik+j-3}{ik} & \cdots & \binom{ik+1}{ik} & \binom{i+j-1}{i}^k \end{vmatrix}$$

[189]

(0.68) 
$$\sum_{k=0}^{2n} (-)^k \frac{\binom{2n}{k}}{\binom{4n}{2k}} = \frac{4n+1}{2n+1}.$$

[189]

(0.69) 
$$\sum_{k=0}^{2n} (-)^k \frac{\binom{4n}{2k}}{\binom{2n}{k}} = -\frac{1}{2n-1}.$$



[189]

$$(0.70) \quad \sum_{k=0}^m \frac{\binom{m}{k}}{\binom{n+m}{p+k}} = \frac{n+m+1}{n+1} \binom{n}{p}^{-1},$$

with  $m, n, p$  nonnegative integers and  $p \leq n$ .

[189]

$$(0.71) \quad \sum_{k=0}^n (-)^k \frac{1}{\binom{n+m}{m+k}} = \frac{n+m+1}{m+n+2} \left( \binom{m+n+1}{m}^{-1} + (-)^n \right),$$

with  $m$  and  $n$  nonnegative integers.

[182]

$$(0.72) \quad \sum_{k=0}^n \left[ \begin{matrix} n \\ k \end{matrix} \right] H_k = n! \sum_{k=1}^n \frac{c_{k-1}}{k!},$$

where  $H_k$  are the harmonic numbers and  $[ \dots ]$  unsigned Stirling numbers of the first kind.

[182]

$$(0.73) \quad \sum_{k=0}^n s(n, k) = s(n-1, 0) + s(n-1, -1),$$

where  $s(\cdot, \cdot)$  are the Stirling numbers of the first kind.

[182]

$$(0.74) \quad \sum_{k=0}^n s(n, k)k = s(n-1, 1) + s(n-1, 0).$$

[182]

$$(0.75) \quad n! \sum_{k=0}^n s(k, m) (-1)^{n-k} / k! = s(n+1, m+1).$$

[182]

$$(0.76) \quad \sum_{k=0}^n s(n, k) k^{\underline{m}} = m! [s(n-1, m) + s(n-1, m-1)],$$

where  $k^{\underline{m}} = k(k-1)(k-2) \cdots (k-m+1)$ .

[182]

$$(0.77) \quad \sum_{k=0}^n s(n, k) k^m = \sum_{j=0}^m \left\{ \begin{matrix} m \\ j \end{matrix} \right\} (s(n-1, j) + s(n-1, j-1)) j!.$$

[182]

$$(0.78) \quad \sum_{k=0}^n \frac{s(n, k)}{k+1} = b_n,$$

where  $b_n = \int_0^1 x^n dx$  are the Cauchy numbers of the first type [175, A006232].

[182]

$$(0.79) \quad n! \sum_{k=0}^n \left[ \begin{matrix} k \\ m \end{matrix} \right] \frac{1}{k!} = \left[ \begin{matrix} n+1 \\ m+1 \end{matrix} \right].$$

where  $[\dots]$  are the unsigned Stirling numbers of the first kind.  
[\[182\]](#)

$$(0.80) \quad \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} k = \begin{bmatrix} n+1 \\ 2 \end{bmatrix}.$$

[\[182\]](#)

$$(0.81) \quad \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} k^m = \begin{bmatrix} n+1 \\ m+1 \end{bmatrix} m!,$$

where  $k^{\underline{m}} = k(k-1)(k-2)\cdots(k-m+1)$ .  
[\[182\]](#)

$$(0.82) \quad \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} k^m = \sum_{j=0}^m \begin{bmatrix} n+1 \\ j+1 \end{bmatrix} \left\{ \begin{matrix} m \\ j \end{matrix} \right\} j!.$$

[\[182\]](#)

$$(0.83) \quad \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{k+1} = c_n,$$

where  $c_n = \int_0^1 (x)_n dx$  are the Cauchy numbers of the second type [\[175, A002657\]](#), using Pochhammer's symbol.  
[\[48\]](#)

$$(0.84) \quad \sum_{k=0}^m \binom{m}{k} (n+k)! \left\{ \begin{matrix} n+k+s \\ q \end{matrix} \right\} = \sum_{k=0}^n \binom{n}{k} (m+k)! (-1)^{n-k} \frac{(m+k+s)^q}{(m+k+s)!} \\ + \sum_{j=0}^{s-1} \sum_{i=0}^{s-1-j} \binom{s-1-j}{i} \frac{(-1)^{n+1+i} \left\{ \begin{matrix} j \\ q \end{matrix} \right\}}{(s-1-j)! (m+n+1+i) \binom{m+n+i}{n}}$$

and

$$(0.85) \quad \sum_{k=0}^m (n+k)! \left\{ \begin{matrix} n+k-s \\ q \end{matrix} \right\} = \sum_{k=0}^n \binom{n}{k} (m+k)! (-1)^{n-k} \frac{(m+k-s)^q}{(m+k-s)!}.$$

[\[138\]](#)

$$(0.86) \quad \left\{ \begin{matrix} n \\ m \end{matrix} \right\}_r = \sum_{k=2}^n \binom{n}{k} \sum_{l=1}^{k-1} (-1)^{l-1} \binom{l+r-2}{l-1} \left\{ \begin{matrix} k-l \\ m-1 \end{matrix} \right\}_{r-1}$$

where  $\{\}_r$  are  $r$ -Stirling numbers of the second kind, namely

$$(0.87) \quad \left\{ \begin{matrix} n \\ m \end{matrix} \right\}_r = \begin{cases} 0, & n < r \\ \delta_{mr}, & n = r \\ m \left\{ \begin{matrix} n-1 \\ m \end{matrix} \right\}_r + \left\{ \begin{matrix} n-1 \\ m-1 \end{matrix} \right\}_r, & n > r. \end{cases}$$

0.2. **Numerical Series.** [56, 52][175, A152649]

$$(0.88) \quad \sum_{n=1}^{\infty} \frac{H_n^{(1)}}{n^3} = \frac{1}{2}\zeta^2(2),$$

where  $H_n^{(r)} \equiv \sum_{k=1}^n \frac{1}{k^r}$ .  
[52]

$$(0.89) \quad \sum_{n=1}^{\infty} \frac{H_n^{(1)^2}}{n^4} = \frac{97}{24}\zeta(6) - 2\zeta^2(3) \approx 1.22187994531988.$$

[52]

$$(0.90) \quad \sum_{n=1}^{\infty} \frac{H_n^{(1)^3}}{n^3} = \zeta^2(3) - \frac{1}{3}\zeta(6) \approx 1.1058264444388.$$

[56]

$$(0.91) \quad \sum_{n=1}^{\infty} \frac{H_n^{(1)}}{n^{2p+1}} = \frac{1}{2} \sum_{j=2}^{2p} (-1)^j \zeta(j) \zeta(2p-j+2),$$

where  $H_n^{(r)} \equiv \sum_{k=1}^n \frac{1}{k^r}$ .  
[35, 137, 179]

$$(0.92) \quad \sum_{n=1}^{\infty} \frac{H_n^{(1)}}{n^m} = \frac{m+2}{2} \zeta(m+1) - \frac{1}{2} \sum_{k=1}^{m-2} \zeta(m-k) \zeta(k+1),$$

where  $H_n^{(r)} \equiv \sum_{k=1}^n \frac{1}{k^r}$ .  
[179]

$$(0.93) \quad \sum_{n=1}^{\infty} \frac{H_n}{n(n+\alpha)} = \frac{1}{2\alpha} [3\zeta(2) + \psi^2(\alpha) + 2\gamma\psi(\alpha) + \gamma^2 - \psi'(\alpha)],$$

and

$$(0.94) \quad \sum_{n=1}^{\infty} \frac{H_n}{(n+\alpha)^2} = \gamma\psi'(\alpha) + \psi(\alpha)\psi'(\alpha) - \frac{1}{2}\psi''(\alpha),$$

and

$$(0.95) \quad \sum_{n=1}^{\infty} \frac{H_n}{(n+x)^q} = \frac{(-)^q}{(q-1)!} \left[ (\psi(x) + \gamma) \psi^{(q-1)}(x) - \frac{1}{2} \psi^{(q)}(x) + \sum_{m=1}^{q-2} \binom{q-2}{m} \psi^{(m)} \psi^{(q-m-1)}(x) \right].$$

The paper also demonstrates a finite expansion of  $\sum_{n \geq 1} H_n / [n^q \binom{an+k}{k}^t]$  in terms of  $\zeta$  and  $\psi$  functions for  $t = 1$  and 2.

[137]

$$(0.96) \quad S(r, m) = S(1, m) + \sum_{k=1}^{r-1} \frac{S(k, m-1) - B(k, m)}{k}.$$

where  $H_n^{(r)} \equiv \sum_{k=1}^n \frac{1}{k^r}$ , where  $S(r, m) \equiv \sum_{n=1}^{\infty} \frac{H_n^{(r)}}{n^m}$ , where  $B(k, m) \equiv {}_{m+1}F_m(1, 1, \dots, 1, k+1; 2, 2, \dots, 2; 1)$ .

[137]

$$(0.97) \quad S(2, 3) = \frac{\pi^4}{72} - \frac{\pi^2}{6} + 2\zeta(3) \approx 2.1120837816098848.$$

[137]

$$(0.98) \quad S(2, 4) = \frac{\pi^4}{72} + 3\zeta(5) - \zeta(3) \left(1 + \frac{\pi^2}{6}\right).$$

[21, 52]

$$(0.99) \quad \sum_{k=1}^{\infty} \frac{H_k^{(1)2}}{k^2} = \frac{17}{4}\zeta(4).$$

[21]

$$(0.100) \quad \sum_{k=1}^{\infty} \frac{H_k^{(1)2}}{(k+1)^n} = \frac{1}{3}n(n+1)\zeta(n+2) + \zeta(2)\zeta(n) - \frac{1}{n} \sum_{k=0}^{n-2} \zeta(n-k)\zeta(k+2) \\ + \frac{1}{3} \sum_{k=2}^{n-2} \zeta(n-k) \sum_{j=1}^{k-1} \zeta(j+1)\zeta(k+1-j) + \sum_{k=1}^{\infty} \frac{H_k^{(2)}}{(k+1)^n}.$$

[21]

$$(0.101) \quad \sum_{k=1}^{\infty} \frac{H_k^{(2)}}{(k+1)^2} = \frac{1}{2}\zeta^2(2) - \frac{1}{2}\zeta(4) = \frac{1}{120}\pi^4.$$

[167, 21][175, A214508]

$$(0.102) \quad A_4 \equiv \sum_{k=1}^{\infty} (-)^{k+1} \frac{1}{(k+1)^2} H_k^{(2)} = -4\text{Li}_4(1/2) + \frac{13\pi^4}{288} + \log(2) \left[ -\frac{7}{2}\zeta(3) + \frac{\pi^2}{6} \log 2 - \frac{\log^3 2}{6} \right].$$

[21]

$$(0.103) \quad \sum_{k=1}^{\infty} \frac{H_k^{(2)}}{(k+1)^4} = -6\zeta(6) + \frac{8}{3}\zeta(2)\zeta(4) + \zeta^2(3) = \zeta^2(3) - \frac{4}{2835}\pi^6.$$

[21]

$$(0.104) \quad \sum_{k=1}^{\infty} \frac{H_k^{(1)}}{(k+1)^n} = \frac{1}{2}n\zeta(n+1) - \frac{1}{2} \sum_{k=1}^{n-2} \zeta(n-k)\zeta(k+1).$$

[21]

$$(0.105) \quad \sum_{k=1}^{\infty} \frac{H_k^{(2)}}{(k+1)^{2n-1}} = -\frac{1}{2}(2n^2+n+1)\zeta(2n+1) + \zeta(2)\zeta(2n-1) + \sum_{k=1}^{n-1} 2k\zeta(2k+1)\zeta(2n-2k).$$

[21]

$$(0.106) \quad \sum_{k=1}^{\infty} \frac{H_k^{(1)2}}{(k+1)^{2n-1}} = \frac{1}{6}(2n^2-7n-3)\zeta(2n+1) + \zeta(2)\zeta(2n-1) - \frac{1}{2} \sum_{k=1}^{n-2} (2k-1)\zeta(2n-1-2k)\zeta(2k+2) \\ + \frac{1}{3} \sum_{k=1}^{n-2} \zeta(2k+1) \sum_{j=1}^{n-2-k} \zeta(2j+1)\zeta(2n-1-2k-2j).$$

[138]

$$(0.107) \quad \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n!} = e \left[ 1 + \frac{1}{4} {}_2F_2 \left( \begin{matrix} 1 & 1 \\ 3 & 3 \end{matrix} \middle| -1 \right) \right],$$

$$(0.108) \quad \sum_{n=1}^{\infty} \frac{H_n^{(4)}}{2^n n!} = \sqrt{e} \left[ \frac{H_1^{(3)}}{2^1 1!} + \frac{H_2^{(2)}}{2^2 2!} + \frac{H_3^{(1)}}{2^3 3!} + \frac{3!}{2^4 (4!)^2} {}_2F_2 \left( \begin{matrix} 1 & 1 \\ 5 & 5 \end{matrix} \middle| -\frac{1}{2} \right) \right],$$

where  $H_n^{(r)}$  are hyperharmonic numbers (1.77).

[74]

$$(0.109) \quad \sum_{-\infty}^{\infty} \frac{1}{n^2 + q^2} = \frac{\pi \coth \pi q}{q}.$$

[74]

$$(0.110) \quad \sum_{-\infty}^{\infty} \frac{1}{(n^2 + q_1^2)((N-n)^2 + q_2^2)} = \frac{2\pi}{2q_1 2q_2} \left[ \frac{1 + n_b(q_1) + n_b(q_2)}{Ni + q_1 + q_2} + \frac{n_b(q_1) - n_b(q_2)}{Ni - q_1 + q_2} \right. \\ \left. - \frac{n_b(q_1) - n_b(q_2)}{Ni + q_1 - q_2} - \frac{1 + n_b(q_1) + n_b(q_2)}{Ni - q_1 - q_2} \right]$$

where  $n_b(z) \equiv 1/(e^{2\pi z} - 1)$ .

[74] Define the Matsubara sum associated to the Graph  $G$  (loopless multigraph such that the degree of each vertex is at least 2, with  $I$  lines) by

$$(0.111) \quad S_G \equiv \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \cdots \sum_{n_I=-\infty}^{\infty} \frac{\delta_g(n_1, n_2, \dots, n_I; \{N_v\})}{(n_1^2 + q_1^2)(n_2^2 + q_2^2) \cdots (n_I^2 + q_I^2)},$$

where

$$(0.112) \quad \delta_g(n_1, \dots; \{N_v\}) = \prod_{v=1}^V \delta_{T_v, N_v}$$

imposes a series of constraints over the vertices  $v$ , and  $T_v \equiv \sum_i s_i^v n_i$  is an algebraic sum at vertex  $v$ , with  $s_i^v$  having values  $\pm 1$  or 0 depending on the orientation of the line  $i$  with respect to the vertex  $v$ . The  $q_i$  are weights associated with the lines  $i$ .

Then the integral

$$(0.113) \quad I(N, q_1, q_2, \dots) \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \cdots \frac{1}{(x_1^2 + q_1^2)(x_2^2 + q_2^2) \cdots ((N - x_1 - x_2 \cdots)^2 + q_I^2)}$$

is related to the sum via

$$(0.114) \quad S_G = \hat{O}_G I_G$$

where the operator  $\hat{O}_G = \prod_{i=1}^f [1 + n_{b_i}(1 - \hat{R}_i)]$  is composed of the functions  $n_b$  of the previous formula and the reflection operator  $\hat{R}_i$  (which switches the sign of the variable  $q_i$ ).

[175, A152416][126]

$$(0.115) \quad \sum_{n=2}^{\infty} \frac{1}{n^s(n-1)} = s - \sum_{l=2}^s \zeta(l).$$

[36]

$$(0.116) \quad \sum_{n \geq 1} \frac{1}{n(n^2+1)} = \gamma + \Re \psi(1+i) \approx 0.67186598552400983.$$

[36]

$$(0.117) \quad \sum_{n \geq 1} \frac{1}{n^2(n^2+1)} = \frac{\pi^2}{6} - \frac{\pi \coth \pi - 1}{2} \approx 0.56826001937964526.$$

[132]

$$(0.118) \quad \sum_{k=1}^{\infty} \frac{1}{(2k)^{2s}(2k+1)^{2s}} = \sum_{t=1}^{2s} \binom{4s-t-1}{2s-1} \left\{ \left[ 1 - \frac{1-(-)^t}{2^t} \right] \zeta(t) - 1 \right\}.$$

[132]

$$(0.119) \quad \sum_{k=1}^{\infty} \frac{1}{(2k)^2(2k+1)^2} = -3 + \frac{\pi^2}{6} + 2 \log 2.$$

[132]

$$(0.120) \quad \sum_{k=1}^{\infty} \frac{1}{(2k)^4(2k+1)^4} = -35 + \frac{\pi^4}{90} + 3\zeta(3) + \frac{5\pi^2}{3} + 20 \log 2.$$

[132]

$$(0.121) \quad \sum_{n=1}^{\infty} \frac{1}{n^{2s}(n+1)^{2s}} = \sum_{t=1}^{2s} \binom{4s-t-1}{2s-1} \{ [1 + (-)^t] \zeta(t) - 1 \}.$$

[132]

$$(0.122) \quad \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)^2} = \frac{\pi^2}{3} - 3.$$

[132]

$$(0.123) \quad \sum_{n=1}^{\infty} \frac{1}{n^4(n+1)^4} = -35 + \frac{10\pi^2}{3} + \frac{\pi^4}{45}.$$

[119, (1)]

$$(0.124) \quad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{8^n} = \sqrt{2}.$$

[119, (1)]

$$(0.125) \quad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{10^n} = \sqrt{5/3}.$$

[119, (1)]

$$(0.126) \quad \sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}}{8^n} = \sqrt{2/3}.$$

[119][175, A145439]

$$(0.127) \quad \sum_{n=0}^{\infty} \frac{\binom{4n}{2n}}{64^n} = \frac{3\sqrt{2} + \sqrt{6}}{6}$$

[119]

$$(0.128) \quad 4 \sum_{n=0}^{\infty} \frac{\binom{8n}{4n}}{8^{4n}} = \frac{3\sqrt{2} + \sqrt{6}}{3} + \frac{2\sqrt{2 + \sqrt{5}}}{\sqrt{5}}.$$

[119, (6)]

$$(0.129) \quad \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{n4^n} = \log 4.$$

[119, (6)][175, A157699]

$$(0.130) \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \binom{2n}{n}}{n4^n} = 2 \log \frac{1 + \sqrt{2}}{2},$$

The formula above corrects a factor 2 in [119], see [127].

[175, A091648]

$$(0.131) \quad \sum_{n=1,3,5,7,\dots}^{\infty} \frac{\binom{2n}{n}}{n4^n} = \log(1 + \sqrt{2}).$$

[119, (7)]

$$(0.132) \quad \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{n(n+1)4^n} = \log 4 - 1.$$

[119, (8)]

$$(0.133) \quad \sum_{n=1}^{\infty} \frac{n \binom{2n}{n}}{8^n} = 1/\sqrt{2}.$$

[119]

$$(0.134) \quad \sum_{n=1}^{\infty} \frac{n^2 \binom{2n}{n}}{8^n} = \frac{5\sqrt{2}}{4}.$$

[119][175, A019670]

$$(0.135) \quad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)16^n} = \frac{\pi}{3}.$$

[175, A086466]

$$(0.136) \quad \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m \binom{2m}{m}} = \frac{2}{\sqrt{5}} \log \frac{1+\sqrt{5}}{2}.$$

This corrects a factor 2 in [119] and two typos in [17, 4.1.42], see [127].  
[119, (15)] [66, 189][175, A073016]

$$(0.137) \quad \sum_{m=1}^{\infty} \frac{1}{\binom{2m}{m}} = \frac{9+2\pi\sqrt{3}}{27}.$$

[189]

$$(0.138) \quad \sum_{k=0}^{\infty} \frac{1}{\binom{mk}{nk}} = \int_0^1 \frac{1+(m-1)t^n(1-t)^{m-n}}{(1-t^n(1-t)^{m-n})^2} dt,$$

where  $m$  and  $n$  are positive integers with  $m > n$ .

[189]

$$(0.139) \quad \sum_{k=0}^{\infty} \frac{1}{\binom{4k}{2k}} = \frac{16}{15} + \frac{\pi\sqrt{3}}{27} - \frac{2\sqrt{5}}{25} \ln \frac{1+\sqrt{5}}{2}.$$

[119][175, A086465]

$$(0.140) \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\binom{2n}{n}} = \frac{1}{5} + \frac{4\sqrt{5}}{25} \log \frac{1+\sqrt{5}}{2}.$$

[119, 66][175, A145429]

$$(0.141) \quad \sum_{m=1}^{\infty} \frac{m}{\binom{2m}{m}} = \frac{2}{27}(\pi\sqrt{3}+9).$$

[39] And by linear combination with (0.137):

$$(0.142) \quad \sum_{n \geq 1} \frac{18-9n}{\binom{2n}{n}} = 2\frac{\pi}{\sqrt{3}}.$$

[119, 66]

$$(0.143) \quad \sum_{m=1}^{\infty} \frac{m^2}{\binom{2m}{m}} = \frac{2}{81}(5\pi\sqrt{3}+54).$$

[119, 66]

$$(0.144) \quad \sum_{m=1}^{\infty} \frac{m^3}{\binom{2m}{m}} = \frac{2}{243}(37\pi\sqrt{3}+405).$$

[119][175, A145433]

$$(0.145) \quad \sum_{m=1}^{\infty} \frac{(-1)^{m-1}m}{\binom{2m}{m}} = \frac{2}{125}(2\sigma+15), \quad \sigma \equiv \sqrt{5} \log \frac{1+\sqrt{5}}{2}.$$

[119]

$$(0.146) \quad \sum_{m=1}^{\infty} \frac{(-1)^{m-1}m^2}{\binom{2m}{m}} = \frac{2}{125}(5-\sigma).$$



Erratum to [17, 4.1.40]:

$$(0.147) \quad \sum_{n=0}^{\infty} \frac{(-1)^{n-1} n^2}{\binom{2n}{n}} = \frac{4}{125} \left[ 5 - \sqrt{5} \ln \left( \frac{1 + \sqrt{5}}{2} \right) \right].$$

$$(0.148) \quad \sum_{m=1}^{\infty} \frac{(-1)^{m-1} m^3}{\binom{2m}{m}} = \frac{2}{625} (14\sigma - 5),$$

which corrects a factor 2 in [119], see [127].

[119, 66]

$$(0.149) \quad \sum_{m=1}^{\infty} \frac{2^m}{m^2 \binom{2m}{m}} = \frac{\pi^2}{8},$$

a special case of (1.17).

[119]

$$(0.150) \quad \sum_{m=1}^{\infty} \frac{2^m}{m \binom{2m}{m}} = \frac{\pi}{2}.$$

[119]

$$(0.151) \quad \sum_{m=1}^{\infty} \frac{2^m}{\binom{2m}{m}} = \frac{\pi}{2} + 1.$$

[119]

$$(0.152) \quad \sum_{m=1}^{\infty} \frac{m 2^m}{\binom{2m}{m}} = \pi + 3.$$

[119, 66]

$$(0.153) \quad \sum_{m=1}^{\infty} \frac{m^2 2^m}{\binom{2m}{m}} = \frac{1}{5} \sum_{m=1}^{\infty} \frac{m^3 2^m}{\binom{2m}{m}} = \frac{7\pi}{2} + 11.$$

[119, 66]

$$(0.154) \quad \sum_{m=1}^{\infty} \frac{m^4 2^m}{\binom{2m}{m}} = 113\pi + 355.$$

[119, 66]

$$(0.155) \quad \sum_{m=1}^{\infty} \frac{m^{10} 2^m}{\binom{2m}{m}} = 229093376\pi + 719718067.$$

[119]

$$(0.156) \quad \sum_{m=1}^{\infty} \frac{3^m}{m^2 \binom{2m}{m}} = \frac{2\pi^2}{9},$$

a special case of (1.17).

[119][175, A186706]

$$(0.157) \quad \sum_{m=1}^{\infty} \frac{3^m}{m \binom{2m}{m}} = \frac{2\pi}{\sqrt{3}} \equiv \nu.$$

[119]

$$(0.158) \quad \sum_{m=1}^{\infty} \frac{3^m}{\binom{2m}{m}} = 2\nu + 3.$$

[119]

$$(0.159) \quad \sum_{m=1}^{\infty} \frac{m3^m}{\binom{2m}{m}} = 10\nu + 18.$$

[119]

$$(0.160) \quad \sum_{m=1}^{\infty} \frac{m^2 3^m}{\binom{2m}{m}} = 2(43\nu + 78).$$

[119]

$$(0.161) \quad \sum_{m=1}^{\infty} \frac{(-1)^{m-1} 2^m}{m \binom{2m}{m}} = \rho/3, \quad \rho \equiv \sqrt{3} \log(2 + \sqrt{3}).$$

[12]

$$(0.162) \quad \sum_{m=1}^{\infty} \frac{(2t)^{2m+2k}}{m(2m+2k) \binom{2m}{m}} = \arcsin^2(t) \binom{2k}{k} + \sum_{j=1}^k \binom{2k}{k-j} \frac{(-)^{j+1}}{j^2} \\ + \sum_{j=1}^k (-)^j \binom{2k}{k-j} \left( \frac{2 \arcsin t \sin(2j \arcsin t)}{j} + \frac{\cos(2j \arcsin t)}{j^2} \right),$$

and a similar logarithmic result for an alternating sign sum on the left hand side.

[12]

(0.163)

$$\sum_{m=1}^{\infty} \frac{(2t)^{2m+2k}}{m^2(2m+2k) \binom{2m}{m}} = -\frac{1}{k} \arcsin^2(t) + \frac{(2t)^{2k}}{k} \arcsin^2 t - \sum_{j=1}^k \binom{2k}{k-j} \frac{(-)^{j+1}}{kj^2} \\ - \sum_{j=1}^k (-)^j \binom{2k}{k-j} \left( \frac{2 \arcsin t \sin(2j \arcsin t)}{kj} + \frac{\cos(2j \arcsin t)}{kj^2} \right),$$

and a similar logarithmic result for an alternating sign sum on the left hand side.

[119]

$$(0.164) \quad \sum_{m=1}^{\infty} \frac{(-1)^{m-1} 2^m}{\binom{2m}{m}} = \frac{\rho + 3}{9}.$$

[119]

$$(0.165) \quad \sum_{m=1}^{\infty} \frac{(-1)^{m-1} m 2^m}{\binom{2m}{m}} = \frac{1}{3}.$$

[119]

$$(0.166) \quad \sum_{m=1}^{\infty} \frac{(-1)^{m-1} m^2 2^m}{\binom{2m}{m}} = \frac{1}{27}(3 - \rho).$$

[119]

$$(0.167) \quad \sum_{m=1}^{\infty} \frac{(-1)^m m^3 2^m}{\binom{2m}{m}} = \frac{1}{81}(\rho + 15).$$

[119]

$$(0.168) \quad \sum_{m=1}^{\infty} \frac{(2 - \sqrt{2})^m}{\binom{2m}{m}} = \frac{3 - 2\sqrt{2}}{4}(\pi\sqrt{2} + 4).$$

[119]

$$(0.169) \quad \sum_{m=1}^{\infty} \frac{(-1)^{m-1} 3^{2m}}{4^m \binom{2m}{m}} = \frac{48}{125}(\log 2 + \frac{15}{16}).$$

[119][175, A152422]

(0.170)

$$\sum_{m=1}^{\infty} \frac{2^m (2 - \sqrt{3})^m}{m^2 \binom{2m}{m}} = 2(\arcsin \tau)^2, \quad \tau = \frac{\sqrt{3} - 1}{2} = \sqrt{2} \sin \frac{\pi}{12} = \sin \frac{\pi}{3} - \sin \frac{\pi}{6},$$

where both right hand sides in [119] are erroneous, see [127].

[39]

$$(0.171) \quad \sum_{n \geq 0} \frac{50n - 6}{\binom{3n}{n} 2^n} = \pi.$$

[39]

$$(0.172) \quad \sum_{n \geq 1} \frac{1}{\binom{3n}{n} 2^n} = \frac{2}{25} - \frac{6}{125} \ln 2 + \frac{11}{250} \pi.$$

[39]

$$(0.173) \quad \sum_{n \geq 1} \frac{n}{\binom{3n}{n} 2^n} = \frac{81}{625} - \frac{18}{3125} \ln 2 + \frac{79}{3125} \pi.$$

[39]

$$(0.174) \quad \sum_{n \geq 1} \frac{n^2}{\binom{3n}{n} 2^n} = \frac{561}{3125} + \frac{42}{15625} \ln 2 + \frac{673}{31250} \pi.$$

[39]

$$(0.175) \quad \sum_{n \geq 1} \frac{-150n^2 + 230n - 36}{\binom{3n}{n} 2^n} = \pi.$$

[39]

$$(0.176) \quad \sum_{n \geq 1} \frac{575n^2 - 965n + 273}{\binom{3n}{n} 2^n} = 6 \log 2.$$

[39]

$$(0.177) \quad \sum_{n \geq 1} \frac{(-1)^n}{\binom{3n}{n} 4^n} = -\frac{1}{28} - \frac{3}{32} \ln 2 + \frac{13}{112} \frac{\arctan(\sqrt{7/5})}{\sqrt{7}}.$$

[39]

$$(0.178) \quad \sum_{n \geq 1} \frac{(-1)^n n}{\binom{3n}{n} 4^n} = -\frac{81}{1568} - \frac{9}{256} \ln 2 + \frac{17}{6272} \frac{\arctan(\sqrt{7/5})}{\sqrt{7}}.$$

[39]

$$(0.179) \quad \sum_{n \geq 1} \frac{1}{n \binom{3n}{n} 2^n} = \frac{1}{10} \pi - \frac{1}{5} \ln 2.$$

[39]

$$(0.180) \quad \sum_{n \geq 1} \frac{1}{n^2 \binom{3n}{n} 2^n} = \frac{1}{24} \pi^2 - \frac{1}{2} \ln^2 2.$$

[34]

$$(0.181) \quad \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} (j/2)! = 1 - \frac{1}{2} e^{1/4} \sqrt{\pi} (1 - \operatorname{erf}(1/2)).$$

Erratum to [17, 4.1.47][175, A145438]: Using twice 9.14.+13, then [91, 9.121.6] and [91, 9.121.26], then substituting  $tt'/4 = z^2$ , then  $y = \sqrt{t'/4}$ , then  $\arcsin y = v$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^3 \binom{2n}{n}} &= \frac{1}{2} {}_4F_3(1, 1, 1, 1; 2, 2, 3/2; 1/4) = \frac{1}{2} \int_0^1 dt {}_3F_2(1, 1, 1; 2, 3/2; t/4) \\ &= \frac{1}{4} \int_0^1 dt \int_0^1 dt' {}_2F_1(1, 1; 3/2; tt'/4) = \frac{1}{4} \int_0^1 dt \int_0^1 dt' \frac{1}{\sqrt{1-tt'/4}} {}_2F_1(1/2, 1/2; 3/2; tt'/4) \\ &= \frac{1}{4} \int_0^1 dt \int_0^1 dt' \frac{1}{\sqrt{1-tt'/4}} \frac{\arcsin(\sqrt{tt'/4})}{\sqrt{tt'/4}} = \int_0^1 \frac{dt'}{t'} \int_0^{\sqrt{t'/4}} dz \frac{\arcsin z}{\sqrt{1-z^2}} = \frac{1}{2} \int_0^1 \frac{dt'}{t'} \arcsin^2(\sqrt{t'/4}) \\ &= \int_0^{\sqrt{1/2}} \frac{dy}{y} \arcsin^2 y = \int_0^{\sin \sqrt{1/2}} v^2 \cot v dv. \end{aligned}$$

This becomes [39, (35)][119][38, Theorem 3.3]

(0.182)

$$\frac{2\pi}{3} \Im \operatorname{Li}_2(e^{i\pi/4}) - \frac{4}{3} \zeta(3) = \frac{2\pi}{3} \operatorname{Cl}_2(\pi/3) - \frac{4}{3} \zeta(3) = \frac{\pi\sqrt{3}}{18} \{\psi'(1/3) - \psi'(2/3)\} - \frac{4\zeta(3)}{3}.$$

So the r.h.s. of [119] is missing a factor 4, and [15, 4.1.47] is in addition misleading to imply that the digamma (instead of the trigamma) functions are in effect [127].

[178]

$$(0.183) \quad \sum_{n=1}^{\infty} \frac{t^n}{n^{k+1} \binom{an+j+1}{j+1}} = \begin{cases} \frac{(j+1)t(-)^k}{k!} \int_0^1 \int_0^1 \frac{(1-x)^j x^a (\ln y)^k}{1-tx^a y} dx dy, & k \geq 1 \\ at \int_0^1 \frac{(1-x)^{j+1} x^{a-1}}{1-tx^a} dx, & k = 1 \end{cases}$$

$$= T_0 {}_{a+k+1}F_{a+k} \left( \begin{matrix} 1, 1, \dots, 1; (a+1)/a, \dots, (2a-1)/a \\ 2, 2, \dots, 2; (a+j+2)/a, \dots, (a+j+a+1)/a \end{matrix} \middle| t \right)$$

where  $T_0 = t(j+1)B(j+1, a+1)$ .

[67]

(0.184)

$$\frac{1}{2^m} \sum_{i=1}^{\infty} \binom{i+1}{2}^{-m} = (-)^{m-1} \binom{2m-1}{m} + (-)^m 2 \sum_{i=1}^{\lfloor m/2 \rfloor} \binom{2m-2i-1}{m-1} \zeta(2i).$$

[39, 119][175, A073010]

$$(0.185) \quad \sum_{n \geq 1} \frac{1}{n \binom{2n}{n}} = \frac{\pi}{3\sqrt{3}}.$$

$$(0.186) \quad \sum_{n \geq 1} \frac{1}{n^2 \binom{2n}{n}} = \frac{1}{3}\zeta(2).$$

a special case of (1.17).

[94]

$$(0.187) \quad \sum_{n \geq 1} \frac{1}{n^4 \binom{2n}{n}} = \frac{17}{36}\zeta(4).$$

[94]

$$(0.188) \quad \sum_{n \geq 1} \frac{1}{n^5 \binom{2n}{n}} = 2\pi \operatorname{Cl}_4(\pi/3) - \frac{19}{3}\zeta(5) + \frac{2}{3}\zeta(3)\zeta(2).$$

[94]

$$(0.189) \quad \sum_{n \geq 1} \frac{1}{n^6 \binom{2n}{n}} = -\frac{4\pi}{3}\mathfrak{S}L_{4,1}(e^{i\pi/3}) + \frac{3341}{1296}\zeta(6) - \frac{4}{3}\zeta^2(3).$$

[94]

$$(0.190) \quad \sum_{n \geq 1} \frac{(-1)^n}{n \binom{2n}{n}} = -2 \frac{\operatorname{arctanh}(1/\sqrt{5})}{\sqrt{5}}.$$

Correction of a sign error in [119], see [127]:

$$(0.191) \quad \sum_{n \geq 1} \frac{(-1)^n}{n^2 \binom{2n}{n}} = -2 \left( \ln \frac{\sqrt{5}-1}{2} \right)^2 = -2 \left( \ln \frac{\sqrt{5}+1}{2} \right)^2.$$

$$(0.192) \quad \sum_{n \geq 1} \frac{(-1)^n}{n^3 \binom{2n}{n}} = -\frac{2}{5}\zeta(3).$$

(0.193)

$$\sum_{n \geq 1} \frac{(-1)^n}{n^4 \binom{2n}{n}} = -4K_4(\rho) + 4K_4(-\rho) + \frac{1}{2}\ln^4 2 + 7\zeta(4), \quad K_k(x) \equiv \sum_{r=0}^{k-1} \frac{(-\ln|x|)^r}{r!} L_{k-r}(x).$$

$$(0.194) \quad \sum_{n \geq 1} \frac{1}{n^3 \binom{3n}{n} 2^n} = -\frac{33}{16}\zeta(3) + \frac{1}{6}\ln^3 2 - \frac{1}{24}\pi^2 \ln 2 + \pi \mathfrak{S}L_2(i).$$

(0.195)

$$\sum_{n \geq 1} \frac{1}{n^4 \binom{3n}{n} 2^n} = -\frac{143}{16}\zeta(3) \ln 2 + \frac{91}{640}\pi^4 - \frac{3}{8}\ln^4 2 + \frac{3}{8}\pi^2 \ln^2 2 - 8L_4(1/2) - 8\Re L_{3,1}\left(\frac{1+i}{2}\right) - 8\Re L_4\left(\frac{1+i}{2}\right).$$

$$\begin{aligned} \sum_{n \geq 1} \frac{(-1)^n}{8^n n \binom{6n}{2n}} &= \left( -\frac{1}{3} + \frac{2}{57}\sqrt{114\sqrt{57}-342} \right) \ln 2 - \frac{1}{114}\sqrt{114\sqrt{57}-342} \ln \left( 13 + \sqrt{57} + \sqrt{-30+26\sqrt{57}} \right) \\ &\quad + \frac{1}{57}\sqrt{114\sqrt{57}+342} \arctan \frac{\sqrt{2\sqrt{57}+9}}{7}. \end{aligned}$$

$$(0.196) \quad \sum_{n \geq 1} \frac{(-1)^n}{2^n n \binom{4n}{n}} = \sum_{13+12r+2r^3=0} \frac{\ln(r+2)}{r+3} - \frac{6}{7} \ln 3 + \frac{1}{7} \ln 2.$$

$$(0.197) \quad \sum_{n \geq 1} \frac{(-1)^n}{n \binom{3n}{n}} = \sum_{8+4r+r^3=0} \frac{\ln(r+2)}{r+3} - \ln 2.$$

[206] Let  $A_k \equiv \sum_{n=0}^{\infty} \binom{n+k}{n} / \binom{2n}{n}$ , then

$$(0.198) \quad A_{k+1} = -\frac{2}{3(k+1)} + \frac{2(2k+3)}{3(k+1)} A_k.$$

[206] Let  $B_k \equiv \sum_{n=0}^{\infty} (-1)^n \binom{n+k}{n} / \binom{2n}{n}$ , then

$$(0.199) \quad B_{k+1} = -\frac{2}{5(k+1)} + \frac{2(2k+3)}{5(k+1)} B_k.$$

and similar recurrences with an additional factor  $n$  or  $n^2$  in the denominator of the sums.

[33]

$$(0.200) \quad \sum_{k=1}^{\infty} \arctan \frac{2}{k^2} = \frac{3\pi}{4}.$$

[33][175, A091007]

$$(0.201) \quad \sum_{k=1}^{\infty} \arctan \frac{1}{k^2} = \arctan \frac{\tan(\pi/\sqrt{2}) - \tanh(\pi/\sqrt{2})}{\tan(\pi/\sqrt{2}) + \tanh(\pi/\sqrt{2})}.$$

[33]

$$(0.202) \quad \sum_{k=1}^{\infty} \arctan \frac{a}{a^2 k^2 + a(a+2b)k + 1 + ab + b^2} = \frac{\pi}{2} - \arctan(a+b).$$

[33]

$$(0.203) \quad \sum_{k=1}^{\infty} \arctan \frac{a^2 k^2 + a^2 k - 1 - b^2}{a^4 k^4 + 2a^3(a+2b)k^3 + a^2(2+a^2+6ab+6b^2)k^2 + 2a(a+2b)(1+ab+b^2) + (1+b^2)(1+[a+b]^2)} = \frac{1}{1+(a+b)^2}.$$

[33]

$$(0.204) \quad \sum_{k=1}^{\infty} \arctan \frac{2ak + a + b}{a_4 k^4 + a_3 k^3 + a_2 k^2 + a_1 k + a_0} = \frac{\pi}{2} - \arctan(a+b+c).$$

[33]

$$(0.205) \quad \sum_{k=1}^n \arctan \frac{f(k+1) - f(k-1)}{1 + f(k+1)f(k-1)} = \arctan f(n+1) - \arctan f(1) + \arctan f(n) - \arctan f(0).$$

[33]

$$(0.206) \quad \sum_{k=1}^{\infty} \arctan \frac{8k}{k^4 - 2k^2 + 5} = \pi - \arctan \frac{1}{2}.$$

[33]

$$(0.207) \quad \sum_{k=1}^{\infty} \arctan \frac{4ak}{k^4 + a^2 + 4} = \arctan \frac{a}{2} + \arctan a.$$

[33]

$$(0.208) \quad \sum_{k=1}^{\infty} \arctan \frac{2xy}{k^2 - x^2 + y^2} = \arctan \frac{y}{x} - \arctan \frac{\tanh \pi y}{\tan \pi x}.$$

[33]

$$(0.209) \quad \sum_{k=1}^{\infty} \arctan \frac{1}{2k^2} = \pi/4.$$

[33]

$$(0.210) \quad \sum_{k=1}^{\infty} \frac{k^2}{k^4 + 4x^4} = \frac{\pi}{4x} \frac{\sin 2\pi x - \sinh 2\pi x}{\cos 2\pi x - \cosh 2\pi x}.$$

[33]

$$(0.211) \quad \sum_{k=1}^{\infty} \frac{k^2}{k^4 + 4} = \frac{\pi}{4} \coth \pi.$$

[33]

$$(0.212) \quad \sum_{k=1}^{\infty} \frac{k^2}{k^4 + x^2 k^2 + x^4} = \frac{\pi}{2x\sqrt{3}} \frac{\sinh \pi x \sqrt{3} - \sqrt{3} \sin \pi x}{\cosh \pi x \sqrt{3} - \cos \pi x}.$$

[153]

$$(0.213) \quad \sum_{k=1}^{\infty} \frac{A_0 + B_0 k + C_0 k^2}{(k^2 - a^2)(k^2 - b^2)} = \sum_{n=1}^{\infty} \frac{d_n}{\prod_{m=1}^n (m^2 - a^2)(m^2 - b^2)}$$

for  $|a| < 1$ ,  $|b| < 1$ , with  $d_n$  defined via a recurrence in the reference.

$$(0.214) \quad \sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2}.$$

$$(0.215) \quad \sum_{n=0}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}.$$

[175, A000578]

$$(0.216) \quad \sum_{n=0}^{\infty} n^3 x^n = \frac{x(1+4x+x^2)}{(1-x)^4}.$$

[175, A000583]

$$(0.217) \quad \sum_{n=0}^{\infty} n^4 x^n = \frac{x(1+x)(1+10x+x^2)}{(1-x)^5}.$$

[175, A000584]

$$(0.218) \quad \sum_{n=0}^{\infty} n^5 x^n = \frac{x(1 + 26x + 66x^2 + 26x^3 + x^4)}{(1-x)^6}.$$

[175, A001014]

$$(0.219) \quad \sum_{n=0}^{\infty} n^6 x^n = \frac{x(x+1)(1 + 56x + 246x^2 + 56x^3 + x^4)}{(1-x)^7}.$$

[175, A001015]

$$(0.220) \quad \sum_{n=0}^{\infty} n^7 x^n = \frac{x(1 + 120x + 1191x^2 + 2416x^3 + 1191x^4 + 120x^5 + x^6)}{(1-x)^8}.$$

See [175, A008292] for coefficients in the numerator polynomial if exponents of  $n$  are higher than 7. The pole at  $x = 1$  in the generating functions obviously delimits the radius of convergence.

[98] Let

$$(0.221) \quad S_{a,p}(n) \equiv \sum_{k=0}^n a^k k^p;$$

then

$$(0.222) \quad S_{a,p}(n) = \frac{a^{n+1}}{a-1} \sum_{r=0}^{p-1} \binom{p}{r} f_r(a) (n+1)^{p-r} + f_p(a) \frac{a^{n+1} - 1}{a-1}$$

with

$$(0.223) \quad f_0(a) = 1, \quad a \sum_{j=0}^r \binom{r}{j} f_j(a) - f_r(a) = 0, \quad r = 1, 2, \dots$$

[6]

$$(0.224) \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = G.$$

[6, 42]

$$(0.225) \quad 2 \sum_{k=0}^{\infty} \frac{1}{(4k+1)^2} = G + \frac{\pi^2}{8}.$$

[6, 42]

$$(0.226) \quad -2 \sum_{k=0}^{\infty} \frac{1}{(4k+3)^2} = G - \frac{\pi^2}{8}.$$

[42]

$$(0.227) \quad \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{\infty} \binom{n}{k} \frac{(-1)^k}{(2k+1)^2} = G.$$

[42]

$$(0.228) \quad 1 - \sum_{n=1}^{\infty} \frac{n\zeta(2n+1)}{16^n} = G.$$



[42]

$$(0.229) \quad \frac{1}{4} \sum_{n=1}^{\infty} n 16^{-n} (3^{2n} - 1) \zeta(2n+1) = G - \frac{1}{6}$$

(and similar  $\zeta$ -sums).

