INVERSE PROBLEMS ASSOCIATED WITH PERFECT CUBOIDS.

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Abstract. A perfect cuboid is a rectangular parallelepiped with integer edges, integer face diagonals, and integer space diagonal. Such cuboids have not yet been found, but nor has their existence been disproved. Perfect cuboids are described by a certain system of Diophantine equations possessing an intrinsic S_3 symmetry. Recently these equations were factorized with respect to this S_3 symmetry and the factor equations were transformed into E-form. As appears, the transformed factor equations are explicitly solvable. Based on this solution, polynomial inverse problems are formulated in the present paper.

1. Introduction.

Perfect cuboids are described by the following four polynomial equations:

$$\begin{aligned} &x_1^2+x_2^2+x_3^2-L^2=0, && x_2^2+x_3^2-d_1^2=0,\\ &x_3^2+x_1^2-d_2^2=0, && x_1^2+x_2^2-d_3^2=0. \end{aligned} \tag{1.1}$$

Here x_1, x_2, x_3 are edges of a cuboid, d_1, d_2, d_3 are its face diagonals, and L is its space diagonal. For the history of perfect cuboids the reader is referred to [1–44].

Recently in [45] the symmetry approach to the equations (1.1) was initiated. It is based on an intrinsic S_3 symmetry of these equations. Indeed, let's consider the following action of the group S_3 upon the variables x_1, x_2, x_3, d_1, d_2, d_3 and L:

$$\sigma(x_i)=x_{\sigma i}, \qquad \sigma(d_i)=d_{\sigma i}, \qquad \sigma(L)=L. \tag{1.2}$$

The first equation (1.1) is invariant with respect to the transformations (1.2). The other three equations are not invariant, but the system as a whole is again invariant, i.e. it possesses S_3 symmetry based on the transformations (1.2). In [46] the equations (1.1) were factorized with respect to their S_3 symmetry and the following system of eight factor equations was derived:

$$x_1^2+x_2^2+x_3^2-L^2=0, \tag{1.3}$$

$$(x_2^2+x_3^2-d_1^2)+(x_3^2+x_1^2-d_2^2)+(x_1^2+x_2^2-d_3^2)=0, \tag{1.4}$$

$$d_1\,(x_2^2+x_3^2-d_1^2)+d_2\,(x_3^2+x_1^2-d_2^2)+d_3\,(x_1^2+x_2^2-d_3^2)=0, \tag{1.5}$$

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$$x_1\,(x_2^2+x_3^2-d_1^2)+x_2\,(x_3^2+x_1^2-d_2^2)+x_3\,(x_1^2+x_2^2-d_3^2)=0, \tag{1.6}$$

$$\begin{aligned} &x_1\,d_1\,(x_2^2+x_3^2-d_1^2)+x_2\,d_2\,(x_3^2+x_1^2-d_2^2)+\\ &\qquad\qquad+\,x_3\,d_3\,(x_1^2+x_2^2-d_3^2)=0, \end{aligned} \tag{1.7}$$

$$x_1^2\,(x_2^2+x_3^2-d_1^2)+x_2^2\,(x_3^2+x_1^2-d_2^2)+x_3^2\,(x_1^2+x_2^2-d_3^2)=0, \tag{1.8}$$

$$d_1^2\,(x_2^2+x_3^2-d_1^2)+d_2^2\,(x_3^2+x_1^2-d_2^2)+d_3^2\,(x_1^2+x_2^2-d_3^2)=0, \tag{1.9}$$

$$\begin{aligned} &x_1^2\,d_1^2\,(x_2^2+x_3^2-d_1^2)+x_2^2\,d_2^2\,(x_3^2+x_1^2-d_2^2)+\\ &\qquad\qquad+\,x_3^2\,d_3^2\,(x_1^2+x_2^2-d_3^2)=0. \end{aligned} \tag{1.10}$$

Each solution of the equations (1.1) is a solution of the equations (1.3), (1.4), (1.5), (1.6), (1.7), (1.8), (1.9), (1.10). But generally speaking, the converse is not true. Fortunately, in [47] the following theorem was proved.

Theorem 1.1. *Each integer or rational solution of the equations* (1.3) *through* (1.10) *such that* $x_1>0$, $x_2>0$, $x_3>0$, $d_1>0$, $d_2>0$, *and* $d_3>0$ *is an integer or rational solution for the equations* (1.1).

Due to Theorem 1.1 the factor equations (1.3) through (1.10) can be applied for studying perfect cuboids.

Note that each of the equations (1.3) through (1.10) is invariant with respect to the transformations (1.2). The left hand sides of these equations are multisymmetric polynomials in the sense of the following definition.

Definition 1.1. A polynomial $p\in\mathbb{Q}[x_1,x_2,x_3,d_1,d_2,d_3,L]$ is called multisymmetric if it is invariant with respect to the action (1.2) of the group S_3.

Multisymmetric polynomials constitute a subring within the polynomial ring $\mathbb{Q}[x_1,x_2,x_3,d_1,d_2,d_3,L]$. For the sake of brevity we introduce the matrix

$$M=\begin{Vmatrix} x_1 & x_2 & x_3\\ d_1 & d_2 & d_3 \end{Vmatrix}$$

and denote $Q[x_1,x_2,x_3,d_1,d_2,d_3,L]=\mathbb{Q}[M,L]$. Similarly, the subring of multisymmetric polynomials is denoted through $\operatorname{Sym