[42]

$$(0.230) \quad \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{n+1/2}{n}^{-2} = {}_4F_3(1, 1, 1, 1; 2, \frac{3}{2}, \frac{3}{2}; 1) = 2\pi G - \frac{7}{2}\zeta(3).$$

[42]

$$(0.231) \quad 2 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n^2} \sum_{k=0}^{n-1} \frac{1}{2k+1} = 2\pi G - \frac{7}{2}\zeta(3).$$

[42]

$$(0.232) \quad \sum_{n=0}^{\infty} \frac{2^n}{(2n+1) \binom{2n}{n}} \sum_{k=0}^n \frac{1}{2k+1} = 2G.$$

[42]

$$(0.233) \quad \sum_{n=0}^{\infty} \frac{4^n}{(2n+1)^2 \binom{2n}{n}} = 2G.$$

[42]

$$(0.234) \quad \frac{3}{8} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2 \binom{2n}{n}} = G - \frac{1}{8}\pi \log(2 + \sqrt{3}).$$

[42]

$$(0.235) \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \sum_{k=1}^n \frac{1}{k} = G - \frac{1}{2}\pi \log 2.$$

[42]

$$(0.236) \quad -2 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \sum_{k=0}^{n-1} \frac{1}{2k+1} = G - \frac{1}{4}\pi \log 2.$$

[42]

$$(0.237) \quad -\frac{1}{32}\pi \sum_{n=0}^{\infty} \frac{(2n+1)^2}{(n+1)^3 16^n} \binom{2n}{n}^2 = G - \frac{1}{2}\pi \log 2.$$

[42]

$$(0.238) \quad \sum_{n=0}^{\infty} \frac{\sqrt{2}}{(2n+1) 28^n} \binom{2n}{n} = G + \frac{1}{4}\pi \log 2.$$

[42]

$$(0.239) \quad \sum_{n=1}^{\infty} \frac{\sin(n\pi/4)}{n^2 2^{n/2}} = G - \frac{1}{8}\pi \log 2.$$

[42]

$$(0.240) \quad \sum_{n=0}^{\infty} \frac{2^{n+1}(n!)^2}{(2n+1)!(n+1)^2} = 2\pi G - \frac{35}{8}\zeta(3) + \frac{1}{4}\pi^2 \log 2.$$

[42]

$$(0.241) \quad \frac{1}{4}\pi {}_3F_2(1/2, 1/2, n+1/2; 1, n+3/2; 1) - \frac{1}{2} \sum_{k=0}^{n-1} \frac{(k!)^2}{\left(\frac{3}{2}\right)_k^2} = G.$$

[76, 77] Let  $T$  denote Tornheim double sums

$$(0.242) \quad T(a, b, c) \equiv \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{1}{r^a s^b (r+s)^c}$$

with  $a+c > 1$ ,  $b+c > 1$  and  $a+b+c > 2$ . Then

(0.243)

$$T(m, k, n) = \sum_{i=1}^m \binom{m+k-i-1}{m-i} T(i, 0, N-i) + \sum_{i=1}^k \binom{m+k-i-1}{k-i} T(i, 0, N-i).$$

$$\begin{aligned} T(m, 0, n) &= (-1)^m \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{m+n-2j-1}{m-1} \zeta(2j) \zeta(m+n-2j) \\ &\quad + (-1)^m \sum_{j=0}^{\lfloor m/2 \rfloor} \binom{m+n-2j-1}{n-1} \zeta(2j) \zeta(m+n-2j) - \frac{1}{2} \zeta(m+n), \end{aligned}$$

if  $N \equiv m+k+n$  is odd.

$$(0.244) \quad T(1, 0, 5) = -\frac{1}{2}\zeta(3)^2 + \frac{3}{4}\zeta(6).$$

$$(0.245) \quad T(2, 0, 4) = \zeta^2(3) - \frac{4}{3}\zeta(6).$$

$$(0.246) \quad T(3, 0, 3) = \frac{1}{2}\zeta(3)^2 - \frac{1}{2}\zeta(6).$$

$$(0.247) \quad T(4, 0, 2) = -\zeta^2(3) + \frac{25}{12}\zeta(6).$$

$$(0.248) \quad T(0, 0, N) = \zeta(N-1) - \zeta(N), \quad N \geq 3; \quad T(0, 0, 4) = \zeta(3) - \frac{\pi^4}{90}.$$

$$(0.249) \quad T(n, 0, n) = \frac{1}{2}\zeta^2(n) - \frac{1}{2}\zeta(2n). \quad T(2, 0, 2) = \frac{\pi^4}{120}.$$

$$(0.250) \quad T(0, 0, 8) = \zeta(7) - \zeta(8).$$

$$(0.251) \quad T(1, 0, 7) = \frac{5}{4}\zeta(8) - \zeta(3)\zeta(5).$$

$$(0.252) \quad T(4, 0, 4) = \frac{1}{12}\zeta(8).$$

$$(0.253) \quad T(0, 0, 14) = \zeta(13) - \zeta(14).$$

$$(0.254) \quad T(1, 0, 13) = \frac{11}{4}\zeta(14) - \zeta(3)\zeta(11) - \zeta(5)\zeta(9) - \frac{1}{2}\zeta(7)^2.$$

$$(0.255) \quad T(1, 0, 2) = \zeta(3).$$

$$(0.256) \quad T(1, 1, 1) = 2\zeta(3).$$

$$(0.257) \quad T(1, 0, 3) = \frac{\pi^4}{360}.$$

$$(0.258) \quad T(1, 1, 2) = \frac{\pi^4}{180}.$$

$$(0.259) \quad T(2, 1, 1) = \frac{\pi^4}{72}.$$

$$(0.260) \quad T(2, 2, 0) = \frac{\pi^4}{36}.$$

$$(0.261) \quad T(1, 0, 5) = \frac{3}{4}\zeta(6) - \frac{1}{2}\zeta^2(3).$$

$$(0.262) \quad T(1, 1, 4) = \frac{3}{2}\zeta(6) - \zeta^2(3).$$

$$(0.263) \quad T(3, 3, 0) = \zeta^2(3).$$

$$(0.264) \quad T(4, 1, 1) = \frac{7}{6}\zeta(6) - \frac{1}{2}\zeta^2(3).$$

$$(0.265) \quad T(4, 2, 0) = \frac{7}{4}\zeta(6).$$

$$(0.266) \quad T(1, 1, 4) = \frac{3}{2}\zeta(6) - \zeta^2(3).$$

$$(0.267) \quad \sum_{n=0}^{\infty} \frac{1}{(n+a)(n+b)} = \frac{\psi(a) - \psi(b)}{a-b}.$$

$$(0.268) \quad \sum_{n=0}^{\infty} \frac{(-)^n}{(n+a)(n+b)} = -\frac{1}{2} \frac{\psi(1/2+a/2) - \psi(a/2) - \psi(1/2+b/2) + \psi(b/2)}{a-b}.$$

[89]

$$(0.269) \quad \log 2 = \frac{1}{2} - \sum_{k \geq 1} \frac{(-1)^k}{k(4k^4 + 1)}.$$

After inserting  $x = 1$  in (1.65), [17, 4.1.13]

$$(0.270) \quad 2 \ln 2 = 1 + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k(k+1)} = 1 + 2 \sum_{k=1,3,5,\dots}^{\infty} \frac{1}{k(k+1)(k+2)}$$

$$= 1 + 4 \sum_{k=1,5,9,13,\dots}^{\infty} \frac{6 + 4k + k^2}{k(k+1)(k+2)(k+3)(k+4)}.$$

[17, 4.1.20]

$$2 \ln 2 = \frac{5}{4} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k(k+1)(k+2)} = \frac{5}{4} + 3 \sum_{k=1,3,5,\dots}^{\infty} \frac{1}{k(k+1)(k+2)(k+3)}$$

$$= \frac{17}{12} - 3 \sum_{k=2,4,6,\dots}^{\infty} \frac{1}{k(k+1)(k+2)(k+3)}.$$

(0.271)

$$\frac{4}{3} \ln 2 = \frac{8}{9} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k(k+1)(k+2)(k+3)} = \frac{8}{9} + 4 \sum_{k=1,3,5,\dots}^{\infty} \frac{1}{k(k+1)(k+2)(k+3)(k+4)}.$$

More irregular denominators follow from hybridization. For example we can multiply the penultimate formula by 4, the previous formula by 3, and subtract

$$(0.272) \quad 4 \ln 2 = \frac{7}{3} + 12 \sum_{k=1,3,5,\dots}^{\infty} \frac{1}{k(k+1)(k+2)(k+4)}.$$

$$\frac{2}{3} \ln 2 = \frac{131}{288} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k(k+1)(k+2)(k+3)(k+4)}$$

$$= \frac{131}{288} + 5 \sum_{k=1,3,5,\dots}^{\infty} \frac{1}{k(k+1)(k+2)(k+3)(k+4)(k+5)}.$$

Inserting  $x = -1/2$  in (1.65) [89, 22] yields

$$(0.273) \quad \ln 2 = 1 - \sum_{k=1}^{\infty} \frac{1}{2^k k(k+1)}.$$

$$(0.274) \quad \ln 2 = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{2^{k-1} k(k+1)(k+2)}.$$

$$(0.275) \quad \ln 2 = \frac{5}{6} - 3 \sum_{k=1}^{\infty} \frac{1}{2^{k-1} k(k+1)(k+2)(k+3)}.$$

More irregular denominators follow from hybridization. For example we can multiply the penultimate formula by 3 and add to the previous formula,

$$(0.276) \quad 4 \ln 2 = \frac{7}{3} + 3 \sum_{k=1}^{\infty} \frac{1}{2^{k-1} k(k+1)(k+3)}.$$

$$(0.277) \quad \ln 2 = \frac{7}{12} + 3 \sum_{k=1}^{\infty} \frac{1}{2^{k-3} k(k+1)(k+2)(k+3)(k+4)}.$$

[89]

$$(0.278) \quad \log 2 = \frac{1327}{1920} + \frac{45}{4} \sum_{k \geq 4} \frac{(-1)^k}{k(k^2-1)(k^2-4)(k^2-9)}.$$

[89]

$$(0.279) \quad \log 2 = \sum_{k \geq 1} \left( \frac{1}{3^k} + \frac{1}{4^k} \right) \frac{1}{k}.$$

$$(0.280) \quad \log 2 = \sum_{k \geq 0} \left( \frac{1}{8k+8} + \frac{1}{4k+2} \right) \frac{1}{4^k}.$$

[89]

$$(0.281) \quad \log 2 = \frac{2}{3} + \sum_{k \geq 1} \left( \frac{1}{2k} + \frac{1}{4k+1} + \frac{1}{8k+4} + \frac{1}{16k+12} \right) \frac{1}{16^k}.$$

[89]

$$(0.282) \quad \log 2 = \frac{2}{3} \sum_{k \geq 0} \frac{1}{(2k+1)9^k}.$$

$$(0.283) \quad \log 2 = \frac{3}{4} \sum_{k \geq 0} \frac{(-1)^k k!^2}{2^k (2k+1)!}.$$

[89]

$$(0.284) \quad \log 2 = \frac{3}{4} + \frac{1}{4} \sum_{k \geq 1} \frac{(-1)^k (5k+1)}{k(2k+1)16^k} \binom{2k}{k}.$$

[175, A154920][23]

$$(0.285) \quad \log 3 = \sum_{k \geq 1} \left[ \frac{9}{2k-1} + \frac{1}{2k} \right] \frac{1}{9^k}.$$

[175, A164985][23]

$$(0.286) \quad 27 \log 5 = 4 \sum_{k \geq 0} \left[ \frac{9}{4k+1} + \frac{3}{4k+2} + \frac{1}{4k+3} \right] \frac{1}{81^k}.$$

$$(0.287) \quad \log 5 = 2 \log 3 - \log 2 + \sum_{k=1}^{\infty} \frac{1}{k10^k}.$$

$$(0.288) \quad 4 \log 7 = 5 \log 2 + \log 3 + 2 \log 5 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k2400^k}.$$

TABLE 1. Formulas of the type  $s \log p = t \log q + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} r^k$ . [128]

$s$	$p$	$t$	$q$	$r$
1	3	1	2	$1/2 \approx 0.50000000$
1	3	2	2	$-1/4 \approx -0.25000000$
2	3	3	2	$1/8 \approx 0.12500000$
5	3	8	2	$-13/256 \approx -0.05078125$
12	3	19	2	$7153/524288 \approx 0.01364326$
1	5	2	2	$1/4 \approx 0.25000000$
3	5	7	2	$-3/128 \approx -0.02343750$
1	7	2	2	$3/4 \approx 0.75000000$
1	7	3	2	$-1/8 \approx -0.12500000$
5	7	14	2	$423/16384 \approx 0.02581787$
1	11	3	2	$3/8 \approx 0.37500000$
2	11	7	2	$-7/128 \approx -0.05468750$
11	11	38	2	$10433763667/274877906944 \approx 0.03795781$
1	13	3	2	$5/8 \approx 0.62500000$
1	13	4	2	$-3/16 \approx -0.18750000$
3	13	11	2	$149/2048 \approx 0.07275391$
7	13	26	2	$-4360347/67108864 \approx -0.06497423$
10	13	37	2	$419538377/137438953472 \approx 0.00305254$
1	17	4	2	$1/16 \approx 0.06250000$
1	19	4	2	$3/16 \approx 0.18750000$
4	19	17	2	$-751/131072 \approx -0.00572968$
1	23	4	2	$7/16 \approx 0.43750000$
1	23	5	2	$-9/32 \approx -0.28125000$
2	23	9	2	$17/512 \approx 0.03320312$
1	29	4	2	$13/16 \approx 0.81250000$
1	29	5	2	$-3/32 \approx -0.09375000$
7	29	34	2	$70007125/17179869184 \approx 0.00407495$
1	31	4	2	$15/16 \approx 0.93750000$
1	31	5	2	$-1/32 \approx -0.03125000$
1	37	5	2	$5/32 \approx 0.15625000$
4	37	21	2	$-222991/2097152 \approx -0.10633039$
5	37	26	2	$2235093/67108864 \approx 0.03330548$
1	41	5	2	$9/32 \approx 0.28125000$
2	41	11	2	$-367/2048 \approx -0.17919922$
3	41	16	2	$3385/65536 \approx 0.05165100$
1	43	5	2	$11/32 \approx 0.34375000$
2	43	11	2	$-199/2048 \approx -0.09716797$
5	43	27	2	$12790715/134217728 \approx 0.09529825$
7	43	38	2	$-3059295837/274877906944 \approx -0.01112965$
1	47	5	2	$15/32 \approx 0.46875000$
1	47	6	2	$-17/64 \approx -0.26562500$
2	47	11	2	$161/2048 \approx 0.07861328$
1	53	5	2	$21/32 \approx 0.65625000$
1	53	6	2	$-11/64 \approx -0.17187500$
3	53	17	2	$17805/131072 \approx 0.13584137$
4	53	23	2	$-498127/8388608 \approx -0.05938137$
1	59	5	2	$27/32 \approx 0.84375000$
1	59	6	2	$-5/64 \approx -0.07812500$
1	61	5	2	$29/32 \approx 0.90625000$
1	61	6	2	$-3/64 \approx -0.04687500$
1	67	6	2	$3/64 \approx 0.04687500$

[23]

$$(0.289) \quad 3^5 \log 7 = \sum_{k \geq 0} \left[ \frac{405}{6k+1} + \frac{81}{6k+2} + \frac{72}{6k+3} + \frac{9}{6k+4} + \frac{5}{6k+5} \right] \frac{1}{3^{6k}}.$$

[23]

(0.290)

$$2 \times 3^9 \log 11 = \sum_{k \geq 0} \left[ \frac{85293}{10k+1} + \frac{10935}{10k+2} + \frac{9477}{10k+3} + \frac{1215}{10k+4} + \frac{648}{10k+5} + \frac{135}{10k+6} \right. \\ \left. + \frac{117}{10k+7} + \frac{15}{10k+8} + \frac{13}{10k+9} \right] \frac{1}{3^{10k}}.$$

(0.291)

$$\log 11 = \log 2 + \log 5 + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k10^k}.$$

(0.292)

$$\log 11 = 2 \log 2 + \log 3 - \sum_{k=1}^{\infty} \frac{1}{k12^k}.$$

(0.293)

$$\log 11 = 2(\log 2 + \log 5 - \log 3) - \sum_{k=1}^{\infty} \frac{1}{k100^k}.$$

(0.294)

$$\log 13 = 2 \log 2 + \log 3 + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k12^k}.$$

(0.295)

$$3 \log 17 = 3 \log 2 + 4 \log 5 - \sum_{k=1}^{\infty} \frac{1}{k} \left( \frac{87}{5000} \right)^k.$$

(0.296)

$$\log 19 = 2 \log 2 + \log 5 - \sum_{k=1}^{\infty} \frac{1}{k20^k}.$$

[115]

$$(0.297) \quad \log 11 = \sum_{k \geq 1} \frac{1}{k} \left( \frac{3}{2^k} + \frac{1}{2^{2k}} + \frac{1}{2^{5k}} - \frac{1}{2^{10k}} \right) = \frac{1}{2} \sum_{k \geq 1} \frac{1}{k} \left( \frac{13}{3^k} - \frac{4}{3^{2k}} - \frac{1}{3^{5k}} \right).$$

[115]

$$(0.298) \quad \log 13 = \sum_{k \geq 1} \frac{1}{k} \left( \frac{3}{2^k} + \frac{1}{2^{2k}} + \frac{1}{2^{3k}} + \frac{1}{2^{4k}} - \frac{1}{2^{12k}} \right) = \sum_{k \geq 1} \frac{1}{k} \left( \frac{7}{3^k} - \frac{2}{3^{2k}} - \frac{1}{3^{3k}} \right).$$

[115]

$$(0.299) \quad \log 17 = \sum_{k \geq 1} \frac{1}{k} \left( \frac{4}{2^k} + \frac{1}{2^{4k}} - \frac{1}{2^{8k}} \right).$$

[115]

$$(0.300) \quad \log 19 = \sum_{k \geq 1} \frac{1}{k} \left( \frac{3}{2^k} + \frac{3}{2^{2k}} + \frac{1}{2^{9k}} - \frac{1}{2^{18k}} \right).$$

Similar formulas as the four above are obtained by inserting

(0.301)

$$\log[(2^s - 1)^\tau] = s\tau \log 2 - \sum_{k \geq 1} \frac{\tau}{k 2^{ks}} = -s\tau \log \frac{1}{2} - \sum_{k \geq 1} \frac{\tau}{k 2^{ks}} = \sum_{k \geq 1} \frac{\tau}{k} \left( \frac{s}{2^k} - \frac{1}{2^{ks}} \right),$$

—immediate consequence of putting  $x = 1/2$  in (1.64)—into the right hand sides of [46]:

$$(0.302) \quad \log 41 = \log(2^{20} - 1) - \log(2^{10} - 1) + \log[(2^2 - 1)^2] - \log[(2^4 - 1)^2].$$

$$(0.303) \quad \log 43 = \log(2^{14} - 1) - \log(2^2 - 1) - \log(2^7 - 1).$$

$$(0.304) \quad \log 73 = \log(2^9 - 1) - \log(2^3 - 1).$$

$$(0.305) \quad \log 151 = \log(2^{15} - 1) - \log(2^3 - 1) - \log(2^5 - 1).$$

$$(0.306) \quad \log 241 = \log(2^{24} - 1) - \log(2^{12} - 1) - \log(2^8 - 1) + \log(2^4 - 1).$$

$$(0.307) \quad \log 257 = \log(2^{16} - 1) - \log(2^8 - 1).$$

$$(0.308) \quad \log 331 = \log(2^{30} - 1) - \log(2^{15} - 1) - \log(2^{10} - 1) + \log(2^5 - 1) - \log(2^2 - 1).$$

$$(0.309) \quad \log 337 = \log(2^{21} - 1) - \log(2^7 - 1) - \log[(2^6 - 1)^2] + \log[(2^2 - 1)^4].$$

$$(0.310) \quad \log 683 = \log(2^{22} - 1) - \log(2^{11} - 1) - \log(2^2 - 1).$$

$$(0.311) \quad \log 2731 = \log(2^{26} - 1) - \log(2^{13} - 1) - \log(2^2 - 1).$$

$$(0.312) \quad \log 5419 = \log(2^{42} - 1) - \log(2^{21} - 1) - \log(2^{14} - 1) + \log(2^7 - 1) - \log(2^2 - 1).$$

$$(0.313) \quad \log 43691 = \log(2^{34} - 1) - \log(2^{17} - 1) - \log(2^2 - 1).$$

$$(0.314) \quad \log 61681 = \log(2^{40} - 1) - \log(2^{20} - 1) - \log(2^8 - 1) + \log(2^4 - 1).$$

$$(0.315) \quad \log 174763 = \log(2^{38} - 1) - \log(2^{18} - 1) - \log(2^2 - 1).$$

$$(0.316) \quad \log 262657 = \log(2^{27} - 1) - \log(2^9 - 1).$$

$$(0.317) \quad \log 599479 = \log(2^{33} - 1) - \log(2^{11} - 1) - \log(2^3 - 1).$$

$$(0.318) \quad \sum_{k \geq 0} \frac{\log(2k+1)}{(2k+1)^l} = -\frac{\log 2}{2^l} \zeta(l) - (1 - 2^{-l}) \zeta'(l).$$

$$(0.319) \quad \sum_{k \geq 0} \frac{\log(ak+b)}{(ak+b)^l} = \frac{\log a}{a^l} \zeta(l, b/a) - \frac{1}{a^l} \zeta'(l, b/a).$$

[34]

$$(0.320) \quad \sum_{k=1}^{\infty} \frac{kk!}{(2k)!} = \frac{1}{8} \left( 2 + 3e^{1/4} \sqrt{\pi} \operatorname{erf}(1/2) \right).$$

$$(0.321) \quad \sum_{k=1}^{\infty} \frac{(-1)^k \alpha^k x^k}{kk!} = -\gamma - \Gamma(0, \alpha x) - \ln \alpha - \ln x.$$



[6]

$$\frac{3}{2} \sum_{k=0}^{\infty} \frac{1}{16^k} \left( \frac{1}{(8k+1)^2} - \frac{1}{(8k+2)^2} + \frac{1}{2(8k+3)^2} - \frac{1}{4(8k+5)^2} + \frac{1}{4(8k+6)^2} - \frac{1}{8(8k+7)^2} \right) - \frac{1}{4} \sum_{k=0}^{\infty} \frac{1}{4096^k} \left( \frac{1}{(8k+1)^2} + \frac{1}{2(8k+2)^2} + \frac{1}{8(8k+3)^2} - \frac{1}{64(8k+5)^2} - \frac{1}{128(8k+6)^2} - \frac{1}{512(8k+7)^2} \right) = G,$$

which is Catalan's constant [175, A006752].

[44]

$$(0.322) \quad \sum_{k \geq 0} \frac{1}{16^k} \left( \frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right) = \pi.$$

[90]

$$(0.323) \quad \frac{1}{16807} \sum_{n=0}^{\infty} \frac{1}{2^n \binom{7n}{2n}} \left( \frac{59296}{7n+1} - \frac{10326}{7n+2} - \frac{3200}{7n+3} - \frac{1352}{7n+4} - \frac{792}{7n+5} + \frac{552}{7n+6} \right) = \pi.$$

[6]

$$(0.324) \quad \frac{1}{2} \sum_{k=0}^{\infty} \frac{4^k k!^2}{(2k)!(2k+1)^2} = G.$$

[6]

$$(0.325) \quad -\frac{\pi}{32} \sum_{k=0}^{\infty} \frac{(2k+1)!^2}{16^k k!^4 (k+1)^3} = G - \frac{\pi}{2} \log 2.$$

[6]

$$(0.326) \quad \sqrt{2} \sum_{k=0}^{\infty} \frac{(2k)!}{8^k k!^2 (2k+1)^2} = G + \frac{\pi}{4} \log 2.$$

[6, 41]

$$(0.327) \quad \frac{3}{8} \sum_{k=0}^{\infty} \frac{k!^2}{(2k)!(2k+1)^2} = G + \frac{\pi}{8} \log(2 + \sqrt{3}).$$

[41]

$$(0.328) \quad \frac{5}{8} \sum_{k=0}^{\infty} \frac{L(2k+1)}{(2k+1)^2 \binom{2k}{k}} = G - \frac{\pi}{8} \log \left( \frac{10 + \sqrt{50 - 22\sqrt{5}}}{10 - \sqrt{50 - 22\sqrt{5}}} \right).$$

where  $L$  are the Lucas numbers.

[155]

$$(0.329) \quad \sqrt{e} = \frac{16}{31} \left[ 1 + \sum_{n=1}^{\infty} \frac{1 + n^3/2 + n/2}{2^n n!} \right],$$

which is a special case of  $\frac{1}{2}P_3(1/2) + \frac{1}{2}P_1(1/2) + P_0(1/2)$  of [100]

$$(0.330) \quad \sum_{n=0}^{\infty} n^j \frac{t^n}{n!} = P_j(t) \exp(t)$$

where

$$(0.331) \quad P_0(t) \equiv 1; \quad P_j(t) \equiv t [P'_{j-1}(t) + P_{j-1}(t)], j > 0.$$

[4]

$$(0.332) \quad \sum_{k \geq 1} \binom{k}{p} \frac{1}{k!k} = \zeta(p+1).$$

[4]

$$(0.333) \quad \sum_{k \geq 2} \binom{k}{2} \frac{1}{k!k^q} = \frac{q+1}{2} \zeta(q+2) - \frac{1}{q} \sum_{k=1}^{q-1} k \zeta(k+1) \zeta(q+1-k).$$

[4]

$$(0.334) \quad \sum_{k \geq 1} \binom{k}{p} \frac{z^k}{k!k} = \zeta(p+1) + \sum_{k=1}^p \frac{\binom{-1}{k-1}}{k!} \text{Li}_{p+1-k}(1-z) \log^k(1-z).$$

[4]

$$(0.335) \quad \sum_{k \geq p} \binom{k}{p} \frac{1}{k!k^q} = \sum_{k \geq q} \binom{k}{q} \frac{1}{k!k^p} = \frac{(-1)^{q-1}}{(q-1)!p!} \lim_{\beta \rightarrow 0} \lim_{\alpha \rightarrow 0} \frac{d^{q+p-1}}{d\alpha^q d\beta^{q-1}} \frac{\Gamma(1-\alpha)\Gamma(1+\beta)}{\Gamma(\beta)\Gamma(1-\alpha+\beta)},$$

which can always be represented in finite terms of zeta functions.

### 0.3. Formulae from Differential Calculus. [112]

$$(0.336) \quad \left(\frac{d}{dx}\right)^m e^{g(x)} = e^{g(x)} Y_m(g'(x), g''(x), \dots, g^{(m)}(x))$$

where  $Y_m$  is the  $m$ th exponential complete Bell polynomial,

$$(0.337) \quad Y_n(x_1, \dots, x_n) = \sum_{\pi(n)} \frac{n!}{k_1! \dots k_n!} \left(\frac{x_1}{1!}\right)^{k_1} \dots \left(\frac{x_n}{n!}\right)^{k_n}.$$

$$(0.338) \quad \frac{d}{dx} [f(x)^{g(x)}] = f(x)^{g(x)} \left[ (\log f) g' + g \frac{f'}{f} \right]$$

[62]

$$(0.339) \quad \sum_{n \geq 0} \frac{g^{(n)}(0)}{n!} f(n) x^n = \sum_{n \geq 0} \frac{f^{(n)}(0)}{n!} \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} x^k g^{(k)}(x).$$

where the braces are Stirling numbers of the second kind.

## 1. ELEMENTARY FUNCTIONS

### 1.1. Powers of Binomials. [30]

$$(1.1) \quad \left(\frac{1 + \sqrt{1+4x}}{2}\right)^n = (1+x)^{n/2} \sum_{k \geq 0} \frac{(-1)^{k+1} \Gamma(\frac{3k-2}{2})}{\Gamma(\frac{3k-n}{2} - k + 1) k!} \left(\frac{x}{(1+x)^{3/2}}\right)^k.$$

[119]

$$(1.2) \quad \sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}.$$

[119][175, A000108]

$$(1.3) \quad \sum_{n=0}^{\infty} \binom{2n}{n} \frac{x^n}{n+1} = \frac{1 - \sqrt{1-4x}}{2x}.$$

[119]

$$(1.4) \quad \sum_{n=1}^{\infty} \frac{1}{n} \binom{2n}{n} x^n = 2 \log \frac{1 - \sqrt{1-4x}}{2x}.$$

[119]

$$(1.5) \quad x \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \binom{2n}{n} x^n = 2x \log \frac{1 - \sqrt{1-4x}}{x} + \frac{\sqrt{1-4x}}{2} - x(\log 4 - 1) - \frac{1}{2},$$

which corrects a sign error in [119], see [127].

[40]

$$(1.6) \quad \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} H_n x^{n+1} = \sqrt{1-4x} \log(2\sqrt{1-4x}) - (1+\sqrt{1-4x}) \log(1+\sqrt{1-4x}) + \log 2,$$

$$(1.7) \quad \sum_{n=0}^{\infty} \binom{2n}{n} H_n x^n = \frac{2}{\sqrt{1-4x}} \log\left(\frac{1+\sqrt{1-4x}}{2\sqrt{1-4x}}\right),$$

$$(1.8) \quad \sum_{n=0}^{\infty} \binom{2n}{n} H_n (-1)^{n-1} x^n = \frac{2}{\sqrt{1+4x}} \log\left(\frac{2\sqrt{1+4x}}{1+\sqrt{1+4x}}\right),$$

where  $H_n = \sum_{k=1}^n 1/k$ .

[40]

$$(1.9) \quad \sum_{n=0}^{\infty} \binom{2n}{n} P_q(n) x^{n+1} = \frac{1}{\sqrt{1-4x}} \sum_{k=0}^q \binom{2k}{k} k! \left(\frac{x}{1-4x}\right)^k \sum_{m=k}^q a_m S(m, k)$$

where  $P_q(z) = a_q z^q + a_{q-1} z^{q-1} + \dots + q_0$  is a polynomial and  $S$  are the Stirling Numbers of the Second Kind.

[175, A005430]

$$(1.10) \quad \sum_{n=1}^{\infty} n \binom{2n}{n} x^n = 2x(1-4x)^{-3/2}.$$

[119][175, A002736]

$$(1.11) \quad \sum_{n=1}^{\infty} n^2 \binom{2n}{n} x^n = \frac{2x(2x+1)}{(1-4x)^{5/2}}.$$

[119]

$$(1.12) \quad \sum_{n=0}^{\infty} \frac{1}{2n+1} \binom{2n}{n} x^n = \frac{1}{2x} \arcsin(2x).$$

[40]

$$(1.13) \quad \sum_{n=0}^{\infty} \frac{1}{n+m+1} \binom{2n}{n} x^n = \frac{1}{2^{2m+1} x^{m+1}} \sum_{k=0}^m \binom{m}{k} \frac{(-)^k}{2k+1} [1 - (1-4x)^{k+1/2}].$$

[40]

$$(1.14) \quad \sum_{n=0}^{\infty} \frac{1}{n^2} \binom{2n}{n} x^n = 2 \operatorname{Li}_2 \left( \frac{1 - \sqrt{1 - 4x}}{2} \right) \\ - \log^2(1 + \sqrt{1 - 4x}) - 2 \log 2 \log \frac{1 - \sqrt{1 - 4x}}{x} + 3 \log^2 2.$$

[119]

$$(1.15) \quad \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{\binom{2m}{m}} (2x/y)^{2m} = \frac{xy^2}{h^3} \left[ \log \frac{x+h}{y} + \frac{xh}{y^2} \right]$$

where  $h \equiv \sqrt{x^2 + y^2}$ .

[119]

$$(1.16) \quad \sum_{m=1}^{\infty} \frac{(2x)^{2m}}{m \binom{2m}{m}} = \frac{2x \arcsin x}{\sqrt{1 - x^2}}.$$

[119]

$$(1.17) \quad \sum_{m=1}^{\infty} \frac{(2x)^{2m}}{m^2 \binom{2m}{m}} = 2(\arcsin x)^2.$$

[119]

$$(1.18) \quad \sum_{m=1}^{\infty} \frac{(2x)^{2m}}{\binom{2m}{m}} = \frac{x^2}{1 - x^2} + \frac{x \arcsin x}{(1 - x^2)^{3/2}}.$$

(1.19)

$$\frac{1}{(\sigma^2 + \sigma_L^2)^{\gamma/2}} = \frac{1}{\sigma_L^\gamma} \exp \left( -\frac{\gamma}{2} \frac{\sigma^2}{\sigma_L^2} \right) \left( 1 + \frac{\gamma}{4} \left( \frac{\sigma}{\sigma_L} \right)^4 - \frac{\gamma}{6} \left( \frac{\sigma}{\sigma_L} \right)^6 + \frac{\gamma(\gamma+4)}{32} \left( \frac{\sigma}{\sigma_L} \right)^8 \right. \\ \left. - \frac{\gamma(5\gamma+12)}{120} \left( \frac{\sigma}{\sigma_L} \right)^{10} + \frac{\gamma(3\gamma^2+52\gamma+96)}{1152} \left( \frac{\sigma}{\sigma_L} \right)^{12} - \frac{\gamma(35\gamma^2+308\gamma+480)}{6720} \left( \frac{\sigma}{\sigma_L} \right)^{14} + \dots \right)$$

[162, 3.41(a)]

$$(1.20) \quad \frac{1}{a^2 - x^2} = \frac{2}{a\sqrt{a^2 - 1}} \sum_{j=0}^{\infty \prime} (a - \sqrt{a^2 - 1})^{2j} T_{2j}(x)$$

where the prime at the sum symbols means taking only half of the value at index zero.

[31]

$$(1.21) \quad (1 + x + x^{-1})^n = \sum_{j=-n}^n \binom{n}{j}_2 x^j, \quad \binom{n}{m}_2 \equiv \sum_{j \geq 0} \frac{n!}{j!(m+j)!(n-2j-m)!}.$$

[153]

$$(1.22) \quad \sum_{s=0}^{\infty} \zeta(4s+3) x^{4s} = \sum_{k=1}^{\infty} \frac{k}{k^4 - x^4} = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-)^{k-1}}{\binom{2k}{k}} \frac{k}{k^4 - x^4} \prod_{m=1}^{k-1} \frac{m^4 + 4x^4}{m^4 - x^4}.$$

[28] Let  $B_0(x)$  be a period function of period 1. Assume  $B_0(x)$  has a continuous derivative in the open interval  $(0,1)$ . Let  $a_0 \equiv \int_0^1 B_0(x)dx$  and define

$$(1.23) \quad B_n(x) = \int_0^2 B_{n-1}(y)dy + \int_0^1 (y-1)B_{n-1}(y)dy.$$

Then

$$(1.24) \quad \sum_{k=0}^{\infty} B_k(x)t^k = \frac{te^{xt}}{e^t-1} \left[ a_0 - \int_0^1 B'_0(1-y) \frac{e^{ty}-1}{t} dy \right] + \int_0^x e^{t(x-y)} B'_0(y)dy,$$

and

$$(1.25) \quad B_n(x) = \Re \left[ 2 \sum_{j=1}^{\infty} \frac{e^{2\pi i j x}}{(2\pi i j)^n} (a_j - a_0 - i b_j) \right]$$

where  $a_j$  and  $b_j$  are the Fourier coefficients of  $B_0(x)$ . For example  $B_0(x) = \cos(\pi x)$  yields  $B_2(x) = (1-2x-\cos(\pi x))/\pi^2$  and

$$(1.26) \quad \frac{\pi^3}{2} B_2(x) = - \sum_{j=1}^{\infty} \frac{\sin(2\pi j x)}{j(4j^2-1)}.$$

Consider also a periodic sequence  $\lambda_{j+T} = \lambda_j$ , the Dirichlet series

$$(1.27) \quad f(s) = \sum_{j=1}^{\infty} \frac{a_j - a_0}{j^s} \lambda_j, \quad g(s) = \sum_{j=1}^{\infty} \frac{b_j}{j^s} \lambda_j,$$

and the Fourier coefficients

$$(1.28) \quad \alpha_j \equiv \sum_{k=1}^T \lambda_k \cos(2\pi j k/T), \quad \beta_j \equiv \sum_{k=1}^T \lambda_k \sin(2\pi j k/T).$$

If  $\lambda_{T-j} = \lambda_j$  for  $1 \leq j < T$ , then

$$(1.29) \quad f(2n) = \frac{(-)^n}{2T} (2\pi)^{2n} \sum_{j=1}^T \alpha_j B_{2n}(j/T),$$

$$(1.30) \quad g(2n+1) = \frac{(-)^{n+1}}{2T} (2\pi)^{2n+1} \sum_{j=1}^T \alpha_j B_{2n+1}(j/T),$$

and similar expressions for other even-odd symmetries of the  $\lambda$ .

[28]

$$(1.31) \quad 8 \log 2 - 4 = \zeta(3) + \sum_{j=1}^{\infty} \frac{1}{j^2(4j^2-1)} = \sum_{k=0}^{\infty} \frac{\zeta(3+2k)}{4^k}.$$

[28]

$$(1.32) \quad \sum_{j=1}^{\infty} \frac{\cos(2\pi j/3)}{j^{2k}(4\pi^2 j^2+1)} = \frac{(-)^k}{4} (2\pi)^{2k} \alpha_{2k},$$

with g.f.

$$(1.33) \quad \sum_{k=0}^{\infty} \alpha_k t^k = \frac{e + e^{4/3} - e^{1+t} - e^{t+4/3} + t^3 [e^{(2t+5)/3} - e^{(2+2t)/3} - e^{(2+t)/3} + e^{(5+t)/3}]}{(e-1)e^{2/3}(e^t-1)(t^2-1)} - \frac{2te^{t/3}}{e^t-2}.$$

### 1.2. The Exponential Function. [84], [175, A001469]

$$(1.34) \quad \frac{2t}{e^t + 1} = \sum_{N=0}^{\infty} \frac{G_N}{N!} t^N;$$

with  $(G+1)^N + G_N = 1$  for  $N > 1$ ,  $G_1 = 1$ ,  $G_2 = -1$ ,  $G_4 = 1$ ,  $G_6 = -3$ ,  $G_8 = 17$ ,  $G_{10} = -155$ ,  $G_{12} = 2073$ , Genocchi numbers.

[31]

$$(1.35) \quad \exp(xt + yt^2) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x, y), \quad H_n(x, y) \equiv n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{x^{n-2k} y^k}{k!(n-2k)!}.$$

### 1.3. Fourier Series. [61, B2a]

$$(1.36) \quad \cos(m\theta) = \sum_{k=0}^{\lfloor m/2 \rfloor} (-1)^k \binom{m}{2k} \cos^{m-2k} \theta \sin^{2k} \theta.$$

$$(1.37) \quad \sin^{2k} \alpha + \cos^{2k} \alpha = 1 - \sum_{l=1}^{k-1} \binom{k}{l} \sin^{2l} \alpha \cos^{2k-2l} \alpha.$$

[60]

$$(1.38) \quad S_\nu(\alpha) \equiv \sum_{k=0}^{\infty} \frac{\sin(2k+1)\alpha}{(2k+1)^\nu};$$

$$(1.39) \quad C_\nu(\alpha) \equiv \sum_{k=0}^{\infty} \frac{\cos(2k+1)\alpha}{(2k+1)^\nu};$$

$$(1.40) \quad S_{2n+1}(\alpha) = \frac{(-1)^n}{4(2n)!} \pi^{2n+1} E_{2n} \left( \frac{\alpha}{\pi} \right);$$

$$(1.41) \quad C_{2n}(\alpha) = \frac{(-1)^n}{4(2n-1)!} \pi^{2n} E_{2n-1} \left( \frac{\alpha}{\pi} \right);$$

$$(1.42) \quad S_\nu(2\pi p/q) = \frac{1}{q^\nu} \sum_{s=1}^{q-1} \zeta(\nu, s/q) \left[ \sin(s2\pi p/q) - \frac{\sin(s4\pi p/q)}{2^\nu} \right];$$

$$(1.43) \quad C_\nu(2\pi p/q) = \frac{1}{q^\nu} \left\{ \zeta(\nu) \left( 1 - \frac{1}{2^\nu} \right) + \sum_{s=1}^{q-1} \zeta(\nu, s/q) \left[ \cos(s2\pi p/q) - \frac{\cos(s4\pi p/q)}{2^\nu} \right] \right\}.$$

[91, 1.448.1]

$$(1.44) \quad \sum_{k=1}^{\infty} \frac{p^k \sin(kx)}{k} = \arctan \frac{p \sin x}{1 - p \cos x}, \quad 0 < x < 2\pi, p^2 \leq 1,$$

with special case

$$(1.45) \quad \sum_{k=1}^{\infty} \frac{\sin(kx)}{2^k k} = \arctan \frac{\sin x}{2 - \cos x}, \quad 0 < x < 2\pi.$$

[91, 1.448.2]

$$(1.46) \quad \sum_{k=1}^{\infty} \frac{p^k \cos(kx)}{k} = \ln \frac{1}{\sqrt{1 - 2p \cos x + p^2}}, \quad 0 < x < 2\pi,$$

with special case

$$(1.47) \quad \sum_{k=1}^{\infty} \frac{\cos(kx)}{2^k k} = -\ln \sqrt{1 - \cos x + 1/4}, \quad 0 < x < 2\pi.$$

[99, (1.10)]

$$(1.48) \quad -\sqrt{2 - 2 \cos \theta} = -\frac{4}{\pi} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\cos k\theta}{(k - 1/2)(k + 1/2)}$$

[184]

$$(1.49) \quad \text{Cl}_{2n+1}(\pi/2) = -2^{-2n-1}(1 - 2^{-2n})\zeta(2n + 1), n \in \mathbb{N}.$$

[184]

$$(1.50) \quad \text{Cl}_{2n+1}(\pi/3) = \frac{1}{2}(1 - 2^{-2n})(1 - 3^{-2n})\zeta(2n + 1), n \in \mathbb{N}.$$

[184]

$$(1.51) \quad \text{Cl}_{2n+1}(2\pi/3) = -\frac{1}{2}(1 - 3^{-2n})\zeta(2n + 1), n \in \mathbb{N}.$$

[184]

$$(1.52) \quad \sum_{k \geq 1} \frac{\cos(k\pi/2)}{k^s} = -2^{-s}(1 - 2^{1-s})\zeta(s), \Re s > 1.$$

[119]

$$(1.53) \quad \sum_{m=1}^{\infty} \frac{m^{k-2}(2x)^{2m}}{\binom{2m}{m}} = \frac{x}{2^{k-2}(1-x^2)^{k-1/2}} \left[ \arcsin x V_x(x^2) + x\sqrt{1-x^2} W_k(x^2) \right], \quad k \geq 0,$$

[119]

$$(1.54) \quad \sum_{m=1}^{\infty} \frac{m^{k-2} 4^m (\sin \theta)^{2m}}{\binom{2m}{m}} = \frac{\sin 2\theta}{(2 \cos^2 \theta)^k} [2\theta V_k(\sin^2 \theta) + \sin 2\theta W_k(\sin^2 \theta)],$$

[119]

$$(1.55) \quad \sum_{m=1}^{\infty} \frac{(-1)^{m-1} m^{k-2} 4^m (\sinh z)^{2m}}{\binom{2m}{m}} \\ = \frac{\sinh 2z}{(2 \cosh^2 z)^k} [2 \log\{\sinh z + \cosh z\} V_k(-\sinh^2 z) + \sinh 2z W_k(-\sinh^2 z)],$$

where  $V_1(t) = 1$ , (see [175, A156919])  $W_1(t) = 0$ ,

$$V_{k+1}(t) = \{(2k-2)t + 1\}V_k(t) + 2(1-t)\delta V_k(t)$$

$$W_{k+1}(t) = \{(2k-4)t + 2\}W_k(t) + 2(1-t)\delta W_k(t) + V_k(t)$$

and  $\delta$  is the operator  $x \frac{d}{dx}$ .

#### 1.4. Expansions of Hyperbolic Functions. [30]

$$(1.56) \quad \cos x \cosh x = \sum_{k \geq 0} \frac{(-)^k (2x^2)^{2k}}{(4k)!}.$$

[30]

$$(1.57) \quad \sin x \sinh x = \sum_{k \geq 0} \frac{(-)^k (2x^2)^{2k+1}}{(4k+2)!}.$$

[30]

$$(1.58) \quad \coth(2x) \tanh x = 1 - \sum_{k \geq 1} \frac{2^{2k-1} (2^{2k} - 1) (2k - 1) B_{2k} x^{2k-2}}{(2k)!}.$$

[33]

$$(1.59) \quad \sum_{j=0}^{\infty} \frac{x}{\sinh 2^{-j} x} - 2^j = 1 - \frac{x}{\tanh x}.$$

[33]

$$(1.60) \quad \sum_{j=0}^{\infty} \frac{2^j - \coth 2^{-j}}{2^j \sinh 2^{-j}} = \frac{1 + 4e^2 - e^4}{1 - 2e^2 + e^4}.$$

[86]

$$(1.61) \quad \sum_{m,n=-\infty}^{\infty} \frac{F(|2m+2n+1|)}{\cosh(2m+1)u \cosh 2nu} = 2 \sum_{n=0}^{\infty} \frac{(2n+1)F(2n+1)}{\sinh(2n+1)u},$$

for any summable function  $F(x)$ .

[86]

$$(1.62) \quad \sum_{m,n=-\infty}^{\infty} \frac{F(m+n+1) + F(m-n)}{\sinh(2m+1)u \cosh(2n+1)u} = 8 \sum_{n=1}^{\infty} \frac{nF(n)}{\cosh(2nu)},$$

and

$$(1.63) \quad \sum_{k,m,n=-\infty}^{\infty} \frac{F(k+m+n+1) + F(k-m+n)}{\cosh(2ku) \cosh(2m+1)u \sinh(2n+1)u} = 8 \sum_{n=1}^{\infty} \frac{n^2 F(n)}{\sinh(2nu)},$$

where  $F$  is any sine transform (and hence odd).

#### 1.5. The Logarithmic Function.

$$(1.64) \quad \ln(1-x) = - \sum_{k=1}^{\infty} \frac{x^k}{k}.$$

(1.65)

$$(1+x) \ln(1+x) = x + \frac{1}{1 \cdot 2} x^2 - \frac{1}{2 \cdot 3} x^3 + \frac{1}{3 \cdot 4} x^4 - \frac{1}{4 \cdot 5} x^5 + \dots = x + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{k+1}}{k(k+1)}.$$

By integration of this w.r.t.  $x$  [91, 2.729], [135]

$$(1.66) \quad \frac{1}{2} (1+x)^2 \ln(1+x) = \frac{x}{2} + \frac{3x^2}{4} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{k+2}}{k(k+1)(k+2)}.$$



$$(1.67) \quad \frac{1}{6}(1+x)^3 \ln(1+x) = \frac{x}{6} + \frac{5x^2}{12} + \frac{11x^3}{36} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{k+3}}{k(k+1)(k+2)(k+3)}.$$

$$(1.68) \quad \frac{1}{24}(1+x)^4 \ln(1+x) = \frac{x}{24} + \frac{7x^2}{48} + \frac{13x^3}{72} + \frac{25x^4}{288} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{k+4}}{k(k+1)(k+2)(k+3)(k+4)}.$$

$$(1.69) \quad x + (1-x)^l \ln(1-x) = \sum_{i=2}^l \tau_{i,l} x^i - (-1)^l l! \sum_{k=1}^{\infty} \frac{x^{k+l}}{k(k+1)(k+2) \cdots (k+l)},$$

where

$$(1.70) \quad \tau_{2,l} = l - 1/2, \quad l \geq 2; \quad i\tau_{i,l} = (-1)^i \binom{l-1}{i-1} - l\tau_{i-1,l-1}, \quad i \geq 3.$$

[116]

$$(1.71) \quad \log n = - \sum_{s \geq 1} \frac{\alpha(s, n)}{s}$$

with

$$(1.72) \quad \alpha(s, n) \equiv \begin{cases} n-1, & n \mid s; \\ -1 & n \nmid s. \end{cases}$$

Superposition of [91, 1.513.1] and [91, 1.511], see [175, A165998]:

$$(1.73) \quad \frac{1}{3x} \ln \frac{1+x}{(1-x)^2} = 1 + \frac{x}{6} + \frac{x^2}{3} + \frac{x^3}{12} + \cdots + \frac{x^{2j}}{2j+1} + \frac{x^{2j+1}}{3(2j+1)} + \cdots$$

[11]

$$(1.74) \quad \sum_{j=k}^{\infty} \frac{|S_j^{(k)}| b^j}{j!} = \frac{\ln^k(1+b)}{k!}.$$

[101]

$$(1.75) \quad \sum_{k=0}^{\infty} \frac{x^k}{\binom{k+L}{L}} = {}_2F_1(1, 1; L+1; x) = L \sum_{j=0}^{L-2} \frac{(x-1)^j}{(L-j-1)x^{j+1}} - L(x-1)^{L-1} \frac{\ln(1-x)}{x^L}; \quad L \geq 1.$$

[43]

$$(1.76) \quad \frac{1}{2x} \left\{ 1 - \ln(1+x) - \frac{1-x}{\sqrt{x}} \arctan \sqrt{x} \right\} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{k-1}}{(2k-1)2k(2k+1)};$$

$0 < x \leq 1$ .

[138]

$$(1.77) \quad - \frac{\ln(1-t)}{(1-t)^r} = \sum_{n=0}^{\infty} H_n^{(r)} t^n,$$

where the Hyperharmonic numbers are defined by  $H_n = \sum_{k=1}^n (1/k)$  and  $H_n^{(r)} = \sum_{k=1}^n H_k^{(r-1)} = \binom{n+r-1}{r-1} (H_{n+r-1} - H_{r-1})$ .

### 1.6. The Inverse Trigonometric and Hyperbolic Function. [33]

$$(1.78) \quad \sum_{k=1}^{\infty} 2^{-k} \tan \frac{x}{2^k} = \frac{1}{2^n} \cot \frac{x}{2^n} - \cot x.$$

[33]

$$(1.79) \quad \sum_{k=1}^{\infty} \csc \frac{x}{2^{k-1}} = \cot \frac{x}{2^n} - \cot x.$$

[33]

$$(1.80) \quad \sum_{k=1}^{\infty} \arctan \frac{2x^2}{k^2} = \frac{\pi}{4} - \arctan \frac{\tanh \pi x}{\tan \pi x}.$$

[33]

$$(1.81) \quad \sum_{k=1}^{\infty} (-1)^{k-1} \arctan \frac{2x^2}{k^2} = -\frac{\pi}{4} - \arctan \frac{\sinh \pi x}{\sin \pi x}.$$

## 2. INDEFINITE INTEGRALS OF ELEMENTARY FUNCTIONS

### 2.1. Rational Functions. Aids to partial fraction decompositions: [205]

$$(2.1) \quad \frac{1}{s^n(s^2 + as + b)} = \frac{-\alpha_{n-1}s + \alpha_n}{b^n(s^2 + as + b)} + \sum_{k=0}^{n-1} \frac{\alpha_k}{b^{k+1}s^{n-k}};$$

where  $b > 0$ ,  $a^2 - 4b < 0$ ,  $\alpha_0 \equiv 1$ ,

$$(2.2) \quad \alpha_m \equiv (-1)^m \sqrt{b^m} U_m \left( \frac{a}{2\sqrt{b}} \right) = \sum_{k=0}^{\lfloor m/2 \rfloor} (-1)^{k+m} \binom{m-k}{k} a^{m-2k} b^k.$$

[205]

$$(2.3) \quad \frac{\alpha s + \beta}{(s^2 + as + b)(s^2 + kas + kb)} = \frac{1}{(k-1)b^2} \left( \frac{L(s+a) + \beta b}{s^2 + as + b} - \frac{L(s+ka) + \beta b}{s^2 + kas + kb} \right),$$

$$k \neq 1, ab \neq 0, L \equiv \begin{vmatrix} \alpha & \beta \\ a & c \end{vmatrix}.$$

[205]

$$(2.4) \quad \frac{1}{(pq + ap + b)(pq + cp + b)} = \begin{vmatrix} a & c \\ b & b \end{vmatrix}^{-1} \left( \frac{q+a}{pq + ap + b} - \frac{q+c}{pq + cp + b} \right),$$

$a \neq c, b \neq 0$ .

[205]

$$(2.5) \quad \frac{\alpha p + \beta}{(pq + ap + b)(pq + cp + b)} = \begin{vmatrix} a & c \\ b & b \end{vmatrix}^{-1} \left( \frac{\begin{vmatrix} \alpha & \beta \\ q+c & b \end{vmatrix}}{pq + cp + b} - \frac{\begin{vmatrix} \alpha & \beta \\ q+a & b \end{vmatrix}}{pq + ap + b} \right),$$

$a \neq c, b \neq 0$ .

[205]  
(2.6)

$$\frac{as + b}{(s^2 + \alpha s + \beta)(s^2 + \gamma s + \delta)} = \frac{1}{k} \left( \frac{\left(b - a \frac{\delta - \beta}{\gamma - \alpha}\right) s + b\alpha - a\beta - b \frac{\delta - \beta}{\gamma - \alpha}}{s^2 + \alpha s + \beta} - \frac{\left(b - a \frac{\delta - \beta}{\gamma - \alpha}\right) s + b\gamma - a\delta - b \frac{\delta - \beta}{\gamma - \alpha}}{s^2 + \gamma s + \delta} \right),$$

$\alpha \neq \gamma.$

[132]  
(2.7)

$$\frac{1}{n^{2s}(n+1)^{2s}} = \sum_{t=1}^{2s} \binom{4s-t-1}{2s-1} \left[ \frac{(-1)^t}{n^t} + \frac{1}{(n+1)^t} \right].$$

[65, 170.]

(2.8)

$$\int \frac{dx}{a^4 + x^4} = \frac{1}{4a^3\sqrt{2}} \log \frac{x^2 + ax\sqrt{2} + a^2}{x^2 - ax\sqrt{2} + a^2} + \frac{1}{2a^3\sqrt{2}} \arctan \frac{ax\sqrt{2}}{a^2 - x^2}.$$

[65, 171.]

(2.9)

$$\int \frac{dx}{a^4 - x^4} = \frac{1}{4a^3} \log \left| \frac{a+x}{a-x} \right| + \frac{1}{2a^3} \arctan \frac{x}{a}.$$

[65, 170.1]

(2.10)

$$\int \frac{xdx}{a^4 + x^4} = \frac{1}{2a^2} \arctan \frac{x^2}{a^2}.$$

[65, 170.2]

(2.11)

$$\int \frac{x^2 dx}{a^4 + x^4} = -\frac{1}{4a\sqrt{2}} \log \frac{x^2 + ax\sqrt{2} + a^2}{x^2 - ax\sqrt{2} + a^2} + \frac{1}{2a\sqrt{2}} \arctan \frac{ax\sqrt{2}}{a^2 - x^2}.$$

[65, 170.3]

(2.12)

$$\int \frac{x^3 dx}{a^4 + x^4} = \frac{1}{4} \log(a^4 + x^4).$$

[65, 171.1]

(2.13)

$$\int \frac{xdx}{a^4 - x^4} = \frac{1}{4a^2} \log \left| \frac{a^2 + x^2}{a^2 - x^2} \right|.$$

[65, 171.2]

(2.14)

$$\int \frac{x^2 dx}{a^4 - x^4} = \frac{1}{4a} \log \left| \frac{a+x}{a-x} \right| - \frac{1}{2a} \arctan \frac{x}{a}.$$

[65, 171.3]

(2.15)

$$\int \frac{x^3 dx}{a^4 - x^4} = -\frac{1}{4} \log |a^4 - x^4|.$$

[65, 173]

(2.16)

$$\int \frac{dx}{x(a + bx^m)} = \frac{1}{am} \log \left| \frac{x^m}{a + bx^m} \right|.$$

## 2.2. Algebraic Functions. [65, 186.11,188.11]

$$(2.17) \quad \int \frac{dx}{(a^2 + b^2x)x^{1/2}} = \frac{2}{ab} \arctan \frac{bx^{1/2}}{a}.$$

$$(2.18) \quad \int \frac{dx}{(a^2 - b^2x)x^{1/2}} = \frac{1}{2ab} \log \left| \frac{a + bx^{1/2}}{a - bx^{1/2}} \right|.$$

[65, 185.11,187.11]

$$(2.19) \quad \int \frac{x^{1/2}dx}{a^2 + b^2x} = \frac{2x^{1/2}}{b^2} - \frac{2a}{b^3} \arctan \frac{bx^{1/2}}{a}.$$

$$(2.20) \quad \int \frac{x^{1/2}dx}{a^2 - b^2x} = -\frac{2x^{1/2}}{b^2} + \frac{a}{b^3} \log \left| \frac{a + bx^{1/2}}{a - bx^{1/2}} \right|.$$

[65, 185.13,187.13]

$$(2.21) \quad \int \frac{x^{3/2}dx}{a^2 + b^2x} = \frac{2}{3} \frac{x^{3/2}}{b^2} - \frac{2a^2x^{1/2}}{b^4} + \frac{2a^3}{b^5} \arctan \frac{bx^{1/2}}{a}.$$

$$(2.22) \quad \int \frac{x^{3/2}dx}{a^2 - b^2x} = -\frac{2}{3} \frac{x^{3/2}}{b^2} - \frac{2a^2x^{1/2}}{b^4} + \frac{a^3}{b^5} \log \left| \frac{a + bx^{1/2}}{a - bx^{1/2}} \right|.$$

[65, 186.21,188.21]

$$(2.23) \quad \int \frac{dx}{(a^2 + b^2x)^2x^{1/2}} = \frac{x^{1/2}}{a^2(a^2 + b^2x)} + \frac{1}{a^3b} \arctan \frac{bx^{1/2}}{a}.$$

$$(2.24) \quad \int \frac{dx}{(a^2 - b^2x)^2x^{1/2}} = \frac{x^{1/2}}{a^2(a^2 - b^2x)} + \frac{1}{2a^3b} \log \left| \frac{a + bx^{1/2}}{a - bx^{1/2}} \right|.$$

[65, 185.21,187.21]

$$(2.25) \quad \int \frac{x^{1/2}dx}{(a^2 + b^2x)^2} = -\frac{x^{1/2}}{b^2(a^2 + b^2x)} + \frac{1}{ab^3} \arctan \frac{bx^{1/2}}{a}.$$

$$(2.26) \quad \int \frac{x^{1/2}dx}{(a^2 - b^2x)^2} = \frac{x^{1/2}}{b^2(a^2 - b^2x)} - \frac{1}{2ab^3} \log \left| \frac{a + bx^{1/2}}{a - bx^{1/2}} \right|.$$

[65, 185.23,187.23]

$$(2.27) \quad \int \frac{x^{3/2}dx}{(a^2 + b^2x)^2} = \frac{2x^{3/2}}{b^2(a^2 + b^2x)} + \frac{3a^2x^{1/2}}{b^4(a^2 + b^2x)} - \frac{3a}{b^5} \arctan \frac{bx^{1/2}}{a}.$$

$$(2.28) \quad \int \frac{x^{3/2}dx}{(a^2 - b^2x)^2} = \frac{3a^2x^{1/2} - 2b^2x^{3/2}}{b^4(a^2 - b^2x)} - \frac{3a}{2b^5} \log \left| \frac{a + bx^{1/2}}{a - bx^{1/2}} \right|.$$

[65, 186.13,188.13]

$$(2.29) \quad \int \frac{dx}{(a^2 + b^2x)x^{3/2}} = -\frac{2}{a^2x^{1/2}} - \frac{2b}{a^3} \arctan \frac{bx^{1/2}}{a}.$$

$$(2.30) \quad \int \frac{dx}{(a^2 - b^2x)x^{3/2}} = -\frac{2}{a^2x^{1/2}} + \frac{b}{a^3} \log \left| \frac{a + bx^{1/2}}{a - bx^{1/2}} \right|.$$

[65, 186.23,188.23]

$$(2.31) \quad \int \frac{dx}{(a^2 + b^2x)^2 x^{3/2}} = -\frac{2}{a^2(a^2 + b^2x)x^{1/2}} - \frac{3b^2x^{1/2}}{a^4(a^2 + b^2x)} - \frac{3b}{a^5} \arctan \frac{bx^{1/2}}{a}.$$

(2.32)

$$\int \frac{dx}{(a^2 - b^2x)^2 x^{3/2}} = -\frac{2}{a^2(a^2 - b^2x)x^{1/2}} + \frac{3b^2x^{1/2}}{a^4(a^2 - b^2x)} + \frac{3b}{2a^5} \log \left| \frac{a + bx^{1/2}}{a - bx^{1/2}} \right|.$$

[65, 189.2,189.4]

$$(2.33) \quad \int \frac{dx}{(a^4 + x^2)x^{1/2}} = \frac{1}{2a^3\sqrt{2}} \log \frac{x + a\sqrt{2x} + a^2}{x - a\sqrt{2x} + a^2} + \frac{1}{a^3\sqrt{2}} \arctan \frac{a\sqrt{2x}}{a^2 - x}.$$

$$(2.34) \quad \int \frac{dx}{(a^4 - x^2)x^{1/2}} = \frac{1}{2a^3} \log \left| \frac{a + x^{1/2}}{a - x^{1/2}} \right| + \frac{1}{a^3} \arctan \frac{x^{1/2}}{a}.$$

[65, 188.23,189.3]

$$(2.35) \quad \int \frac{x^{1/2}dx}{a^4 + x^2} = -\frac{1}{2a\sqrt{2}} \log \frac{x + a\sqrt{2x} + a^2}{x - a\sqrt{2x} + a^2} + \frac{1}{a\sqrt{2}} \arctan \frac{a\sqrt{2x}}{a^2 - x}.$$

$$(2.36) \quad \int \frac{x^{1/2}dx}{a^4 - x^2} = \frac{1}{2a} \log \left| \frac{a + x^{1/2}}{a - x^{1/2}} \right| - \frac{1}{a} \arctan \frac{x^{1/2}}{a}.$$

$$(2.37) \quad \int \frac{x^n}{\sqrt{a + bx}} dx = \frac{2\sqrt{a + bx}}{b^{n+1}} (-a)^n \sum_{k=0}^n \binom{n}{k} \frac{(-z/a)^k}{2k + 1}.$$

This corrects two sign errors in [91] at the  $b$ :

$$(2.38) \quad \int \frac{z^m dx}{t^n \sqrt{z}} = -z^m \sqrt{z} \left\{ \frac{1}{(n-1)\Delta} \frac{1}{t^{n-1}} + \sum_{k=2}^{n-1} \frac{(2n-2m-3)(2n-2m-5)\cdots(2n-2m-2k+1)(-b)^{k-1}}{2^{k-1}(n-1)(n-2)\cdots(n-k)\Delta^k} \frac{1}{t^{n-k}} \right\} - \frac{(2n-3m-3)(2n-3m-5)\cdots(-2m+3)(-2m+1)(-b)^{n-1}}{2^{n-1}(n-1)!\Delta^n} \int \frac{z^m dx}{t\sqrt{z}}$$

where  $z = a + bx$  and  $t = \alpha + \beta x$  and  $\Delta \equiv a\beta - \alpha b$ .

[191]

$$(2.39) \quad \int \frac{f_m(z)}{\sqrt{D_m(z)}} dz = \log(x_m(z) + y_m(z)\sqrt{D_m(z)}),$$

if for example

$$(2.40) \quad f = 4z+2; \quad D = z^4+8(z+1); \quad x = z^4-2z^3+2z^2+4z-4; \quad y = z^2-2z+2.$$

or

$$(2.41) \quad f = 5z + 1; \quad D = (z^2 + 1)^2 + 4z; \quad x = z^5 - z^4 + 3z^3 + z^2 + 2; \quad y = z^3 - z^2 + 2z.$$

or

$$(2.42) \quad f = 6z+2; \quad D = (z^2+2)^2+8z; \quad x = z^6-2z^5+8z^4-4z^3+8z^2+8z; \quad y = z^4-2z^3+6z^2-4z+4.$$

or

(2.43)

$$f = 3z - s; \quad D = (z^2 - s^2)^2 + t(z - s); \quad x = 1 + 2(z + s)(z^2 - s^2)/t; \quad y = 2(z + s)/t.$$

### 2.3. The Exponential Function.

$$(2.44) \quad \int x^{m+2} e^{-ax^2} dx = \frac{m+1}{2a} \left[ -x^{m+1} e^{-ax^2} + \int x^m e^{-ax^2} dx \right];$$

$m \neq -1$ .

### 2.4. Hyperbolic Functions.

### 2.5. Trigonometric Functions.

$$\begin{aligned} \int \sin x \frac{(a + \sin x)^2}{(a + \sin x)^2 + (b + \cos x)^2} dx &= \frac{1}{4} \left[ \frac{2b^2}{a^2 + b^2} + \frac{a^2 - b^2}{(a^2 + b^2)^2} - 3 \right] \cos(x) \\ &\quad - \frac{ab}{2} \left[ \frac{1}{a^2 + b^2} - \frac{1}{(a^2 + b^2)^2} \right] \sin(x) \\ + \frac{1}{8} \frac{b \cos(2x) - a \sin(2x)}{a^2 + b^2} + \frac{b}{4} \left[ 1 + \frac{2(a^2 - b^2)}{(a^2 + b^2)^2} + \frac{-3a^2 + b^2}{(a^2 + b^2)^3} \right] \ln \sqrt{(\cos x + b)^2 + (\sin x + a)^2} \\ + \frac{a}{4} \left[ 1 - \frac{4b^2}{(a^2 + b^2)^2} - \frac{a^2 - 3b^2}{(a^2 + b^2)^3} \right] \arctan \frac{\sin x + a}{\cos x + b} + \frac{a}{4} \left[ \frac{4b^2}{(a^2 + b^2)^2} + \frac{-3b^2 + a^2}{(a^2 + b^2)^3} \right] x. \end{aligned}$$

### 2.6. Rational Functions of Trigonometric Functions.

### 2.7. The Logarithm. [137]

$$(2.45) \quad \int \frac{\log z}{(1-z)z} dz = \text{Li}_2(1-z) + \frac{1}{2} \log^2 z.$$

[17, 3.1.6.]

$$(2.46) \quad \int \frac{\ln x}{1+ax} dx = \frac{1}{a} [\ln x \ln(1+ax) + \text{Li}_2(-ax)].$$

Correcting a sign error in [17, 3.1.7]:

$$(2.47) \quad \int \frac{\ln(a+bx)}{c+hx} dx = \frac{1}{h} \left[ \ln \left( \frac{ah-bc}{h} \right) \ln(c+hx) - \text{Li}_2 \left( \frac{bc+bhx}{bc-ah} \right) \right].$$

(2.48)

$$\begin{aligned} \int z \ln(c+z+1/z) dz &= \frac{z^2 - c^2/2 + 1}{2} \ln(z^2 + cz + 1) - \frac{c\sqrt{c^2/4 - 1}}{2} \ln \frac{z + c/2 + \sqrt{c^2/4 - 1}}{z + c/2 - \sqrt{c^2/4 - 1}} \\ &\quad - \frac{(z-c)^2}{4} - \frac{1}{2} z^2 \ln z. \end{aligned}$$

## 3. DEFINITE INTEGRALS OF ELEMENTARY FUNCTIONS I.

## 3.1. General formulae. [8]

$$(3.1) \quad \int_0^\infty f([ax - b/x]^2) dx = \frac{1}{a} \int_0^\infty f(y^2) dy, \quad a, b > 0.$$

[34] Let

$$\varphi(x) = \sum_{k=0}^{\infty} A_k x^k,$$

then

$$(3.2) \quad \int_0^\infty x^{\beta-1} e^{-x} \varphi(x) dx = \sum_{k=0}^{\infty} A_k \Gamma(k + \beta), \quad \beta > 0.$$

Let

$$\varphi(x) = \sum_{k=0}^{\infty} A_k x^{k/p},$$

then

$$(3.3) \quad \int_0^\infty x^{\beta-1} e^{-x} \varphi(x) dx = \sum_{k=0}^{\infty} A_k \Gamma(k/p + \beta), \quad \beta > 0.$$

Let

$$\varphi(x) = \sum_{k=0}^{\infty} A_k x^k,$$

then

$$(3.4) \quad p \int_0^\infty x^{\beta-1} e^{-x^p} \varphi(x) dx = \sum_{k=0}^{\infty} A_k \Gamma\left(\frac{\beta+k}{p}\right), \quad \beta > 0.$$

[117]

$$(3.5) \quad \int_a^u \frac{xdx}{\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} [(a-b)\Pi(\mu, 1, q) + bF(\mu, q)], \quad u > a > b > c.$$

[117]

$$(3.6) \quad \int_u^c \frac{dx}{(r-x)\sqrt{(a-x)(b-x)(c-x)}} = \frac{2(c-b)}{(r-b)(r-c)\sqrt{a-c}} \\ \times \Pi\left(\beta, \frac{r-b}{r-c}, p\right) + \frac{2}{(r-b)\sqrt{a-c}} F(\beta, p), \quad a > b > c > u, r \neq c.$$

[117]

$$(3.7) \quad \int_a^u \frac{dx}{(x-r)\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{(b-r)(a-r)\sqrt{a-c}} \\ \times \left[ (b-a)\Pi\left(\mu, \frac{b-r}{a-b}, q\right) + (a-r)F(\mu, q) \right], \quad u > a > b > c, r \neq a.$$

[117]

$$(3.8) \quad \int_u^b \sqrt{\frac{(x-c)(b-x)}{a-x}} dx = \frac{2}{3} \sqrt{a-c} [2(b-a)F(\delta, q) \\ + (2a-b-c)E(\delta, q)] + \frac{2}{3}(b+c-a-u) \sqrt{\frac{(b-u)(u-c)}{a-u}}, \quad a > b > u \geq c.$$

[117]

$$(3.9) \quad \int_a^u \sqrt{\frac{(x-b)(x-c)}{x-a}} dx = \frac{2}{3} \sqrt{a-c} [2(a-b)F(\mu, q) \\ + (b+c-2a)E(\mu, q)] + \frac{2}{3}(u+2a-2b-c) \sqrt{\frac{(u-a)(u-c)}{u-b}}, \quad u > a > b > c.$$

[117]

$$(3.10) \quad \int_a^u \sqrt{\frac{(x-a)(x-c)}{x-b}} dx = \frac{2}{3} \sqrt{a-c} [(a+c-2b)E(\mu, q) \\ - (a-b)F(\mu, q)] + \frac{2}{3}(u+b-a-c) \sqrt{\frac{(u-a)(u-c)}{u-b}}, \quad u > a > b > c.$$

[58]

$$(3.11) \quad \int_0^\infty \frac{u^\alpha du}{(au+1)^\beta (bu+1)^\gamma} = A_0 \log a + B_0 \log b + \sum_{r=1}^{\beta-1} \frac{A_r}{r} + \sum_{s=1}^{\gamma-1} \frac{B_s}{s};$$

where

(3.12)

$$A_r = (-1)^{\alpha+\beta+r+1} a^{\gamma-\alpha-1} \sum_{j=0}^{\beta-\gamma-1} \binom{\alpha}{j} \binom{\beta+\gamma-r-j-2}{\gamma-1} \frac{b^{\beta-r-j-1}}{(a-b)^{\beta+\gamma-r-j-1}};$$

$$(3.13) \quad B_s = (-1)^\beta b^{\beta-\alpha-1} \sum_{j=0}^{\gamma-s-1} \binom{\alpha}{j} \binom{\beta+\gamma-s-j-2}{\beta-1} \frac{a^{\gamma-s-j-1}}{(a-b)^{\beta+\gamma-s-j-1}}.$$

$$(3.14) \quad r = 0, 1, \dots, \beta-1; \quad s = 0, 1, \dots, \gamma-1.$$

### 3.2. Powers of $x$ , of binomials of the form $\alpha + \beta x^p$ , and of polynomials in

 $x$ . [64]

(3.15)

$${}_3F_2(-n, b/2, (b+1)/2; c/2, (c+1)/2; x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-xt^2)^n dt, \quad \Re c > \Re b > 0.$$

[32]

$$(3.16) \quad \int_0^\infty \left[ \frac{x^2}{x^4 + 2ax^2 + 1} \right]^r \cdot \frac{x^2 + 1}{x^b + 1} \frac{dx}{x^2} \\ = \int_0^\infty \left[ \frac{x^2}{x^4 + 2ax^2 + 1} \right]^r \frac{dx}{x^2} = \int_0^\infty \left[ \frac{x^2}{x^4 + 2ax^2 + 1} \right]^r dx \\ = \int_0^\infty \left[ \frac{x^2}{x^4 + 2ax^2 + 1} \right]^r \frac{x^2 + 1}{x^2} dx = 2^{-1/2-r} (1+a)^{1/2-r} B(r-1/2, 1/2),$$



with  $B$  Euler's beta function,  $a >= 1, r > 1/2$ , any  $b$ .

From this by specialization [32]

$$(3.17) \quad \int_0^\infty \frac{x^4}{(x^4 + x^2 + 1)^3} dx = \frac{\pi}{48\sqrt{3}}.$$

[32]

$$(3.18) \quad \int_0^\infty \frac{x^3}{(x^4 + 7x^2 + 1)^{5/2}} dx = \frac{2}{243}.$$

[32]

$$(3.19) \quad \int_0^\infty \frac{\sqrt{x}}{(x^4 + 14x^2 + 1)^{5/4}} dx = \frac{\Gamma^2(3/4)}{4\sqrt{2\pi}}.$$

[32]

$$(3.20) \quad \int_0^\infty \left[ \frac{x^2}{bx^4 + 2ax^2 + c} \right]^r dx = \frac{B(r - 1/2, 1/2)}{2^{r+1/2}\sqrt{b}[a + \sqrt{bc}]^{r-1/2}}$$

with  $b > 0, c \geq 0, a > -\sqrt{bc}$  and  $r > 1/2$ .

[32]

$$(3.21) \quad \int_0^\infty \left[ \frac{x^2}{x^4 - x^2 + 1} \right]^r \frac{dx}{x^2} = \frac{\sqrt{\pi}}{2} \frac{\Gamma(r - 1/2)}{\Gamma(r)}$$

[10, 143]

$$(3.22) \quad \int_0^\infty \frac{dx}{(x^4 + 2ax^2 + 1)^{m+1}} = \frac{\pi}{2} \frac{\sum_{l=0}^m d_l(m) a^l}{[2(a+1)]^{m+1/2}}$$

where

$$(3.23) \quad d_l(m) \equiv 2^{-2m} \sum_{k=l}^m 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l}.$$

[143]

$$(3.24) \quad \int_0^\infty \frac{dx}{bx^4 + 2ax^2 + 1} = \frac{\pi}{2\sqrt{2}} \frac{1}{\sqrt{a + \sqrt{b}}}.$$

[121]

$$(3.25) \quad \frac{(-)^n}{4^{n-1}} \int_0^1 \frac{x^{4n}(1-x)^{4n}}{1+x^2} dx = \pi - \sum_{k=0}^{n-1} (-)^k \frac{2^{4-2k}(4k)!(4k+3)!}{(8k+7)!} (820k^3 + 1533k^2 + 902k + 165).$$

[121]

$$(3.26) \quad \int_0^1 \frac{x^m(1-x)^n}{1+x^2} dx = \frac{\sqrt{\pi}\Gamma(m+1)\Gamma(n+1)}{2^{m+n+1}} {}_3F_2 \left( \begin{matrix} 1, (m+1)/2, (m+2)/2 \\ (m+n+2)/2, (m+n+3)/2 \end{matrix} \middle| -1 \right).$$

[19]

$$(3.27) \quad \int_0^1 \frac{x^m(1-x)^n}{1+x^2} dx = R_{m,n} + A_{m,n}\pi + B_{m,n} \ln \sqrt{2}$$

induced by the partial fraction decomposition

$$(3.28) \quad \frac{x^m(1-x)^n}{1+x^2} = Q_{m,n}(x) + \frac{A_{m,n}}{1+x^2} + \frac{B_{m,n}x}{1+x^2}.$$

[170][175, A093341]

$$(3.29) \quad \int_0^\infty \frac{dx}{\sqrt{1+x^4}} = \mathbf{K}\left(\frac{1}{\sqrt{2}}\right).$$

[170]

$$(3.30) \quad \int_0^\infty \frac{dx}{\sqrt{x+x^4}} = \mathbf{K}\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right).$$

[170]

$$(3.31) \quad \int_0^1 \frac{dx}{\sqrt[4]{1-x^2}} = \sqrt{3} \left[ 2\mathbf{E}\left(\frac{1}{\sqrt{2}}\right) - \mathbf{K}\left(\frac{1}{\sqrt{2}}\right) \right].$$

[170][175, A062539]

$$(3.32) \quad \int_0^1 \frac{dx}{(1-x^2)^{3/4}} = \sqrt{2}\mathbf{K}\left(\frac{1}{\sqrt{2}}\right).$$

[170]

$$(3.33) \quad \int_0^1 \frac{x^2}{(1-x^2)^{3/4}} dx = \frac{2^{3/2}}{3} \mathbf{K}\left(\frac{1}{\sqrt{2}}\right).$$

[170]

$$(3.34) \quad \int_0^1 \frac{1}{\sqrt{1-x^6}} dx = \frac{1}{\sqrt[4]{3}} \mathbf{K}\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right).$$

[170]

$$(3.35) \quad \int_0^\infty \frac{1}{\sqrt{1+x^6}} dx = \frac{2}{\sqrt[4]{27}} \mathbf{K}\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right).$$

[170]

$$(3.36) \quad \int_0^1 \frac{1}{\sqrt{1-x^8}} dx = \frac{1}{\sqrt{2}} \mathbf{K}(\sqrt{2}-1).$$

[170]

$$(3.37) \quad \int_0^1 \frac{x^2}{\sqrt{1-x^8}} dx = \left(1 - \frac{1}{\sqrt{2}}\right) \mathbf{K}(\sqrt{2}-1).$$

[170]

$$(3.38) \quad \int_0^1 \frac{x^{a-1}}{\sqrt{1-x^n}} dx = \cos(a\pi/n) \int_0^\infty \frac{z^{a-1}}{\sqrt{1+z^n}} dz, \quad 2a < n.$$

[170]

$$(3.39) \quad \int_0^\infty \frac{z^{n-a-1}}{\sqrt{1+z^n}} dz \cdot \int_0^1 \frac{x^{a-1}}{\sqrt{1-x^n}} dx = \frac{2\pi}{n(2a-n)\sin(\pi a/n)}, \quad n/2 < a < n.$$

[170]

$$(3.40) \quad \int_0^\infty \frac{1}{\sqrt{1+x^8}} dx = \int_0^\infty \frac{x^2}{\sqrt{1+x^8}} dx = \sqrt{2-\sqrt{2}} \mathbf{K}(\sqrt{2}-1).$$

[170]

$$(3.41) \quad \int_0^1 \frac{x^4}{\sqrt{1-x^8}} dx = \frac{\pi}{8} \frac{\sqrt{2}}{\mathbf{K}(\sqrt{2}-1)}.$$

[170]

$$(3.42) \quad \int_0^1 \frac{x^6}{\sqrt{1-x^8}} dx = \frac{\pi}{24} \frac{2+\sqrt{2}}{\mathbf{K}(\sqrt{2}-1)}.$$

[170]

$$(3.43) \quad \int_0^\infty \frac{1}{\sqrt[3]{1+x^6}} dx = \frac{\sqrt[3]{4}}{\sqrt[4]{3}} \mathbf{K}\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right).$$

[170]

$$(3.44) \quad \int_0^1 \frac{1}{\sqrt[3]{1-x^6}} dx = \frac{\sqrt[3]{4}}{\sqrt[4]{27}} \mathbf{K}\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right).$$

[170]

$$(3.45) \quad \int_1^\infty \frac{1}{\sqrt[m]{(x^n+a)(x^n+b)}} dx = \frac{m}{2n-m} F_1\left(\begin{matrix} \frac{2n-m}{m}; 1/m, 1/n \\ \frac{2n-m+mn}{mn} \end{matrix} \middle| -a, -b\right), \quad 2n-m > 0, \quad a, b > 0.$$

[170]

$$(3.46) \quad \int_0^1 \frac{1}{\sqrt[m]{(x^n+a)(x^n+b)}} dx = \frac{1}{\sqrt[m]{ab}} F_1\left(\begin{matrix} 1/n; 1/m, 1/m \\ 1+1/n \end{matrix} \middle| -1/a, -1/b\right), \quad a, b > 0.$$

**3.3. The Exponential function. [11]**

$$(3.47) \quad -\int_0^2 \frac{x^p - x^q}{1-x} dx = \psi(p+1) - \psi(q+1)$$

[110]

$$(3.48) \quad \int_u^\infty \exp\left(-\frac{x^2}{4\beta} - \gamma x\right) dx = \sqrt{\pi\beta} e^{\beta\gamma^2} \left[1 - \Phi\left(\gamma\sqrt{\beta} + \frac{u}{2\sqrt{\beta}}\right)\right], \quad \Re\beta > 0, u \geq 0.$$

[110]

$$(3.49) \quad \int_{-\infty}^\infty \exp(-p^2 x^2 \pm qx) dx = \exp\left(\frac{q^2}{4p^2}\right) \frac{\sqrt{\pi}}{|p|}.$$

[110]

$$(3.50) \quad \int_0^\infty \frac{x^n e^{-\mu x}}{x+\beta} dx = (-1)^{n-1} \beta^n e^{\beta\mu} \text{Ei}(-\beta\mu) + \sum_{k=1}^n (k-1)! (-\beta)^{n-k} \mu^{-k}, \quad |\arg \beta| < \pi, \Re\mu > 0, n \geq 0.$$

**3.4. Rational functions of powers and exponentials. [110]**

$$(3.51) \quad \int_0^\infty \frac{x^{\nu-1} e^{-\mu x}}{1-\beta e^{-x}} dx = \Gamma(\nu) \sum_{n=0}^\infty (\mu+n)^{-\nu} \beta^n.$$

[110]

$$(3.52) \quad \int_0^\infty \frac{(1+ix)^{2n-1} - (1-ix)^{2n-1}}{i} \frac{dx}{e^{\pi x} + 1} = \frac{1}{2n} [1 - 2^{2n} B_{2n}].$$

[110]

$$(3.53) \quad \int_0^\infty \frac{x^q e^{-px} dx}{(1-ae^{-px})^2} = \frac{\Gamma(q+1)}{ap^{q+1}} \sum_{k=1}^\infty \frac{a^k}{k^q}, \quad -1 \leq a < 1, q > -1, p > 0.$$

[110]

$$(3.54) \quad \int_0^\infty \frac{(1+a)e^x + a}{(1+e^x)^2} e^{-ax} x^n dx = -n! \sum_{k=1}^\infty \frac{(-1)^k}{(a+k)^n}, \quad a > -1, \quad n = 1, 2, \dots$$

[34]

$$(3.55) \quad p \int_0^\infty x^{\beta-1} e^{ax-x^p} dx = \sum_{k=0}^\infty \frac{a^k}{k!} \Gamma\left(\frac{k+\beta}{p}\right).$$

$$(3.56) \quad p^2 \int_0^\infty t^{p-1} e^{at-t^p} dt = \sum_{k=1}^\infty \frac{a^k}{(k-1)!} \Gamma(k/p).$$

$$(3.57) \quad \int_0^\infty x^{\beta-1} e^{-x-x^p} dx = \sum_{j=0}^\infty \frac{(-1)^j}{j!} \Gamma(\beta + jp).$$

[171][175, A002161]

$$(3.58) \quad \int_0^\infty (1 - e^{-1/x^2}) dx = \sqrt{\pi}.$$

[110][175, A155739]

$$(3.59) \quad \int_0^\infty \left\{ e^{-x^2} - \frac{1}{1+x^{2n}} \right\} \frac{dx}{x} = -\frac{1}{2}C \approx -0.288607.$$

[192]

$$(3.60) \quad \int_0^\infty \exp(nx - \beta \operatorname{sh} x) dx = \frac{1}{2} [S_n(\beta) - \pi \mathbf{E}_n(\beta) - \pi N_n(\beta)].$$

[192]

$$(3.61) \quad \int_{-\infty}^\infty \frac{\exp(\nu \operatorname{Arsh} x - iax)}{\sqrt{1+x^2}} dx = \begin{cases} 2 \exp\left(-\frac{i\nu\pi}{2}\right) K_\nu(a) & \text{if } a > 0, \\ 2 \exp\left(\frac{i\nu\pi}{2}\right) K_\nu(-a) & \text{if } a < 0. \end{cases}$$

[ $\Re\nu < 1$ ]

### 3.5. Hyperbolic Functions. [6, 42]

$$(3.62) \quad \frac{1}{2} \int_0^\infty \frac{x}{\cosh x} dx = G.$$

[14]

$$(3.63) \quad \int_0^\infty \frac{du}{(b^2 + u^2) \sinh(au)} = \frac{1}{2b} \left[ \psi\left(\frac{ab}{2\pi} + \frac{3}{4}\right) - \psi\left(\frac{ab}{2\pi} + \frac{1}{4}\right) \right]$$

for  $\Re a > 0$ ,  $\Re b > \max(-\Re a, -\Re 3a)$ .

[14]

$$(3.64) \quad \int_0^\infty \frac{udu}{(b^2 + u^2) \sinh(au)} = \frac{1}{2} \left[ \psi\left(\frac{ab}{2\pi} + \frac{1}{2}\right) - \psi\left(\frac{ab}{2\pi}\right) \right] - \frac{\pi^2}{4a^2b}$$

for  $\Re a > 0$ ,  $\Re b > 0$ .

[34]

$$(3.65) \quad \int_0^\infty x^{\beta-1} e^{-x} \frac{\sinh b\sqrt{x}}{b\sqrt{x}} dx = \sum_{j=0}^\infty \frac{\Gamma(j+\beta)}{(2j+1)!} b^{2j}.$$

$$(3.66) \quad \int_0^\infty t^{2\beta-2} e^{-t^2/b^2} \sinh t dt = \frac{1}{2} \sum_{j=0}^\infty \frac{\Gamma(j+\beta)}{(2j+1)!} b^{2(j+\beta)}.$$

$$(3.67) \quad \int_0^\infty x^{\beta-1} e^{-x} \sinh \sqrt{x} dx = \sum_{k=1}^\infty \frac{\Gamma(\beta+k)}{\Gamma(2k)}.$$

[42]

$$(3.68) \quad \int_0^{\pi/2} \sinh^{-1}(\sin x) dx = \int_0^{\pi/2} \sinh^{-1}(\cos x) dx = G.$$

[42]

$$(3.69) \quad \int_0^{\pi/2} \operatorname{csch}^{-1}(\csc x) dx = \int_0^{\pi/2} \operatorname{csch}^{-1}(\sec x) dx = G.$$

**3.6. Rational Functions of Sines and Cosines.** [110]

$$(3.70) \quad \int_0^{\pi/2} \frac{\sin 2nx \cos^{2m+1} x}{\sin x} dx = \frac{\pi}{2}, \quad n > m \geq 0.$$

[61, B2b]

$$(3.71) \quad \int_0^{2\pi} \sin^m \theta \cos^n \theta d\theta = 2\pi \epsilon_m \epsilon_n \frac{(m-1)!(n-1)!!}{(m+n)!!}$$

where  $\epsilon_j = 1$  if  $j$  is even and  $\epsilon_j = 0$  otherwise.

[26]

$$(3.72) \quad \int_0^z \sin^\mu t \sin^\nu(z-t) dt = \frac{\sqrt{\pi} \Gamma(\mu+1) \Gamma(\nu+1)}{2^{(\mu+\nu+1)/2} \Gamma(\mu/2 + \nu/2 + 1)} \sin^{(\mu+\nu+1)/2} z P_{(\mu-\nu-1)/2}^{-(\mu+\nu+1)/2}(\cos z), \quad \Re \mu > -1, \Re \nu > -1.$$

[26]

$$(3.73) \quad \frac{2^m \Gamma(m+1/2)}{\sqrt{\pi} \Gamma(m+n+1) \Gamma(m-n)} \int_0^z \sin^{m+n} t \sin^{m-n-1}(z-t) dt = \sin^m z P_n^{-m}(\cos z).$$

[26]

$$(3.74) \quad \int_0^z \left( \frac{\sin t}{\sin(z-t)} \right)^\mu dt = \frac{\pi \sin \mu z}{\sin \mu \pi}, \quad -1 < \Re \mu < 1.$$

[124]

$$(3.75) \quad \int_0^{\arcsin q} \left( \cos \phi \pm \sqrt{q^2 - \sin^2 \phi} \right)^{n+2} T_n(\cos \phi) d\phi = \dots$$

[6]

$$(3.76) \quad \int_0^{\pi/2} \sinh^{-1}(\sin x) dx = G.$$

[6]

$$(3.77) \quad \int_0^{\pi/2} \sinh^{-1}(\cos x) dx = G.$$

[6]

$$(3.78) \quad \int_0^{\pi/2} \operatorname{csch}^{-1}(\csc x) dx = G.$$

[6]

$$(3.79) \quad \int_0^{\pi/2} \operatorname{csch}^{-1}(\sec x) dx = G.$$

[122, p40]

$$(3.80) \quad \int_0^\infty \cos(T_n(t, -x)) dt = \frac{\pi\sqrt{x}}{2n \sin \frac{\pi}{2n}} [J_{1/n}(2x^{n/2}) - J_{-1/n}(2x^{n/2})],$$

(3.81)

$$\int_0^\infty \cos(T_{2m}(t, x)) dt = \frac{\pi\sqrt{x}}{4m \sin \frac{\pi}{4m}} [J_{-1/(2m)}(2x^m) - J_{1/(2m)}(2x^m)], \quad m = 1, 2, 3 \dots$$

(3.82)

$$\int_0^\infty \cos(T_{2m+1}(t, x)) dt = \frac{2\sqrt{x} \cos \frac{\pi}{4m+2}}{2m+1} K_{1/(2m+1)}(2x^{m+1/2}), \quad m = 1, 2, 3 \dots$$

where  $x$  real positive, where

$$(3.83) \quad T_n(t, x) \equiv t^n {}_2F_1\left(-\frac{n}{2}, \frac{1-n}{2}; 1-n; -\frac{4x}{t^2}\right), \quad n = 2, 3, 4, \dots$$

for example  $T_2 = t^2 + 2x$ ,  $T_3 = t^3 + 3tx$ .

### 3.7. Trigonometric and Rational Functions. [6, 42]

$$(3.84) \quad \frac{1}{2} \int_0^{\pi/2} \frac{x}{\sin x} = G.$$

[6]

$$(3.85) \quad \frac{3}{4} \int_0^{\pi/6} \frac{x}{\sin x} = G - \frac{\pi}{8} \log(1 + \sqrt{3}).$$

[6, 42]

$$(3.86) \quad -\frac{\pi^2}{4} \int_0^1 \left(x - \frac{1}{2}\right) \sec(\pi x) dx = G.$$

[11]

$$(3.87) \quad \frac{2^{s+2}}{\pi} \int_0^{\pi/2} x \cos^s x \sin(sx) dx - \gamma = \psi(s+1).$$

[110]

$$(3.88) \quad \int_0^{\pi/4} x^m \tan x dx = 2\pi G - \frac{7}{2} \zeta(3), \quad m = 2.$$

[57]

$$(3.89) \quad \int_0^{1/2} x^n \cot \pi x dx = \frac{n!}{2^n} \sum_{k=1, \text{ k odd}}^n \frac{(-)^{(k-1)/2}}{\pi^k} \frac{\eta(k)}{(n-k+1)!} + \frac{(-)^n + 1}{2} \frac{4n!(1-2^{-n-1})}{(2\pi)^{n+1}} \zeta(n+1),$$

where  $\eta(s) \equiv (1-2^{1-s})\zeta(s)$ .

[149, p 6]

$$(3.90) \quad \int_0^\infty \frac{\cos(xy)}{(a^2 + x^2)^{\nu+1/2}} dx = \sqrt{\pi} \left(\frac{y}{2a}\right)^\nu \frac{1}{\Gamma(\nu + 1/2)} K_\nu(ay).$$

[149, p 10]

$$(3.91) \quad \int_0^\infty \frac{x^\nu}{(a^2 + x^2)^{\mu+1}} \cos(xy) dx = \frac{a^{\nu-2\mu-1}}{2} B\left(\frac{1}{2} + \frac{1}{2}\nu, \mu - \frac{1}{2}\nu + \frac{1}{2}\right) {}_1F_2\left(\frac{\nu+1}{2}; \frac{\nu+1}{2} - \mu, \frac{1}{2}, \frac{a^2 y^2}{4}\right)$$

(3.92)

$$+ \sqrt{\pi} \frac{2^{-2\mu+\nu-2}}{\Gamma(1 + \mu - \frac{1}{2}\nu)} y^{2\mu-\nu+1} \Gamma\left(\frac{1}{2}\nu - \mu - \frac{1}{2}\right) {}_1F_2\left(\mu + 1 - \frac{\nu}{2}; \mu - \frac{\nu}{2} + \frac{3}{2}, \frac{a^2 y^2}{4}\right).$$

[110]

$$(3.93) \quad \int_0^\infty \frac{x^{2m} \cos(ax) dx}{(\beta^2 + x^2)^{n+1/2}} = \frac{(-1)^m \sqrt{\pi}}{2^n \beta^n \Gamma(n + 1/2)} \cdot \frac{d^{2m}}{da^{2m}} \{a^n K_n(a\beta)\}, \quad a > 0, \Re\beta > 0, 0 \leq m \leq n.$$

[149, p 116]

$$(3.94) \quad \int_0^\infty \frac{\sin(xy)}{x(a^2 + x^2)^{\nu+1/2}} dx = \frac{\pi y}{2a^{2\nu}} [K_\nu \mathbf{L}_{\nu-1}(ay) + \mathbf{L}_\nu(ay) K_{\nu-1}(ay)]$$

where  $\mathbf{L}$  are Struve functions.

[6]

$$(3.95) \quad \frac{1}{2\pi} \int_0^{\pi/2} \frac{x^2}{\sin x} = G - \frac{7}{4\pi} \zeta(3).$$

[42]

$$(3.96) \quad \int_0^{\pi/4} \frac{x^2}{\sin^2 x} dx = G - \frac{1}{16} \pi^2 + \frac{1}{4} \pi \log 2.$$

[6, 42]

$$(3.97) \quad \int_0^{\pi/2} \frac{x \csc x}{\cos x + \sin x} = G + \frac{\pi}{4} \log 2.$$

[6, 42]

$$(3.98) \quad -2 \int_0^{\pi/2} \frac{x \cos x}{\cos x + \sin x} = G - \frac{\pi^2}{8} - \frac{\pi}{4} \log 2.$$

[6, 42]

$$(3.99) \quad 2 \int_0^{\pi/2} \frac{x \sin x}{\cos x + \sin x} = G + \frac{\pi^2}{8} - \frac{\pi}{4} \log 2.$$

[42]

$$(3.100) \quad \frac{3}{4} \int_0^{\pi/6} \frac{x}{\sin x} dx = G - \frac{1}{8} \pi \log(2 + \sqrt{3}).$$

[9]

$$(3.101) \quad \int_0^\infty x^{-p} \cos^{2n+1}(x+b) dx = \frac{\Gamma(1-p)}{2^{2n}} \sum_{k=0}^n \binom{2n+1}{n-k} \frac{\sin[\pi p/2 - (2k+1)b]}{(2k+1)^{1-p}}.$$

[9]

$$(3.102) \quad \int_0^\infty x^{-p} \sin^{2n+1}(x+b) dx = \frac{\Gamma(1-p)}{2^{2n}} \sum_{k=0}^n \binom{2n+1}{n-k} \frac{\cos[\pi p/2 - (2k+1)b]}{(2k+1)^{1-p}}$$

[9]

$$(3.103) \quad \int_0^\infty x^{-p} \cos^{2n+1} x dx = \frac{\Gamma(1-p)}{2^{2n}} \sin\left(\frac{\pi p}{2}\right) \sum_{k=0}^n \frac{\binom{2n+1}{n-k}}{(2k+1)^{1-p}}$$

for  $0 < p < 1$ .

[9]

$$(3.104) \quad \int_0^\infty x^{-p} \sin^{2n+1} x dx = \frac{\Gamma(1-p)}{2^{2n}} \cos\left(\frac{\pi p}{2}\right) \sum_{k=0}^n (-1)^k \frac{\binom{2n+1}{n-k}}{(2k+1)^{1-p}}$$

for  $0 < p < 1$ .

[9]

$$(3.105) \quad \int_0^\infty \cos^{2n+1} x^p dx = \frac{1}{2^{2n}} \Gamma\left(\frac{p+1}{p}\right) \cos\left(\frac{\pi}{2p}\right) \sum_{k=0}^n \frac{\binom{2n+1}{n-k}}{(2k+1)^{1/p}}$$

for  $p > 1$ .

[9]

$$(3.106) \quad \int_0^\infty \sin^{2n+1} x^p dx = \frac{1}{2^{2n}} \Gamma\left(\frac{p+1}{p}\right) \sin\left(\frac{\pi}{2p}\right) \sum_{k=0}^n (-1)^k \frac{\binom{2n+1}{n-k}}{(2k+1)^{1/p}}$$

for  $p > 1$ .

[9]

$$(3.107) \quad \int_0^{\pi/2} x^p \cos^{2n} x dx = \sum_{j=0}^{\lfloor p/2 \rfloor} a_{n,p,p+1-2j} \pi^{p+1-2j} + \delta_{\text{odd},p} \cdot a_{n,p}^*,$$

where for  $p \geq 2$  and  $0 \leq j \leq \lfloor p/2 \rfloor$ 

$$(3.108) \quad a_{n,p,p+1-2j} = \frac{(-)^j \binom{2n}{n} p!}{2^{2n+p+1} (p+1-2j)!} \sum_{1 \leq k_1 \leq k_2 \leq \dots \leq k_j \leq n} \frac{1}{k_1^2 k_2^2 \dots k_j^2}.$$

and  $a_{n,p}^*$  is a similar multinomial sum. A similar form exists for odd powers of the cosine.

[110]

$$(3.109) \quad \int_0^\infty \sin^{2m+1} x \frac{xdx}{a^2 + x^2} = \frac{\pi}{2^{2m+1}} e^{-(2m+1)a} \sum_{k=0}^m (-1)^{m+k} \binom{2m+1}{k} e^{2ka}.$$

[110]

$$(3.110) \quad \int_0^\infty \cos^{2m} x \frac{xdx}{a^2 + x^2} = \frac{\pi}{2^{2m+1} a} \binom{2m}{m} + \frac{\pi}{2^{2m} a} \sum_{k=1}^m \binom{2m}{m+k} e^{-2ka}, \quad a > 0.$$

[69]

$$I(a, b) \equiv \int_0^\infty x^{-a} \left(1 - \frac{\sin^b x}{x^b}\right) dx.$$



then

$$(3.111) \quad I(a, b) = \frac{\pi \sec(\pi a/2)}{2^b \Gamma(a+b)} \sum_{k=0}^{\lfloor (b-1)/2 \rfloor} (-1)^{k+1} \binom{b}{k} (b-2k)^{a+b-1},$$

with special cases

$$(3.112) \quad I(2, b) = \frac{\pi}{2^b (b+1)!} \sum_{k=0}^{\lfloor (b-1)/2 \rfloor} (-1)^{k+1} \binom{b}{k} (b-2k)^{b+1},$$

$$(3.113) \quad I(a, 1) = -\frac{\pi \sec(\pi a/2)}{2\Gamma(1+a)} \int_0^\infty x^{-3/2} \left(1 - \frac{\sin x}{x}\right) dx = \frac{2\sqrt{2\pi}}{3},$$

$$(3.114) \quad I(a, 2) = -\frac{\pi 2^{a-1} \sec(\pi a/2)}{\Gamma(2+a)}, \int_0^\infty x^{-3/2} \left(1 - \frac{\sin^2 x}{x^2}\right) dx = \frac{16\sqrt{\pi}}{15},$$

$$(3.115) \quad I(a, 3) = \frac{(3 - 3^{2+a})\pi \sec(\pi a/2)}{8\Gamma(3+a)} \int_0^\infty x^{-3/2} \left(1 - \frac{\sin^3 x}{x^3}\right) dx = \frac{2}{35}(9\sqrt{3} - 1)\sqrt{2\pi}.$$

### 3.8. Trigonometric Functions and Exponentials. [120]

$$(3.116) \quad \int_0^\infty e^{-tx^2} \cos x^2 dx = \sqrt{\pi/8} \sqrt{\frac{\sqrt{1+t^2} + t}{1+t^2}}.$$

[120]

$$(3.117) \quad \int_0^\infty e^{-tx^2} \sin x^2 dx = \sqrt{\pi/8} \sqrt{\frac{\sqrt{1+t^2} - t}{1+t^2}}.$$

## 4. DEFINITE INTEGRALS OF ELEMENTARY FUNCTIONS II

[167] In terms of the constant (0.102) we have

$$(4.1) \quad \int_0^1 \frac{\log^2 u}{u} \log(1+u) du = \frac{7\pi^4}{360},$$

$$(4.2) \quad \int_0^1 \frac{\log^2 u}{u} \log(1-u) du = -\frac{\pi^4}{45},$$

$$(4.3) \quad \int_0^1 \frac{\log^2 u}{u} \log \frac{1+u}{1-u} du = \frac{\pi^4}{24},$$

$$(4.4) \quad \int_0^1 \frac{\log u}{u} \log^2(1+u) du = A_4 - \frac{\pi^4}{288},$$

$$(4.5) \quad \int_0^1 \frac{\log u}{u} \log^2(1-u) du = -\frac{\pi^4}{180},$$

$$(4.6) \quad \int_0^1 \frac{\log u}{u} \log^2 \frac{1+u}{1-u} du = 2A_4 - \frac{\pi^4}{60},$$

$$(4.7) \quad \int_0^1 \frac{1}{u} \log^3(1+u) du = \frac{3}{2}A_4 - \frac{\pi^4}{960},$$

$$(4.8) \quad \int_0^1 \frac{1}{u} \log^3(1-u) du = -\frac{\pi^4}{15},$$

[167]

$$(4.9) \quad \int_0^1 \frac{\log u}{u} \log(1+u) du = -\frac{3}{4}\zeta(3),$$

$$(4.10) \quad \int_0^1 \frac{\log u}{u} \log(1-u) du = \zeta(3),$$

$$(4.11) \quad \int_0^1 \frac{\log u}{u} \log \frac{1+u}{1-u} du = -\frac{7}{4}\zeta(3),$$

$$(4.12) \quad \int_0^1 \frac{1}{u} \log^2(1+u) du = \frac{1}{4}\zeta(3),$$

$$(4.13) \quad \int_0^1 \frac{1}{u} \log^2(1-u) du = 2\zeta(3).$$

[110]

$$(4.14) \quad 2 \int_0^{\pi/2} \ln|1 - \sin x| dx = -\pi \ln 2 - 4G.$$

[42]

$$(4.15) \quad -2 \int_0^{\pi/4} \log(2 \sin x) dx = G.$$

A factor 2 is missing in [6].

[42]

$$(4.16) \quad \frac{1}{4} \int_0^{\pi/2} \log \frac{1 + \cos x}{1 - \cos x} dx = G.$$

[42]

$$(4.17) \quad \frac{1}{4} \int_0^{\pi/2} \log \frac{1 + \sin x}{1 - \sin x} dx = G.$$

[110]

$$(4.18) \quad \int_0^{\pi/2} (\ln \tan x)^{2n} dx = (\pi/2)^{2n+1} |E_{2n}|.$$

[110]

$$(4.19) \quad \int_0^\infty \frac{\ln x dx}{(x+a)^2} = \frac{\ln a}{a}, \quad 0 < a.$$

[6, 42]

$$(4.20) \quad 2 \int_0^{\pi/4} \log(2 \cos x) dx = G.$$

[11, 144][175, A115252]

$$(4.21) \quad \int_{\pi/4}^{\pi/2} \ln \ln \tan x dx = \int_0^1 \ln \ln \left( \frac{1}{x} \right) \frac{dx}{1+x^2} = \frac{\pi}{2} \ln \left( \frac{\Gamma(3/4)\sqrt{2\pi}}{\Gamma(1/4)} \right)$$

[142]

$$(4.22) \quad \frac{1}{4} \int_0^1 \frac{x^4 - 6x^2 + 1}{(1+x^2)^3} \log \log(1/x) dx = -(\gamma + \log 4) [\zeta(-1, 1/4) - \zeta(-1, 3/4)] + \zeta'(-1, 1/4) - \zeta'(-1, 3/4).$$

[6]

$$(4.23) \quad - \int_0^1 \frac{\log(x)}{x^2 + 1} dx = G.$$

[56]

$$(4.24) \quad \int_0^1 \frac{\log t}{1 - (1-t)(1-u)} dt = \frac{1}{1-u} \operatorname{Li}_2 \left( -\frac{u-1}{u} \right)$$

[11]

$$(4.25) \quad \int_0^b \frac{\ln t}{(1+t)^{n+1}} dt = \frac{1}{n} [1 - (1+b)^{-n}] \ln b - \frac{1}{n} \ln(1+b) - \frac{1}{n(1+b)^{n-1}} \sum_{j=1}^{n-1} \frac{1}{j!} \binom{n-1}{j} |S_{j+1}^{(2)}| b^j$$

[4]

$$(4.26) \quad \frac{z^p}{(p-1)!} \int_0^1 \frac{t^{z-1}}{(1-t)^z} \log^{p-1} \frac{1}{t} dt = {}_{p+1}F_p \left( \begin{matrix} z, z, \dots, z \\ z+1, \dots, z+1 \end{matrix} \middle| 1 \right).$$

[4]

$$(4.27) \quad \frac{1}{\Gamma(1-z)\Gamma(p)} \int_0^1 \frac{t^{z-1}}{(1-t)^z} \log^{p-1}(t) dt = \left[ \begin{matrix} z \\ p \end{matrix} \right].$$

[11]

$$(4.28) \quad \int_0^x \frac{\ln t}{(1+t^2)^{n+1}} dt = \frac{\binom{2n}{n}}{2^{2n}} \left[ g_0(x) + p_n(x) \ln x - \sum_{k=0}^{n-1} \frac{\tan^{-1} x + p_k(x)}{2k+1} \right]$$

where

$$(4.29) \quad g_0(x) \equiv \ln x \tan^{-1} x - \int_0^x \frac{\tan^{-1} t}{t} dt.$$

[144]

$$(4.30) \quad \int_0^\infty \frac{\ln^{n-1} x dx}{(x-1)(x+a)} = \frac{(-)^n (n-1)!}{1+a} \{ [1 + (-)^n] \zeta(n) - \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} (2^{2j} - 2) (-)^j B_{2j} \pi^{2j} \log^{n-2j} a \}$$

for  $n \geq 2, a > 0$ .

[177]

$$(4.31) \quad -2t \int_0^1 \frac{(1-x)^{j+1} \log(1-x)}{(1-tx(1-x))^3} dx = \sum_{n \geq 1} \frac{t^n}{C_n(j)} \sum_{r=1}^n \frac{1}{r+j+n}$$

where  $C_n(j) = \binom{2n+j}{n}/(n+1)$  are Catalan related numbers. For  $t = 2$  the integral becomes a sum of  $G$ ,  $\zeta(2)$ ,  $\pi \ln 2$ ,  $\pi$  and 1 with rational coefficients, and similar results are given for  $t = 1/2$ .

[20]

$$(4.32) \quad \int_0^\infty \frac{\log(1+x)}{1+x+x^2} dx = - \int_0^1 \frac{(1+t)\log t}{1+t^3} dt = -\frac{3}{2}L_{-3}(2),$$

where  $L_{-3}(s)$  is a Dirichlet series [175, A086724].

[11]

$$(4.33) \quad \int_0^1 \frac{\ln t}{(1+t^2)^{n+1}} dt = -2^{-2n} \binom{2n}{n} \left( G + \sum_{k=0}^{n-1} \frac{\frac{\pi}{4} + p_k(1)}{2k+1} \right)$$

where

$$(4.34) \quad p_k(1) = \sum_{j=1}^k k \frac{2^j}{2^j \binom{2j}{j}}.$$

[6]

$$(4.35) \quad \int_0^1 \left( \frac{2}{x^2 - 4x + 8} - \frac{3}{x^2 + 2x + 2} \right) \log x dx = C.$$

[56]

$$(4.36) \quad (-1)^{p+1} n \int_0^1 (1-t)^{n-1} \log^p t dt = p! \sum_{k=1}^n \binom{n}{k} \frac{(-1)^k}{k^p}.$$

[56][175, A152648]

$$(4.37) \quad \int_0^1 \frac{\log^2 t}{1-t} dt = \sum_{n=1}^{\infty} \frac{H_n^{(1)}}{n^2} = 2\zeta(3) = 2\text{Li}_3(t) - 2\text{Li}_2(t) \log t - \log(1-t) \log^2 t + c$$

where  $H_n^{(r)} \equiv \sum_{k=1}^n \frac{1}{k^r}$ .

[56]

$$(4.38) \quad \int_0^1 \frac{\log^3 u}{1-u} du = -6\zeta(4).$$

[196]

$$(4.39) \quad \int_0^{\pi/2} \frac{\ln(2 \cos x)}{x^2 + \ln^2(2 \cos x)} dx = \pi/4.$$

[196]

$$(4.40) \quad \int_0^{\pi/2} \ln[x^2 + \ln^2(2 \cos x)] dx = 0.$$

[196]

$$(4.41) \quad \int_0^{\pi/2} \ln[x^2 + \ln^2(2e^{-a} \cos x)] dx = x \ln \frac{a}{e^b - 1},$$

and

$$(4.42) \quad \int_0^{\pi/2} \ln[x^2 + \ln^2(2e^{-a} \cos x)] \cos 2x dx = \frac{\pi}{2} \left( 1 - \frac{1}{a} - e^b + \frac{1}{e^b - 1} \right)$$

where  $b = \min(a, \ln 2)$ .

[196]

$$(4.43) \quad \int_0^{\pi/2} \ln[x^2 + \ln^2(\cos x)] dx = \frac{\pi}{2} \ln \ln 2.$$

[196]

$$(4.44) \quad \int_0^{\pi/2} \ln[x^2 + \ln^2(\cos x)] \cos 2x dx = -\frac{\pi}{\ln 2}.$$

[196]

$$(4.45) \quad \int_0^{\pi/2} \frac{\ln \cos x}{x^2 + \ln^2(\cos x)} dx = \frac{\pi}{2} \left(1 - \frac{1}{\ln 2}\right).$$

[196]

$$(4.46) \quad \int_0^{\pi/2} \frac{x \sin 2x}{x^2 + \ln^2(\cos x)} dx = \frac{\pi}{4 \ln^2 2}.$$

[196]

$$(4.47) \quad \int_0^{\pi/2} \frac{x \sin 2x}{x^2 + \ln^2(2 \cos x)} dx = \frac{13\pi}{48}.$$

[196]

$$(4.48) \quad \int_{-\pi/2}^{\pi/2} \frac{(1 + e^{-2ix})^\beta}{\ln(1 + e^{-2ix}) - a} dx = -\frac{\pi}{a} + \pi \frac{e^{(\beta+1)a}}{e^a - 1} H(\ln 2 - a),$$

and

$$(4.49) \quad \int_0^{\pi/2} \frac{x \sin x}{x^2 + \ln^2(2e^{-a} \cos x)} dx = \frac{\pi}{4a^2} + \frac{\pi e^a}{4} \left(1 - \frac{1}{(e^a - 1)^2}\right) H(\ln 2 - a),$$

where  $H$  is the unit step function.

[11, 196]

$$(4.50) \quad \frac{4}{\pi} \int_0^{\pi/2} \frac{x^2 dx}{x^2 + \ln^2(2 \cos x)} = \frac{1}{2}(1 + \ln(2\pi) - \gamma).$$

[11]

$$(4.51) \quad \frac{4}{\pi} \int_0^{\pi/2} \frac{x^2 dx}{x^2 + \ln^2(2e^{-a} \cos x)} = \frac{\gamma}{a} + \frac{a + \ln(1 - e^{-a}) - \gamma - \ln a}{1 - e^{-a}} + \frac{a}{1 - e^{-a}} \int_0^1 e^{-at} \ln \Gamma(t) dt$$

$$(4.52) \quad = \frac{\gamma}{a} + \frac{a + \ln(1 - e^{-a}) + \Gamma(0, a)}{1 - e^{-a}} + \frac{1}{1 - e^{-a}} \int_0^1 e^{-at} \psi(t + 1) dt$$

where  $0 < a < \ln 2$ .

[11]

$$(4.53) \quad \frac{4}{\pi} \int_0^{\pi/2} \frac{x^2 dx}{x^2 + \ln^2(2e^{-a} \cos x)} = \frac{\gamma}{a} + \int_0^\infty e^{-at} \psi(t + 1) dt$$

where  $a > \ln 2$ .

[11]

$$(4.54) \quad \int_0^{\pi/2} \frac{x^2 \ln(2 \cos x) dx}{(x^2 + \ln^2(2 \cos x))^2} = \frac{7\pi}{192} + \frac{\pi \ln 2\pi}{96} - \frac{\zeta'(2)}{16\pi}.$$

[196]

$$(4.55) \quad \frac{1}{2i} \int_{-\pi/2}^{\pi/2} \frac{x(1 + \exp(-2ix))^\beta}{\ln(1 + \exp(-2ix))} dx = \frac{\pi}{8} [1 + \ln 2\pi - \gamma(2\beta + 1) - 2 \ln \Gamma(\beta + 1)]$$

with  $\Re\beta > -1$ .

[196]

$$(4.56) \quad \int_{-\pi/2}^{\pi/2} \frac{(1 + \exp(-2ix))^\beta}{\ln(1 + \exp(-2ix))} dx = \frac{\pi}{2} (1 + 2\beta).$$

## 4.1. Logarithmic functions of compound arguments and powers. [6]

$$(4.57) \quad - \int_0^1 \frac{\log(\frac{1}{\sqrt{2}}(1-x))}{x^2+1} = G.$$

[6]

$$(4.58) \quad - \int_0^1 \frac{\log[\frac{1}{2}(1-x^2)]}{x^2+1} = G.$$

[32]

$$(4.59) \quad \int_0^\infty \left[ \frac{1}{x^4 - x^2 + 1} \right]^r \ln \frac{x^2}{x^4 - x^2 + 1} \frac{dx}{x^2} = \frac{\sqrt{\pi}}{2} \frac{\Gamma(r)\Gamma'(r-1/2) - \Gamma(r-1/2)\Gamma'(r)}{\Gamma^2(r)}$$

[32]

$$(4.60) \quad \int_0^\infty \frac{1}{x^4 - x^2 + 1} \ln^2 \frac{x^2}{x^4 - x^2 + 1} \frac{dx}{x^2} = \frac{\pi}{2} \left( \frac{\pi^2}{3} + 4 \ln^2 2 \right).$$

[32]

$$(4.61) \quad \int_0^\infty \left[ \frac{x}{x^2+1} \right]^{2r} \ln \frac{x}{x^2+1} \frac{dx}{x^2} = \frac{\sqrt{\pi}}{4} G(r) [\psi(r-1/2) - \psi(r) - 2 \ln 2],$$

where  $G \equiv 2^{1-2r} B(r-1/2, 1/2) / \sqrt{\pi}$ .

[32]

$$(4.62) \quad \int_0^\infty \left[ \frac{x}{x^2+1} \right]^{2r} \ln \frac{x}{x^2+1} \frac{x^2+1}{x^2(x^b+1)} dx = \frac{\sqrt{\pi}}{4} G'(r)$$

where  $G \equiv 2^{1-2r} B(r-1/2, 1/2) / \sqrt{\pi}$ .

[32]

$$(4.63) \quad \int_0^\infty \left[ \frac{x}{x^2+1} \right]^{2r} \ln^2 \frac{x}{x^2+1} \frac{dx}{x^2} = \frac{\sqrt{\pi}}{8} G''(r)$$

where  $G'' \equiv G(r) [\psi'(r-1/2) - \psi'(r) + (\psi(r-1/2) - \psi(r) - 2 \ln 2)^2]$ 

[11]

$$(4.64) \quad \int_0^\infty e^{-ax} \ln x dx = -\frac{\gamma + \ln a}{a}.$$

[11]  
 (4.65) 
$$\int_0^\infty \frac{\ln x}{e^x + e^{-x} - 1} dx = \int_0^1 \ln \ln \left( \frac{1}{x} \right) \frac{dx}{1-x+x^2} = \frac{2\pi}{\sqrt{3}} \left( \frac{5}{6} \ln 2\pi - \ln \Gamma \left( \frac{1}{6} \right) \right).$$

[112]  
 (4.66) 
$$\frac{a^\nu}{\Gamma(\nu)} \int_0^\infty e^{-ax} x^{\nu-1} \ln^m x dx = \phi^m(a, \nu) + \sum_{j=1}^m \binom{m}{j} \eta_j \phi^{m-j}(a, \nu),$$

where

(4.67) 
$$\phi(a, \nu) \equiv \psi(\nu) - \ln a,$$

(4.68) 
$$\eta_j(\nu) \equiv (-1)^j \sum_{\pi_0(j)} (j; 0, k_2, \dots, k_j)^* \zeta^{k_2}(2, \nu) \cdots \zeta^{k_j}(j, \nu),$$

(4.69) 
$$\zeta(k, \nu) \equiv \sum_{l=0}^\infty \frac{1}{(l+\nu)^k}, \quad k \geq 2.$$

[9]

$$\int_0^\infty \log x \cos^{2n+1} x^2 dx = -\frac{\sqrt{\pi}}{2^{2n+3}} (\pi + 2\gamma + 4 \log 2) \sum_{k=0}^n \binom{2n+1}{n-k} \frac{1}{\sqrt{4k+2}} - \frac{\sqrt{\pi}}{2^{2n+2}} \sum_{k=0}^n \binom{2n+1}{n-k} \frac{\log(2k+1)}{\sqrt{4k+2}}.$$

[11]

(4.70) 
$$\int_0^{\pi/2} x \ln(2 \cos x) dx = -\frac{7}{16} \zeta(3).$$

[11]

(4.71) 
$$\int_0^{\pi/2} x^2 \ln(2 \cos x) dx = -\frac{\pi}{4} \zeta(3).$$

[11]

(4.72) 
$$\int_0^{\pi/2} x^2 \ln^2(2 \cos x) dx = \frac{11\pi}{16} \zeta(4) = \frac{11\pi^5}{1440}.$$

[52]

(4.73) 
$$\int_0^{\pi/2} x^4 \ln^2(2 \cos x) dx = \frac{5\pi^7}{8064} + \frac{3\pi}{4} \zeta^2(3).$$

[52]

(4.74) 
$$\int_0^{\pi/2} x^2 \ln^4(2 \cos x) dx = \frac{33\pi^7}{4480} + \frac{3\pi}{2} \zeta^2(3).$$

[52]

(4.75)

$$\int_0^{\pi/2} x^3 \ln(\cos x) \sin[(p-1)x] \cos^{p-1} x dx = -\frac{\pi}{15} 2^{-(p+6)} [60\gamma^4 - 60\gamma^2\pi^2 + \pi^4 + 60\gamma(\pi^2 - 2\gamma^2) \ln 2 + 60\psi'''(p) + 60\{-(\pi^2 + 6\gamma(-\gamma + \ln 2))\psi^2(p) + (4\gamma - 2 \ln 2)\psi^3(p) + \psi^4(p) + 6\gamma \ln 2\psi'(p) - 3[\psi'(p)]^2 - 2(\gamma + \ln 2)\psi''(p) + (8\gamma - 4 \ln 2)\zeta(3) + \psi(p)(4\gamma^3 - 2\gamma\pi^2 + (\pi^2 - 6\gamma^2) \ln 2 + 6 \ln 2\psi'(p) - 2\psi''(p) + 8\zeta(3))\}].$$

[6, 42]

$$(4.76) \quad \int_0^{\pi/4} \log(\cot x) dx = -\int_0^{\pi/4} \log(\tan x) dx = G.$$

[75, 134]

$$(4.77) \quad \int_{\pi/4}^{\pi/2} \ln \ln \tan x dx = \frac{\pi}{2} \ln \frac{\Gamma(3/4)\sqrt{2\pi}}{\Gamma(1/4)}.$$

[134, 3]

$$(4.78) \quad \int_0^1 x^j \log \log \frac{1}{x} dx = -\frac{\gamma + \log(j+1)}{j+1}.$$

[3]

(4.79)

$$\int_0^1 \frac{x^{p-1}}{1+x^n} \log \log \frac{1}{x} dx = \frac{1}{2n} [\log(2n) + \gamma] \left( \psi\left(\frac{p}{2n}\right) - \psi\left(\frac{n+p}{2n}\right) \right) + \frac{1}{2n} \left( \zeta'\left(1, \frac{p}{2n}\right) - \zeta'\left(1, \frac{n+p}{2n}\right) \right)$$

for  $\Re p > 0$  and  $\Re n > 0$ , and a similar expression if  $(1+x^n)^2$  or  $(1+x^n)^3$  are in the denominator.

[3]

(4.80)

$$\int_0^1 \frac{x^{nr-1}}{1+x^n} \log \log \frac{1}{x} dx = \frac{1}{2n} [\log(2n) + \gamma] \left( \psi(r/2) - \psi\left(\frac{r+1}{2}\right) \right) + \frac{1}{2n} (\zeta'(1, r/2) - \zeta'(1, (r+1)/2)).$$

[3]

(4.81)

$$\int_0^1 x^{p-1} \frac{1-x}{1-x^n} \log \log \frac{1}{x} dx = \frac{1}{n} [\log(n) + \gamma] \left( \psi(p/n) - \psi\left(\frac{p+1}{n}\right) \right) + \frac{1}{n} (\zeta'(1, p/n) - \zeta'(1, (p+1)/n)).$$

[134] Let

$$(4.82) \quad R_{m,j}(a) \equiv \int_0^1 \frac{x^j \log \log 1/x}{(x+a)^{m+1}} dx,$$

and (Eulerian numbers)

$$(4.83) \quad A_{m,j} = \sum_{k=0}^j (-)^k \binom{m+1}{k} (j-k)^m$$

and

$$(4.84) \quad E_m \equiv \int_0^1 \frac{T_{m-1}(x) \log \log 1/x}{(x+1)^{m+1}} dx$$



defined via polynomials

$$(4.85) \quad T_m(x) \equiv \sum_{j=0}^m (-)^j A_{m+1,j+1} x^j,$$

then

$$(4.86) \quad E_m = (1 - 2^m)\zeta'(1 - m) - (-)^m[\gamma(2^m - 1) + 2^m \log 2] \frac{B_m}{m}$$

where  $B_m$  are the Bermoulli numbers. The  $R_{m,j}$  are then recursively

$$(4.87) \quad R_{0,0}(1) = -(\log^2 2)/2; \quad R_{m,0}(1) = \frac{E_m}{b_0(m)} - \sum_{k=1}^{m-1} \frac{b_k(m)}{b_0(m)} R_{m-k,0}(1),$$

$$(4.88) \quad R_{0,0}(a) = -\gamma \log(1 + 1/a) - \text{Li}'_1(-1/a); \quad \text{Li}'_c(x) = - \sum_{n>=1} \frac{\log n}{n^c} x^n,$$

and for  $m > 0$

$$(4.89) \quad R_{m,0}(a) = -\frac{\gamma}{a^m(1+a)m} - \frac{\gamma}{a^{m+1}m!} \sum_{j=2}^m \frac{S_1(m,j)T_{j-2}(1/a)}{(1+1/a)^j} - \frac{1}{a^m m!} \sum_{j=1}^m S_1(m,j) \text{Li}'_{1-j}(-1/a),$$

where

$$(4.90) \quad b_k(m) \equiv (-)^k \sum_{j=0}^{m-1} \binom{j}{k} A_{m,j+1},$$

and the unsigned Stirling numbers of the first kind are  $S_1(m,j)$  as in  $(t)_m = \sum_{j=1}^m S_j(m,j)t^j$ . For larger parameters  $m$  then

$$(4.91) \quad R_{m,0}(a) = \sum_{j=0}^r \alpha_{j,r}(a) R_{m-r+j,j}(a), \quad \alpha_{j,r}(a) \equiv (-)^j \binom{r}{j} a^{-r}.$$

[134] Let

$$(4.92) \quad D_{m,j}(r, \theta) \equiv \int_0^1 \frac{x^j \log \log 1/x}{(x^2 - 2rx \cos \theta + r^2)^{m+1}} dx,$$

then

$$(4.93) \quad D_{0,0}(1, \theta) = \frac{\pi}{2 \sin \theta} \left[ (1 - \theta/\pi) \log 2\pi + \log \frac{\Gamma(1 - \theta/2\pi)}{\Gamma(\theta/2\pi)} \right],$$

$$(4.94) \quad D_{0,0}(r, \theta) = -\frac{\gamma}{r \sin \theta} \tan^{-1} \frac{\sin \theta}{r - \cos \theta} + \frac{1}{2ri \sin \theta} (\text{Li}'_1(e^{i\theta}/r) - \text{Li}'_1(e^{-i\theta}/r)),$$

(4.95)

$$D_{0,1}(r, \theta) = -\frac{\gamma}{2} \log \frac{r^2 - 2r \cos \theta + 1}{r^2} - \gamma \cot \theta \tan^{-1} \frac{\sin \theta}{r - \cos \theta} + \frac{1}{2ri \sin \theta} [\Phi'(e^{i\theta}/r, 1, 1) - \Phi'(e^{-i\theta}/r, 1, 1)],$$

$$(4.96) \quad D_{m,j}(r\theta) = -\frac{1}{2rm \sin \theta} \frac{\partial}{\partial \theta} D_{m-1,j-1}(r, \theta), \quad m, j > 0.$$

[134]

$$(4.97) \quad \int_0^1 \frac{\log(1-x)}{x} \log \log 1/x dx = \int_0^\infty \log t \log(1 - e^{-t}) dt = \frac{\gamma \pi^2}{6} - \zeta'(2).$$

[134]

$$(4.98) \quad \int_0^1 \frac{\log(1+x)}{x} \log \log 1/x dx = \frac{\pi^2}{12} (\log 2 - \gamma) + \zeta'(2)/2,$$

and other examples involving the kernel  $\log \log 1/x$ .

[75]

$$(4.99) \quad \int_0^1 q^n \ln(\sin \pi q) dq = -\frac{\ln 2}{n+1} + n! \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{(-1)^k (\zeta(2k+1))}{(2\pi)^{2k} (n+1-2k)!}.$$

[80, 129] by differentiation of [91, 3.761.4] w.r.t. the parameter:

$$(4.100) \quad \int_0^\infty \frac{\sin x}{x^s} \ln x dx = \frac{\pi}{2} \frac{1}{[\Gamma(s) \sin \frac{\pi s}{2}]^2} \left[ \Gamma'(s) \sin \frac{s\pi}{2} + \frac{\pi}{2} \Gamma(s) \cos \frac{\pi s}{2} \right].$$

$$(4.101) \quad \int_0^\infty \frac{\sin x}{x^s} \ln^2 x dx = \pi \frac{1}{[\Gamma(s) \sin \frac{\pi s}{2}]^3} \left\{ \Gamma'(s) \sin \frac{s\pi}{2} + \frac{\pi}{2} \Gamma(s) \cos \frac{\pi s}{2} \right\} \\ - \frac{\pi}{2} \frac{1}{[\Gamma(s) \sin \frac{\pi s}{2}]^3} \left\{ \Gamma''(s) \sin \frac{s\pi}{2} + \pi \Gamma'(s) \cos \frac{\pi s}{2} - \frac{\pi^2}{4} \Gamma(s) \sin \frac{s\pi}{2} \right\}.$$

[108] Define

$$(4.102) \quad s_{n,p} \equiv \frac{(-)^{n+p-1}}{(n-1)!p!} \int_0^1 t^{-1} \log^{n-1} t \log^p(1-t) dt$$

then

$$(4.103) \quad s_{n,p} = s_{p,n} = \sum_{k=1}^p \frac{(-)^{k+1}}{k!} \sum_{m_i} \frac{H_p(m_1, \dots, m_k)}{m_1 \dots m_k} \zeta(m_1) \dots \zeta(m_k),$$

where

$$(4.104) \quad H_p(m_1, \dots, m_k) = \sum_{p_i} \binom{m_1}{p_1} \dots \binom{m_k}{p_k},$$

the sum over  $m_i$  over all sets of integers which satisfy  $m_i \geq 2$ ,  $\sum_{i=1}^k m_i = n+p$ , and the sum over  $p_i$  over all sets of integers which satisfy  $1 \leq p_i \leq m_i - 1$ ,  $\sum_{i=1}^k p_i = p$ . Examples with  $s_{n,p} = \sum_{k=1}^p (-)^{k+1} \alpha_k(n,p)/k!$  are  $\alpha_1(n,p) = (n+p-1)! \zeta(n+p)/(n!p!)$  or  $\alpha_2(n,2) = \sum_{\nu=2}^n \zeta(\nu) \zeta(n-\nu+2)$ . The reference provides an explicit table for  $n, p \leq 4$ .

[109]

$$(4.105) \quad r_{np} \equiv \int_0^{\pi/2} \log^n \cos x \log^p \sin x dx;$$

$$(4.106) \quad r_{10} = -\frac{\pi}{2} \log 2,$$

$$(4.107) \quad r_{11} = \frac{\pi}{2} \left( -\frac{\pi^2}{24} + \log^2 2 \right),$$

$$(4.108) \quad r_{20} = \frac{\pi}{2} \left( \frac{\pi^2}{12} + \log^2 2 \right),$$

$$(4.109) \quad r_{21} = \frac{\pi}{2} \left( -\log^3 2 + \frac{1}{4} \zeta(3) \right),$$

$$(4.110) \quad r_{22} = \frac{\pi}{2} \left( \frac{\pi^4}{160} + \log^4 2 - \zeta(3) \log 2 \right).$$

[56]

$$(4.111) \quad \int_0^1 \frac{\log u}{1-u} \operatorname{Li}_2 \left( \frac{u-1}{u} \right) du = \frac{17}{4} \zeta(4).$$

[56]

$$(4.112) \quad - \int_0^1 \frac{\operatorname{Li}_{q-1}(1-t) \log t}{1-t} dt = \sum_{n=1}^{\infty} \frac{H_n^{(1)}}{n^q},$$

where  $H_n^{(r)} \equiv \sum_{k=1}^n \frac{1}{k^r}$ .

[56]

$$(4.113) \quad \int \frac{\operatorname{Li}_2(1-t) \log t}{1-t} dt = \frac{1}{2} [\operatorname{Li}_2(1-t)]^2 + c.$$

[56]

$$(4.114) \quad \int_0^1 \frac{\operatorname{Li}_{2p}(1-t) \log t}{1-t} dt = \frac{1}{2} \sum_{j=2}^{2p} (-1)^j \zeta(j) \zeta(2p-j+2),$$

[42]

$$(4.115) \quad -\frac{1}{4} \int_0^1 \frac{\log x}{(x+1)\sqrt{x}} dx = \frac{1}{4} \int_1^{\infty} \frac{\log x}{(x+1)\sqrt{x}} dx = G.$$

[42]

$$(4.116) \quad \frac{1}{\sqrt{2}} \int_0^{\pi/2} \log \left( \frac{1 + \frac{1}{\sqrt{2}} \sin x}{1 - \frac{1}{\sqrt{2}} \sin x} \right) \frac{dx}{1 + \cos^2 x} = G.$$

[42]

$$(4.117) \quad \frac{1}{2} \int_0^{\pi/4} \log \left( \frac{1 + \sin x}{1 - \sin x} \right) \frac{dx}{\cos x \sqrt{\cos 2x}} = G.$$

[42]

$$(4.118) \quad \int_0^{\frac{\sqrt{2}+1}{\sqrt{2}-1}} \frac{(x+1) \log x}{4x\sqrt{6x-x^2-1}} dx = G.$$

[42]

$$(4.119) \quad - \int_0^1 \frac{\log x}{1+x^2} dx = \int_1^\infty \frac{\log x}{1+x^2} dx = G.$$

[42]

$$(4.120) \quad - \int_0^1 \log\left(\frac{1-x}{\sqrt{2}}\right) \frac{dx}{1+x^2} = G.$$

[42]

$$(4.121) \quad - \int_0^1 \log\left(\frac{1-x^2}{2}\right) \frac{dx}{1+x^2} = G.$$

[42]

$$(4.122) \quad \int_1^\infty \log\left(\frac{x+1}{\sqrt{2}}\right) \frac{dx}{1+x^2} = G.$$

[42]

$$(4.123) \quad \int_0^\infty \frac{\log(1+x)}{1+x^2} dx = G + \frac{1}{4}\pi \log 2.$$

[42]

$$(4.124) \quad - \int_0^1 \frac{\log(1+x^2)}{1+x^2} dx = G - \frac{1}{2}\pi \log 2.$$

[42]

$$(4.125) \quad - \int_1^{\sqrt{2}} \frac{2 \log x}{x\sqrt{x^2-1}} dx = G - \frac{1}{2}\pi \log 2.$$

[42]

$$(4.126) \quad \int_0^{\pi/2} \log(\cos x + \sin x) dx = G - \frac{1}{4}\pi \log 2.$$

[42]

$$(4.127) \quad - \frac{3}{2} \int_0^{2-\sqrt{3}} \frac{\log x}{1+x^2} dx = G.$$

[42]

$$(4.128) \quad \frac{3}{2} \int_{2+\sqrt{3}}^\infty \frac{\log x}{1+x^2} dx = G.$$

#### 4.2. Inverse Trigonometric Functions. [6, 42]

$$(4.129) \quad \frac{2}{\pi} \int_0^1 \frac{\tan^{-1} x}{x} dx = G - \frac{7}{4\pi} \zeta(3).$$

[42]

$$(4.130) \quad \frac{3}{2} \int_0^{2-\sqrt{3}} \frac{\tan^{-1} x}{x} dx = G - \frac{1}{8}\pi \log(2 + \sqrt{3}).$$

[42]

$$(4.131) \quad - \int_0^1 (\tan^{-1} x)^2 dx = G - \frac{1}{16}\pi^2 - \frac{1}{4}\pi \log 2.$$

[6, 42]

$$(4.132) \quad \int_0^1 \frac{\tan^{-1} x}{x} dx = G.$$

[42]

$$(4.133) \quad 2 \int_0^1 \left(\frac{1}{4}\pi - \tan^{-1} x\right) \frac{dx}{1-x^2} = G.$$

[42]

$$(4.134) \quad - \int_0^1 \frac{\sin^{-1} x}{\sqrt{1+x^2}} dx = G - \frac{1}{2}\pi \log(1+\sqrt{2}).$$

[88]

$$(4.135) \quad \int_0^b \tan^{-1} \frac{a}{\sqrt{1+x^2}} \cdot \frac{dx}{\sqrt{1+x^2}} = \frac{1}{2}[g(a,b) + g(b,a)]$$

where

$$(4.136) \quad g(a,b) = \tan^{-1} \frac{b}{a} \ln[4(a^2+1) - \frac{4a\sqrt{a^2+1}}{2a^2+1}] - 2\eta \ln(\sqrt{a^2+1} - a) \\ - \text{Cl}_2(2 \tan^{-1} \frac{b}{a}) + \frac{1}{2} \text{Cl}_2(4 \tan^{-1} \frac{b}{a} - 2\eta) + \frac{1}{2} \text{Cl}_2(2\eta)$$

and

$$(4.137) \quad \eta = \tan^{-1} \frac{ab}{(\sqrt{a^2+1} - a + 1)a^2 + (\sqrt{a^2+1} - a - 1)b^2}.$$

4.3. Multiple Integrals. [20] Let

$$(4.138) \quad C_n = \frac{4}{n!} \int_0^\infty \cdots \int_0^\infty \frac{1}{[\sum_{j=1}^n (u_j + 1/u_j)]^2} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n}$$

then

$$(4.139) \quad C_2 = 1; \quad C_3 = L_{-3}(s); \quad C_4 = 7\zeta(3)/12$$

see (4.32).

[20] Let

$$(4.140) \quad D_n = \frac{4}{n!} \int_0^\infty \cdots \int_0^\infty \frac{\prod_{i<j} (\frac{u_i - u_j}{u_i + u_j})^2}{[\sum_{j=1}^n (u_j + 1/u_j)]^2} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n} = \frac{1}{n!} \int d^n x \frac{\prod_{i<j} \tanh^2[(x_i - x_j)/2]}{(\cosh x_1 + \cdots + \cosh x_n)^2}$$

then

$$(4.141) \quad D_1 = 2; \quad D_2 = 1/3; \quad D_3 = 8 + 4\pi^2/3 - 27L_{-3}(2); \quad D_4 = 4\pi^2/9 - 1/6 - 7\zeta(3)/2,$$

see (4.32).

[42]

$$(4.142) \quad \frac{2}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \tan^{-1}(\sin x \sin y) \frac{dx dy}{\sin x} = G.$$

[42]

$$(4.143) \quad \frac{1}{2} \int_0^1 \int_0^{\pi/2} \frac{d\theta dx}{\sqrt{1-x^2 \sin^2 \theta}} = G.$$

[42]

$$(4.144) \quad \int_0^1 \int_0^{\pi/2} \sqrt{1-x^2 \sin^2 \theta} d\theta dx = G + \frac{1}{2}.$$

[42]

$$(4.145) \quad 8 \int_0^1 \int_0^1 \frac{\tan^{-1}(xy) dx dy}{1+x^2 y^2} = 2\pi G - \frac{7}{2} \zeta(3).$$

[42]

$$(4.146) \quad 4 \int_0^1 \int_0^1 \frac{\tan^{-1} x}{1+x^2 y^2} dx dy = 2\pi G - \frac{7}{2} \zeta(3).$$

[42]

$$(4.147) \quad - \int_0^1 \int_0^1 \frac{\log(1-x^2 y^2)}{xy \sqrt{(1-x^2)(1-y^2)}} dx dy = 2\pi G - \frac{7}{2} \zeta(3).$$

[30]

$$(4.148) \quad \int_0^{\pi/2} \int_0^{\pi/2} \frac{\tan(\phi/2) d\theta d\phi}{\sqrt{1-x \cos^2 \theta} \cos^2 \phi} = \frac{\pi}{4} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-(1-x) \sin^2 \phi}} + \frac{1}{4} \log x \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-x \sin^2 \phi}}.$$

[6, 42]

$$(4.149) \quad \frac{1}{4} \int_0^1 \int_0^1 \frac{1}{\sqrt{(1-x)(1-y)}} \frac{dx dy}{x+y} = G.$$

[96]

$$(4.150) \quad \int_0^1 \cdots \int_0^1 \prod_{1 \leq i < j \leq n} |t_i - t_j|^{2z} \prod_{j=1}^n t_j^{x-1} (1-t_j)^{y-1} dt_j = \prod_{j=1}^n \frac{\Gamma(x+(j-1)z) \Gamma(y+(j-1)z) \Gamma(jz+1)}{\Gamma(x+y+(n+j-2)z) \Gamma(z+1)},$$

where  $n$  is a positive integer,  $x, y, z$  are in  $\mathbb{C}$ , and  $\Re x, \Re y > 0, \Re z > -\max\{1/n, \Re x/(n-1), \Re y/(n-1)\}$ .

[56]

$$(4.151) \quad \int_0^1 \int_0^1 \frac{\text{Li}_{q-2}[(1-t)(1-u)] \log t \log u du dt}{(1-t)(1-u)} = \sum_{n=1}^{\infty} \frac{[H_n^{(1)}]^2}{n^q}$$

where  $H_n^{(r)} \equiv \sum_{k=1}^n \frac{1}{k^r}$ .

[56]

$$(4.152) \quad \int_0^1 \int_0^1 \frac{\log t \log u du dt}{1-(1-t)(1-u)} = \sum_{n=1}^{\infty} \frac{[H_n^{(1)}]^2}{n^2}$$

where  $H_n^{(r)} \equiv \sum_{k=1}^n \frac{1}{k^r}$ .

[56]

$$(4.153) \quad - \int_0^1 \int_0^1 \frac{\log[1 - (1-t)(1-u)] \log t \log u \, dudt}{(1-t)(1-u)} = \sum_{n=1}^{\infty} \frac{[H_n^{(1)}]^2}{n^3}$$

where  $H_n^{(r)} \equiv \sum_{k=1}^n \frac{1}{k^r}$ .

[56]

$$(4.154) \quad n^2 \int_0^1 (1-t)^{n-2} \log t \, dt \int_0^1 (1-u)^{n-1} \log u \, du = [H_n^{(1)}]^2,$$

where  $H_n^{(r)} \equiv \sum_{k=1}^n \frac{1}{k^r}$ .

[9]

$$(4.155) \quad \int_0^{\infty} \int_0^{\infty} \frac{\cos^{2n+1}(x+y)}{x^p y^q} dx ds = -\Gamma(1-p)\Gamma(1-q) \cos \frac{\pi(p+q)}{2} \sum_{k=0}^n \binom{2n+1}{n-k} \frac{(2k+1)^{p+1-2}}{2^{2n}}$$

[9]

$$(4.156) \quad \int_0^{\infty} \int_0^{\infty} \frac{\cos(x+y)}{x^p y^q} dx dy = -\Gamma(1-p)\Gamma(1-q) \cos \frac{\pi(p+q)}{2}.$$

[9]

$$(4.157) \quad \int_0^{\infty} \int_0^{\infty} \frac{\log x \log y}{\sqrt{xy}} \cos(x+y) dx dy = (\gamma + 2 \log 2)\pi^2.$$

[9]

$$(4.158) \quad \int_{R_+^n} (\cos ||x||^2) \cdot \prod_{j=1}^n \log x_j dV = \frac{(-)^{\Delta_n} \pi^{n/2}}{2^{2n}} \times \begin{cases} \Re \psi_n & n \text{ even} \\ \Im \psi_n & n \text{ odd} \end{cases}$$

where  $\Delta_n \equiv n(n+1)/2$  and  $\psi_n \equiv (\gamma + 2 \log 2 + \pi i/2)^n e^{\pi i n/4}$ .

[95]

$$(4.159) \quad \int_0^1 \int_0^1 \frac{x^{u-1} y^{v-1}}{1-xyz} (-\ln xy)^s dx dy = \Gamma(s+1) \frac{\Phi(z, s+1, v) - \Phi(z, s+1, u)}{u-v},$$

[95]

$$(4.160) \quad \int_0^1 \int_0^1 \frac{(xy)^{u-1}}{1-xyz} (-\ln xy)^s dx dy = \Gamma(s+1) \Phi(z, s+2, u),$$

where

$$(4.161) \quad \Phi(z, s, u) = \sum_{k=0}^{\infty} \frac{z^k}{(u+k)^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{e^{-(u-1)t}}{e^t - z} t^{s-1} dt$$

is the Lerch transcendent. [95]

$$(4.162) \quad \int_0^1 \int_0^1 \frac{1}{1+xyi} dx dy = G - \frac{\pi^2 i}{48},$$

[95]

$$(4.163) \quad \int_0^1 \int_0^1 \frac{-x \ln xy}{1+x^2 y^2} dx dy = G - \frac{\pi^2}{48},$$

[95]

$$(4.164) \quad \int_0^1 \int_0^1 \frac{-x \ln xy}{1-x^2y^2} dx dy = \frac{\pi^2}{12},$$

[95]

$$(4.165) \quad \int_0^1 \int_0^1 \frac{(-\ln xy)^n}{1-xyz} dx dy = \frac{(n+1)! \text{Li}_{n+2}(z)}{z},$$

[95]

$$(4.166) \quad \int_0^1 \int_0^1 \frac{-1}{(1-xyz) \ln xy} dx dy = -\frac{\ln(1-z)}{z},$$

[95]

$$(4.167) \quad \int_0^1 \int_0^1 \frac{-1}{(2-xyz) \ln xy} dx dy = \ln 2,$$

[95]

$$(4.168) \quad \int_0^1 \int_0^1 \frac{1}{2-xy} dx dy = \frac{\pi^2}{12} - \frac{\ln^2 2}{2}.$$

[95]

$$(4.169) \quad \int_0^1 \int_0^1 \frac{-\ln xy}{2-xy} dx dy = \frac{7\zeta(3)}{4} - \frac{\pi^2 \ln 2}{6} + \frac{\ln^3 2}{3}.$$

[95]

$$(4.170) \quad \int_0^1 \int_0^1 \frac{-1}{(\varphi-xy) \ln xy} dx dy = \ln \varphi,$$

[95]

$$(4.171) \quad \int_0^1 \int_0^1 \frac{1}{\varphi-xy} dx dy = \frac{\pi^2}{10} - \ln^2 \varphi,$$

[95]

$$(4.172) \quad \int_0^1 \int_0^1 \frac{-1}{(\varphi^2-xy) \ln xy} dx dy = \ln \varphi,$$

etc where

$$(4.173) \quad \varphi \equiv (1 + \sqrt{5})/2.$$

[95]

$$(4.174) \quad \int_0^1 \int_0^1 \frac{1-2xy}{(8+xy)(9-xy)} dx dy = \frac{1}{2} \ln^2 \frac{9}{8}.$$

[95]

$$(4.175) \quad \int_0^1 \int_0^1 \frac{52-7xy}{(2+xy)(9-xy)} dx dy = \frac{\pi^3}{3} + 3 \ln^2 2 + 2 \ln^2 3 - 6 \ln 2 \ln 3.$$

etc [95]

$$(4.176) \quad \int_0^1 \int_0^1 \frac{(-\ln xy)^s}{1-xy} dx dy = \Gamma(s+2) \zeta(s+2), \Re s > -1.$$



[95]  
 (4.177) 
$$\int_0^1 \int_0^1 \frac{(-\ln xy)^s}{1+xy} dx dy = \Gamma(s+2)\zeta^*(s+2), \Re s > -2, \quad \zeta^*(s) \equiv (1-2^{1-s})\zeta(s).$$

[95]  
 (4.178) 
$$\int_0^1 \int_0^1 \frac{(-\ln xy)^s}{1+x^2y^2} dx dy = \Gamma(s+2)\beta(s+2), \Re s > -2.$$

[95]  
 (4.179) 
$$\int_0^1 \int_0^1 \frac{1}{1-xy} dx dy = \zeta(2).$$

[95]  
 (4.180) 
$$\int_0^1 \int_0^1 \frac{-\ln xy}{1-xy} dx dy = 2\zeta(3).$$

[95]  
 (4.181) 
$$\int_0^1 \int_0^1 \frac{-1}{(1+x^2y^2)\ln xy} dx dy = \pi/4.$$

[95, 42]  
 (4.182) 
$$\int_0^1 \int_0^1 \frac{1}{1+x^2y^2} dx dy = G.$$

[95]  
 (4.183) 
$$\int_0^1 \int_0^1 \frac{\ln xy}{1+x^2y^2} dx dy = \frac{\pi^3}{16}.$$

[95]  
 (4.184) 
$$\int_0^1 \int_0^1 \frac{(-\ln xy)^s}{1+x^2y^2z^2} dx dy = \Gamma(s+1) \frac{\chi_{s+2}(z)}{z}, \quad \Re s > -2, \Re z \neq 0.$$

[95]  
 (4.185) 
$$\int_0^1 \int_0^1 \frac{1}{1-x^2y^2 \tan^2(\pi/8)} dx dy = \frac{\pi^2}{16 \tan \frac{\pi}{8}} - \frac{\ln^2 \tan \frac{\pi}{8}}{4 \tan \frac{\pi}{8}}.$$

[95]  
 (4.186) 
$$\int_0^1 \int_0^1 \frac{x^{u-1}y^{v-1}}{-\ln xy} dx dy = \frac{1}{u-v} \ln \frac{u}{v}.$$

[95]  
 (4.187) 
$$\int_0^1 \int_0^1 \frac{(xy)^{u-1}}{-\ln xy} dx dy = \frac{1}{u}.$$

[95]  
 (4.188) 
$$\int_0^1 \int_0^1 \frac{-x^{u-1}y^{v-1}}{(1+xy)\ln xy} dx dy = \frac{1}{u-v} \ln \frac{\Gamma(u/2)\Gamma(\frac{u+1}{2})}{\Gamma(v/2)\Gamma(\frac{v+1}{2})}, \quad u > 0, v > 0.$$

[95]

$$(4.189) \quad \int_0^1 \int_0^1 \frac{-(xy)^{u-1}}{(1+xy) \ln xy} dx dy = \frac{1}{2} \left[ \psi\left(\frac{u+1}{2}\right) - \psi\left(\frac{u}{2}\right) \right].$$

[95][175, A053510]

$$(4.190) \quad \int_0^1 \int_0^1 \frac{1+x}{-(1+xy) \ln xy} dx dy = \ln \pi.$$

[95][175, A094640]

$$(4.191) \quad \int_0^1 \int_0^1 \frac{1-x}{-(1+xy) \ln xy} dx dy = \ln \frac{4}{\pi}.$$

[95]

$$(4.192) \quad \int_0^1 \int_0^1 \frac{-x}{(1+x^2y^2) \ln xy} dx dy = \ln \frac{\sqrt{2\pi}}{\Gamma^2(3/4)}.$$

[95]

$$(4.193) \quad \int_0^1 \int_0^1 \frac{x^{u-1}y^{v-1}}{1-xy} dx dy = \frac{\psi(u) - \psi(v)}{u-v}; \quad \int_0^1 \int_0^1 \frac{(xy)^{u-1}}{1-xy} dx dy = \psi'(u).$$

[95][175, A073010]

$$(4.194) \quad \int_0^1 \int_0^1 \frac{y}{(1-x^3y^3)} dx dy = \frac{\pi}{3\sqrt{3}}.$$

[95]

$$(4.195) \quad \int_0^1 \int_0^1 x^{u-1}y^{v-1}(-\ln xy)^s dx dy = \Gamma(s+1) \frac{v^{-s-1} - u^{-s-1}}{u-v}$$

and others of similar shape.

[37]

$$(4.196) \quad W_n(2k) = \sum_{a_1+a_2+\dots+a_n=k} \binom{k}{a_1, \dots, a_n}^2,$$

where the sum is over all compositions (unordered partitions) with  $n$  terms, and

$$(4.197) \quad W_n(s) \equiv \int_{[0,1]^n} \left| \sum_{k=1}^n e^{2\pi i x_k} \right|^s d^n x.$$

[37]

$$(4.198) \quad W_3(k) = \Re {}_3F_2 \left( \begin{matrix} 1/2, -k/2, -k/2 \\ 1, 1 \end{matrix} \middle| 4 \right).$$

[37]

$$(4.199) \quad W_n(s) = n^s \sum_{m \geq 0} (-1)^m \binom{s/2}{m} \sum_{k=0}^m \frac{(-)^k}{n^{2k}} \binom{m}{k} \sum_{a_1+a_2+\dots+a_n=k} \binom{k}{a_1, \dots, a_n}^2.$$

## 5. INDEFINITE INTEGRALS OF SPECIAL FUNCTIONS

### 5.1. Elliptic Integrals and Functions.

5.2. The Exponential Integral.

5.3. The Sine Integral and Cosine Integral.

5.4. The Error Function and Fresnel Integrals.

5.5. Cylinder Functions.

$$(5.1) \quad \int x^2 Z_{\nu+1}(x) dx = -2x^2 Z_{\nu}(x) + 4 \int x Z_{\nu}(x) dx + \int x^2 Z_{\nu-1}(x) dx,$$

by partial integration of  $\int x Z_{\nu} dx$  with [91, 8.471.2], where  $Z$  is a Bessel Function.

$$(5.2) \quad \int x^{\mu+1} Z_{\nu-1}(x) dx = x^{\mu+1} Z_{\nu}(x) + (\nu - \mu - 1) \int x^{\mu} Z_{\nu}(x) dx,$$

$\mu \neq -1$ , by partial integration of  $\int x^{\mu} Z_{\nu} dx$  with [91, 8.472.1], where  $Z$  is a Bessel Function. Equivalent formula for spherical Bessel functions  $j_n(z) \equiv \sqrt{\pi/(2z)} J_{n+1/2}(z)$ :

$$(5.3) \quad (n - m) \int x^m j_n(x) dx = \int x^{m+1} j_{n-1}(x) dx - x^{m+1} j_n(x).$$

$$(5.4) \quad \int x^{\mu+1} Z_{\nu+1}(x) dx = -x^{\mu+1} Z_{\nu}(x) + (\mu + 1 + \nu) \int x^{\mu} Z_{\nu}(x) dx,$$

$\mu \neq -1$ , by partial integration of  $\int x^{\mu} Z_{\nu} dx$  with [91, 8.472.2], where  $Z$  is a Bessel Function.

[154]

$$(5.5) \quad \int \frac{\sin(x) Z_{\nu}(x)}{x^{3/2}} dx = \frac{2[(2\nu + 1) \sin(x) - 2x \cos(x)]}{x^{1/2}(2\nu - 1)(2\nu + 1)} Z_{\nu}(x) - \frac{4x^{1/2} \sin(x)}{(2\nu - 1)(2\nu + 1)} Z_{\nu+1}(x)$$

where  $Z$  is a Bessel function  $J$  or  $Y$ .

[154]

$$(5.6) \quad \int \frac{\cos(x) Z_{\nu}(x)}{x^{3/2}} dx = \frac{2[(2\nu + 1) \cos(x) + 2x \sin(x)]}{x^{1/2}(2\nu - 1)(2\nu + 1)} Z_{\nu}(x) - \frac{4x^{1/2} \cos(x)}{(2\nu - 1)(2\nu + 1)} Z_{\nu+1}(x)$$

where  $Z$  is a Bessel function  $J$  or  $Y$ .

[154]

$$(5.7) \quad \int \frac{Z_{\mu}(x) \bar{Z}_{\nu}(x)}{x^2} dx = -\frac{1 + \mu + \nu + 2\mu\nu + \mu\nu^2 + \mu^2\nu - \mu^2 - \mu^3 - \nu^2 - \nu^3 + 2x^2}{x(-1 + \mu - \nu)(-1 + \mu + \nu)(1 + \mu - \nu)(1 + \mu + \nu)} Z_{\mu} \bar{Z}_{\nu}$$

$$+ \frac{1}{(1 - \mu - \nu)(1 - \mu + \nu)} Z_{\mu} \bar{Z}_{\nu+1} + \frac{1}{(1 - \mu - \nu)(1 + \mu - \nu)} Z_{\mu+1} \bar{Z}_{\nu}$$

$$- \frac{2x}{(1 - \mu - \nu)(1 - \mu + \nu)(1 + \mu - \nu)(1 + \mu + \nu)} Z_{\mu+1} \bar{Z}_{\nu+1}.$$

[154]

(5.8)

$$\begin{aligned} \int \frac{Z_\mu(x)\bar{Z}_\nu(x)}{x^3} dx = & -\frac{(2+\mu+\nu)(4\mu+4\nu+\mu\nu^2+\mu^2\nu-\mu^3-\nu^3+4x^2)}{x^2(\mu+\nu)[\nu^2-(2-\mu)^2][\nu^2-(2+\mu)^2]} Z_\mu\bar{Z}_\nu \\ & -\frac{4\mu\nu^2+2\mu^2\nu^2-4\mu^2-4\nu^3-\mu^4+4\nu^2-\nu^4+8x^2}{x(\mu^2-\nu^2)[\nu^2-(2-\mu)^2][\nu^2-(2+\mu)^2]} Z_\mu\bar{Z}_{\nu+1} \\ & +\frac{4\mu^2\nu+2\mu^2\nu^2+4\mu^2-\mu^4-4\nu^2-4\nu^3-\nu^4+8x^2}{x(\mu^2-\nu^2)[\nu^2-(2-\mu)^2][\nu^2-(2+\mu)^2]} Z_{\mu+1}\bar{Z}_\nu \\ & -\frac{4}{[-\nu^2+(2-\mu)^2][-\nu^2+(2+\mu)^2]} Z_{\mu+1}\bar{Z}_{\nu+1}. \end{aligned}$$

[154]

(5.9)

$$\int \frac{Z_\nu^2(x)}{x^2} dx = \frac{1+2\nu+2x^2}{(4\nu^2-1)x} Z_\nu^2(x) - \frac{2}{-1+2\nu} Z_\nu(x)Z_{\nu+1}(x) - \frac{2x}{1-4\nu^2} Z_{\nu+1}^2(x).$$

[154]

(5.10)

$$\begin{aligned} \int \frac{Z_\nu^2(x)}{x^4} dx = & \frac{-9-6\nu+x^2(6+16\nu+8\nu^2)+36\nu^2+24\nu^3+16x^4}{3x^3(1-4\nu^2)(9-4\nu^2)} Z_\nu^2(x) \\ & -\frac{2(-3+4\nu+4\nu^2+8x^2)}{3x^2(1-2\nu)(9-4\nu^2)} Z_\nu(x)Z_{\nu+1}(x) - \frac{2(1-4\nu^2-8x^2)}{3x(1-4\nu^2)(9-4\nu^2)} Z_{\nu+1}^2(x). \end{aligned}$$

[154]

(5.11)

$$\int x^2 Z_{1/3}^3(x) dx = \left(-\frac{4}{9}x - \frac{16}{81x}\right) Z_{1/3}^3(x) - \frac{4x}{3} Z_{1/3}(x) Z_{4/3}^2(x) + \left(\frac{8}{9} + x^2\right) Z_{1/3}^2(x) Z_{4/3}(x) + \frac{2}{3} x^2 Z_{4/3}^3(x).$$

[154]

(5.12)

$$\int \frac{Z_1^4(x)}{x} dx = \frac{x^2}{4} Z_2^4(x) + \left(\frac{3}{4} + \frac{x^2}{4}\right) Z_1^4(x) - \frac{3x}{2} Z_1(x) Z_2^3(x) + 6\left(\frac{1}{2} + \frac{x^2}{12}\right) Z_1^2(x) Z_2^2(x) + 4\left(-\frac{3x}{8} - \frac{1}{2x}\right) Z_1^3(x) Z_2(x).$$

[154]

(5.13)

$$\begin{aligned} \int \frac{Z_3^4(x)}{x^3} dx = & \left(\frac{1}{24} + \frac{1}{2x^2} + \frac{2}{x^4} + \frac{x^2}{378}\right) Z_3^4(x) + \left(\frac{5}{216} + \frac{2}{27x^2} + \frac{x^2}{378}\right) Z_4^4(x) \\ & + 4\left(-\frac{x}{108} - \frac{5}{54x} - \frac{1}{3x^3}\right) Z_3(x) Z_4^3(x) + 6\left(\frac{7}{216} + \frac{1}{3x^2} + \frac{4}{3x^4} + \frac{x^2}{1134}\right) Z_3^2(x) Z_4^2(x) \\ & + 4\left(-\frac{x}{108} - \frac{1}{8x} - \frac{1}{x^3} - \frac{4}{x^5}\right) Z_3^3(x) Z_4(x). \end{aligned}$$

where  $Z$  and  $\bar{Z}$  are Bessel functions  $J$  or  $Y$ .

[154]

(5.14)

$$\begin{aligned} \int x^l Z_\mu(x)\bar{Z}_\nu(x) dx = & A_{00}(x)Z_\mu(x)\bar{Z}_\nu(x) + A_{01}(x)Z_\mu(x)\bar{Z}_{\nu+1}(x) + A_{10}(x)Z_{\mu+1}(x)\bar{Z}_\nu(x) \\ & + A_{11}(x)Z_{\mu+1}(x)\bar{Z}_{\nu+1}(x), \end{aligned}$$

where  $Z$  and  $\bar{Z}$  are Bessel functions  $J$  or  $Y$ , where

$$A_{00} = \frac{x}{2(\mu + \nu)} D^3 A_{11} + \frac{3 + \mu + \nu}{2(\mu + \nu)} D^2 A_{11} + \frac{-7 - 3\mu - 3\nu - 2\mu\nu + \mu^2 + \nu^2 - 4x^2}{2x(\mu + \nu)} DA_{11} \\ + \frac{(-2 - \mu - \nu)(-4 - 2\mu\nu + \mu^2 + \nu^2 - 2x^2)}{2x^2(\mu + \nu)} A_{11} + \frac{x^{l+1}}{\mu + \nu},$$

$$A_{01} = \frac{-x^2}{2(\mu^2 - \nu^2)} D^3 A_{11} + \frac{3x}{2(\mu^2 - \nu^2)} D^2 A_{11} - \frac{7 - 3\mu^2 - \nu^2 + 4x^2}{2(\mu^2 - \nu^2)} DA_{11} \\ + \frac{4 + \mu\nu^2 - 3\mu^2 - \mu^3 - \nu^2 + 2x^2}{x(\mu^2 - \nu^2)} A_{11} + \frac{x^{l+2}}{\mu^2 - \nu^2},$$

$$A_{10} = \frac{x^2}{2(\mu^2 - \nu^2)} D^3 A_{11} - \frac{3x}{2(\mu^2 - \nu^2)} D^2 A_{11} + \frac{7 - \mu^2 - 3\nu^2 + 4x^2}{2(\mu^2 - \nu^2)} DA_{11} \\ - \frac{4 + \mu^2\nu - \mu^2 - 3\nu^2 - \nu^3 + 2x^2}{x(\mu^2 - \nu^2)} A_{11} - \frac{x^{l+2}}{\mu^2 - \nu^2},$$

$$A_{11}(x) = x^{l+3} \sum_{n=0}^{n'} d_n x^{2n},$$

$$d_0 = \frac{2(l+1)}{(l+3)^4 - 8(l+3)^3 + 2(12 - \mu^2 - \nu^2)(l+3)^2 - 8(l+3)(4 - \mu^2 - \nu^2) + [(2 - \mu)^2 - \nu^2][(2 + \mu)^2 - \nu^2]},$$

$$\{(3 + 2n + l)^4 - 8(3 + 2n + l)^3 + 2(12 - \mu^2 - \nu^2)(3 + 2n + l)^2 - 8(3 + 2n + l)(4 - \mu^2 - \nu^2) + [(2 - \mu)^2 - \nu^2][(2 + \mu)^2 - \nu^2]\} d_n = -4(1 + 2n + l)(2n + l)d_{n-1}, n' \geq n > 0, \\ \text{and } d_n = 0 \text{ if } n > n'.$$

$$n' = \begin{cases} 0, & l = -1 \\ \frac{|l|}{2} - 1, & l < -1, \text{ even} \\ \frac{|l|-3}{2}, & l < -1, \text{ odd} \\ \infty, & l \geq 0 \end{cases}$$

[154]

$$(5.15) \quad \int x^l Z_\nu^2(x) dx = A_{00}(x) Z_\nu^2(x) + 2A_{01}(x) Z_\nu(x) Z_{\nu+1}(x) + A_{11}(x) Z_{\nu+1}^2(x)$$

where

$$(5.16) \quad A_{00}(x) = \frac{1}{2} D^2 A_{11}(x) - \frac{3 + 2\nu}{2x} DA_{11}(x) + \frac{(2 + 2\nu + x^2)}{x^2} A_{11}(x),$$

$$(5.17) \quad A_{01}(x) = \frac{1}{2} DA_{11}(x) - \frac{1 + \nu}{x} A_{11}(x),$$

$$(5.18) \quad A_{11}(x) = xy(x),$$

$$(5.19) \quad y = \sum_{n=0}^{(l-1)/2} b_n x^{2n+1}, \quad \frac{b_{l-1}}{2} = 1/2l,$$

$$(5.20) \quad \frac{b_{l-1}}{2} = 1/2l, \quad b_n = \frac{2(n+1)[\nu^2 - (n+1)^2]}{2n+1} b_{n+1},$$

if  $0 \leq n \leq (l-3)/2$ , and if  $l \geq 3$  a positive odd integer.

[198, 136,1]

(5.21)

$$\int z^{-\mu-\nu-1} \mathcal{C}_{\mu+1}(z) \mathcal{D}_{\nu+1}(z) dz = -\frac{z^{-\mu-\nu}}{2(\mu+\nu+1)} \{ \mathcal{C}_{\mu}(z) \mathcal{D}_{\nu}(z) + \mathcal{C}_{\mu+1}(z) \mathcal{D}_{\nu+1}(z) \}$$

where  $\mathcal{C}$  and  $\mathcal{D}$  are arbitrary Bessel functions.

[198, 136,2]

$$(5.22) \quad \int z^{\mu+\nu+1} \mathcal{C}_{\mu}(z) \mathcal{D}_{\nu}(z) dz = \frac{z^{\mu+\nu+2}}{2(\mu+\nu+1)} \{ \mathcal{C}_{\mu}(z) \mathcal{D}_{\nu}(z) + \mathcal{C}_{\mu+1}(z) \mathcal{D}_{\nu+1}(z) \}$$

where  $\mathcal{C}$  and  $\mathcal{D}$  are arbitrary Bessel functions.

[198]

$$(\rho + \mu + \nu) \int z^{\rho-1} \mathcal{C}_{\mu}(z) \mathcal{D}_{\nu}(z) dz + (\rho - \mu - \nu - 2) \int z^{\rho-1} \mathcal{C}_{\mu+1}(z) \mathcal{D}_{\nu+1}(z) dz = z^{\rho} \{ \mathcal{C}_{\mu}(z) \mathcal{D}_{\nu}(z) + \mathcal{C}_{\mu+1}(z) \mathcal{D}_{\nu+1}(z) \}$$

where  $\mathcal{C}$  and  $\mathcal{D}$  are arbitrary Bessel functions.

[198, 136,5]

$$(\mu + \nu) \int \mathcal{C}_{\mu}(z) \mathcal{D}_{\nu}(z) \frac{dz}{z} - (\mu + \nu + 2n) \int \mathcal{C}_{\mu+n}(z) \mathcal{D}_{\nu+n}(z) \frac{dz}{z} = \mathcal{C}_{\mu}(z) \mathcal{D}_{\nu}(z) + 2 \sum_{m=1}^{n-1} \mathcal{C}_{\mu+m}(z) \mathcal{D}_{\nu+m}(z) + \mathcal{C}_{\mu+n}(z) \mathcal{D}_{\nu+n}(z)$$

where  $\mathcal{C}$  and  $\mathcal{D}$  are arbitrary Bessel functions.

[198, 137,1]

(5.23)

$$\int \mathcal{C}_n(z) \mathcal{D}_n(z) \frac{dz}{z} = -\frac{1}{2n} \left[ \mathcal{C}_0(z) \mathcal{D}_0(z) + 2 \sum_{m=1}^{n-1} \mathcal{C}_m(z) \mathcal{D}_m(z) + \mathcal{C}_n(z) \mathcal{D}_n(z) \right]$$

where  $\mathcal{C}$  and  $\mathcal{D}$  are arbitrary Bessel functions.

[198, 138]

$$(5.24) \quad (\mu + 2) \int z^{\mu+2} \mathcal{C}_{\nu}^2(z) dz = (\mu + 1) \left\{ \nu^2 - \frac{1}{4}(\mu + 1)^2 \right\} \int z^{\mu} \mathcal{C}_{\nu}^2(z) dz + \frac{1}{2} \left[ z^{\mu+1} \left\{ z \mathcal{C}'_{\nu}(z) - \frac{1}{2}(\mu + 1) \mathcal{C}_{\nu}(z) \right\}^2 + z^{\mu+1} \left\{ z^2 - \nu^2 + \frac{1}{4}(\mu + 1)^2 \right\} \mathcal{C}_{\nu}^2(z) \right]$$

where  $\mathcal{C}$  and  $\mathcal{D}$  are arbitrary Bessel functions.

With [1, 10.121], then partial integration for a product of three spherical Bessel functions

$$(5.25) \quad (n + m + l + 2) \int j_n(x) j_m(x) j_l(kx) dx = -(2n - 1) \int j_{n-1}(x) j_m(x) j_l(kx) dx + (n - 3 - m - l) \int j_{n-2}(x) j_m(x) j_l(kx) dx + (2n - 1) \int j_{n-1}(x) j_{m-1}(x) j_l(kx) dx + (2n - 1) k \int j_{n-1}(x) j_m(x) j_{l-1}(kx) dx.$$

$$(5.26) \quad \int x J_1(x) dx = \frac{\pi}{2} x [J_1(x) \mathbf{H}_0(x) - J_0(x) \mathbf{H}_1(x)]$$

where  $\mathbf{H}$  are Struve functions [1, §12].

### 5.6. Orthogonal Polynomials. [154]

$$(5.27) \quad \int \ln(1 \pm x) P_\nu(x) dx = \left[ -\frac{x}{\nu} \ln(1 \pm x) \pm \frac{1}{\nu(\nu+1)} - \frac{x}{\nu^2} \right] P_\nu + \left[ \frac{1}{\nu} \ln(1 \pm x) + \frac{1}{\nu^2(\nu+1)} \right] P_{\nu+1}.$$

where  $P$  are Legendre functions.

[154]

$$(5.28) \quad \int P_\mu(x) \bar{P}_\nu(x) dx = -\frac{x}{1+\mu+\nu} P_\mu(x) \bar{P}_\nu(x) - \frac{1+\nu}{(\mu-\nu)(1+\mu+\nu)} P_\mu(x) \bar{P}_{\nu+1}(x) + \frac{1+\mu}{(\mu-\nu)(1+\mu+\nu)} P_{\mu+1}(x) \bar{P}_\nu(x).$$

[154]

$$(5.29) \quad \int x [P_\nu(x)]^2 dx = -\frac{1+\nu}{2\nu} \left[ \frac{x^2+\nu}{1+\nu} [P_\nu(x)]^2 - 2x P_\nu(x) P_{\nu+1}(x) + [P_{\nu+1}(x)]^2 \right].$$

[154]

$$(5.30) \quad \int [P_{1/2}(x)]^2 dx = \frac{9}{2} x [P_{3/2}(x)]^2 + \frac{x(-7+16x^2)}{2} [P_{1/2}(x)]^2 - 3(-1+2x)(1+2x) P_{1/2}(x) P_{3/2}(x).$$

[154]

$$(5.31) \quad \int [P_{3/2}(x)]^2 dx = \frac{x(93-480x^2+512x^4)}{18} x [P_{3/2}(x)]^2 + \frac{25x(-3+8x^2)}{18} [P_{5/2}(x)]^2 - \frac{5}{9} (3-42x^2+64x^4) P_{3/2}(x) P_{5/2}(x).$$

[154]

$$(5.32) \quad \int x^5 P_{1/3}(x) P_{2/3}(x) dx = \left( -\frac{335}{2352} - \frac{512x^2}{392} + \frac{235x^4}{336} + \frac{x^6}{3} \right) P_{1/3}(x) P_{2/3}(x) + \left( \frac{685x}{784} - \frac{573x^3}{1176} - \frac{5x^5}{48} \right) P_{1/3}(x) P_{5/3}(x) + \left( \frac{235x}{196} - \frac{295x^3}{588} - \frac{2x^5}{21} \right) P_{2/3}(x) P_{4/3}(x) + \left( -\frac{5}{6} + \frac{65x^2}{196} + \frac{25x^4}{588} \right) P_{4/3}(x) P_{5/3}(x).$$

[154]

$$(5.33) \quad \int x [P_{1/3}(x)]^3 dx = \left( \frac{125x^4}{12} - \frac{14}{3} x^2 - \frac{5}{12} \right) [P_{1/3}(x)]^3 + (-4+20x^2) P_{1/3}(x) [P_{4/3}(x)]^2 + (9x-25x^3) [P_{1/3}(x)]^2 P_{4/3}(x) - \frac{16}{3} x [P_{4/3}(x)]^3.$$

[154]

$$(5.34) \quad \int x[P_{1/2}(x)]^4 dx = \left(-\frac{5}{16} - \frac{19}{4}x^2\right)[P_{1/2}(x)]^4 + \frac{81}{4}xP_{1/2}(x)[P_{3/2}(x)]^3 \\ + 6\left(-\frac{9}{16} - \frac{9}{2}x^2\right)[P_{1/2}(x)]^2[P_{3/2}(x)]^2 + 4\left(\frac{33x}{16} + 3x^3\right)[P_{1/2}(x)]^3P_{3/2}(x) - \frac{81}{16}[P_{3/2}(x)]^4.$$

[154]

$$(5.35) \quad \int x^l P_\mu(x) P_\nu(x) dx = A_{00}(x)P_\mu(x)P_\nu(x) + A_{01}(x)P_\mu(x)P_{\nu+1}(x) \\ + A_{10}(x)P_{\mu+1}(x)P_\nu(x) + A_{11}(x)P_{\mu+1}(x)P_{\nu+1}(x),$$

[154]

$$(5.36) \quad \int e^{-x^2} H_\nu(x) dx = -e^{-x^2} H_{\nu-1}(x).$$

[154]

$$(5.37) \quad \int H_\nu(x)x^{-(\nu+3)} dx = \left[\frac{2x^{-\nu}}{(\nu+1)(\nu+2)} - \frac{x^{-(\nu+2)}}{\nu+2}\right] H_\nu(x) - \frac{2\nu}{(\nu+1)(\nu+2)} x^{-(\nu+1)} H_{\nu-1}(x).$$

[154]

$$(5.38) \quad \int x H_\nu(x) dx = \frac{1+2x^2}{2(\nu+2)} H_\nu(x) - \frac{\nu x}{\nu+2} H_{\nu-1}(x).$$

[154]

$$(5.39) \quad \int e^{-x^2} H_\mu(x) \bar{H}_\nu dx = \frac{e^{-x^2}}{2(\mu-\nu)} [-H_\mu(x) \bar{H}_{\nu+1}(x) + H_{\mu+1}(x) \bar{H}_\nu(x)].$$

[154]

(5.40)

$$\int x e^{-x^2} H_\mu(x) \bar{H}_\nu dx = \frac{e^{-x^2}}{2} \left[-\frac{1+\mu+\nu}{(1-\mu+\nu)(1+\mu-\nu)} H_\mu(x) \bar{H}_{\nu+1}(x) + \frac{x}{1+\mu-\nu} H_{\mu+1}(x) \bar{H}_\nu(x) \right. \\ \left. + \frac{x}{1-\mu+\nu} H_\mu(x) \bar{H}_{\nu+1}(x) - \frac{1}{(1-\mu+\nu)(1+\mu-\nu)} H_{\mu+1}(x) \bar{H}_{\nu+1}(x)\right].$$

[154]

$$(5.41) \quad \int x^2 e^{-x^2} H_\mu(x) \bar{H}_\nu dx = -e^{-x^2} \frac{H_\mu(x) \bar{H}_\nu(x) x(\mu+\nu)}{(2-\mu+\nu)(2+\mu-\nu)} \\ + H_{\mu+1}(x) \bar{H}_\nu(x) \frac{2+\mu+3\nu+2\mu x^2-2\nu x^2-\mu^2 x^2-\nu^2 x^2+2\mu\nu x^2}{2(\mu-\nu)(2-\mu+\nu)(2+\mu-\nu)} \\ + H_\mu(x) \bar{H}_{\nu+1}(x) \frac{2+3\mu+\nu-2\mu x^2+2\nu x^2-\mu^2 x^2-\nu^2 x^2+2\mu\nu x^2}{2(\mu-\nu)(2-\mu+\nu)(2+\mu-\nu)} \\ - H_{\mu+1}(x) \bar{H}_{\nu+1}(x) \frac{x}{(2-\mu+\nu)(2+\mu-\nu)}.$$



[154]

$$(5.42) \quad \int e^{-3x^2} x^2 H_{2/3}^3(x) dx = e^{-3x^2} \left[ -\frac{1}{12} x(5+6x^2) H_{2/3}^3(x) + \frac{1}{8} (1+6x^2) H_{2/3}^2(x) H_{5/3}(x) - \frac{3}{8} x H_{2/3}(x) H_{5/3}^2(x) + \frac{1}{16} H_{5/3}^3(x) \right].$$

where  $H$  and  $\bar{H}$  are Hermite functions.

[154]

$$(5.43) \quad \int x e^{-(\nu+1)x} L_\nu(x) dx = \frac{e^{-(\nu+1)x}}{\nu+1} [-(1+x)L_\nu(x) + L_{\nu-1}(x)].$$

[154]

$$(5.44) \quad \int x(1+x)^{-(\nu+3)} L_\nu(x) dx = \frac{(1+x)^{-(\nu+1)}}{\nu+2} \left[ \left( \frac{x-\nu}{\nu+1} - \frac{x}{1+x} \right) L_\nu(x) + \frac{\nu}{\nu+1} L_{\nu-1}(x) \right].$$

[154]

$$(5.45) \quad \int e^{-x} L_\mu(x) \bar{L}_\nu(x) dx = e^{-x} \left[ L_\mu(x) \bar{L}_\nu(x) + \frac{1+\nu}{\mu-\nu} L_\mu(x) \bar{L}_{\nu+1}(x) - \frac{1+\mu}{\mu-\nu} L_{\mu+1}(x) \bar{L}_\nu(x) \right].$$

[154]

$$(5.46) \quad \int x e^{-x} L_\mu(x) \bar{L}_\nu(x) dx = e^{-x} \left[ -\frac{1+\mu+\nu-x+2\mu\nu+\mu^2x+\nu^2x-2\mu\nu x}{(1-\mu+\nu)(1+\mu-\nu)} L_\mu(x) \bar{L}_\nu(x) + \frac{(1+\nu)(1+2\mu-\mu x+\nu x)}{(\mu-\nu)(1-\mu+\nu)} L_\mu(x) \bar{L}_{\nu+1}(x) - \frac{(1+\mu)(1+2\nu+\mu x-\nu x)}{(\mu-\nu)(1+\mu-\nu)} L_{\mu+1}(x) \bar{L}_\nu(x) - \frac{2(1+\mu)(1+\nu)}{(1-\mu+\nu)(1+\mu-\nu)} L_{\mu+1}(x) \bar{L}_{\nu+1}(x) \right].$$

[154]

$$(5.47) \quad \int e^{-3x} x L_{2/3}^3(x) dx = e^{-3x} \left\{ L_{2/3}^3(x) \left[ \frac{125}{24} - \frac{625x}{24} + \frac{853x^2}{16} - \frac{675x^3}{16} + \frac{225x^4}{16} - \frac{27x^5}{16} \right] + L_{5/3}^3(x) \left[ -\frac{125}{24} + \frac{125x}{12} - \frac{125x^2}{16} \right] + 3 \left[ \frac{125}{24} - \frac{125x}{8} + \frac{275x^2}{16} - \frac{75x^3}{16} \right] L_{2/3}(x) L_{5/3}^2(x) + 3 \left[ -\frac{125}{24} + \frac{125x}{6} - \frac{515x^2}{16} + \frac{135x^3}{8} - \frac{45x^4}{16} \right] L_{2/3}^2(x) L_{5/3}(x) \right\}.$$

where  $L$  and  $\bar{L}$  are Laguerre functions.

## 6. DEFINITE INTEGRALS OF SPECIAL FUNCTIONS

## 6.1. Elliptic Integrals and Functions. [6]

$$(6.1) \quad \frac{1}{2} \int_0^1 K(x^2) dx = G.$$

[6]

$$(6.2) \quad \int_0^1 E(x^2) dx = G + \frac{1}{2}.$$

6.2. **The Exponential Integral and Related Functions.** [34]

$$(6.3) \quad \int_0^\infty t^{2\beta} e^{-3t^2} \operatorname{erf}(t) dt = \frac{\Gamma(1+\beta)}{\sqrt{\pi} 3^{\beta+1}} {}_2F_1\left(\frac{1}{2}, 1+\beta; \frac{3}{2}; -\frac{1}{3}\right).$$

6.3. **The Gamma Function and Related Functions.** [11][175, A075700]

$$(6.4) \quad \int_0^1 \ln \Gamma(t) dt = \frac{1}{2} \ln 2\pi.$$

[11]

$$(6.5) \quad \int_0^1 t \ln \Gamma(t) dt = \frac{\zeta'(2)}{2\pi^2} + \frac{1}{6} \ln 2\pi - \frac{\gamma}{12}.$$

[11]

$$(6.6) \quad \int_0^\infty 2^{-t} \ln \Gamma(t) dt = 2 \int_0^1 2^{-t} \ln \Gamma(t) dt - \frac{\gamma + \ln \ln 2}{\ln 2}.$$

[11]

$$(6.7) \quad \int_0^\infty 2^{-t} t \ln \Gamma(t) dt = 2 \int_0^1 2^{-t} (t+1) \ln \Gamma(t) dt - \frac{(\gamma + \ln \ln 2)(1 + 2 \ln 2) - 1}{\ln^2 2}.$$

[75]

$$(6.8) \quad \int_0^1 \ln \Gamma(q) \cos((2n+1)\pi q) dq = \frac{2}{\pi^2} \left( \frac{\gamma + 2 \ln \sqrt{2\pi}}{(2n+1)^2} + 2 \sum_{k=1}^\infty \frac{\ln k}{4k^2 - (2n+1)^2} \right).$$

[75]

$$(6.9) \quad \int_0^1 B_{2m}(q) \ln \Gamma(q) dq = (-)^{m+1} \frac{(2m)! \zeta(2m+1)}{2(2\pi)^{2m}}.$$

[75]

$$(6.10) \quad \int_0^1 B_{2m-1}(q) \ln \Gamma(q) dq = \frac{B_{2m}}{2m} \left[ \frac{\zeta'(2m)}{\zeta(2m)} - A \right],$$

where  $A = 2 \ln \sqrt{2\pi} + \gamma$ .

[75]

$$(6.11) \quad \int_0^1 q^n \ln \Gamma(q) dq = \frac{1}{n+1} \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} (-)^k \binom{n+1}{2k-1} \frac{(2k!)}{k(2\pi)^{2k}} [A\zeta(2k) - \zeta'(2k)] \\ - \frac{1}{n+1} \sum_{k=1}^{\lfloor n/2 \rfloor} (-)^k \binom{n+1}{2k} \frac{(2k!)}{2(2\pi)^{2k}} \zeta(2k+1) + \frac{\ln \sqrt{2\pi}}{n+1},$$

where  $A = 2 \ln \sqrt{2\pi} + \gamma$ .

[75]

$$(6.12) \quad \int_0^1 (q-1/2) \ln \Gamma(q) dq = \frac{1}{12} \left( \frac{6\zeta'(2)}{\pi^2} - 2 \ln \sqrt{2\pi} - \gamma \right).$$

[75]

$$(6.13) \quad \int_0^{1/2} \ln \Gamma(q+1) dq = \frac{\gamma}{8} + \frac{3 \ln \sqrt{2\pi}}{4} - \frac{13 \ln 2}{24} - \frac{3\zeta'(2)}{4\pi^2} - \frac{1}{2}.$$

[75]

(6.14)

$$\int_0^1 q^n \psi^{(m)}(q) dq = (-1)^m \frac{n!}{(n-m)!} \left[ \frac{\gamma}{n-m+1} + (n-m)! \sum_{k=0}^{m-2} \frac{\Gamma(m-k)\zeta(m-k)}{(n-k)!} \right. \\ \left. + \sum_{k=0}^{n-m-1} (-)^k \binom{n-m}{k} [H_k \zeta(-k) + \zeta'(-k)] \right],$$

where  $H_k$  are harmonic sums.

[75]

$$(6.15) \quad \int_0^1 q^n \psi(q) dq = \zeta'(0) + \sum_{k=1}^{n-1} (-)^k \binom{n}{k} [H_k \zeta(-k) + \zeta'(-k)],$$

where  $H_k$  are harmonic sums.

[16]

$$(6.16) \quad \int_0^\infty \frac{t^{\alpha+z} \psi(\alpha+z+1)}{\Gamma(\alpha+z+1)} dz = \frac{t^\alpha}{\Gamma(\alpha+1)} + \nu(t, \alpha) \ln t, \quad \Re \alpha > -1.$$

$$(6.17) \quad \int_0^\infty \frac{\psi(\alpha+z+1)}{\Gamma(\alpha+z+1)} dz = \frac{1}{\Gamma(\alpha+1)}, \quad \Re \alpha > -1.$$

$$(6.18) \quad \int_1^\infty \frac{\psi(z)}{\Gamma(z)} dz = 1.$$

$$(6.19) \quad \int_0^\infty \frac{t^z \psi(z+1)}{\Gamma(z+1)} dz = 1 + \nu(t) \ln t, \quad t \neq 1.$$

[16]

(6.20)

$$\int_0^\infty \frac{t^{\alpha+z}}{\Gamma(\alpha+z+1)} \{\psi(\alpha+z+1)^2 - \psi^{(1)}(\alpha+z+1)\} dz = \frac{2t^\alpha \ln t}{\Gamma(\alpha+1)} + (\ln t)^2 \nu(t, \alpha) + L^{-1} \left\{ \frac{\ln s}{s^{\alpha+1}} \right\}, \quad \Re \alpha > -1$$

where  $L^{-1}$  is the inverse Laplace transform.

$$(6.21) \quad \int_0^\infty \frac{t^z}{\Gamma(z+1)} \{\psi(z+1)^2 - \psi^{(1)}(z+1)\} dz = -\gamma + \ln t (1 + \nu(t) \ln t).$$

$$(6.22) \quad \int_1^\infty \frac{1}{\Gamma(z)} \{\psi^{(1)}(z) - \psi(z)^2\} dz = \gamma.$$

(6.23)

$$\int_0^\infty \frac{t^{\alpha+z}}{\Gamma(\alpha+z+1)} \{\psi(\alpha+z+1)^2 - \psi^{(1)}(\alpha+z+1)\} dz = \frac{t^\alpha}{\Gamma(\alpha+1)} [\psi(\alpha+1) + \ln t] + (\ln t)^2 \nu(t, \alpha).$$

(6.24)

$$\int_0^\infty \frac{t^{\alpha+z} z^\beta}{\Gamma(\alpha+z+1)} \frac{\psi(\alpha+z+1)}{\Gamma(\beta+1)} dz = \mu(t, \beta-1, \alpha) + \ln t \mu(t, \beta, \alpha), \quad \Re \alpha > -1, \Re \beta > -1.$$

$$(6.25) \quad \int_0^\infty \frac{t^{\alpha+z} z^\beta}{\Gamma(\alpha+z+1)} \frac{\psi(\alpha+z+1)^2 - \psi^{(1)}(\alpha+z+1)}{\Gamma(\beta+1)} dz$$

$$= \mu(t, \beta-2, \alpha) + 2 \ln t \mu(t, \beta-1, \alpha) + (\ln t)^2 \mu(t, \beta, \alpha), \quad \Re \alpha > -1, \Re \beta > 1.$$

[16] Let  $L\{f(t)\} \equiv \int_0^\infty e^{-st} f(t) dt = F(s)$  be the Laplace transform and

$$(6.26) \quad \nu(z) \equiv \int_0^\infty \frac{z^t dt}{\Gamma(t+1)},$$

$$(6.27) \quad \nu(z, \alpha) \equiv \int_0^\infty \frac{z^{\alpha+t} dt}{\Gamma(\alpha+t+1)},$$

$$(6.28) \quad \mu(z, \beta) \equiv \int_0^\infty \frac{z^t t^\beta dt}{\Gamma(\beta+1)\Gamma(t+1)},$$

$$(6.29) \quad \mu(z, \beta, \alpha) \equiv \int_0^\infty \frac{z^{\alpha+t} t^\beta dt}{\Gamma(\beta+1)\Gamma(\alpha+t+1)},$$

then

$$(6.30) \quad L\{\nu(t)\} = \frac{1}{s \ln s}.$$

[16]

$$(6.31) \quad L\{\nu(t, \alpha)\} = \frac{1}{s^{1+\alpha} \ln s}.$$

[16]

$$(6.32) \quad L\{\mu(t, \beta)\} = \frac{1}{s(\ln s)^{\beta+1}}.$$

[16]

$$(6.33) \quad L\{\mu(t, \beta, \alpha)\} = \frac{1}{s^{\alpha+1} (\ln s)^{\beta+1}}.$$

Above  $\Re \alpha > -1, \Re \beta > -1, \Re s > 1$ . [16]

$$(6.34) \quad \frac{F(\ln s)}{s \ln s} = L\left\{\int_0^\infty \nu(t, x) f(x) dx\right\}.$$

[16]

$$(6.35) \quad \frac{F(\ln s)}{s^{\alpha+1} \ln s} = L\left\{\int_0^\infty \nu(t, \alpha+x) f(x) dx\right\}.$$

[16]

$$(6.36) \quad \frac{F(\ln s)}{s(\ln s)^{\beta+1}} = L\left\{\int_0^\infty \mu(t, \beta, x) f(x) dx\right\}.$$

[16]

$$(6.37) \quad \frac{F(\ln s)}{s^{\alpha+1} (\ln s)^{\beta+1}} = L\left\{\int_0^\infty \mu(t, \beta, \alpha+x) f(x) dx\right\}.$$

[16]

$$(6.38) \quad \int_0^\infty x^{\beta-1} \nu(t, \alpha+x) dx = \Gamma(\beta) \mu(t, \beta, \alpha), \quad \Re \alpha > -1, \Re \beta > 0.$$

[16]

$$(6.39) \quad \int_0^\infty x^{\beta-1} \nu(t, x) dx = \Gamma(\beta) \mu(t, \beta).$$

[16]

$$(6.40) \quad \int_0^\infty \nu(t, \alpha + x) dx = \mu(t, 1, \alpha).$$

[16]

$$(6.41) \quad \int_0^\infty \nu(t, x) dx = \mu(t, 1).$$

[16]

$$(6.42) \quad \int_0^\infty t^{\lambda-1} \mu(t, \beta, \alpha + x) dx = \Gamma(\lambda) \mu(t, \beta + \lambda, \alpha), \quad \Re \alpha > -1, \Re \beta > -1, \Re \lambda > 0.$$

[16]

$$(6.43) \quad \frac{1}{3\pi} \int_0^\infty \left(\frac{x}{t}\right)^{1/2} K_{1/2}(\phi) \mu(x, \beta, \alpha) dx = 3^{\beta+1} \mu(t, \beta, \alpha/3), \phi \equiv 2(x^3/27t)^{1/2},$$

and others,

6.4. **Cylinder Functions.** [147]

(6.44)

$$2\pi(-1)^{(n-m)/2} \int_0^\infty dk J_{n+1}(2\pi k) J_m(2\pi kr) = R_n^m(r) = \sum_{s=0}^{(n-m)/2} (-1)^s \binom{n-s}{s} \binom{n-2s}{(n-m)/2-s} r^{n-2s},$$

for  $n \geq 0, 0 \leq m \leq n, n - m$  even.

[27]

$$(6.45) \quad \int_0^\infty J_\mu(ct \sin \phi) J_\nu(ct \sin \Phi) K_\rho(ct \cos \phi \cos \Phi) dt = \dots$$

[27]

(6.46)

$$\int_0^\infty J_\mu(ct \sin \phi \sin \Phi) J_\nu(ct \cos \phi \sin \Phi) J_\rho(ct) dt = \frac{\Gamma(\frac{1+\mu+\nu+\rho}{2}) \sin^\mu \phi \cos^\mu \Phi \cos^\nu \phi \sin^\nu \Phi}{c^{\mu+\nu+1} \Gamma(\mu+1) \Gamma(\nu+1) \Gamma(\frac{1-\mu-\nu+\rho}{2})}$$

(6.47)

$$\times {}_2F_1\left(\frac{1+\mu+\nu-\rho}{2}, \frac{1+\mu+\nu+\rho}{2}; \mu+1; \sin^2 \phi\right) {}_2F_1\left(\frac{1+\mu+\nu-\rho}{2}, \frac{1+\mu+\nu+\rho}{2}; \nu+1; \sin^2 \Phi\right)$$

where  $\phi$  and  $\Phi$  are positive angles whose sum is acute.

6.4.1. *Cylinder Functions combined with  $x$  and  $x^2$ .* [203]

$$(6.48) \quad \int_0^1 x J_\nu(\lambda_n x) J_\nu(\lambda_m x) dx = \frac{1}{2} \{ (1 - \nu^2/\lambda_n^2) J_\nu^2(\lambda_n) + J_\nu'^2(\lambda_n) \} \delta_{nm}$$

for  $\{\lambda_n\}$  ( $n = 1, 2, \dots$ ) a sequence of successive positive roots of the equation  $xJ'_\nu(x) + HJ_\nu(x) = 0$ , where  $H$  is a real number and  $\nu \geq -1$ .

[125]

$$(6.49) \quad \int_0^1 x^{2+l+2n} j_l(2\pi\sigma x) dx = \frac{1}{2\pi\sigma} \sum_{k=0}^n \frac{(-n)_k}{(\pi\sigma)^k} j_{l+k+1}(2\pi\sigma); \quad l, n = 0, 1, 2, 3, \dots$$

[152]

$$(6.50) \quad \int_0^\infty k^2 j_l(kr) j_l(kr') dk = \frac{\pi\delta(r-r')}{2r^2}.$$

[93, 136]

$$(6.51) \quad \frac{4p_1 p_2 p_3}{\pi} \int_0^\infty x^2 j_1(p_1 x) j_2(p_2 x) j_3(p_3 x) dx = \Delta(p_1, p_2, p_3) (-1)^{(l_1+l_2+l_3)/2} \\ \times \frac{1}{2} \sum_{k_1=0}^{l_1} \sum_{k_2=0}^{l_2} \sum_{k_3=0}^{l_3} \frac{(-1)^{k_1+k_2+k_3}}{(k_1+k_2+k_3)!} \prod_{i=1}^3 \frac{(l_i+k_i)!}{k_i!(l_i-k_i)!} (2p_i)^{-k_i} \\ \times [(-1)^{l_1+k_1} (p_2+p_3-p_1)^{k_1+k_2+k_3} + (-1)^{l_2+k_2} (p_3+p_1-p_2)^{k_1+k_2+k_3} \\ + (-1)^{l_3+k_3} (p_1+p_2-p_3)^{k_1+k_2+k_3} - (p_1+p_2+p_3)^{k_1+k_2+k_3}]$$

supposed that  $p_1$ ,  $p_2$  and  $p_3$  can be the sides of a plane triangle, that is where  $\Delta(\cdot) = 1$  if they form a non-degenerate triangle,  $\Delta(\cdot) = 1/2$  if they form a degenerate triangle, and  $\Delta(\cdot) = 0$  otherwise.

$$(6.52) \quad \int_0^\infty x j_0(ax) j_0(bx) j_1(cx) dx = \frac{\pi}{8abc^2} (c^2 - (a-b)^2)$$

for  $c > 0$ ,  $a > 0$ ,  $|c-a| < b < c+a$ .

[102]

$$(6.53) \quad \int_0^\infty J_{n_1}(k_1\rho) J_{n_2}(k_2\rho) J_{n_3}(k_3\rho) \rho d\rho = \frac{\Delta}{6\pi A} [\cos(n_1\alpha_2 - n_2\alpha_1) + \cos(n_2\alpha_3 - n_3\alpha_2) + \cos(n_3\alpha_1 - n_1\alpha_3)]$$

if  $n_1 + n_2 + n_3 = 0$ ,  $\Delta$  as above, the area of the triangle of  $k_1$ ,  $k_2$  and  $k_3$  given by  $2A = k_1 k_3 \sin \theta_{13}$ , and  $\alpha$  three external angles in that triangle.

#### 6.4.2. Cylinder Functions and Rational Functions. [82]

$$(6.54) \quad \int_0^\infty \frac{J_\nu(x)}{x^2 + a^2} dx = \frac{i}{a} [S_{0,\nu}(ia - e^{-i\nu\pi/2} K_\nu(a))] = \frac{1}{a} [i s_{0,\nu}(ia) + \frac{\pi}{2} \sec \frac{\nu\pi}{2} I_\nu(a)].$$

[180]

$$(6.55) \quad \int_0^\infty x^{1-2n} J_\nu(ax) J_\nu(bx) \frac{dx}{x^2 + c^2} = (-1)^n c^{-2n} \left\{ I_\nu(bc) K_\nu(ac) \right. \\ \left. - \frac{1}{2} \left( \frac{b}{a} \right)^\nu \frac{\pi}{\sin \pi\nu} \sum_{p=0}^{n-1} \frac{(ac/2)^{2p}}{p! \Gamma(1-\nu+p)} \sum_{k=0}^{n-1-p} \frac{(bc/2)^{2k}}{k! \Gamma(1+\nu+k)} \right\}$$

for  $0 < b < a$ ,  $\Re c > 0$ ,  $\Re \nu > n-1$ ,  $n = 1, 2, \dots$ . For  $0 < a < b$ , the arguments  $a$  and  $b$  should be interchanged.

6.4.3. *Cylinder Functions and Powers.* [110, 114]

(6.56)

$$\int_0^1 x^\mu J_\nu(ax) dx = 2^\mu \frac{\Gamma(\frac{1}{2} + \frac{1}{2}\mu + \frac{1}{2}\nu)}{a^{\mu+1}\Gamma(\frac{\nu}{2} + \frac{1}{2} - \frac{1}{2}\mu)} + a^{-\mu} \{(\mu + \nu - 1)J_\nu(a)S_{\mu-1,\nu-1}(a) - J_{\nu-1}(a)S_{\mu,\nu}(a)\}$$

$$[a > 0, \Re(\mu + \nu) > -1].$$

(6.57)

$$\int_0^u z^\nu K_\nu(x) dx = \frac{2^\nu - 1}{1 + 2\nu} [(1 + 2\nu)\Gamma(\nu)u {}_1F_2(1/2; 3/2, 1 - \nu; u^2/4) + 2(u/2)^{1+2\nu}\Gamma(-\nu) {}_1F_2(\nu + 1/2; 1 + \nu, 3/2 + \nu; u^2/4)].$$

[147]

(6.58)

$$\int_0^\infty x^{-P} [1 - J_0(bx)] dx = \frac{\pi b^{P-1}}{2^P \Gamma^2(\frac{P+1}{2}) \sin[\pi(P-1)/2]}.$$

[72, p 22]

(6.59)

$$\int_0^1 x^\mu J_\nu(xy) \sqrt{xy} dx = y^{-\mu-1} [(\nu + \mu - \frac{1}{2})y J_\nu(y) S_{\mu-1/2,\nu-1}(y) - y J_{\nu-1}(y) S_{\mu+1/2,\nu}(y) + 2^{\mu+1/2} \frac{\Gamma(\frac{\mu}{2} + \frac{\nu}{2} + \frac{3}{4})}{\Gamma(\frac{\nu}{2} - \frac{\mu}{2} + \frac{1}{4})}]$$

for  $\Re(\mu + \nu) > -3/2$ .

[82]

(6.60)

$$\int_0^\infty \frac{x^{\rho-1} J_\nu(ax)}{(x^2 + k^2)^{\mu+1}} dx = \frac{a^\nu k^{\rho+\nu-2\mu-2} \Gamma(\rho/2 + \nu/2) \Gamma(\mu + 1 - \rho/2 - \nu/2)}{2^{\nu+1} \Gamma(\mu + 1) \Gamma(\nu + 1)} \times {}_1F_2\left(\frac{\rho + \nu}{2}; \frac{\rho + \nu}{2} - \mu, \nu + 1; \frac{a^2 k^2}{4}\right) + \frac{a^{2\mu+2-\rho} \Gamma(\nu/2 + \rho/2 - \mu - 1)}{2^{2\mu+3-\rho} \Gamma(\mu + 2 + \nu/2 - \rho/2)} \times {}_1F_2\left(\mu + 1; \mu + 2 + \frac{\nu - \rho}{2}, \mu + 2 - \frac{\nu + \rho}{2}; \frac{a^2 k^2}{4}\right), a > 0, -\Re\nu < \Re\rho < 2\Re\mu + \frac{7}{2}, \Re k > 0.$$

[204]

$$\int_0^\infty \frac{J_{n+1}(k) J_{n'+1}(k)}{k(k^2 + k_0^2)^{1+\gamma}} dk = k_0^{n+n'-\gamma} \frac{\Gamma(\frac{\gamma-n-n'}{2}) \Gamma(\frac{n+n'}{2} + 1)}{2^{n+n'+3} \Gamma(n+2) \Gamma(n'+2) \Gamma(1 + \gamma/2)} \times {}_3F_4\left(\begin{matrix} \frac{n+n'}{2} + 1, \frac{n+n'}{2} + 2, \frac{n+n'+3}{2} \\ n+2, n'+2, n+n'+3, 1 + \frac{n+n'-\gamma}{2} \end{matrix} \middle| k_0^2\right) + \frac{\Gamma(\frac{n+n'-\gamma}{2}) \Gamma(3 + \gamma)}{2^{3+\gamma} \Gamma(3 + \frac{\gamma+n+n'}{2}) \Gamma(2 + \frac{\gamma+n-n'}{2}) \Gamma(2 + \frac{\gamma+n'-n}{2})} {}_3F_4\left(\begin{matrix} 2 + \frac{\gamma}{2}, 1 + \frac{\gamma}{2}, \frac{3+\gamma}{2} \\ 2 + \frac{\gamma+n-n'}{2}, 2 + \frac{\gamma+n'-n}{2}, 3 + \frac{\gamma+n+n'}{2}, 1 + \frac{\gamma-n-n'}{2} \end{matrix} \middle| k_0^2\right)$$

[122, p50]  
(6.61)

$$\int_0^\infty J_{\nu+n}(at)J_{\nu-n-1}(bt)dt = \begin{cases} \frac{b^{\nu-n-1}\Gamma(\nu)}{a^{\nu-n}\Gamma(\nu-n)} {}_2F_1(\nu, -n; \nu-n; \frac{b^2}{a^2}), & 0 < b < a, \\ (-1)^n/(2a), & 0 < b = a, \\ 0, & 0 < a < b. \end{cases}$$

where  $n = 0, 1, 2, \dots$ ,  $\Re\nu > 0$ .

[122, p50]

$$(6.62) \quad \int_0^\infty J_\nu(at)J_{\nu+1}(bt)dt = \begin{cases} a^\nu b^{-\nu-1}, & 0 < b < a \\ 1/(2a), & 0 < b = a \\ 0, & 0 < a < b \end{cases}$$

where  $a, b$  real positive,  $\Re\nu > -1$ .

[122, p50]

$$(6.63) \quad \int_0^\infty J_\mu(at)J_\nu(at)dt = \frac{2 \sin(\frac{\nu-\mu}{2}\pi)}{\pi \nu^2 - \mu^2},$$

$\Re(\nu + \mu) > 0$ ,  $a > 0$ .

[147]

$$(6.64) \quad \int_0^\infty x^{-P} \left(1 - \frac{4J_1^2(x)}{x^2}\right) dx = \frac{\pi\Gamma(P+2)}{2^P\Gamma^2(\frac{P+3}{2})\Gamma(\frac{P+5}{2})\Gamma(\frac{P+1}{2})\sin[\pi(P-1)/2]}.$$

[79]

$$(6.65) \quad \int_0^\infty t^{\rho-\mu-\nu-3} J_\mu(at)J_\nu(bt)J_\rho(ct)dt = \frac{2^{\rho-\mu-\nu-3} a^\mu b^\nu \Gamma(\rho-1)}{c^{\rho-2} \Gamma(\mu+1)\Gamma(\nu+1)} \left(1 - \frac{\rho-1}{\mu-1} \frac{a^2}{c^2} - \frac{\rho-1}{\nu-1} \frac{b^2}{c^2}\right).$$

[190] Let

$$(6.66) \quad G_{lmn} \equiv \int_0^\infty dx x^\alpha J_l(x)J_m(x)J_n(\beta x)$$

with  $\Re\alpha < 3/2$  and  $\Re(\alpha + l + m + n + 1) > 0$ , then for  $\beta/2 < 1$

(6.67)

$$\begin{aligned} G_{lmn} &= \frac{\Gamma(\frac{\alpha+n}{2})(\beta/2)^{-\alpha}}{2\Gamma(\frac{-l+m+1}{2})\Gamma(\frac{l-m+1}{2})\Gamma(\frac{-\alpha+n+2}{2})} {}_4F_3 \left( \begin{matrix} \frac{l+m+1}{2}, \frac{l-m+1}{2}, \frac{-l-m+1}{2}, \frac{-l+m+1}{2} \\ -\frac{\alpha+n+2}{2}, \frac{1}{2}, \frac{-\alpha-n+2}{2} \end{matrix} \middle| \beta^2/4 \right) \\ &- \frac{\Gamma(\frac{l+m+2}{2})\Gamma(\frac{\alpha+n-1}{2})(\beta/2)^{-\alpha+1}}{\Gamma(\frac{-l+m}{2})\Gamma(\frac{l+m}{2})\Gamma(\frac{l-m}{2})\Gamma(\frac{-\alpha+n+3}{2})} {}_4F_3 \left( \begin{matrix} \frac{l+m+2}{2}, \frac{l-m+2}{2}, \frac{-l-m+2}{2}, \frac{-l+m+2}{2} \\ -\frac{\alpha+n+3}{2}, \frac{3}{2}, \frac{-\alpha-n+3}{2} \end{matrix} \middle| \beta^2/4 \right) \\ &+ \frac{2^{\alpha+n}\Gamma(-\alpha-n)\Gamma(\frac{\alpha+l+m+n+1}{2})(\beta/2)^n}{\Gamma(\frac{-\alpha-l+m-n+1}{2})\Gamma(\frac{-\alpha+l+m-n+1}{2})\Gamma(\frac{-\alpha+l-m-n+1}{2})\Gamma(n+1)} \\ &\times {}_4F_3 \left( \begin{matrix} \frac{\alpha+l+m+n+1}{2}, \frac{\alpha+l-m+n+1}{2}, \frac{\alpha-l-m+n+1}{2}, \frac{\alpha-l+m+n+1}{2} \\ n+1, \frac{n+\alpha+2}{2}, \frac{\alpha+n+1}{2} \end{matrix} \middle| \beta^2/4 \right), \end{aligned}$$

and for  $\beta/2 > 1$

(6.68)

$$G_{lmn}(\alpha, \beta) = \frac{2^{-l-m-1}\Gamma(\frac{\alpha+l+m+n+1}{2})(\beta/2)^{-\alpha-l-m-1}}{\Gamma(\frac{-\alpha-l-m+n+1}{2})\Gamma(l+1)\Gamma(m+1)} {}_4F_3 \left( \begin{matrix} \frac{\alpha+l+m+n+1}{2}, \frac{l+m+1}{2}, \frac{l+m+2}{2}, \frac{\alpha+l+m-n+1}{2} \\ m+1, l+1, m+l+1 \end{matrix} \middle| 4/\beta^2 \right)$$



[187]  
 (6.69) 
$$\int_0^\infty t \left( \frac{t}{\sqrt{u^2+t^2}} - 1 \right) J_0(\gamma t) dt = \frac{u^2}{2} [I_1(u\gamma/2)K_1(u\gamma/2) - I_0(u\gamma/2)K_0(u\gamma/2)].$$

6.4.4. *Cylinder Functions and Exponentials.*

6.4.5. *Cylinder Functions, Exponentials and Powers.* With

$$l_1(a, b, c) \equiv \frac{1}{2} \left[ \sqrt{(a+b)^2+c^2} - \sqrt{(a-b)^2+c^2} \right]$$

$$l_2(a, b, c) \equiv \frac{1}{2} \left[ \sqrt{(a+b)^2+c^2} + \sqrt{(a-b)^2+c^2} \right]$$

at  $a > 0, b > 0, c > 0$  [79]

(6.70) 
$$\int_0^\infty e^{-cx} J_1(ax) J_{1/2}(bx) \frac{dx}{x^{3/2}} = \frac{\sqrt{2}}{\sqrt{\pi ba}} \left[ \frac{l_1}{2} \sqrt{a^2 - l_1^2} + \frac{a^2}{2} \sin^{-1} \left( \frac{l_1}{a} \right) + c(\sqrt{b^2 - l_1^2} - b) \right].$$

(6.71) 
$$\int_0^\infty e^{-cx} J_1(ax) J_{1/2}(bx) \frac{dx}{\sqrt{x}} = \frac{\sqrt{2}}{\sqrt{\pi ba}} (b - \sqrt{b^2 - l_1^2}).$$

(6.72) 
$$\int_0^\infty e^{-cx} J_1(ax) J_{1/2}(bx) \sqrt{x} dx = \frac{\sqrt{2}}{\sqrt{\pi ba}} \frac{l_1 \sqrt{a^2 - l_1^2}}{l_2^2 - l_1^2}.$$

(6.73) 
$$\int_0^\infty e^{-cx} J_1(ax) J_{3/2}(bx) \sqrt{x} dx = \frac{2l_1^2 \sqrt{b^2 - l_1^2}}{\sqrt{2\pi} b^{3/2} a (l_2^2 - l_1^2)}.$$

(6.74) 
$$\int_0^\infty e^{-cx} J_1(ax) J_{3/2}(bx) \frac{dx}{\sqrt{x}} = \frac{1}{\sqrt{2\pi} b^{3/2} a} \left( -l_1 \sqrt{a^2 - l_1^2} + a^2 \sin^{-1} (l_1/a) \right).$$

(6.75) 
$$\int_0^\infty e^{-cx} J_1(ax) J_{5/2}(bx) \frac{dx}{\sqrt{x}} = \frac{2^{-1/2} c}{\sqrt{\pi} b^{5/2} a} \left( l_1 \sqrt{a^2 - l_1^2} + \frac{2a^2 l_1}{\sqrt{a^2 - l_1^2}} - 3a^2 \sin^{-1} (l_1/a) \right).$$

and similar combinations of even and odd-indexed  $J(ax)$  and  $J(bx)$ .

[7]

(6.76) 
$$\int_0^\infty x^\nu e^{-x/2} J_\nu(\mu x) L_n^{2\nu}(x) dx = 2^\nu \Gamma(\nu + \frac{1}{2}) \frac{1}{\sqrt{\pi \mu}} (\sin \theta)^{\nu + \frac{1}{2}} C_n^{\nu + \frac{1}{2}}(\cos \theta),$$

$\mu \geq 0, \nu > -\frac{1}{2}, \cos \theta \equiv \frac{\mu^2 - 1/4}{\mu^2 + 1/4}, C_n^\lambda(x)$  ultraspherical polynomial.

[150]

(6.77) 
$$\int_0^\infty \frac{x^{\nu+1}}{a^2+x^2} J_\nu(xy) dx = \frac{a^\nu}{\sqrt{y}} K_\nu(ay), \quad 1 < \Re \nu < 3/2.$$

[150]

(6.78) 
$$\int_0^\infty \frac{x^{\nu+1}}{(a^2+x^2)^\mu} J_\nu(xy) dx = \frac{2^{1-\mu} a^{\nu-\mu+1} y^{\mu-1}}{\Gamma(\mu)} K_{\nu-\mu+1}(ay), \quad \Re \nu > -1, \Re(2\mu-\nu) > 1/2.$$

[150]

$$(6.79) \quad \int_0^\infty \frac{x^{1-\nu}}{(a^2+x^2)^\mu} J_\nu(xy) dx = a^{-\mu-\nu+1} y^{\mu-1} \left[ 2^{-\mu} \frac{\Gamma(1-m\mu)}{1-\nu} I_{\nu+\mu-1}(ay) \right. \\ \left. - 2 \frac{1-\mu}{\Gamma(\nu)} e^{-i\frac{\pi}{2}(\nu-\mu+1)} s_{-\mu+\nu, -\mu-\nu+1}(iay) \right], \quad \Re(\nu+2\mu) > 1/2.$$

[34]

$$(6.80) \quad \int_0^\infty x^{\beta-1} e^{-x} I_n(x) dx = \sum_{j=0}^{\infty} \frac{\Gamma(\beta+2j+n)}{2^{2j+n} j! (j+n)!}.$$

[34]

$$(6.81) \quad \int_0^\infty x^{2n+1} e^{-x^2/4a} I_0(x) dx = 2^{2n+1} a^{n+1} n! e^a L_n(-a) = 2^{2n+1} a^{n+1} n! {}_1F_1(n+1; 1; a).$$

$$(6.82) \quad \int_0^\infty x^3 e^{-x^2/4a} I_0(x) dx = 8a^2(1+a)e^a.$$

[2]

$$(6.83) \quad \int_0^\infty x e^{-x^2/z} J_2(x) Y_2(x) dx = -\frac{2}{\pi} + \frac{4}{\pi z} - \frac{zK_2(z/2)}{2\pi \exp(z/2)}.$$

[2]

$$(6.84) \quad \int_0^\infty x e^{-x^2/z} I_3(x) K_3(x) dx = -\frac{32+16z+3z^2}{2z^2} + \frac{\exp(z/2)zK_3(z/2)}{4}.$$

[2]

$$(6.85) \quad \int_0^\infty x^3 e^{-x^2/z} J_2(x) Y_2(x) dx = -\frac{4}{\pi} + \frac{z^2(2+z)K_0(z/2)}{4\pi \exp(z/2)} + \frac{z(8+4z+z^2)K_1(z/2)}{4\pi \exp(z/2)}.$$

[2]

$$(6.86) \quad \int_0^\infty x^5 e^{-x^2/z} I_3(x) K_3(x) dx = -32 + \frac{1}{8} e^{z/2} z^2 (32 - 16z + 5z^2 - z^3) K_0(z/2) \\ + \frac{1}{8} e^{z/2} z (128 - 64z + 24z^2 - 6z^3 + z^4) K_1(z/2).$$

[68] Let

$$(6.87) \quad I(\mu, \nu, \lambda) \equiv \int_0^\infty J_\mu(at) J_\nu(bt) e^{-ct} t^\lambda dt,$$

then

$$(6.88) \quad I(n, n; 0) = \frac{(-)^n k}{\pi \sqrt{ab}} \int_0^{\pi/2} \frac{\cos(2n\psi) d\psi}{\sqrt{1-k^2 \sin^2 \psi}},$$

$$(6.89) \quad I(n, n; 1) = \frac{(-)^n c k^3}{4\pi (ab)^{3/2}} \int_0^{\pi/2} \frac{\cos(2n\psi) d\psi}{(1-k^2 \sin^2 \psi)^{3/2}},$$

in particular

$$(6.90) \quad I(0, 0; 0) = \frac{k}{2\sqrt{ab}}F_0(k),$$

$$(6.91) \quad I(1, 1; 0) = \frac{1}{k\sqrt{ab}}[(1 - k^2/2)F_0(k) - E_0(k)],$$

$$(6.92) \quad I(0, 0; 1) = \frac{ck^3E_0(k)}{8k'^2(ab)^{3/2}},$$

and more results on  $I(n + 1, n; \pm 1)$  and  $I(n + 1, n; 0)$  in terms of Elliptic Integrals. Associated recurrences:

$$(6.93) \quad a[I(\mu + 1, \nu; \lambda) + I(\mu - 1, \nu; \lambda)] = 2\mu I(\mu, \nu; \lambda - 1);$$

$$(6.94) \quad b[I(\mu, \nu + 1; \lambda) + I(\mu, \nu - 1; \lambda)] = 2\nu I(\mu, \nu; \lambda - 1);$$

$$(6.95) \quad aI(\mu + 1, \nu; \lambda) - bI(\mu, \nu - 1; \lambda) = C_{\mu, \nu} + (\mu - \nu + \lambda)I(\mu, \nu; \lambda - 1) - cI(\mu, \nu; \lambda),$$

with

$$(6.96) \quad C_{\mu, \nu} \equiv \begin{cases} \frac{a^\mu b^\nu}{2^{\mu+\nu}\Gamma(\mu+1)\Gamma(\nu+1)}, & \text{if } \lambda + \mu + \nu = 0; \\ 0 & \text{if } \lambda + \mu + \nu > 0. \end{cases}$$

6.4.6. *Cylinder and Trigonometric Functions and Powers.* [79]

$$(6.97) \quad \int_0^\infty \sin(cx)x^{\nu-\mu-4}J_\mu(ax)J_\nu(bx)dx = \frac{\Gamma(\nu)a^\mu b^{-\nu}c}{2^{\mu-\nu+3}\Gamma(\mu+1)} \left( \frac{b^2}{\nu-1} - \frac{a^2}{\mu+1} - \frac{2c^2}{3} \right).$$

[79]

$$(6.98) \quad \int_0^\infty \cos(cx)x^{\nu-\mu-3}J_\mu(ax)J_\nu(bx)dx = \frac{\Gamma(\nu)a^\mu b^{-\nu}}{2^{\mu-\nu+3}\Gamma(\mu+1)} \left( \frac{b^2}{\nu-1} - \frac{a^2}{\mu+1} - 2c^2 \right).$$

[165]

$$(6.99) \quad \int_0^\infty J_{2n}(a\sqrt{t^2+2bt})e^{-pt}dt = (-1)^n \left\{ \frac{e^{b(p-\sqrt{p^2+a^2})}}{\sqrt{p^2+a^2}} + \frac{2e^{bp}}{a} \right. \\ \left. \times \sum_{k=1}^n (-)^k (2k-1)I_{k-1/2}[b(\sqrt{p^2+a^2}-a)/2]K_{k-1/2}[b(\sqrt{p^2+a^2}+a)/2] \right\}.$$

[165]

$$(6.100) \quad \int_0^\infty J_{2n}(a\sqrt{t^2+2bt})e^{-pt}dt = (-1)^n \left\{ \frac{e^{b(p-\sqrt{p^2+a^2})}}{\sqrt{p^2+a^2}} + 2n \sum_{\lambda=1}^n \frac{(n-1+\lambda)!}{\lambda!(n-\lambda)!} \frac{1}{b^{2\lambda-1}a^{2\lambda}} \right. \\ \left. \times \sum_{\mu=0}^{\lambda-1} \frac{(2\lambda-2-\mu)!}{\mu!(\lambda-1-\mu)!} (2b\sqrt{p^2+a^2})^\mu \left( e^{b(p-\sqrt{p^2+a^2})} - \sum_{\nu=0}^{2\lambda-2-\mu} \frac{[b(p-\sqrt{p^2+a^2})]^\nu}{\nu!} \right) \right\}.$$

[34]

$$(6.101) \quad \int_0^\infty x^{\beta-1}e^{-x}I_0(2\sqrt{ax})dx = \sum_{k=0}^\infty \frac{a^k}{k!^2}\Gamma(\beta+k).$$

## 7. DEFINITE INTEGRALS OF SPECIAL FUNCTIONS II.

## 7.1. Associated Legendre Functions and Powers. [159]

$$\int_{-1}^1 P_{l+\alpha}^{-\alpha}(x)P_{k+\beta}^{-\beta}(x)(1-x^2)^{-p-1}dx = 2^{-(\alpha+\beta)}$$

$$\times \frac{\Gamma[\frac{1}{2}(k+1)]\Gamma(-\frac{1}{2}k-\beta)\Gamma[\frac{1}{2}(\alpha+\beta)-p]\Gamma[\frac{1}{2}(\alpha-\beta-p)]\Gamma[\frac{1}{2}(l-k)]\Gamma[\frac{1}{2}(l+k+1)+\beta]}{\Gamma(\beta+1)\Gamma(-\beta)\Gamma(\alpha+1)\Gamma(-\frac{1}{2}k)\Gamma[\frac{1}{2}(k+1)+\beta]\Gamma[\frac{1}{2}(l-k+\alpha-\beta)-p]\Gamma[\frac{1}{2}(l+k+\alpha+\beta+1)-p]}$$

$$\times {}_4F_3\left(\begin{matrix} \frac{1}{2}(\alpha-\beta)+p+1, & \frac{1}{2}(\alpha-\beta)-p, & -\frac{1}{2}(l-1), & -\frac{1}{2}l \\ \frac{1}{2}(k-l)+1, & -\frac{1}{2}(l+k-1)-\beta, & \alpha+1 & \end{matrix}; 1\right)$$

for  $\Re[\frac{1}{2}(\alpha+\beta)-p] > 0$ , for  $k, l$  both even or both odd.  
[159]

$$(7.1) \quad \int_{-1}^1 P_l^m(x)P_k^n(x)(1-x^2)^{-p-1}dx = (-1)^{(l-k+m-n)/2}2^{-(m+n)}$$

$$\times \frac{(l+m)!\Gamma[\frac{1}{2}(m+n)-p]\Gamma[\frac{1}{2}(m-n)-p]\Gamma[\frac{1}{2}(l+k+1-m+n)]}{(l-m)!m![\frac{1}{2}(k-l+m-n)]!\Gamma[\frac{1}{2}(l-k)-p]\Gamma[\frac{1}{2}(l+k+1)-p]}$$

$$\times {}_4F_3\left(\begin{matrix} \frac{1}{2}(m-n)+p+1, & \frac{1}{2}(m-n)-p, & -\frac{1}{2}(l-m-1), & -\frac{1}{2}(l-m) \\ \frac{1}{2}(k-l+m-n)+1, & -\frac{1}{2}(l+k-1-m+n), & m+1 & \end{matrix}; 1\right)$$

for  $\Re[\frac{1}{2}(m+n)-p] > 0$ , for  $k-n \geq l-m$ .  
[159]

$$(7.2) \quad \int_{-1}^1 P_l^m(x)P_k^m(x)(1-x^2)^{-1}dx = \frac{1}{m} \frac{(l+m)!}{(l-m)!}, \quad k \geq l \geq m > 0.$$

[159]

$$\int_{-1}^1 P_l^m(x)P_k^m(x)(1-x^2)^{-2}dx = \frac{(l+m)!}{2(l-m)!(m-1)m(m+1)}$$

$$\times [l(l+1) + (m-1)(m+1) + \frac{1}{2}(k-l)(m+1)(k+l+1)], \quad k \geq l \geq m > 1.$$

[159]

$$(7.3) \quad \int_{-1}^1 P_l^m(x)P_k^m(x) = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{kl}$$

[159]

$$\int_{-1}^1 P_l^m(x)P_k^m(x)(1-x^2)dx =$$

$$(-1)^{(k-l)/2} \frac{(l+m)![l(l+1) + (m-1)(m+1) + \frac{1}{2}(k-l)(m+1)(k+l+1)]}{(l-m)![\frac{1}{2}(k-l)+1]!\Gamma[\frac{1}{2}(l-k)+2](k+l-1)(k+l+1)(k+l+3)}$$

for  $k \geq l \geq m \geq 0$ , zero for  $k-l \geq 4$ .

## 7.2. Associated Legendre functions, powers, and trigonometric functions.

[26]

$$(7.4) \quad \int_0^z P_n[\cos(z-t)]P_n^{-m}(\cos t)\frac{dt}{\sin t} = \frac{P_n^{-m}(\cos z)}{m}, \quad \Re m > 0.$$

$$(7.5) \quad \int_0^z \sin^m t P_{n-m-1}[\cos(z-t)]P_n^{-m}(\cos t)dt = \frac{\Gamma(m+1/2)}{2^{1/2}\Gamma(m+1)} \sin^{m+1/2} z P_{n-1/2}^{-m-1/2}(\cos z), \quad \Re m > -1/2.$$

$$(7.6) \quad \int_0^z \sin^{m+1} t P_{n-m-2}[\cos(z-t)]P_n^{-m}(\cos t)dt = \frac{\Gamma(m+3/2)}{2^{1/2}\Gamma(m+2)} \sin^{m+3/2} z P_{n-1/2}^{-m-1/2}(\cos z), \quad \Re m > -1.$$

$$(7.7) \quad \int_0^z \sin^{-1-k} t \sin^k(z-t)P_n^{-m}(\cos t)P_n^{-k}[\cos(z-t)] = \frac{2^k \Gamma(m-k)\Gamma(k+1/2)}{\sqrt{\pi}\Gamma(k+m+1)} \sin^k z P_n^{-m}(\cos z), \quad \Re m > \Re k > -1/2.$$

$$(7.8) \quad \int_0^z \sin^m t \sin^{-m}(z-t)P_n^{-m}(\cos t)P_{n-1}^m[\cos(z-t)] = \frac{\sin[(m+n)z]}{(m+n)\cos m\pi}, \quad -1/2 < \Re m < 1/2.$$

$$(7.9) \quad \int_0^z P_{n+1}^{-m}(\cos t) \sin[n(z-t)]dt = 2^{3/2} \sin^{1/2} z \sum_{r=0}^{\infty} \frac{(-1)^r (m+n+2)_{2r} (3/2)_r \Gamma(m+r+3/2)}{r! \Gamma(m+r+1)} \\ \times \frac{m+2r+3/2}{(m+2r+1)(m+2r+2)} P_{n-1/2}^{-(m+3/2+2r)}(\cos z).$$

[146]

$$(7.10) \quad \int_0^\pi d\theta \sin^{|\mathfrak{m}|+1} \theta \exp(\pm iR \cos \theta) P_n^{|\mathfrak{m}|}(\cos \theta) = 2(\pm i)^{n+|\mathfrak{m}|} \frac{(n+|\mathfrak{m}|)!}{(n-|\mathfrak{m}|)!} \frac{j_n(R)}{R^{|\mathfrak{m}|}}.$$

[146, (23)]

$$(7.11) \quad \int_0^\pi d\theta \sin \theta \exp(iR \cos \alpha \cos \theta) P_n^m(\cos \theta) J_m(R \sin \alpha \sin \theta) = 2i^{n-m} P_n^m(\cos \alpha) j_n(R).$$

[15, §11.4,13][123]

$$(7.12) \quad \int_0^\infty e^{-c^2 x^2} H_{2m}(ax) H_{2k}(bx) dx = \\ (-)^{m+k} 2^{2m+2k-1} \frac{\Gamma(m+1/2)\Gamma(k+1/2)}{\pi c^{2m+2k+1}} (c^2 - a^2)^m (c^2 - b^2)^k {}_2F_1(-m, -k; \frac{1}{2}; \frac{a^2 b^2}{(c^2 - a^2)(c^2 - b^2)}),$$

$$c^2 - a^2 - b^2 > 0.$$

[15, §11.4,14]

$$(7.13) \quad \int_0^\infty e^{-c^2 x^2} H_{2m+1}(ax) H_{2k+1}(bx) dx = (-)^{m+k} 2^{2m+2k+1} \frac{\Gamma(m+3/2)\Gamma(k+3/2)}{\Gamma(3/2)} \\ \times \frac{ab(c^2 - a^2)^m (c^2 - b^2)^k}{c^{2m+2k+3}} {}_2F_1(-m, -k; \frac{3}{2}; \frac{a^2 b^2}{(c^2 - a^2)(c^2 - b^2)}).$$

[15, §11.4,11]

$$(7.14) \quad \int_{-\infty}^{\infty} e^{-2x^2} [H_n(x)]^2 dx = 2^{n-1/2} \Gamma(n+1/2).$$

[15, §11.4,20]

$$(7.15) \quad \int_{-\infty}^{\infty} e^{-(a^2+b^2)x^2} H_{2m}(ax) H_{2k}(bx) dx = (-)^{m+k} 2^{2(m+k)} \Gamma(m+k+\frac{1}{2}) \frac{a^{2k} b^{2m}}{(a^2+b^2)^{m+k+1/2}}.$$

[15, §11.4,21]

$$(7.16) \quad \int_{-\infty}^{\infty} e^{-(a^2+b^2)x^2} H_{2m+1}(ax) H_{2k+1}(bx) dx = (-)^{m+k} 2^{2(m+k+1)} \Gamma(m+k+\frac{3}{2}) \frac{a^{2k+1} b^{2m+1}}{(a^2+b^2)^{m+k+3/2}}.$$

[111]

$$(7.17) \quad \int_0^{\infty} x^{\nu+1} e^{-\alpha x^2} L_m^{\nu-\sigma}(\alpha x^2) L_n^{\sigma}(\alpha x^2) J_{\nu}(xy) dx = \frac{(-1)^{m+n}}{2\alpha} \left(\frac{y}{2\alpha}\right)^{\nu} \exp\left(-\frac{y^2}{4\alpha}\right) L_m^{\sigma-m+n}\left(\frac{y^2}{4\alpha}\right) L_n^{\nu-\sigma+m-n}\left(\frac{y^2}{4\alpha}\right),$$

for  $y > 0$ ,  $\Re\alpha > 0$ ,  $\Re\nu > -1$ .

(7.18)

$$\int_0^{\infty} x^{(\alpha+\beta)/2} e^{-x} L_m^{\alpha}(x) L_n^{\beta}(x) J_{\alpha+\beta}(2\sqrt{ax}) dx = (-1)^{m+n} a^{(\alpha+\beta)/2} L_m^{\beta-m+n}(a) L_n^{\alpha+m-n}(a); \quad \Re(\alpha+\beta) > -1.$$

### 7.3. Hypergeometric Functions. [73, p 238]

(7.19)

$$\frac{1}{\Gamma(2\lambda+2\nu)} \int_0^{\infty} dt e^{-pt} t^{2\lambda+2\nu-1} {}_1F_2(\nu; \lambda+\nu, \lambda+\nu+\frac{1}{2}; -\frac{1}{4}a^2t^2) = \frac{1}{p^{2\lambda}} \frac{1}{(p^2+a^2)^{\nu}}; \quad \Re(\lambda+\nu) > 0.$$

previous formula at  $\lambda = 0$  with [1, 9.1.69] gives [91, 6.623.1]

(7.20)

$$\begin{aligned} \frac{1}{\Gamma(2\nu)} \int_0^{\infty} dt e^{-pt} t^{2\nu-1} {}_0F_1(; \nu+\frac{1}{2}; -\frac{1}{4}a^2t^2) &= \frac{\Gamma(\nu+1/2)}{\Gamma(2\nu)} \left(\frac{a}{2}\right)^{1/2-\nu} \int_0^{\infty} e^{-pt} t^{\nu-1/2} J_{\nu-1/2}(at) \\ &= \frac{1}{(p^2+a^2)^{\nu}}; \quad \Re(\nu) > 0. \end{aligned}$$

[133]

(7.21)

$$\int_0^{\infty} J_{\mu}^2(\omega\rho) {}_3F_2\left(\begin{matrix} 3/2, (\sigma+\nu)/2, (\sigma-\nu)/2 \\ 1-\mu, 1+\mu \end{matrix} \mid -4\omega^2\right) \omega d\omega = -\frac{\mu\rho^{\sigma-2} K_{\nu}(\rho)}{2^{\sigma-1} \Gamma[(\sigma+\nu)/2] \Gamma[(\sigma-\nu)/2]}$$

for  $\Re\sigma > 1 + |\Re\nu|$ ,  $\mu = l + 1/2$  with  $l \in \mathbb{N}$ ,  $\Re(\sigma+\nu) > 0$ ,  $\Re\rho > 0$ .

[133]

(7.22)

$$\int_0^{\infty} J_{\mu}^2(\omega\rho) J_{\mu+1}(\omega\rho) {}_3F_2\left(\begin{matrix} 3/2, (\sigma+\nu)/2, (\sigma-\nu)/2 \\ 1-\mu, 2+\mu \end{matrix} \mid -4\omega^2\right) \omega^2 d\omega = -\frac{\mu(\mu+1)\rho^{\sigma-3} K_{\nu}(\rho)}{2^{\sigma-1} \Gamma[(\sigma+\nu)/2] \Gamma[(\sigma-\nu)/2]}$$

for  $\Re\sigma > 2 + |\Re\nu|$ ,  $\mu = l + 1/2$  with  $l \in \mathbb{N}$ ,  $\Re(\sigma+\nu) > 0$ ,  $\Re\rho > 0$ .

## 8. SPECIAL FUNCTIONS

## 8.1. The exponential integral and related functions. [30] If

$$(8.1) \quad \int_0^x \frac{\sin u}{u} du = \frac{\pi}{2} - r \cos(x - \theta); \quad \int_0^x \frac{1 - \cos u}{u} du = \gamma + \log x - r \sin(x - \theta)$$

then

$$(8.2) \quad r \cos \theta \sim \sum_{k \geq 0} \frac{(-)^k (2k)!}{x^{2k+1}}; \quad r \sin \theta \sim \sum_{k \geq 1} \frac{(-)^{k+1} (2k-1)!}{x^{2k}}; \quad r \sim \sum_{k \geq 1} \frac{(-)^{k+1} (2k-1)!}{kx^{2k}}$$

for  $x \rightarrow \infty$ .

## 8.2. The error function and Fresnel integrals. [34]

$$(8.3) \quad 2\sqrt{\pi}e^{b^2} \operatorname{erf}(b) = \sum_{j=0}^{\infty} \frac{j! b^{2j+1}}{(2j+1)! 2^{2j+2}}.$$

## 8.3. The gamma function.

$$(8.4) \quad \frac{1}{a+r} = \frac{1}{a} \frac{(a)_r}{(a+1)_r}.$$

[174, 2.2.3.1]

$$(8.5) \quad (a)_{m-r} = \frac{(-1)^r (a)_m}{(1-a-m)_r}.$$

[174, (2.4.5.2.)][64, 166]

$$(8.6) \quad (a)_{2r} = (a/2)_r \left(\frac{a+1}{2}\right)_r 2^{2r}; \quad (a)_{qr} = (a/q)_r \left(\frac{a+1}{q}\right)_r \cdots \left(\frac{a+q-1}{q}\right)_r q^{qr}$$

$$(8.7) \quad \Gamma(n+1-k) = \frac{(-1)^k \Gamma(n+1)}{(-n)_k}.$$

[174, (I.4)]

$$(8.8) \quad (a+kn)_n = \frac{(a)_{(k+1)n}}{(a)_{kn}}.$$

[174, (I.6)]

$$(8.9) \quad (a-kn)_n = \frac{(-1)^n (1-a)_{kn}}{(1-a)_{(k-1)n}}.$$

[174, (I.9)][166]

$$(8.10) \quad (a)_{N-n} = \frac{(-1)^n (a)_N}{(1-a-N)_n}.$$

[174, (I.11)]

$$(8.11) \quad (a+kn)_{N-n} = \frac{(a)_N (a+N)_{(k-1)n}}{(a)_{kn}}.$$

[174, (I.13)]

$$(8.12) \quad (a-kn)_{N-n} = \frac{(-1)^n (a)_N (1-a)_{kn}}{(1-a-N)_{(k+1)n}}.$$

[70, (§1.2)]

$$(8.13) \quad \frac{\Gamma(n + \frac{1}{2} + z)\Gamma(n + \frac{1}{2} - z)}{\Gamma^2(n + \frac{1}{2})} = \frac{1}{\cos(\pi z)} \prod_{l=1}^n \left[ 1 - \frac{4z^2}{(2l-1)^2} \right],$$

for  $n = 1, 2, 3, \dots$

[194][175, A073006]

$$(8.14) \quad \Gamma(2/3) = \frac{2\pi}{\sqrt{3}} \frac{1}{\Gamma(1/3)}.$$

[194][175, A068465]

$$(8.15) \quad \Gamma(3/4) = \pi\sqrt{2} \frac{1}{\Gamma(1/4)}.$$

[194][175, A175379]

$$(8.16) \quad \Gamma(1/6) = \frac{\sqrt{3}}{\sqrt{\pi}2^{1/3}} \Gamma^2(1/3).$$

[194]

$$(8.17) \quad \Gamma(3/5) = \frac{\pi\sqrt{2}\sqrt{\phi^*}}{\sqrt{5}} \frac{1}{\Gamma(2/5)},$$

where  $\phi^* = 5 - \sqrt{5}$ .

[194][175, A203145]

$$(8.18) \quad \Gamma(5/6) = \frac{\pi^{3/2}2^{4/3}}{\sqrt{3}} \frac{1}{\Gamma^2(1/3)}.$$

[194]

$$(8.19) \quad \Gamma(4/5) = \frac{\pi\sqrt{2}\sqrt{\phi}}{\sqrt{5}} \frac{1}{\Gamma(1/5)},$$

where  $\phi = 5 + \sqrt{5}$ .

[194][175, A203143]

$$(8.20) \quad \Gamma(3/8) = \sqrt{\pi}\sqrt{\sqrt{2}-1} \frac{\Gamma(1/8)}{\Gamma(1/4)}.$$

[194][175, A203144]

$$(8.21) \quad \Gamma(5/8) = \sqrt{\pi}2^{3/4} \frac{\Gamma(1/4)}{\Gamma(1/8)}.$$

[194][175, A203146]

$$(8.22) \quad \Gamma(7/8) = \pi 2^{3/4} \sqrt{\sqrt{2}+1} \frac{1}{\Gamma(1/8)}.$$

[194]

$$(8.23) \quad \Gamma(1/10) = \frac{\sqrt{\phi}}{\sqrt{\pi}2^{7/10}} \Gamma(1/5)\Gamma(2/5),$$

where  $\phi = 5 + \sqrt{5}$ .

[194]

$$(8.24) \quad \Gamma(3/10) = \frac{\sqrt{\pi}\phi^*}{2^{3/5}\sqrt{5}} \frac{\Gamma(1/5)}{\Gamma(2/5)},$$



where  $\phi^* = 5 - \sqrt{5}$ .

[194]

$$(8.25) \quad \Gamma(7/10) = \sqrt{\pi} 2^{3/5} \frac{\Gamma(2/5)}{\Gamma(1/5)}.$$

[194]

$$(8.26) \quad \Gamma(9/10) = \frac{\pi^{3/2} 2^{7/10} \sqrt{\phi}}{\sqrt{5}} \frac{1}{\Gamma(2/5)\Gamma(1/5)}.$$

[194]

$$(8.27) \quad \Gamma(1/12) = \frac{3^{3/8} \sqrt{\sqrt{3} + 1}}{\sqrt{\pi} 2^{1/4}} \Gamma(1/3)\Gamma(1/4).$$

[194]

$$(8.28) \quad \Gamma(5/12) = \frac{\sqrt{\pi} 2^{1/4} \sqrt{\sqrt{3} - 1}}{3^{1/8}} \frac{\Gamma(1/4)}{\Gamma(1/3)}.$$

[164]

$$(8.29) \quad (N + \frac{1}{2}, K)(N + \frac{1}{2}, L) = \sum_{j=\max(K,L)}^{\min(N,K+L)} \binom{2j - K - L}{j - K} (j + \frac{1}{2}, K + L - j)(N + \frac{1}{2}, j),$$

where  $(n + \frac{1}{2}, j) = \frac{(n+j)!}{j!(n-j)!}$  is Hankel's symbol.

[164]

$$(8.30) \quad (-)^N \frac{(N + \frac{1}{2}, K)(N + \frac{1}{2}, L)}{\binom{K+L}{K}} = \sum_{j=\max(K,L)}^{\min(N,K+L)} (-)^j \binom{2j - K - L}{j - K} \frac{(j + 1/2, K + L - j)(N + 1/2, j)}{\binom{K+L}{j}},$$

where  $(n + \frac{1}{2}, j) = \frac{(n+j)!}{j!(n-j)!}$  is Hankel's symbol.

[104]

$$(8.31) \quad \log \frac{\Gamma(z + 1/2)}{\sqrt{z}\Gamma(z)} = - \sum_{r=1}^k \frac{(1 - 2^{-2r})B_{2r}}{r(2r - 1)z^{2r-1}} + O(z^{-2k+1/2}).$$

[104]

$$(8.32) \quad \log \frac{\Gamma(z + 3/4)}{\sqrt{z}\Gamma(z + 1/4)} = - \sum_{r=1}^k \frac{E_{2r}}{4r(4z)^{2r}} + O(z^{-2k+1/2}).$$

[104]

$$(8.33) \quad \frac{1}{z} \left[ \frac{\Gamma(z + 3/4)}{\Gamma(z + 1/4)} \right]^2 = 1 + \frac{2u}{1+} \frac{9u}{1+} \frac{25u}{1+} \frac{49u}{1+} \dots$$

where  $u = 1/(64z^2)$ .

8.4. The psi function  $\psi$ . [6]

$$(8.34) \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} [\psi(k+1) + \gamma] = G - \frac{\pi}{2} \log 2.$$

[6]

$$(8.35) \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} [\psi(k+3/2) + \gamma] = G - \frac{\pi}{4} \log 2.$$

[6]

$$(8.36) \quad \frac{(-1)^n}{q^n (n-1)!} \sum_{k=1}^{q-1} e^{2\pi i k p/q} \psi^{n-1}(k/q) = \text{Li}_n(e^{2\pi i p/q}) - \frac{\zeta(n)}{q^n}.$$

[6]

$$(8.37) \quad \frac{1}{4} \sum_{k=0}^{\infty} \frac{2^k k!^2}{(2k+1)!} [\psi(k+3/2) + \gamma] = G - \frac{\pi}{4} \log 2.$$

[6]

$$(8.38) \quad q \sum_{k=0}^{q-1} \text{Li}_2(e^{2\pi i k/q} t) = \text{Li}_2(t^q).$$

8.5. Integral representations of the functions  $J_\nu$  and  $N_\nu$ . [185, La. 4.13]

$$(8.39) \quad J_{\mu+\nu+1}(t) = \frac{t^{\nu+1}}{2^\nu \Gamma(\nu+1)} \int_0^1 J_\mu(ts) s^{\mu+1} (1-s^2)^\nu ds.$$

[71, 7.2.7][118]

$$(8.40) \quad \Gamma(\nu+1)\Gamma(\mu+1)J_\nu(z)J_\mu(z) = (z/2)^{\nu+\mu} {}_2F_3\left(\frac{1+\nu+\mu}{2}, 1+\frac{\nu+\mu}{2}; 1+\nu, 1+\mu, 1+\nu+\mu; -z^2\right).$$

[24]

$$(8.41) \quad \begin{aligned} \frac{z}{2} J_\mu(z \cos \phi \cos \Phi) J_\nu(z \sin \phi \sin \Phi) &= (\cos \phi \cos \Phi)^\mu (\sin \phi \sin \Phi)^\nu \\ &\times \sum_{n=0}^{\infty} (-1)^n (\mu + \nu + 2n + 1) J_{\mu+\nu+2n+1}(z) \frac{\Gamma(\mu + \nu + n + 1) \Gamma(\nu + n + 1)}{n! \Gamma(\mu + n + 1) \Gamma^2(\nu + 1)} \\ &\times F(-n, \mu + \nu + n + 1; \nu + 1; \sin^2 \phi) F(-n, \mu + \nu + n + 1; \nu + 1; \sin^2 \Phi), \end{aligned}$$

where  $\nu$  and  $\mu$  are not negative integers.

[63]

$$(8.42) \quad J_\mu(Xz) J_\nu(xz) = \frac{X^\mu x^\nu}{\pi} \int_{-\pi/2}^{\pi/2} e^{i(\mu-\nu)\theta} (\lambda_1/\lambda_2)^{\mu+\nu} J_{\mu+\nu}(z\lambda_1\lambda_2) d\theta$$

where  $\lambda_1 = +\sqrt{(e^{i\theta} + e^{-i\theta})}$ ,  $\lambda_2 = (X^2 e^{i\theta} + x^2 e^{-i\theta})^{1/2}$ .

[71, p 99]

$$(8.43) \quad (z/2)^{\gamma-\mu-\nu} J_\mu(\alpha z) J_\nu(\beta z) = \frac{\alpha^\mu \beta^\nu}{\Gamma(\mu+1)\Gamma(\nu+1)} \sum_{m=0}^{\infty} \frac{(\gamma+2m)\Gamma(\gamma+m)}{m!} \\ \times F_4(-m, \gamma+m; \mu+1, \nu+1; \alpha^2, \beta^2) J_{\gamma+2m}(z) \\ (8.44) \\ = \frac{\alpha^\mu \beta^\nu}{\Gamma(\nu+1)} \sum_{m=0}^{\infty} (\gamma+2m) J_{\gamma+2m}(z) \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\gamma+m+n) \alpha^{2n}}{n!(m-n)! \Gamma^2(n+\mu+1)} {}_2F_1(-n, -n-\mu; \nu+1; \frac{\beta^2}{\alpha^2}).$$

Application of [81, (3.5)] to [1, (9.1.14)]

$$(8.45) \quad J_\rho(cz) J_\nu(z) J_\mu(z) = \frac{c^\rho (z/2)^{\nu+\mu+\rho}}{\Gamma(\rho+1)\Gamma(\nu+1)\Gamma(\mu+1)} \sum_{n \geq 0} \frac{(-c^2 z^2/4)^n}{(\rho+1)_n n!} \\ \times {}_4F_3\left(-n, -n-\rho, 1 + \frac{\nu+\mu}{2}, \frac{1+\nu+\mu}{2}; \nu+1, \mu+1, \nu+\mu+1; \frac{4}{c^2}\right).$$

[31]

$$(8.46) \quad x^{-\alpha} I_\alpha(2\sqrt{x}) = \sum_{r=0}^{\infty} \frac{x^r}{r! \Gamma(r+\alpha+1)}.$$

[31][175, A002426]

$$(8.47) \quad \exp(t) I_0(2t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} c_n; \quad c_n \equiv \sum_{k=0}^{[n/2]} \frac{n!}{(k!)^2 (n-2k)!} = i^n \sqrt{3^n} P_n\left(-\frac{i}{\sqrt{3}}\right).$$

[31]

$$(8.48) \quad \exp(yt) \exp(xt^2)^{-\alpha/2} I_\alpha(2t\sqrt{x}) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \Pi_n^\alpha(x, y); \quad \Pi_n^\alpha(x, y) \equiv n! \sum_{k=0}^{[n/2]} \frac{x^k y^{n-2k}}{(n-2k)! k! \Gamma(k+\alpha+1)}.$$

$$(8.49) \quad \frac{\exp(t)}{t} I_1(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \Pi_n^1(1, 1).$$

[146]

$$(8.50) \quad j_s(R) = \frac{R^s}{2^{s+1} s!} \int_0^\pi d\theta \sin \theta \cos(R \cos \theta) (\sin \theta)^{2s}.$$

[175, A122848]

$$(8.51) \quad \frac{d^s}{dt^s} t^\nu Z_\nu(t) = \sum_{l=\lfloor (s+1)/2 \rfloor}^s \alpha_{l,s} t^{\nu-(s-l)} Z_{\nu-l}(t).$$

$$(8.52) \quad \frac{d^s}{dt^s} t^\nu K_\nu(t) = (-)^s \sum_{l=\lfloor (s+1)/2 \rfloor}^s \alpha_{l,s} t^{\nu-(s-l)} K_{\nu-l}(t)$$

with Bessel polynomial coefficients [97]

$$\alpha_{l,s} = \frac{s!}{(s-l)! (2l-s)! 2^{s-l}}.$$

[164]  
(8.53)

$$\sum_{k=1}^n \frac{2k-1}{\sqrt{\pi}} K_{k-1/2}(x) K_{k-1/2}(y) = \sum_{k=1}^n \frac{n(n-1+k)!}{k!(n-k)!} \left(\frac{x+y}{2xy}\right)^{k-1/2} K_{k-1/2}(x+y).$$

[164]  
(8.54)

$$\frac{1}{\sqrt{\pi}} K_{n+1/2}(x) K_{n+1/2}(y) = \sum_{\mu=0}^n \frac{(n+\mu)!}{\mu!(n-\mu)!} \left(\frac{x+y}{2xy}\right)^{\mu+1/2} K_{k+1/2}(x+y).$$

[87]  
(8.55)

$$\sum_{n=1}^{\infty} (-1)^n \frac{J_{2m}(n\pi)}{a^2 - n^2} = \frac{\pi J_{2m}(a\pi)}{2a \sin a\pi}.$$

[87]  
(8.56)

$$\sum_{n=1}^{\infty} (-1)^n \frac{n J_{2m-1}(n\pi)}{a^2 - n^2} = \frac{\pi J_{2m-1}(a\pi)}{2 \sin a\pi}.$$

[168]  
(8.57)

$$z^m [K_{\nu+m}(z) - K_{\nu-m}(z)] = \sum_{j=0}^{m-1} (-1)^{m-j-1} b_m(j) z^j K_{\nu+j}(z),$$

(8.58)

$$z^m [I_{\nu-m}(z) - I_{\nu+m}(z)] = \sum_{j=0}^{m-1} b_m(j) z^j I_{\nu+j}(z),$$

(8.59)

$$z^m [J_{\nu-m}(z) - (-1)^m I_{\nu-m}(z)] = \sum_{j=0}^{m-1} b_m(j) (-z)^j J_{\nu+j}(z),$$

with  $m = 1, 2, 3, \dots$  and

$$b_m(j) = 2^{m-j} \binom{m}{j} \nu(\nu-1) \cdots (\nu+j-m+1) = \Gamma(\nu+1) \frac{2^{m-j} \binom{m}{j}}{\Gamma(\nu+j-m+1)}.$$

[71, §7.10.1]

(8.60)

$$f(z) = \frac{1}{z^\nu} \sum_{n=0}^{\infty} a_n J_{\nu+n}(z); \quad a_n = (\nu+n) 2^{\nu+n} \sum_{s=0}^{\lfloor n/2 \rfloor} \frac{\Gamma(\nu+n-s)}{s! 2^{2s}} b_{n-2s},$$

where  $f(z) = \sum_{n=0}^{\infty} b_n z^n$ .

[71, §7.10.1]

(8.61)

$$f(z) = \frac{1}{z^\nu} \sum_{n=0}^{\infty} a_n z^n J_{\nu+n}(z); \quad a_n = \sum_{s=0}^n \frac{\Gamma(\nu+s+1)}{(n-s)!} 2^{2s-n+\nu} b_s,$$

(8.62)

$$\Gamma(\nu+n+1) b_n = \sum_{s=0}^{\infty} (-1)^s 2^{-\nu-n-s} \frac{a_{n-s}}{s!}.$$

where  $f(z) = \sum_{l=0}^{\infty} b_l z^{2l}$ .

[25]  
 (8.63) 
$$[z^{2n}] \frac{1}{z^{2\nu}} \sum_{r=0}^{\infty} A_r J_{\nu+r}^2(z) = \frac{(-)^n \Gamma(\nu + n + 1/2)}{n! \sqrt{\pi} \Gamma(2\nu + n + 1) \Gamma(\nu + n + 1)} \sum_{r=0}^n \frac{A_r (-n)_r}{(2\nu + n + 1)_r}.$$

[25]  
 (8.64) 
$$\frac{1}{z^{2\nu}} \sum_{r=0}^{\infty} \frac{(2\nu)_r (\nu + 1)_r (a_3)_r (a_4)_r}{r! (\nu)_r (1 + 2\nu - a_3)_r (1 + 2\nu - a_4)_r} J_{\nu+r}^2(z) = \frac{1}{4\nu \Gamma^2(1 + \nu)} {}_2F_3 \left( \begin{matrix} \nu + 1/2, 1 + 2\nu - a_3 - a_4 \\ \nu + 1, 1 + 2\nu - a_3, 1 + 2\nu - a_4 \end{matrix} \mid -z^2 \right)$$

[25]  
 (8.65) 
$$z^\nu J_\nu(2z) = \frac{2\sqrt{\pi}}{\Gamma(\nu + 1/2)} \sum_{r=0}^{\infty} (-)^r (\nu + r) \frac{\Gamma(2\nu + r)}{r!} J_{\nu+r}^2(z).$$

[25]  
 (8.66) 
$$z^{-1/2} J_{2\nu+1/2}(2z) = -\frac{\Gamma(\nu + 1)}{\Gamma(\nu + 1/2)} \sum_{r=0}^{\infty} (\nu + r) \frac{\Gamma(2\nu + r) \Gamma(r - 1/2)}{r! \Gamma(2\nu + r + 3/2)} J_{\nu+r}^2(z).$$

[25]  
 (8.67) 
$$[z^{2n}] \frac{1}{z^{2\nu}} \sum_{r=0}^{\infty} A_r J_{\nu+2r}^2(z) = \frac{(-)^n \Gamma(\nu + n + 1/2)}{n! \sqrt{\pi} \Gamma(2\nu + n + 1) \Gamma(\nu + n + 1)} \sum_{r=0}^n \frac{A_r (-n)_{2r}}{(2\nu + n + 1)_{2r}}.$$

[25]  
 (8.68) 
$$\frac{1}{z^{2\nu}} \sum_{r=0}^{\infty} \frac{(\nu)_r (1 + \nu/2)_r (a_3)_r}{r! (\nu/2)_r (1 + \nu - a_3)_r} J_{\nu+2r}^2(z) = \frac{1}{4\nu \Gamma^2(1 + \nu)} {}_1F_2 \left( \begin{matrix} 1/2 + \nu - a_3 \\ \nu + 1, 1 + 2\nu - 2a_3 \end{matrix} \mid -z^2 \right).$$

[25]  
 (8.69) 
$$\frac{J_\lambda(2z \sin \frac{1}{2}\phi)}{(2z \sin \frac{1}{2}\phi)^{\lambda-2\nu}} = \sum_{n=0}^{\infty} \frac{2^{1-\lambda+2\nu} \sqrt{\pi} \Gamma(\nu + 1)}{n! \Gamma(\nu + 1/2) \Gamma(\lambda + 1)} \sin^{2\nu} \frac{1}{2}\phi (\nu+n) \Gamma(2\nu+n) J_{\nu+n}^2(z) {}_3F_2 \left( \begin{matrix} \nu + 1, 2\nu + n, -n \\ \nu + 1/2, \lambda + \end{matrix} \mid \sin^2 \frac{1}{2}\phi \right)$$

(8.70) 
$$J_0(z \cos a) + J_0(z \sin a) \equiv 2 \sum_{n=0}^{\infty} b_{2n}(a) J_{2n}(z);$$

where

$2n$	$b_{2n}$
0	1
2	1
4	$1 + 6 \cos^4 a - 6 \cos^2 a = \frac{1}{4} + \frac{3}{4} \cos(4a)$
6	$1 + 6 \cos^4 a - 6 \cos^2 a = \frac{1}{4} + \frac{3}{4} \cos(4a)$
8	$1 + 70 \cos^8 a - 20 \cos^2 a - 140 \cos^6 a + 90 \cos^4 a = \frac{9}{64} + \frac{35}{64} \cos(8a) + \frac{5}{16} \cos(4a)$
10	$1 + 70 \cos^8 a - 20 \cos^2 a - 140 \cos^6 a + 90 \cos^4 a = \frac{9}{64} + \frac{35}{64} \cos(8a) + \frac{5}{16} \cos(4a)$
12	$1 - 42 \cos^2 a + 420 \cos^4 a - 1680 \cos^6 a + 3150 \cos^8 a - 2772 \cos^{10} a + 924 \cos^{12} a$ $= \frac{25}{256} + \frac{104}{512} \cos(4a) + \frac{63}{256} \cos(8a) + \frac{231}{512} \cos(12a)$

Apparently, the  $b_{2n}$  count numbers of Delannoy paths, A109983 in [175].

[122, p31]

$$(8.71) \quad e^{ik \cos \varphi} = 2^\nu \Gamma(\nu) \sum_{m=0}^{\infty} (\nu + m) i^m J_{\nu+m}(k\rho) (k\rho)^{-\nu} C_m^{(\nu)}(\cos \varphi),$$

$\nu \neq 0, -1, -2, -3, \dots$

[82]

(8.72)

$$S_{\mu,\nu}(z) \sim z^{\mu-1} \sum_{m=0}^{\infty} (-1)^m \left(\frac{1-\mu+\nu}{2}\right)_m \left(\frac{1-\mu-\nu}{2}\right)_m (z/2)^{-2m}, \quad |z| \rightarrow \infty, |\arg z| < \pi.$$

The series terminates and is equal to  $S_{\mu,\nu}(z)$  when  $\mu \pm \nu$  is a positive odd integer.

[158]

(8.73)

$$\left(\frac{\sin \beta}{\sin \alpha}\right)^{m+n} P_{m+n}^m(\cos \alpha) = \sum_{r=0}^n \binom{2m+n}{r} \left(\frac{\sin(\beta-\alpha)}{\sin \alpha}\right)^r P_{n+m-r}^m(\cos \beta).$$

### 8.6. Orthogonal Polynomials. [197]

(8.74)

$$L_m^{(\alpha)} L_n^{(\alpha)} = \sum_{M=|m-n|}^{m+n} \frac{L_M^{(\alpha)}(z)(-)^{m+n-M} 2^{2m+n-M} M!}{(M-m)!(M-n)!(m+n-M)!} {}_3F_2 \left( \begin{matrix} \alpha + M + 1, \frac{M-m-n}{2}, \frac{M-m-n+1}{2} \\ M-m+1, M-n+1 \end{matrix} \middle| 1 \right).$$

[54]

$$(8.75) \quad \sum_{n \geq 0} t^n L_n^{a+bn}(x) = \frac{(1+v)^{a+1}}{1-bv} \exp(-xv),$$

where  $v = t(1+v)^{b+1}$ ,  $v(0) = 0$ .

[54]

$$(8.76) \quad \sum_{n \geq 0} t^n L_n^{v+bn}(x(1+an)) = \frac{(1-z)^{1-v}}{1-z(b+2-ax) + z^2(b+1)} e^{xz/(z-1)},$$

where  $t = z(1-z)^b \exp[axz/(1-z)]$  and  $|t| < 1$ .

[54]

(8.77)

$$\sum_{n \geq 0} \frac{t^n}{v+bn+n} L_n^{v+bn}(x(1+an)) = \frac{\exp[xz/(z-1)]}{v(1-z)^v} {}_1F_1 \left( \begin{matrix} 1 \\ \frac{v+1+b}{1+b} \end{matrix} \middle| \frac{xz(1+b-av)}{(1-z)(1+b)} \right),$$

where  $t = z(1-z)^b \exp[axz/(1-z)]$  and  $|t| < 1$ .

[54]

$$(8.78) \quad \sum_{n \geq 0} \frac{t^n}{1+an} L_n^{v+avn-n}(x(1+an)) = \frac{1}{(1-z)^v} e^{xz/(z-1)},$$

where  $t = z(1-z)^{av-1} \exp[axz/(1-z)]$  and  $|t| < 1$ .

[54]

$$(8.79) \quad \sum_{n \geq 0} \frac{t^n}{1+an} L_n^{v+bn}(x(1+an)) = \frac{\exp[xz/(z-1)]}{(1-z)^v} {}_2F_1 \left( \begin{matrix} 1, \frac{1+b-av}{a} \\ \frac{a+1}{a} \end{matrix} \middle| z \right),$$

where  $t = z(1-z)^b \exp[axz/(1-z)]$  and  $|t| < 1$ .

[54]

$$(8.80) \quad \sum_{n \geq 0} \frac{t^n}{1+n} L_n^{v+bn}(x(1+an)) = \frac{\exp[x(1-a-z)/(1-z)]}{(1-z)^v z(v-b)} \left\{ {}_1F_1 \left( \begin{matrix} v-b \\ v-b+1 \end{matrix} \middle| \frac{x(a-1)}{1-z} \right) - (1-z)^{v-b} {}_1F_1 \left( \begin{matrix} v-b \\ v-b+1 \end{matrix} \middle| x(a-1) \right) \right\},$$

where  $t = z(1-z)^b \exp[axz/(1-z)]$  and  $|t| < 1$ .

[54]

$$(8.81) \quad \sum_{n \geq 0} \frac{(1+bn)^{n/2}}{n!} t^n H_n[x(1+an)/(1+bn)^{1/2}] = \frac{e^{-z^2-2xz}}{1+2bz^2+2axz},$$

where  $t = (-z)e^{bz^2+2axz}$  and  $|2axz \exp[bz^2+2axz+1]| < 1$ .

[54]

$$(8.82) \quad \sum_{n \geq 0} \frac{(1+bn)^{n/2-1}}{n!} t^n H_n[x(1+an)/(1+bn)^{1/2}] = e^{-z^2-2xz} {}_1F_1 \left( \begin{matrix} 1 \\ 1+1/b \end{matrix} \middle| 2xz(b-a)/b \right)$$

where  $t = (-z)e^{bz^2+2axz}$  and  $|2axz \exp[bz^2+2axz+1]| < 1$ .

[54]

$$(8.83) \quad \sum_{n \geq 0} \frac{(1+an)^{n/2-1}}{n!} t^n H_n[x(1+an)] = e^{-z^2-2xz}$$

where  $t = (-z)e^{az^2+2axz}$  and  $|2axz \exp[az^2+2axz+1]| < 1$ .

[83]

$$(8.84) \quad \int_0^1 f(t)t^r dt = c_r, \quad (r = 0, 1, \dots, n) \rightsquigarrow f(t) = \sum_{i=0}^n (2i+1) \sum_{r=0}^i \left\{ (-)^r \frac{(i+r)!}{(i-r)!} \frac{1}{(r!)^2} c_r \right\} P_i(1-2t).$$

## 9. SPECIAL FUNCTIONS II.

### 9.1. Hypergeometric Functions. [29]

$$(9.1) \quad {}_2F_1(a, b; c; z) = (a)_m (b)_m z^m {}_2F_1(a+m, b+m; 1+m; z).$$

[29]

$$(9.2) \quad {}_2F_1(a, b; c; z) = (1-z)^{c-a-b} {}_2F_1(c-a, c-b; c; z).$$

[29]

$$(9.3) \quad {}_2F_1(a, b; c; z) = (a)_m (b)_m z^m (1-z)^{1-a-b-m} {}_2F_1(1-b, 1-a; 1+m; z).$$

[29]

$$(9.4) \quad {}_2F_1(a, b; c; z) = (1-z)^{-a} {}_2F_1(a, c-b; c; z/(z-1)).$$

[145]

$$(9.5) \quad \frac{(z/2)^c}{\Gamma(1+c)} {}_2F_1(a, b; c+1; -z^2/(4ab)) = \sum_{\nu \geq 0} \frac{1}{\nu!} {}_3F_0(-\nu, a, b; 1/(ab))(z/2)^\nu J_{c+\nu}(z).$$

[29]

$$(9.6) \quad {}_2F_1(a, b; c; z) = (1-z)^{-a} \left( \frac{z}{z-1} \right)^{1-c} \left\{ \frac{\Gamma(|m|)}{\Gamma(a+\bar{m})\Gamma(c-a-\underline{m})} \left( \frac{1}{1-z} \right)^{\bar{m}} \right. \\ \left. \sum_{n=0}^{|\underline{m}|-1} \frac{(1-a-\bar{m})_n(1-c+a+\underline{m})_n}{(1-|m|)_n\Gamma(1+n)} \left( \frac{1}{1-z} \right)^n - \frac{(-)^m}{\Gamma(a+\underline{m})\Gamma(c-a-\bar{m})} \left( \frac{1}{1-z} \right)^{\bar{m}} \right. \\ \left. \sum_{n=0}^{\infty} \frac{(1-a-\underline{m})_n(1-c+a+\bar{m})_n}{\Gamma(1+|m|+n)\Gamma(1+n)} \left( \frac{1}{1-z} \right)^n \right. \\ \left. [-\ln(1-z) - \pi \cot \pi(c-a) - \pi \cot \pi a + \psi(1-a-\underline{m}+n) + \psi(1-c+a+\bar{m}+n) - \psi(1+|m|+n) - \psi(1+n)] \right\},$$

where  $\bar{m} \equiv \max(0, m)$ ,  $\underline{m} \equiv \min(0, m)$ .

[163]

$$(9.7) \quad {}_1F_1(1/2; 9/2; z) = -\frac{525 + 280z + 140z^2}{128z^3} e^z + \frac{525 + 630z + 420z^2 + 280z^3}{256z^{7/2}} \sqrt{\pi} \operatorname{erfi}(\sqrt{z}).$$

[173]

$$(9.8) \quad {}_1F_1\left(\frac{1}{2} + \frac{1}{2}a - b, 1 + a - b; x\right) = e^{x/2} \sum_{r=0}^{\infty} \frac{(a)_r (b)_r (-x/4)^r}{r!(a/2)_r (1+a-b)_r} {}_0F_1\left(\frac{1}{2}a + r + 1; (x/4)^2\right).$$

[174, 1.7.7]

$$(9.9) \quad {}_2F_1(a, -m; c, 1) = \frac{(c-a)_m}{(c)_m}.$$

[47]

$$(9.10) \quad {}_2F_1(a, -n; c; p) = \frac{(a)_n}{(c)_n} (-p)^n {}_2F_1(1-c-n, -n; 1-a-n; 1/p).$$

[193]

$$(9.11) \quad {}_2F_1(a+n, b; a-b; -1) = P(n) \frac{\Gamma(a-b)\Gamma(\frac{a+1}{2})}{\Gamma(a)\Gamma(\frac{a+1}{2}-b)} + Q(n) \frac{\Gamma(a-b)\Gamma(\frac{a}{2})}{\Gamma(a)\Gamma(\frac{a}{2}-b)}$$

where

$$P(n) = \frac{1}{2^{n+1}} {}_3F_2(-n/2, -(n+1)/2, a/2-b; 1/2, a/2; 1); \\ Q(n) = \frac{n+1}{2^{n+1}} {}_3F_2(-(n-1)/2, -n/2, (a+1)/2-b; 3/2, (a+1)/2; 1).$$

[13]

$$(9.12) \quad {}_2F_1(-2n, b; -2n+2r-b; -1) = \frac{(1/2)_n (b+1-r)_n}{(b/2+1-r)_n (b/2+1/2-r)_n} \sum_{i=0}^{r-1} \frac{2^{2i} i! \binom{r+i-1}{2i}}{(b-r+1)_i} \binom{n}{i}.$$



[193]

(9.13)

$${}_2F_1(-a, 1/2; 2a+3/2+n; 1/4) = \frac{2^{n+3/2}}{3^{n+1}} \frac{\Gamma(a+5/4+n/2)\Gamma(a+3/4+n/2)\Gamma(a+1/2)}{\Gamma(a+7/6+n/3)\Gamma(a+5/6+n/3)\Gamma(a+1/2+n/3)} K(n) \\ - (-3)^{n-2} 2^{3/2} \frac{\Gamma(a+5/4+n/2)\Gamma(a+3/4+n/2)\Gamma(a+1)}{\Gamma(a+3/2)\Gamma(a+1/2+n/2)\Gamma(a+1+n/2)} L(n)$$

where  $K(n)$  and  $L(n)$  are defined in the reference.

[30]

(9.14)

$$\pi {}_2F_1(1/2, 1/2; 1; 1-x) = \log \frac{16}{x} {}_2F_1(1/2, 1/2; 1; x) - 4 \sum_{k \geq 1} \frac{(1/2)_k^2}{(k!)^2} \sum_{j=1}^k \frac{x^k}{(2j-1)(2j)}.$$

[30]

(9.15)

$${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{1+x}{2}\right) = \frac{\sqrt{\pi}}{\Gamma^2(3/4)} {}_2F_1(1/4, 1/4; 1/2; x^2) + \frac{\Gamma^2(3/4)}{\pi^{3/2}} {}_2F_1(3/4, 3/4; 3/2; x^2).$$

[30]

(9.16)

$${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{1}{2} + \frac{x}{1+x^2}\right) = \frac{\sqrt{\pi}}{\Gamma^2(3/4)} \sqrt{1+x^2} {}_2F_1(1/4, 1/2; 3/4; x^4) + \frac{\Gamma^2(3/4)}{\pi^{3/2}} x(1+x^2)^{3/2} {}_2F_1(1/2, 3/4; 5/4; x^4).$$

[30]

(9.17)

$${}_2F_1(n, -n; 1/2; x^2) = \cos(2n \sin^{-1} x).$$

[30]

(9.18)

$$2nx {}_2F_1\left(\frac{1}{2} + n, \frac{1}{2} - n; 3/2; x^2\right) = \sin(2n \sin^{-1} x).$$

[30]

(9.19)

$${}_2F_1\left(\frac{1}{2} + n, \frac{1}{2} - n; 1/2; x^2\right) = (1-x^2)^{-1/2} \cos(2n \sin^{-1} x).$$

[163]

(9.20)

$${}_2F_1(-3/2, -1/2; 1/2, z) = \frac{2+z}{2} \sqrt{1-z} + \frac{3\sqrt{z}}{2} \arcsin \sqrt{z}.$$

[163]

(9.21)

$$\frac{z^{\mu+1}}{2^\mu \sqrt{\pi} \Gamma(\frac{3}{2} + \mu)} {}_2F_1(1, 3/2; 3/2 + \mu, z^2/4) = L_\mu(z).$$

[163]

(9.22)

$${}_2F_1(-3/2, 1/2; 3/2, z) = \frac{5-2z}{8} \sqrt{1-z} + \frac{3}{8\sqrt{z}} \arcsin \sqrt{z}.$$

[163]

(9.23)

$${}_2F_1(-3/2, -1/2; 3/2, z) = \frac{13+2z}{16} \sqrt{1-z} + \frac{3+12z}{16\sqrt{z}} \arcsin \sqrt{z}.$$

[163]  
 (9.24)  ${}_1F_2(-3/2; -1/2, 1/2; z) = (1 + 2z) \cosh(2\sqrt{z}) + \sqrt{z} \sinh(2\sqrt{z}) - 4z^{3/2} \text{Shi}(2\sqrt{z}).$

[163]  
 (9.25)  ${}_1F_2(-3/2; -1/2, 2; z) = -\frac{4 + 24z - 28z^2}{15z\pi} K(\sqrt{z}) + \frac{4 + 56z + 4z^2}{15z\pi} E(\sqrt{z}).$

[195]  
 (9.26)  ${}_2F_1\left(1/4, -1/12; 2/3; \frac{x(4+x)^3}{4(2x-1)^3}\right) = (1-2x)^{-1/4}.$

[195]  
 (9.27)  ${}_2F_1\left(5/4, -1/12; 5/3; \frac{x(4+x)^3}{4(2x-1)^3}\right) = \frac{1+x}{(1+\frac{1}{4}x)^2} (1-2x)^{-1/4}.$

[195]  
 (9.28)  ${}_2F_1\left(1/4, 7/12; 4/3; \frac{x(4+x)^3}{4(2x-1)^3}\right) = \frac{1}{1+\frac{1}{4}x} (1-2x)^{3/4}.$

[195]  
 (9.29)  ${}_2F_1\left(1/4, -5/12; 1/3; \frac{x(4+x)^3}{4(2x-1)^3}\right) = (1+\frac{5}{2}x)(1-2x)^{-5/4}.$

[195]  
 (9.30)  ${}_2F_1\left(1/2, -1/6; 2/3; \frac{x(2+x)^3}{(2x+1)^3}\right) = (1+2x)^{-1/2}.$

[195]  
 (9.31)  ${}_2F_1\left(1/2, 5/6; 2/3; \frac{x(2+x)^3}{(2x+1)^3}\right) = \frac{1}{(1-x)^2} (1+2x)^{3/2}.$

[195]  
 (9.32)  ${}_2F_1\left(1/6, 5/6; 4/3; \frac{x(2+x)^3}{(2x+1)^3}\right) = \frac{1}{1+\frac{1}{2}x} (1+2x)^{1/2} (1+x)^{1/3}.$

[195]  
 (9.33)  ${}_2F_1\left(1/6, -1/6; 1/3; \frac{x(2+x)^3}{(2x+1)^3}\right) = (1+2x)^{-1/2} (1+x)^{1/3}.$

[195]  
 (9.34)  ${}_2F_1\left(7/24, -1/24; 3/4; \frac{108x(x-1)^4}{(x^2+14x+1)^3}\right) = (1+14x+x^2)^{-1/8}.$

[195]  
 (9.35)  ${}_2F_1\left(7/24, 23/24; 7/4; \frac{108x(x-1)^4}{(x^2+14x+1)^3}\right) = \frac{1+2x-\frac{1}{11}x^2}{(1-x)^2} (1+14x+x^2)^{7/8}.$

[195]  
 (9.36)  ${}_2F_1\left(5/24, 13/24; 5/4; \frac{108x(x-1)^4}{(x^2+14x+1)^3}\right) = \frac{1}{1-x} (1+14x+x^2)^{5/8}.$

[195]

$$(9.37) \quad {}_2F_1\left(5/24, -11/24; 1/4; \frac{108x(x-1)^4}{(x^2+14x+1)^3}\right) = \frac{1-22x-11x^2}{(1+14x+x^2)^{11/8}}.$$

[195]

$$(9.38) \quad {}_2F_1(19/60, -1/60; 4/5; \varphi_1(x)) = (1-228x+494x^2+228x^3+x^4)^{-1/20}.$$

(9.39)

$${}_2F_1(19/60, 59/60; 4/5; \varphi_1(x)) = \frac{(1+66x-11x^2)(1-228x+494x^2+228x^3+x^4)^{19/20}}{(1+x^2)(1+522x-10006x^2-522x^3+x^4)}.$$

$$(9.40) \quad {}_2F_1(11/60, 31/60; 6/5; \varphi_1(x)) = \frac{(1-228x+494x^2+228x^3+x^4)^{11/20}}{1+11x-x^2}.$$

$$(9.41) \quad {}_2F_1(11/60, -29/60; 1/5; \varphi_1(x)) = \frac{1+435x-6670x^2-3335x^4-87x^5}{(1-228x+494x^2+228x^3+x^4)^{29/20}}.$$

$$(9.42) \quad {}_2F_1(13/60, -7/60; 3/5; \varphi_1(x)) = \frac{1-7x}{(1-228x+494x^2+228x^3+x^4)^{7/20}}.$$

(9.43)

$${}_2F_1(13/60, 53/60; 3/5; \varphi_1(x)) = \frac{(1+119x+187x^2+17x^3)(1-228x+494x^2+228x^3+x^4)^{13/20}}{(1+x^2)(1+522x-10006x^2-522x^3+x^4)}.$$

(9.44)

$${}_2F_1(17/60, 37/60; 7/5; \varphi_1(x)) = \frac{(1+\frac{1}{7}x)(1-228x+494x^2+228x^3+x^4)^{17/20}}{(1+11x-x^2)^2}.$$

$$(9.45) \quad {}_2F_1(17/60, -23/60; 2/5; \varphi_1(x)) = \frac{(1+107x-391x^2+1173x^3+46x^4)}{(1-228x+494x^2+228x^3+x^4)^{23/20}}.$$

Where

$$(9.46) \quad \varphi_1(x) = \frac{1728x(x^2-11x-1)^5}{(x^4+228x^3+494x^2-228x+1)^3}.$$

[195]

$$(9.47) \quad {}_2F_1(7/20, -1/20; 4/5; \varphi_2(x)) = \frac{(1+x)^{7/20}}{(1-x)^{1/20}(1-4x-x^2)^{1/4}}.$$

(9.48)

$${}_2F_1(7/20, 19/20; 4/5; \varphi_2(x)) = \frac{(1+3x)(1+x)^{7/20}(1-x)^{19/20}(1-4x-x^2)^{7/4}}{(1+x^2)(1+22x-6x^2-22x^3+x^4)}.$$

Where

$$(9.49) \quad \varphi_2(x) = \frac{64x(x^2-x-1)^5}{(x^2-1)(x^2+4x-1)^5}.$$

[151, 51]

$$(9.50) \quad {}_2F_2(a, d; b, c; x) = e^x \sum_{n \geq 0} \frac{(c-d)_n}{(c)_n n!} (-x)^n {}_2F_2(b-a, d; b, c+n; -x).$$

[140]

$$(9.51) \quad {}_{r+1}F_{r+1}(a, (f_r+1); b, (f_r); y) = e^y {}_{r+1}F_{r+1}(b-a-r, (\xi_r+1); b, (\xi_r); -y),$$

where  $\xi_r$  are nonvanishing zeros of an associated parametric polynomial of  $Q$  degree  $r$ ,

$$(9.52) \quad Q_r(t) = \sum_{j=0}^r s_{r-j} \sum_{l=0}^j \left\{ \begin{matrix} j \\ l \end{matrix} \right\} (a)_l (t)_l (b-a-r-t)_{r-l},$$

and the  $s_{r-j}$  are determined by

$$(9.53) \quad (f_1 + x) \cdots (f + r + x) = \sum_{j=0}^r s_{r-j} x^j.$$

[11]

$$(9.54) \quad {}_3F_2(1, 1, 2-t; 2, 3; 1) = \frac{2(1-\gamma-\psi(t+1))}{1-t}.$$

[166]

$$(9.55) \quad {}_3F_2(-n, -a, -b; c, 2-n-a-b-c; 1) = \frac{(c+b-1)_n (c+a)_n}{(c+a+b-1)_n (c)_n} \left[ 1 - \frac{a}{(c+b-1)(a+c+n-1)} \right].$$

for  $n = 1, 2, 3, \dots$

[113][174, (2.3.1.3)]

$$(9.56) \quad {}_3F_2(-m, b, c; e, -m+b+c-e+1; 1) = \frac{(e-b)_m (e-c)_m}{(e)_m (e-b-c)_m}; \quad m = 0, 1, 2, \dots, \quad e, e-b-c \neq 0, -1, -2, \dots$$

[64]

$$(9.57) \quad {}_3F_2(a, b/2, (b+1)/2; c/2, (c+1)/2; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-b)\Gamma(c-a)} {}_2F_1(a, b; c-a; -1), \quad \Re c > \Re b > 0, \Re(c-a-b) > 0.$$

[64]

$$(9.58) \quad {}_3F_2(a, b/2, (b+1)/2; c/2, (c+1)/2; 1/2) = 2^a \sum_{k=0}^{\infty} \binom{-a}{k} \frac{(c-b)_k}{(c)_k} {}_2F_1(-k, b; c+k; -1), \quad \Re c > \Re b > 0.$$

[47]

$$(9.59) \quad {}_0F_1(a; px) {}_0F_1(c'; qx) = \sum_{n \geq 0} \frac{(px)^n}{n!(c)_n} {}_2F_1(1-c-n, -n; c'; q/p).$$

[47]

$$(9.60) \quad {}_1F_1(a; c; px) {}_1F_1(a'; c'; qx) = \sum_{n \geq 0} \frac{(a)_n (px)^n}{n!(c)_n} {}_3F_2(a', 1-c-n, -n; c', 1-a-n; -q/p).$$

[47]

$$(9.61) \quad {}_2F_0(a, b; px) {}_2F_0(a', b'; qx) = \sum_{n \geq 0} \frac{(a)_n (b)_n (px)^n}{n!} {}_3F_2(a', b', -n; 1-a-n, 1-b-n; -q/p).$$

[47]  
 (9.62)

$${}_2F_1(a, b; c; px) {}_2F_1(a', b'; c'; qx) = \sum_{n \geq 0} \frac{(a)_n (b)_n (px)^n}{n! (c)_n} {}_4F_3(a', b', 1-c-n, -n; c', 1-a-n, 1-b-n; q/p).$$

[202]  
 (9.63)

$$(a+b+1) {}_3F_2[-c, -a, 1; b+1, \frac{1-a-c}{2}; \frac{1}{2}] = (b+1) {}_3F_2[-\frac{a}{2}, \frac{1-a}{2}, 1; \frac{1-a-b}{2}, \frac{1-a-c}{2}; 1]$$

where  $a, b$  and  $c$  are positive integers of the same parity.

[64]  
 (9.64)

$${}_3F_2(-n, b/q, (b+1)/2; c/2, (c+1)/2; 1) = \frac{(c-b)_n}{(c)_n} {}_2F_1(-n, b; c+n; -1), \quad \Re c > \Re b > 0.$$

[113]

$$\begin{aligned} & {}_3F_2(a, b, c; e, a+b+c-e+1; 1) + \frac{\Gamma(e-1)\Gamma(a-e+1)\Gamma(b-e+1)\Gamma(c-e+1)\Gamma(a+b+c-e+1}{\Gamma(1-e)\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(a+b+c-2e+2)} \\ & \quad \times {}_3F_2(a-e+1, b-e+1, c-e+1; 2-e, a+b+c-2e+2; 1) \\ & \quad = \frac{\Gamma(a-e+1)\Gamma(b-e+1)\Gamma(c-e+1)\Gamma(a+b+c-e+1)}{\Gamma(1-e)\Gamma(b+c-e+1)\Gamma(a+c-e+1)\Gamma(a+b-e+1)}. \end{aligned}$$

[113]  
 (9.65)

$${}_3F_2(a, b, c; e, a+b+c-e+1; 1) = \frac{\Gamma(e)\Gamma(a+b+c-e+1)}{\Gamma(a)\Gamma(b+1)\Gamma(c+1)} {}_3F_2(e-a, b+c-e+1, 1; b+1, c+1; 1); \quad \Re a > 0.$$

[160][139]

$${}_3F_2(a, b, c; f, e; 1)$$

(9.66)

$$= \frac{\Gamma(f)\Gamma(e)\Gamma(f+e-a-b-c)}{\Gamma(e-b+f-c)\Gamma(f+e-a-c)} {}_3F_2(e+f-a-b-c, f-c, e-c; e-b+f-c, e+f-a-c; 1)$$

(9.67)

$$= \frac{\Gamma(f)\Gamma(e)\Gamma(f+e-a-b-c)}{\Gamma(e-b+f-c)\Gamma(e-b+f-a)} {}_3F_2(e+f-a-b-c, f-b, e-b; e-b+f-c, e-b+f-a; 1)$$

(9.68)

$$= \frac{\Gamma(e)\Gamma(f+e-a-b-c)}{\Gamma(e-a)\Gamma(e-b+f-c)} {}_3F_2(f-c, f-b, a; e-b+f-c, f; 1)$$

(9.69)

$$= \frac{\Gamma(f)\Gamma(f+e-a-b-c)}{\Gamma(f-a)\Gamma(e-b+f-c)} {}_3F_2(e-c, e-b, a; e-b+f-c, e; 1)$$

(9.70)

$$= \frac{\Gamma(e)\Gamma(f)\Gamma(f+e-a-b-c)}{\Gamma(e+f-a-c)\Gamma(e-b+f-a)} {}_3F_2(e+f-a-b-c, f-a, e-a; e+f-a-c, e-b+f-a; 1)$$

(9.71)

$$= \frac{\Gamma(e)\Gamma(f+e-a-b-c)}{\Gamma(e-b)\Gamma(e+f-a-c)} {}_3F_2(f-c, f-a, b; e+f-a-c, f; 1)$$

(9.72)

$$= \frac{\Gamma(f)\Gamma(f+e-a-b-c)}{\Gamma(f-b)\Gamma(e+f-a-c)} {}_3F_2(e-c, e-a, b; e+f-a-c, e; 1)$$

(9.73)

$$= \frac{\Gamma(e)\Gamma(f+e-a-b-c)}{\Gamma(e-c)\Gamma(e-b+f-a)} {}_3F_2(f-b, f-a, c; e+f-a-b, f; 1)$$

(9.74)

$$= \frac{\Gamma(f)\Gamma(f+e-a-b-c)}{\Gamma(f-c)\Gamma(e-b+f-a)} {}_3F_2(e-b, e-a, c; e+f-a-b, e; 1).$$

[157, (6)]

(9.75)

$${}_3F_2(-n, \alpha, \beta; \gamma, \delta; 1) = \frac{\Gamma(\gamma)\Gamma(\gamma+n-\alpha)}{\Gamma(\gamma+n)\Gamma(\gamma-\alpha)} {}_3F_2(-n, \alpha, \delta-\beta; 1+\alpha-\gamma-n, \delta; 1).$$

[30]

(9.76)

$${}_3F_2(-2\alpha, -2\beta, \gamma; \gamma+1/2, 2\gamma; x) = {}_2F_1^2(-\alpha, -\beta; \gamma+1/2; x).$$

[30]

(9.77)

$${}_3F_2(\alpha, \beta, \gamma; \delta, \epsilon; 1) = \frac{\Gamma(\delta)\Gamma(\delta-\alpha-\beta)}{\Gamma(\delta-\alpha)\Gamma(\delta-\beta)} {}_3F_2(\alpha, \beta, \epsilon-\gamma; \alpha+\beta-\delta+1, \epsilon; 1) \\ + \frac{\Gamma(\delta)\Gamma(\epsilon)\Gamma(\alpha+\beta-\gamma)\Gamma(\delta+\epsilon-\alpha-\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\epsilon-\gamma)\Gamma(\delta+\epsilon-\alpha-\beta)} {}_3F_2(\delta-\alpha, \delta-\beta, \delta+\epsilon-\alpha-\beta-\gamma; \delta-\alpha-\beta+1, \delta+\epsilon-\alpha-\beta; 1).$$

[174, (2.2.3.2)]

(9.78)

$${}_{A+1}F_B[(a), -m; (b); z] = \frac{((a))_m (-z)^m}{((b))_m} {}_{B+1}F_A[1-(b), 1-m; 1-(a); \frac{(-1)^{A+B}}{z}].$$

[174, (4.3.5.1)][200][105]

(9.79)

$${}_4F_3(x, y, z, -n; u, v, w; 1) = \frac{(v-z)_n (w-z)_n}{(v)_n (w)_n} {}_4F_3(u-x, u-y, z, -n; 1-v+z-n, 1-w+z-n, w; 1),$$

if  $u + v + w = 1 + x + y + z - n$ .

[174]

(9.80)

$$[\delta(\delta+b_1-1)(\delta+b_2-1)\cdots(\delta+b_B-1)-z(\delta+a_1)(\delta+a_2)\cdots(\delta+a_A)] {}_A F_B((a), (b), z) = 0,$$

where  $\delta = z \frac{d}{dz}$ .

[47]

$$(9.81) \quad \frac{(\delta+h)_m}{(\delta+k)_m} {}_r F_s \left( \begin{matrix} a, \dots \\ c, \dots \end{matrix} \middle| x \right) = \frac{(h)_m}{(k)_m} {}_{r+2} F_{s+2} \left( \begin{matrix} a, \dots, h+m, k \\ c, \dots, h, k+m \end{matrix} \middle| x \right).$$

[47]

$$(9.82) \quad \sum_{n \geq 0} \frac{(a)_n}{n!} F \left( \begin{matrix} -n, A, \dots \\ C, \dots \end{matrix} \middle| p \right) x^n = (1-x)^{-a} F \left( \begin{matrix} a, A, \dots \\ C, \dots \end{matrix} \middle| -\frac{px}{1-x} \right).$$

[47]

(9.83)

$$\sum_{n \geq 0} \frac{(a)_n}{n!} F \left( \begin{matrix} -n, a+n, A, \dots \\ C, \dots \end{matrix} \middle| p \right) x^n = (1-x)^{-a} F \left( \begin{matrix} a/2, (1+a)/2, A, \dots \\ C, \dots \end{matrix} \middle| -\frac{4px}{(1-x)^2} \right).$$

[47]

$$(9.84) \quad \sum_{n \geq 0} \frac{(a)_n}{n!} F \left( \begin{matrix} -n, A, \dots \\ 1-a-n, C, \dots \end{matrix} \middle| p \right) x^n = (1-x)^{-a} F \left( \begin{matrix} A, \dots \\ C, \dots \end{matrix} \middle| px \right).$$

[47]

$$(9.85) \quad [\delta(\delta+c-1) - px(\delta+a)(\delta-n)] F_n = 0,$$

where  $F_n \equiv \frac{(c)_n}{n!} {}_2 F_1(a, -n; c; px)$ .

[47]

$$(9.86) \quad nF_n - [2n+c-2-p(n+a-1)]F_{n-1} + (1-p)(n+c-2)F_{n-2} = 0$$

where  $F_n \equiv \frac{(c)_n}{n!} {}_2 F_1(a, -n; c; p)$ .

[47]

$$(9.87) \quad [\delta(\delta+c-1)(\delta+c'-1) - (\delta+a)(\delta+a')(\delta-n)] F_n = 0,$$

where  $F_n \equiv \frac{(c)_n}{n!} {}_3 F_2(a, a', -n; c, c'; x)$ .

[47]

(9.88)

$$n(n+c'-1)F_n - [2(n-1)^2 + (2c+2c'-a-a'-1)(n-1) + cc' - aa']F_{n-1} + (n+c-2)(n+c+c'-a-a'-2)F_{n-2} = 0$$

where  $F_n \equiv \frac{(c)_n}{n!} {}_3 F_2(a, a', -n; c, c'; 1)$ .

[174]

$$(9.89) \quad \left[ \sum_{\nu=1}^B z^{\nu-1} (a_\nu z - b_\nu) \frac{d^\nu}{dz^\nu} + a_0 + z^B (1-z) \frac{d^{B+1}}{dz^{B+1}} \right] {}_{B+1} F_B((a), (b), z) = 0.$$

[163]

$$(9.90) \quad {}_2 F_2(-3/2, -1/2; -5/2, 1; z) = \frac{5-4z}{5} e^{z/2} I_0(z/2) + \frac{4z}{5} e^{z/2} I_1(z/2).$$

[163]

$${}_2F_3(-1/2, 1; 1/4, 1/2, 3/4; z) = 1 + z^{1/4}\sqrt{2}\sqrt{\pi}e^{2\sqrt{z}}\operatorname{erf}(\sqrt{2}z^{1/4}) \\ - z^{1/4}\sqrt{2}\sqrt{\pi}e^{-2\sqrt{z}}\operatorname{erfi}(\sqrt{2}z^{1/4}) - 2\sqrt{z}\pi\operatorname{erf}(\sqrt{2}z^{1/4})\operatorname{erfi}(\sqrt{2}z^{1/4}).$$

[163]

$${}_1F_2(3/2; 5/2, 5; z) = -\frac{432 - 24z + 96z^2}{5z^3}I_0(2\sqrt{z}) + \frac{432 + 192z + 48z^2}{5z^{7/2}}I_1(2\sqrt{z}) \\ - \frac{48}{5z}\pi(I_0(2\sqrt{z})L_1(2\sqrt{z}) - I_1(2\sqrt{z})L_0(2\sqrt{z})).$$

[163]

$${}_3F_2(-1/2, 1, 2; 3, 4; z) = -\frac{480 + 3472z - 2100z^2}{525z^3} + \frac{480 + 3712z - 1024z^2 + 192z^3}{525z^3}\sqrt{1-z} \\ - \frac{32}{5z^2}\log\left(\frac{1}{2} + \frac{\sqrt{1-z}}{2}\right).$$

[30]

$$(9.91) \quad {}_2F_3(-\beta, \beta + \gamma; \gamma; \gamma/2, \frac{1+\gamma}{2}; x^2/4) = {}_1F_1(-\beta; \gamma; -x){}_1F_1(-\beta; \gamma; x).$$

[30]

$$(9.92) \quad {}_2F_3(1, n; n + 1; (n + 1)/2, 2 + n2; x^2/4) = {}_1F_1(1; n + 1; -x){}_1F_1(1; n + 1; x).$$

[30]

$$(9.93) \quad {}_4F_1(-\alpha, -\beta, -\frac{\alpha + \beta}{2} - \frac{\alpha + \beta - 1}{2}; -\alpha - \beta; 4x^2) = {}_2F_0(-\alpha, -\beta; x){}_2F_0(-\alpha, -\beta; -x).$$

if  $\alpha$  or  $\beta$  a nonnegative integer.

[174]

(9.94)

$${}_{A+1}F_{B+1}(c, (a); d, (b); z) = \frac{\Gamma(d)}{\Gamma(c)\Gamma(d-c)} \int_0^1 t^{c-1}(1-t)^{d-c-1} {}_A F_B((a), (b), tz) dt.$$

[53, 64]

(9.95)

$${}_{q+1}F_q\left(\begin{matrix} a, \frac{b}{q}, \frac{b+1}{q}, \dots, \frac{b+q-1}{q} \\ \frac{c}{q}, \frac{c+1}{q}, \dots, \frac{c+q-1}{q} \end{matrix} \middle| x\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt^q)^{-a} dt \\ = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k \geq 0} \binom{c-b-1}{k} \frac{(-)^k}{b+k} {}_2F_1(a, (b+k)/q; 1 + (b+k)/q; x)$$

[64]

(9.96)

$${}_{p+k}F_{q+k}\left(\begin{matrix} a_1, \dots, a_p, \frac{\alpha}{k}, \frac{\alpha+1}{k}, \dots, \frac{\alpha+k-1}{k} \\ b_1, \dots, b_q, \frac{\alpha+\beta}{k}, \frac{\alpha+\beta+1}{k}, \dots, \frac{\alpha+\beta+k-1}{k} \end{matrix} \middle| ct^k\right) = \frac{t^{1-\alpha-\beta}}{B(\alpha, \beta)} \int_0^t x^{\alpha-1}(t-x)^{\beta-1} {}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| cx^k\right) dx.$$



$$(9.97) \quad \begin{matrix} [141] \\ r+2F_{r+1} \left( \begin{matrix} a, b, & (f_r + 1) \\ c, & (f_r) \end{matrix} \mid x \right) = (1-x)^{-a} r+2F_{r+1} \left( \begin{matrix} a, \lambda, & (\xi_r + 1) \\ c, & (\xi_r) \end{matrix} \mid \frac{x}{x-1} \right) \end{matrix}$$

$$(9.98) \quad \begin{matrix} [141] \\ r+2F_{r+1} \left( \begin{matrix} a, b, & (f_r + 1) \\ c, & (f_r) \end{matrix} \mid x \right) = (1-x)^{c-a-b-r} r+2F_{r+1} \left( \begin{matrix} \lambda, \lambda', & (\xi_r + 1) \\ c, & (\xi_r) \end{matrix} \mid x \right) \end{matrix}$$

$$(9.99) \quad \begin{matrix} [4] \\ p+1F_p \left( \begin{matrix} a, a, \dots, a \\ a+1, \dots, a+1 \end{matrix} \mid 1 \right) = \frac{a^p}{(a-1)!} \sum_{k=0}^{a-1} (-)^{a-k-1} \zeta(p-k) \left[ \begin{matrix} a \\ k+1 \end{matrix} \right] \end{matrix}$$

for  $a$  a positive integer, where the  $\left[ \begin{matrix} a \\ k+1 \end{matrix} \right]$  is Stirling numbers of the first kind.

$$(9.100) \quad \begin{matrix} [148] \\ p+1F_p(a_1, \dots, a_{p+1}; b_1 \dots b_p \mid z\zeta) = (1-z)^{-a_1} \sum_{k \geq 0} \frac{(a_1)_k}{k!} p+1F_p(-k, a_2, \dots, a_{p+1}; b_1, \dots, b_p \mid \zeta) \left( \frac{z}{z-1} \right)^k. \end{matrix}$$

$$(9.101) \quad \begin{matrix} [4] \\ p+1F_p \left( \begin{matrix} a, a, \dots, a \\ a+1, \dots, a+1 \end{matrix} \mid 1 \right) = \frac{(-)^{p-1} \pi a^p}{\sin(a\pi)(p-1)!} w(a, p-1) \end{matrix}$$

where

$$(9.102) \quad w(n, m) = \frac{1}{(n-1)!} \sum_{i=m+1}^n \left[ \begin{matrix} n \\ i \end{matrix} \right] (i-m)_m (-1)^{n-i} n^{i-m-1},$$

recursively

$$(9.103) \quad w(n, 0) = 1, \quad w(n, m) = \sum_{k=0}^{m-1} (1-m)_k H_{n-1}^{(k+1)} w(n, m-1-k)$$

and the Harmonic numbers defined in (0.88).

$$(9.104) \quad \begin{matrix} [78] \\ q+1F_q \left( \begin{matrix} a_1, \dots, a_{q+1} \\ b_1, \dots, b_q \end{matrix} \mid 1 \right) + q+1F_q \left( \begin{matrix} a_1, \dots, a_{q+1} \\ b_1, \dots, b_q \end{matrix} \mid -1 \right) \\ = 2 {}_2q+2F_{2q+1} \left( \begin{matrix} a_1/2, a_1/2 + 1/2, \dots, a_{q+1}/2 + 1/2 \\ b_1/2, b_1/2 + 1/2, \dots, b_q/2 + 1/2, 1/2 \end{matrix} \mid 1 \right). \end{matrix}$$

$$(9.105) \quad \begin{matrix} [78] \\ q+1F_q \left( \begin{matrix} a_1, \dots, a_{q+1} \\ b_1, \dots, b_q \end{matrix} \mid 1 \right) - q+1F_q \left( \begin{matrix} a_1, \dots, a_{q+1} \\ b_1, \dots, b_q \end{matrix} \mid -1 \right) \\ = 2 \frac{a_1 a_2 \dots a_{q+1}}{b_1 b_2 \dots b_q} q+1F_q \left( \begin{matrix} a_1/2 + 1/2, a_1/2 + 1, \dots, a_{q+1}/2 + 1/2, a_{q+1}/2 + 1 \\ b_1/2 + 1/2, b_1/2 + 1, \dots, b_q/2 + 1, 3/2 \end{matrix} \mid 1 \right). \end{matrix}$$

[78]

(9.106)

$${}_{q+1}F_q \left( \begin{matrix} a, a+2, a+4, \dots, a+2q \\ a+1, a+3, a+5, \dots, a+2q-1 \end{matrix} \mid 1 \right) + {}_{q+1}F_q \left( \begin{matrix} a, a+2, a+4, \dots, a+2q \\ a+1, a+3, a+5, a+2q-1 \end{matrix} \mid -1 \right) \\ = {}_2F_1(a/2, a+q+1/2; 1/2; 1).$$

[78]

(9.107)

$${}_{q+1}F_q \left( \begin{matrix} a, a+2, a+4, \dots, a+2q \\ a+1, a+3, a+5, \dots, a+2q-1 \end{matrix} \mid 1 \right) - {}_{q+1}F_q \left( \begin{matrix} a, a+2, a+4, \dots, a+2q \\ a+1, a+3, a+5, a+2q-1 \end{matrix} \mid -1 \right) \\ = \frac{2a(a+2)(a+4) \cdots (a+2q)}{(a+1)(a+3) \cdots (a+2q-1)} {}_2F_1(a/2+1/2, a/2+q+1; 3/2; 1).$$

[78]

(9.108)

$${}_{q+1}F_q \left( \begin{matrix} a, a+2, a+4, \dots, a+2q \\ a+1, a+3, a+5, \dots, a+2q-1 \end{matrix} \mid 1 \right) \\ = \frac{\sqrt{\pi}\Gamma(-a-q)}{\Gamma(1/2-a)\Gamma(-a/2-q)} + \frac{2(a/2)_q}{(a/2+1/2)_{q-1}\Gamma(1-a/2)\Gamma(1/2-a/2-q)},$$

if  $\Re(a+q) < 0$ , and a similar expression for argument  $-1$ .

[139]

(9.109)  ${}_3F_2(a, m, b; c, m-n; 1) = \Gamma(a+n-m+1)$ 

$$\times \left( \sum_{L=0}^{m-1} \frac{(-1)^L \Gamma(-b+c-a-n+L)\Gamma(1-b+L)}{\Gamma(m-L)\Gamma(-b-n+1+L)\Gamma(-b+c-m+1+L)\Gamma(L+1)} \right) \\ \times \frac{\Gamma(c)\Gamma(m-n)(-1)^n}{\Gamma(c-a)\Gamma(a)}, \quad 0 < n < m.$$

(9.110)  ${}_3F_2(a, b, -n; c, m-n; 1) = \Gamma(a+n-m+1)$ 

$$\times \left( \sum_{L=0}^{m-1} \frac{(-1)^L \Gamma(c-a+L)\Gamma(1+b-m+n+L)}{\Gamma(m-L)\Gamma(b-m+1+L)\Gamma(c+n-m+1+L)\Gamma(L+1)} \right) \\ \times \frac{\Gamma(c)\Gamma(m-n)(-1)^n}{\Gamma(c-a)\Gamma(a)}, \quad 0 < n < m.$$

(9.111)

$${}_3F_2(a, -n, b; c, a-c+b-n+m; 1) = \Gamma(c-b+n-m+1)\Gamma(-a+c-b-m+1) \\ \times \left( \sum_{L=0}^{m-1} \frac{(-1)^L \Gamma(b+L)\Gamma(1-a+c-m+n+L)}{\Gamma(m-L)\Gamma(-a+c-m+1+L)\Gamma(c+n-m+1+L)\Gamma(L+1)} \right) \\ \times \frac{\Gamma(c)\Gamma(m)}{\Gamma(n+1-a+c-b-m)\Gamma(b)\Gamma(c-b)}, \quad 0 < n, m.$$

(9.112)

$$\begin{aligned}
 {}_3F_2(a, 1, b; n+1, c; 1) &= \frac{\Gamma(a-n)\Gamma(b-n)\Gamma(n+1)\Gamma(c)\Gamma(-b+c-a+n)\Gamma(1-b)\Gamma(1+n)\Gamma(c)}{\Gamma(b)\Gamma(c-b)\Gamma(a)\Gamma(c-a)\Gamma(a)} \\
 &\times \left( \sum_{L=0}^{n-1} \frac{(-1)^L \Gamma(a-1-L)}{\Gamma(c-1-L)\Gamma(n-L)\Gamma(2+L-b)} \right) \\
 &, \quad 0 < n, m.
 \end{aligned}$$

and 60 others.

[64]

(9.113)  ${}_4F_3(a, b/3, (b+1)/3, (b+2)/3; c/3, (c+1)/3, (c+2)/3; 1)$

$$= \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-a)\Gamma(c-b)} \sum_{k=0}^{\infty} \frac{(a)_k (-1)^k (b)_k}{k! (c-a)_k} {}_2F_1(-k, b+k; c-a+k; -1).$$

[105]

(9.114)  ${}_4F_3\left(\frac{\alpha}{2}, \frac{\alpha+1}{2}, \beta+n, -n; 1+\alpha, \frac{\beta}{2}, \frac{\beta+1}{2}; 1\right) = \frac{(\beta-\alpha)_n}{(\beta)_n}$ .

[105]

(9.115)  ${}_4F_3\left(\alpha, -\alpha, -\frac{m}{2}, \frac{1-m}{2}; \frac{1}{2}, \beta, 1-m-\beta; 1\right) = \frac{(\alpha+\beta)_m + (\beta-\alpha)_m}{2(\beta)_m}$ .

[105]

(9.116)  ${}_4F_3\left(a, b, \frac{1}{2}-a-b-n, -n; a+b-\frac{1}{2}, 1-a-n, 1-b-n; 1\right) = \frac{(2a)_n (2b)_n (a+b)_n}{(2a+2b-1)_n (a)_n (b)_n}$ .

[105]

(9.117)  ${}_4F_3\left(a-\frac{1}{2}, b-\frac{1}{2}, \frac{1}{2}-a-b-n, -n; a+b-\frac{1}{2}, \frac{1}{2}-a-n, \frac{1}{2}-b-n; 1\right) = \frac{(2a)_n (2b)_n (a+b)_n}{(2a+2b-1)_n (a+\frac{1}{2})_n (b+\frac{1}{2})_n}$ .

[105]

(9.118)  ${}_4F_3\left(a+1, b, \frac{1}{2}-a-b-n, -n; a+b+\frac{1}{2}, 1-a-n, 1-b-n; 1\right) = \frac{(2a+1)_n (2b)_n (a+b)_n}{(2a+2b)_n (a)_n (b)_n}$ .

[105]

(9.119)  ${}_4F_3\left(a+\frac{1}{2}, b-\frac{1}{2}, \frac{1}{2}-a-b-n, -n; a+b+\frac{1}{2}, \frac{1}{2}-a-n, \frac{1}{2}-b-n; 1\right) = \frac{(2a+1)_n (2b)_n (a+b)_n}{(2a+2b)_n (a+\frac{1}{2})_n (b+\frac{1}{2})_n}$ .

[105]

(9.120)  ${}_4F_3\left(a, b, \frac{1}{2}-a-b-n, -n; a+b+\frac{1}{2}, 1-a-n, 1-b-n; 1\right) = \frac{(2a)_n (2b)_n (a+b)_n}{(2a+2b)_n (a)_n (b)_n}$ .

$$\begin{aligned} & \text{[105]} \\ (9.121) \quad & {}_4F_3\left(a, b, a+b-\frac{1}{2}-n, -n; a+b+\frac{1}{2}, \frac{1}{2}+a-n, \frac{1}{2}+b-n; 1\right) = \frac{(1/2)_n(a-b+\frac{1}{2})_n(b-a+\frac{1}{2})_n}{(a+b+\frac{1}{2})_n(\frac{1}{2}-a)_n(\frac{1}{2}-b)_n}. \end{aligned}$$

$$\begin{aligned} & \text{[105]} \\ (9.122) \quad & {}_4F_3\left(a, -a, \frac{1}{2}-n, -n; \frac{1}{2}, \frac{1}{2}+b-n, \frac{1}{2}-b-n; 1\right) = \frac{(\frac{1}{2}+a+b)_n(\frac{1}{2}-a-b)_n + (\frac{1}{2}+a-b)_n(\frac{1}{2}-a+b)_n}{2(\frac{1}{2}+b)_n(\frac{1}{2}-b)_n}. \end{aligned}$$

$$\begin{aligned} & \text{[105]} \\ (9.123) \quad & {}_4F_3\left(a, -a, -\frac{1}{2}-n, -n; \frac{1}{2}, b-n, -b-n; 1\right) = \frac{(a+b)_{n+1}(1-a-b)_n + (b-a)_{n+1}(1+a-b)_n}{2b(1+b)_n(1-b)_n}. \end{aligned}$$

$$\begin{aligned} & \text{[105]} \\ (9.124) \quad & {}_4F_3\left(\frac{1}{2}+a, \frac{1}{2}-a, -\frac{1}{2}-n, -n; \frac{1}{2}, \frac{1}{2}+b-n, \frac{1}{2}-b-n; 1\right) = \frac{(a+b)_{n+1}(1-a-b)_n + (b-a)_{n+1}(1+a-b)_n}{2b(\frac{1}{2}+b)_n(\frac{1}{2}-b)_n}. \end{aligned}$$

$$\begin{aligned} & \text{[105]} \\ (9.125) \quad & {}_4F_3\left(a, -a, -\frac{1}{2}-n, -n; \frac{1}{2}, \frac{1}{2}+b-n, \frac{1}{2}-b-n; 1\right) \\ & = \frac{(a+b)(\frac{1}{2}+a+b)_n(\frac{1}{2}-a-b)_n + (b-a)(\frac{1}{2}+b-a)_n(\frac{1}{2}+a-b)_n}{2b(\frac{1}{2}+b)_n(\frac{1}{2}-b)_n}. \end{aligned}$$

$$\begin{aligned} & \text{[105]} \\ (9.126) \quad & {}_4F_3\left(a, -a, \frac{1}{2}-n, -n; \frac{1}{2}, 1+b-n, 1-b-n; 1\right) = \frac{(a+b)_n(-a-b)_n + (a-b)_n(b-a)_n}{2(b)_n(-b)_n}. \end{aligned}$$

$$\begin{aligned} & \text{[30]} \\ (9.127) \quad & {}_4F_3\left(\alpha, \beta, \frac{\alpha+\beta}{2}, \frac{\gamma+\delta}{2}; \gamma, \delta, \alpha+\beta; x\right) = {}_2F_1\left(\alpha, \beta; \gamma; \frac{1-\sqrt{1-x}}{2}\right) {}_2F_1\left(\alpha, \beta; \delta; \frac{1-\sqrt{1-x}}{2}\right) \end{aligned}$$

if  $\alpha + \beta + 1 = \gamma + \delta$ .

$$\begin{aligned} & \text{[45]} \\ (9.128) \quad & \frac{1}{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)} {}_4F_3\left(\begin{matrix} a_1, a_2, a_3, a_4 \\ b_1, b_2, b_3 \end{matrix} \mid 1\right) = \frac{\Gamma(s)}{\Gamma(a_1+s)\Gamma(a_2+s)\Gamma(a_3)\Gamma(a_4)} \\ & \times \sum_{k=0}^{\infty} \frac{(b_1+b_3-a_3-a_4)_k (b_2+b_3-a_3-a_4)_k (s)_k}{(a_1+s)_k (a_2+s)_k k!} {}_3F_2\left(\begin{matrix} b_3-a_3, b_3-a_4, -k \\ b_1+b_3-a_3-a_4, b_2+b_3-a_3-a_4 \end{matrix} \mid 1\right). \end{aligned}$$

$$\begin{aligned} & \text{[78]} \\ (9.129) \quad & {}_4F_3\left(a/2, a/2+1/2, b/2, b/2+1/2; 1/2+a/2-b/2, 1+a/2-b/2, 1/2; 1\right) \\ & = \frac{\Gamma(1+a-b)\Gamma(1-2b)}{\Gamma(1-b)\Gamma(1+a-2b)} + \frac{\Gamma(1+a-b)\Gamma(1+a/2)}{\Gamma(1+a)\Gamma(1+a/2-b)}. \end{aligned}$$

where  $\Re b < 1/2$ .

[78]

$$(9.130) \quad \frac{2ab}{1+a-b} {}_4F_3(a/2+1/2, a/2+1, b/2+1/2, b/2+1; 1+a/2-b/2, 3/2+a/2-b/2, 3/2; 1) \\ = \frac{\Gamma(1+a-b)\Gamma(1-2b)}{\Gamma(1-b)\Gamma(1+a-2b)} + \frac{\Gamma(1+a-b)\Gamma(1+a/2)}{\Gamma(1+a)\Gamma(1+a/2-b)}.$$

where  $\Re b < 1/2$ .

[45]

$$(9.131) \quad \frac{1}{\prod_{j=1}^p \Gamma(b_j)} {}_{p+2}F_{p+1} \left( \begin{matrix} a_1, a_2, \dots, a_{p+1}, -m \\ b_1, b_2, \dots, b_p, 1-s-m \end{matrix} \mid 1 \right) \\ = \frac{(a_1+s)_m (a_2+s)_m \prod_{j=3}^{p+1} (a_j)_m}{(s)_m \prod_{j=1}^p \Gamma(b_j+m)} \sum_{k=0}^m \frac{(s)_k (-m)_k}{(a_1+s)_k (a_2+s)_k k!} \\ \times (S)_k {}_{p+1}F_p \left( \begin{matrix} 1-b_1-m, 1-b_2-m, \dots, 1-b_p-m, -k \\ 1-a_3-m, 1-a_4-m, \dots, 1-a_{p+1}-m, 1-S-k \end{matrix} \mid 1 \right)$$

where  $s = \sum_{j=1}^p b_j - \sum_{j=1}^{p+1} a_j$  and  $S = a_1 + a_2 + s + m - 1$  and  $s$  not a negative integer or zero.

[45]

$$(9.132) \quad \frac{1}{\Gamma(b_1)\Gamma(b_2)} {}_4F_3 \left( \begin{matrix} a_1, a_2, a_3, -m \\ b_1, b_2, 1-s-m \end{matrix} \mid 1 \right) = \frac{(a_1+s)_m (a_2+s)_m (a_3)_m}{(s)_m \Gamma(b_1+m)\Gamma(b_2+m)} {}_4F_3 \left( \begin{matrix} b_1-a_3, b_2-a_3, s, -m \\ a_1+s, a_2+s, 1-a_3-m \end{matrix} \mid 1 \right)$$

where  $s = b_1 + b_2 - a_1 - a_2 - a_3$  not a negative integer or zero,  $m = 0, 1, 2, \dots$

[30]

$$(9.133) \quad {}_0F_2(m+n+1, n+1; x) {}_0F_2(m+1, 1-n; -x) = 1 + \sum_{k \geq 1} \frac{\alpha_k \binom{2m+n+k+2}{2}_k (2x)^k}{(m+n+1)_k (m+1)_k k!}$$

where

$$(9.134) \quad \alpha_k = \begin{cases} \frac{n}{(n^2-1^2)(n^2-3^2)\dots(n^2-k^2)}, & \text{if } k \text{ odd;} \\ \frac{1}{(n^2-2^2)(n^2-4^2)\dots(n^2-k^2)}, & \text{if } k \text{ even} \end{cases}$$

[54]

$$(9.135) \quad \sum_{k \geq 0} t^k \frac{(\alpha+1+sk)_k}{k!} {}_rF_r \left( \begin{matrix} \Delta(-k, r) \\ \Delta(-\alpha-sk-k, r) \end{matrix} \mid x(\beta+sk) \right) = (1-z)^{\alpha+1} \frac{1}{1+sz+rsy} e^{-\beta y},$$

where  $t = (-z)(1-z)^{-s-1} \exp(sy)$ ,  $x = (-y)[1-1/z]^r$ ,  $(a)_k = \Gamma(a+k)/\Gamma(a)$  and  $\Delta(-k, r) \equiv -k/r, (-k+1)/r, \dots, (-k+r-1)/r, |t| < 1$ .

[54]

$$(9.136) \quad \sum_{k \geq 0} t^k \frac{(\alpha + 1 + sk)_k}{(\alpha + l + 1 + sk)k!((\alpha + l + 1 + s'' + sk)/s'')^{l'}} {}_rF_r \left( \begin{matrix} \Delta(-k, r) \\ \Delta(-\alpha = sk - k, r) \end{matrix} \mid x(\beta + sk) \right) \\ = \frac{(1-z)^{\alpha+l+1} e^{-\beta y}}{l!} \sum_{q=0}^{l'} \sum_{k=0}^{l+s''q} \frac{(-z)^k (1-z)^{s''q-k} (-l')_q (-l-s''q)_k}{k!q!(\alpha+l+1+s''q+sk)} \\ \times {}_1F_1 \left( \begin{matrix} 1 \\ (\alpha+l+1+s'+s''r+sk)/s' \end{matrix} \mid y(\beta - \alpha - l - 1 - s''q) \right),$$

where  $t = (-z)(1-z)^{-s-1} e^{sy}$ ,  $x = -y[1-1/z]^r$  and  $|t| < 1$ .

[54]

$$(9.137) \quad \sum_{p \geq 0} t^p \frac{(\beta + s'p)^p}{p!} {}_rF_0 \left( \begin{matrix} \Delta(-p, r) \\ - \end{matrix} \mid \frac{xr^r(\alpha + s'p)}{(\beta + s'p)^r} \right) = \frac{e^{-\beta y - \alpha z}}{1 + s'y + rs'z},$$

where  $t = -ye^{s'y+s'z}$ ,  $x = -z/y^r$  and  $|s'y \exp(s'y + s'z + 1)| < 1$ .

[54]

$$(9.138) \quad \sum_{p \geq 0} t^p \frac{(\beta + s'p)^p}{p!(\alpha + s'p)} {}_rF_0 \left( \begin{matrix} \Delta(-p, r) \\ - \end{matrix} \mid \frac{xr^r(\alpha + s'p)}{(\beta + s'p)^r} \right) = \frac{e^{-\beta y - \alpha z}}{\alpha} {}_1F_1 \left( \begin{matrix} 1 \\ \alpha/s' + 1 \end{matrix} \mid y(\beta - \alpha) \right)$$

where  $t = -ye^{s'y+s'z}$ ,  $x = -z/y^r$  and  $|s'y \exp(s'y + s'z + 1)| < 1$ .

[170]

$$(9.139) \quad F_1 \left( \begin{matrix} a; b, b \\ 1 + a - b \end{matrix} \mid e^{2\pi i/3}, e^{-2\pi i/3} \right) = \frac{\Gamma(1+a-b)\Gamma(1+a/3)}{\Gamma(1+a)\Gamma(1+a/3-b)}.$$

[170]

$$(9.140) \quad F_D \left( \begin{matrix} a; b, b, \dots, b \\ c \end{matrix} \mid \omega_{1,n}, \dots, \omega_{n-1,n} \right) = \frac{\Gamma(a-b+1)\Gamma(1+a/n)}{\Gamma(1+a)\Gamma(1+a/n-b)},$$

where  $\omega_{k,n} \equiv e^{2k\pi i/n}$ .

[170]

$$(9.141) \quad F_D \left( \begin{matrix} 2mb - a; b, b, \dots, b \\ 2mb \end{matrix} \mid x_1, \dots, x_{2m} \right) = \frac{1}{2m} \frac{\Gamma(a/(2m))\Gamma(2mb)\Gamma((2mb-a)/(2m))}{\Gamma(a)\Gamma(b)\Gamma(2mb-a)},$$

where  $x_k = 1 + e^{(2k-1)\pi i/(2m)}$ , for  $k = 1, \dots, 2m$ , and  $2m$  is an even integer,  $a > 0$ ,  $b > 0$ ,  $nb > a$ .

[170]

$$(9.142) \quad F_D \left( \begin{matrix} (2m-1)b - a; b, b, \dots, b \\ (2m-1)b \end{matrix} \mid y_1, \dots, y_{2m-1} \right) \\ = \frac{1}{2m-1} \frac{\Gamma(a/(2m-1))\Gamma((2m-1)b)\Gamma(((2m-1)b-a)/(2m-1))}{\Gamma(a)\Gamma(b)\Gamma((2m-1)b-a)},$$

where  $y_k = 1 + e^{(2k-1)\pi i/(2m-1)}$ , and  $2m-1$  is an odd integer,  $a > 0$ ,  $b > 0$ ,  $nb > a$ .

[18, p 20]

$$(9.143) \quad \alpha F_4(\alpha + 1, \beta; \gamma, \gamma'; x, y) - \beta F_4(\alpha, \beta + 1; \gamma, \gamma'; x, y) \\ = (\alpha - \beta) F_4(\alpha, \beta; \gamma, \gamma'; x, y).$$

[18, p 21]

$$(9.144) \quad \frac{\beta}{\gamma} x F_4(\alpha + 1, \beta + 1; \gamma + 1, \gamma'; x, y) + \frac{\beta}{\gamma'} y F_4(\alpha + 1, \beta + 1; \gamma, \gamma' + 1; x, y) \\ = F_4(\alpha + 1, \beta; \gamma, \gamma'; x, y) - F_4(\alpha, \beta; \gamma, \gamma'; x, y).$$

[18, p 26]

$$(9.145) \quad F_4(\alpha, \beta; \gamma, \gamma'; x, y) = \sum \frac{(\alpha)_m (\beta)_m}{(\gamma)_m m!} \frac{\Gamma(\gamma') \Gamma(\beta - \alpha)}{\Gamma(\gamma' - \alpha - m) \Gamma(\beta + m)} (-y)^{-\alpha - m} \\ \times {}_2F_1\left(\alpha + m, \alpha + m + 1 - \gamma'; \alpha + 1 - \beta; \frac{1}{y}\right) x^m \\ + \sum \frac{(\alpha)_m (\beta)_m}{(\gamma)_m m!} \frac{\Gamma(\gamma') \Gamma(\alpha - \beta)}{\Gamma(\gamma' - \beta - m) \Gamma(\alpha + m)} (-y)^{-\beta - m} \\ \times {}_2F_1\left(\beta + m, \beta + m + 1 - \gamma'; \beta + 1 - \alpha; \frac{1}{y}\right) x^m.$$

[18]

$$(9.146) \quad F_4(\alpha, \beta; \gamma, \gamma'; x, y) = \frac{\Gamma(\gamma') \Gamma(\beta - \alpha)}{\Gamma(\gamma' - \alpha) \Gamma(\beta)} (-y)^{-\alpha} F_4\left(\alpha, \alpha + 1 - \gamma'; \gamma, \alpha + 1 - \beta; \frac{x}{y}, \frac{1}{y}\right) \\ + \frac{\Gamma(\gamma') \Gamma(\alpha - \beta)}{\Gamma(\gamma' - \beta) \Gamma(\alpha)} (-y)^{-\beta} F_4\left(\beta + 1 - \gamma', \beta; \gamma, \beta + 1 - \alpha; \frac{x}{y}, \frac{1}{y}\right).$$

[27, 85]

$$(9.147) \quad F_4[\alpha, \beta; \gamma, 1 + \alpha + \beta - \gamma; x(1 - y), y(1 - x)] = F(\alpha, \beta; \gamma; x) F(\alpha, \beta; 1 + \alpha + \beta - \gamma; y).$$

$$(9.148) \quad F_4\left[\alpha, \beta; \gamma, \beta; -\frac{x}{(1-x)(1-y)}, -\frac{y}{(1-x)(1-y)}\right] = (1-x)^\alpha (1-y)^\alpha F_1[\alpha; \gamma - \beta, 1 + \alpha - \gamma; \gamma; x, xy].$$

$$(9.149) \quad F_4\left[\alpha, \beta; \alpha, \beta; -\frac{x}{(1-x)(1-y)}, -\frac{y}{(1-x)(1-y)}\right] = (1-xy)^{-1} (1-x)^\beta (1-y)^\alpha.$$

$$(9.150) \quad F_4\left[\alpha, \beta; \beta, \beta; -\frac{x}{(1-x)(1-y)}, -\frac{y}{(1-x)(1-y)}\right] = (1-x)^\alpha (1-y)^\alpha F[\alpha, 1 + \alpha - \beta; \beta; xy].$$

$$(9.151) \quad F_4\left[\alpha, \beta; 1 + \alpha - \beta, \beta; -\frac{x}{(1-x)(1-y)}, -\frac{y}{(1-x)(1-y)}\right] = (1-y)^\alpha F\left[\alpha, \beta; 1 + \alpha - \beta; -\frac{x(1-y)}{1-x}\right].$$

[105]

$$(9.152) \quad F_{q:1:1}^{p:2:2} \left[ \begin{matrix} \alpha_1, \dots, \alpha_p : a_1, b_1 & a_2, b_2 \\ \gamma_1, \dots, \gamma_q : c_1 & c_2 \end{matrix} \middle| X, X \right] = {}_{p+3}F_{q+2}(\alpha_1, \dots, \alpha_p, \beta_1, \beta_2, \beta_3; \gamma_1, \dots, \gamma_q, \delta_1, \delta_2; X)$$

$$\begin{array}{c|ccc|ccc|ccc|cc} a_1 & b_1 & c_1 & a_2 & b_2 & c_2 & \beta_1 & \beta_2 & \beta_3 & \delta_1 & \delta_2 \\ \hline a & b & a+b-\frac{1}{2} & a & b & a+b+\frac{1}{2} & 2a & 2b & a+b & 2a+2b-1 & a+b+\frac{1}{2} \\ a-\frac{1}{2} & b-\frac{1}{2} & a+b-\frac{1}{2} & a+\frac{1}{2} & b+\frac{1}{2} & a+b+\frac{1}{2} & 2a & 2b & a+b & 2a+2b-1 & a+b+\frac{1}{2} \end{array}$$

and more of that format.

[183]  
(9.153)

$$F_{1:1:1}^{1:2:2} \left[ \begin{array}{c} \alpha : \quad -M, \beta : \quad -N, \beta' : \\ \beta + \beta' : \quad \alpha - \beta' - M + 1 \quad \alpha - \beta - N + 1 \end{array} \mid 1, 1 \right] = \frac{(\beta + \beta' - \alpha)_{M+N} (\beta')_M (\beta)_N}{(\beta + \beta')_{M+N} (\beta' - \alpha)_M (\beta - \alpha)_N}$$

for  $M, N = 0, 1, 2, \dots$

[183]

$$(9.154) \quad F_{1:1:1}^{1:2:2} \left[ \begin{array}{c} \beta + \beta' - \alpha : \quad \beta, \gamma : \quad \beta', \gamma' : \\ \beta + \beta' : \quad \delta \quad \delta' \end{array} \mid 1, 1 \right] \\ = \frac{\Gamma(\delta)\Gamma(\delta')\Gamma(\alpha - \beta - \gamma + \delta)\Gamma(\alpha - \beta' - \gamma' + \delta')}{\Gamma(\delta - \gamma)\Gamma(\delta' - \gamma')\Gamma(\alpha - \beta + \delta)\Gamma(\alpha - \beta' + \delta')} F_{1:1:1}^{1:2:2} \left[ \begin{array}{c} \alpha : \quad \beta', \gamma : \quad \beta, \gamma' : \\ \beta + \beta' : \quad \alpha - \beta + \delta \quad \alpha - \beta' + \delta' \end{array} \mid 1, 1 \right].$$

[183]

(9.155)

$$F_{1:s:v}^{1:r:u} \left[ \begin{array}{c} \alpha : (a_r) : (c_u) : \\ \gamma : (b_s) : (d_v); \end{array} \mid x, y \right] = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\gamma - \alpha)_n \prod_{j=1}^r (a_j)_n \prod_{j=1}^u (c_j)_n}{(\gamma + n - 1)_n (\gamma)_{2n} \prod_{j=1}^s (b_j)_n \prod_{l=1}^v (d_l)_n} \frac{(xy)^n}{n!} \\ \times {}_{r+1}F_{s+1} \left( \begin{array}{c} (a_r) + n, \quad \alpha + n \\ (b_s) + n, \quad \gamma + 2n \end{array} \mid x \right) {}_{u+1}F_{v+1} \left( \begin{array}{c} (c_u) + n, \quad \alpha + n; \\ (d_v) + n, \quad \gamma + 2n; \end{array} \mid y \right).$$

[85]

$$(9.156) \quad {}_3F_2 \left( \begin{array}{c} -n, n + a, b \\ c, d \end{array} \mid 1 \right) {}_3F_2 \left( \begin{array}{c} -n, n + a, e \\ c, f \end{array} \mid 1 \right) \\ = \frac{(-)^n (a - c + 1)_n}{(c)_n} F_{2:1:1}^{2:2:2} \left[ \begin{array}{c} -n, n + a : \quad b, e; \quad d - b, f - e \\ d, f : \quad c; \quad a - c + 1 \end{array} \mid 1, 1 \right].$$

[85]

$$(9.157) \quad F_{2:1:1}^{2:2:2} \left[ \begin{array}{c} a, b : \quad -x, y + e; \quad -y, x + d \\ d, e : \quad c; \quad b \end{array} \mid 1, 1 \right] \\ = \frac{(d - a)_x (e - a)_y}{(d)_x (e)_y} F_{2:1:0}^{2:2:1} \left[ \begin{array}{c} a, -x : \quad 1 + a - c, -y; \quad c - b \\ c, 1 + a - d - x : \quad 1 + a - e - y; \quad - \end{array} \mid 1, 1 \right]$$

if  $x, y = 0, 1, \dots$

[85]

(9.158)

$$F_{2:1:1}^{2:2:2} \left[ \begin{array}{c} a, b : \quad -x, y + e; \quad -y, x + d \\ d, e : \quad c; \quad c' \end{array} \mid 1, 1 \right] = \sum_{r=0}^{\min(x,y)} \frac{(a)_r (b)_r (a + b - c - c' + 1)_r}{r! (c)_r (c')_r} \frac{(-x)_r (-y)_r}{(d)_r (e)_r} \\ \times {}_3F_2 \left( \begin{array}{c} r + a, r + b, r - x \\ r + c, r + c \end{array} \mid 1 \right) {}_3F_2 \left( \begin{array}{c} r + a, r + b, r - y \\ r + c', r + e \end{array} \mid 1 \right)$$

if  $x, y = 0, 1, \dots$

## 9.2. The Confluent Hypergeometric Function.



9.3. The Meijer G-Function. [2]

$$(9.159) \quad G_{p+1,q+2}^{m+1,n+1} \left( z \mid \begin{matrix} a, a_p \\ a, b_q, b \end{matrix} \right) = (-)^{a-b} G_{p,q+1}^{m+1,n} \left( z \mid \begin{matrix} a_p \\ b_q, b \end{matrix} \right) \\ - (-1)^{a-b} \sum_{k=1}^{a-b} \text{Res}_{s=k-a} \left[ \frac{\Gamma(b+s) \prod_{i=1}^m \Gamma(b_i+s) \prod_{i=1}^n \Gamma(1-a_i-s)}{\prod_{i=n+1}^p \Gamma(a_i+s) \prod_{i=m+1,q} \Gamma(1-b_i-s)} z^{-s} \right]$$

if  $a - b > 0$ .

[2]

$$(9.160) \quad G_{p+1,q+2}^{m+1,n+1} \left( z \mid \begin{matrix} a, a_p \\ a, b_q, b \end{matrix} \right) = (-)^{a-b} G_{p,q+1}^{m+1,n} \left( z \mid \begin{matrix} a_p \\ b_q, b \end{matrix} \right)$$

if  $a - b \leq 0$ .

[2]

$$(9.161) \quad G_{p+2,q+1}^{m,n+1} \left( z \mid \begin{matrix} a, a_p, b \\ b_q, b \end{matrix} \right) = (-)^{a-b} G_{p+1,q}^{m,n} \left( z \mid \begin{matrix} a_p, a \\ b_q \end{matrix} \right)$$

if  $a - b$  is an integer.

9.4. The MacRobert E-function.

9.5. Riemann and Hurwitz zeta functions. [188, (2.4.1)]

$$(9.162) \quad \zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx, \quad \sigma > 1.$$

[184, 5]

$$(9.163) \quad \zeta(3) = \frac{5}{2} \sum_{k \geq 1} \frac{(-)^{k-1}}{k^3 \binom{2k}{k}} = -\frac{4\pi^2}{7} \sum_{k \geq 0} \frac{\zeta(2k)}{(2k+1)(2k+2)2^{2k}}.$$

[153]

$$(9.164) \quad \zeta(4) = \sum_{n=1}^\infty \frac{1}{n!^4} \left( \frac{4n+1}{2n^2} L_n + \frac{7n^3}{4} L_{n-1} \right)$$

where  $L_0 = 0$ ,  $L_1 = 1/3$  and

$$(9.165) \quad 4(4n+3)(4n+5)L_{n+1} + 2(n+1)^3(6n^3+9n^2+5n+1)L_n - n^6(n+1)^3L_{n-1} = 0.$$

[142]

$$(9.166) \quad \zeta'(2k) = (-)^{k+1} \frac{(2\pi)^{2k}}{2(2k)!} \{2k\zeta'(1-2k) - [\psi(2k) - \log(2\pi)]B_{2k}\}.$$

[142]

(9.167)

$$\zeta'(1-2k, p/q) = \frac{[\psi(2k) - \log(2\pi q)]B_{2k}(p/q)}{2k} - \frac{[\psi(2k) - \log(2\pi)]B_{2k}(p/q)}{q^{2k}2k} \\ + (-)^{k+1} \frac{\pi}{(2\pi q)^{2k}} \sum_{n=1}^{q-1} \sin\left(\frac{2\pi pn}{q}\right) \psi^{(2k-1)}(n/q) \\ + (-)^{k+1} \frac{2(2k-1)!}{(2\pi q)^{2k}} \sum_{n=1}^{q-1} \cos\left(\frac{2\pi pn}{q}\right) \zeta'(2k, n/q) + \frac{\zeta'(1-2k)}{q^{2k}}.$$

[142]

$$(9.168) \quad \zeta'(1-2k, 1/2) = -\frac{B_{2k} \log 2}{k4^k} - \frac{(2^{2k-1}-1)\zeta'(1-2k)}{2^{2k-1}}.$$

[142]

$$(9.169) \quad \zeta'(1-2k, 1/3) = -\frac{(9^k-1)B_{2k}\pi}{\sqrt{3}(3^{2k-1}-1)8k} - \frac{B_{2k} \log 3}{3^{2k-1}4k} \\ + (-)^k \frac{\psi^{(2k-1)}(1/3)}{2\sqrt{3}(6\pi)^{2k-1}} - \frac{(3^{2k-1}-1)\zeta'(1-2k)}{2 \times 3^{2k-1}}.$$

[153]

$$(9.170) \quad \sum_{k=1}^{\infty} \frac{k}{k^4 - x^2 k^2 - y^4} = \sum_{n,m=0}^{\infty} \binom{n+m}{n} \zeta(2n+4m+3) x^{2n} y^{4m} \\ = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-)^{n-1} r(n)}{n \binom{2n}{n}} \frac{\prod_{m=1}^{n-1} ((m^2 - x^2)^2 + 4y^4)}{\prod_{m=n}^{2n} (m^4 - x^2 m^2 - y^4)}$$

where

(9.171)

$$r(n) = 205n^6 - 160n^5 + (32 - 62x^2)n^4 + 40x^2n^3 + (x^4 - 8x^2 - 25y^4)n^2 + 10y^4n + y^4(x^2 - 2).$$

[5]

$$(9.172) \quad \sum_{k \geq 1} \frac{(-)^k}{k} [\zeta(nk) - 1] = \log \left( \prod_{j=0}^{n-1} \Gamma[2 - (-)^{(2j+1)/n}] \right).$$

[5]

(9.173)

$$\sum_{k=2}^{\infty} (-)^k [\zeta(k) - 1] k^n = -1 + \frac{1 - 2^{n+1}}{n+1} B_{n+1} - \sum_{k=1}^n (-)^k k! \zeta(k+1) S(n+1, k+1),$$

with  $S$  the Stirling numbers of the second kind.

[5]

$$(9.174) \quad \sum_{k=2}^{\infty} [\zeta(k) - 1] k^n = 1 + \sum_{k=1}^n k! \zeta(k+1) S(n+1, k+1),$$

[6]

$$(9.175) \quad -\sum_{k=1}^{\infty} \frac{k}{4^{2k}} \zeta(2k+1) = G - 1.$$

[184]

$$(9.176) \quad \sum_{k \geq 0} \frac{\zeta(2k)}{(2k+1)4^{2k}} = -\frac{G}{\pi} - \frac{1}{4} \log 2.$$

[184]

$$(9.177) \quad \sum_{k \geq 0} \frac{\zeta(2k)}{2k+1} \left(\frac{3}{4}\right)^{2k} = \frac{G}{3\pi} - \frac{1}{4} \log 2.$$

[6]

$$(9.178) \quad \frac{1}{16} \sum_{k=1}^{\infty} \frac{3^k - 1}{4^k} (k+1) \zeta(k+2) = G.$$

[5]

$$(9.179) \quad \sum_{k=1}^{\infty} \frac{\zeta(2k+1) - 1}{k+2} = \frac{2}{3} - \frac{\gamma}{2} + \log 2 + 6\zeta'(-1).$$

[5]

$$(9.180) \quad \sum_{k=2}^{\infty} \frac{\zeta(k)}{(k+1)(k+2)} = \frac{1-\gamma}{6} - 2\zeta'(-1).$$

[5]

$$(9.181) \quad \sum_{k=2}^{\infty} \frac{k^2}{k+1} [\zeta(k) - 1] = \frac{3}{2} - \frac{\gamma}{2} + \frac{\pi^2}{6} - \frac{1}{2} \log(2\pi).$$

[5]

$$(9.182) \quad \sum_{k=2}^{\infty} [\zeta(4k) - 1] = \frac{7}{8} - \frac{\pi}{4} \coth \pi$$

[142]

$$(9.183) \quad 4 \sum_{k=0}^{\infty} \frac{1 - \zeta(2k)}{(2k+1)3^{2k}} = \log(192) - \pi \frac{2\sqrt{3}}{9} + \frac{\sqrt{3}}{3\pi} \psi^{(1)}(1/3).$$

[5]

$$(9.184) \quad \sum_{k=1}^{\infty} [\zeta(4k) - 1] z^{4k} = \frac{3z^4 - 1}{2(z^4 - 1)} - \frac{\pi z}{4} [\cot(\pi z) + \coth(\pi z)], \quad |z| < 2.$$

[5]

$$(9.185) \quad \sum_{k=1}^{\infty} [\zeta(2k) - 1] \sin k = -\frac{1}{2} \cot(1/2) + \frac{\pi \sin(1/2) \sin[2\pi \cos(1/2)] - \cos(1/2) \sinh[2\pi \sin(1/2)]}{\cos[2\pi \cos(1/2)] - \cosh[2\pi \sin(1/2)]}$$

[5]

$$(9.186) \quad \sum_{k=1}^{\infty} \binom{p+k}{k} \zeta(p+k+1, a) z^k = \frac{(-)^p}{p!} [\psi^{(p)}(a) - \psi^{(p)}(a-z)].$$

[5]

$$(9.187) \quad \sum_{k=1}^{\infty} \frac{t^k}{k^2} \zeta(2k) = \log[\pi \sqrt{t} \csc(\pi \sqrt{t})].$$

[6]

$$(9.188) \quad \frac{1}{8} \sum_{k=2}^{\infty} \frac{k}{2^k} \zeta(k+1, 3/4) = G.$$

[6]

$$(9.189) \quad -\frac{1}{8} \sum_{k=2}^{\infty} \frac{k}{2^k} \zeta(k+1, 5/4) = G - 1.$$

[3]

(9.190)

$$\zeta'(1, p/q) - \zeta'(1, 1 - p/q) = \pi \cot \frac{\pi p}{q} [\log(2\pi q) + \gamma] - 2\pi \sum_{j=1}^{q-1} \log(\Gamma(j/q)) \sin \frac{2\pi j p}{q}.$$

[75]

$$(9.191) \quad \int_0^1 \sin(2\pi q) \zeta(z, q) dq = \frac{(2\pi)^z}{4\Gamma(z)} \csc \frac{z\pi}{2}.$$

[75]

$$(9.192) \quad \int_0^1 \sin(2k\pi q) \zeta(z, q) dq = \frac{(2\pi)^z k^{z-1}}{4\Gamma(z)} \csc \frac{z\pi}{2}.$$

[75]

$$(9.193) \quad \int_0^1 \cos(2k\pi q) \zeta(z, q) dq = \frac{(2\pi)^z k^{z-1}}{4\Gamma(z)} \sec \frac{z\pi}{2}.$$

[75]

$$(9.194) \quad \int_0^1 \zeta(z', q) \zeta(z, q) dq = -\zeta(z + z' - 1) B(1 - z, 1 - z') \frac{\cos \frac{\pi(z-z')}{2}}{\cos \frac{\pi(z+z')}{2}}.$$

[75]

$$(9.195) \quad \int_0^1 q^n \zeta(z, q) dq = -n! \sum_{j=1}^n \frac{\zeta(z-j)}{(z-j)_j (n-j+1)!}.$$

[75]

$$(9.196) \quad \int_0^1 \ln(\sin \pi q) \zeta(z, q) dq = -\frac{\Gamma(1-z)}{(2\pi)^{1-z}} \sin \frac{\pi z}{2} \zeta(2-z).$$

[95]

$$(9.197) \quad \Phi(z, s, u) = 2^{-s} [\Phi(z^2, s, u/2) + z\Phi(z^2, s, (u+1)/2)].$$

[57]

$$(9.198) \quad \zeta(r, s) \equiv \sum_{m < n} \frac{1}{n^s m^r}.$$

[57]

$$(9.199) \quad \zeta(r, s) + \zeta(s, r) = \zeta(r)\zeta(s) - \zeta(r+s).$$

[57]

$$(9.200) \quad \zeta(r, s) = -\frac{1}{2} \zeta(r+s) + \sum_{j=1, \text{ odd}} \left( \binom{j-1}{s-1} + \binom{j-1}{r-1} \right) \zeta(j) \zeta(r+s-j).$$

for  $r$  even and  $s$  odd.

[57]

$$(9.201) \quad \zeta(1, s) = \frac{s}{2}\zeta(s+1) - \frac{1}{2} \sum_{j=1}^{s-2} \zeta(j+1)\zeta(s-j).$$

[57]

$$(9.202) \quad \zeta(r, s) = -\frac{1}{2}\zeta(r+s) + \sum_{k=3, k \text{ odd}}^{\infty} \Phi_k \sum_{j=0, j \text{ even}}^{k-1} \binom{k}{j} \eta(r-j)\eta(s-k+j),$$

where

$$(9.203) \quad \Phi_k \equiv -\frac{2}{\pi} \sum_{d=1, d \text{ odd}}^{k-2} (-1)^{(d-1)/2} \frac{\pi^d}{d!} \zeta(k-d+1),$$

and

$$(9.204) \quad \eta(s) = (1 - 2^{1-s})\zeta(s).$$

9.6. Bernoulli Numbers and Polynomials. [95]

$$(9.205) \quad B_m(x) = \sum_{n=0}^m \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} (x+k)^m.$$

[48]

$$(9.206) \quad \sum_{k=0}^m \frac{\binom{m}{k}}{\binom{n+k+s}{s}} y^{m-k} B_{n+k+s}(x) = \sum_{k=0}^n \frac{\binom{n}{k}}{\binom{m+k+s}{s}} (-y)^{n-k} B_{m+k+s}(x+y) \\ + \sum_{j=0}^{s-1} \sum_{i=0}^{s-1-j} \binom{s-1-j}{i} \binom{s-1}{j} \frac{(-1)^{n+1+i} s y^{m+n+s-j} B_j(x)}{(m+n+1+i) \binom{m+n+l}{n}}$$

and

$$(9.207) \quad \sum_{k=0}^m \binom{m}{k} \binom{n+k}{s} y^{m-k} B_{n+k-s}(x) = \sum_{k=0}^n \binom{n}{k} \binom{m+k}{s} (-y)^{n-k} B_{m+k-s}(x+y).$$

[48]

$$(9.208) \quad \sum_{j=0}^k \binom{k+1}{j} (k+j+1) B_{k+j} = 0, \quad k \geq 1.$$

[57]

$$(9.209) \quad B_s(t)B_r(t) = \sum_{j>0, j \equiv r+s \pmod{2}} \frac{1}{j} \left( r \binom{s}{j-r} + s \binom{r}{j-s} \right) B_{r+s-j} B_j(t) + \frac{1}{2} ((-1)^{r+s} + 1) \frac{(-)^r B_{r+s}}{\binom{r+s}{s}}$$

[95]

$$(9.210) \quad E_m(x) = \sum_{n=0}^m \frac{1}{2^n} \sum_{k=0}^n (-1)^k \binom{n}{k} (x+k)^m.$$

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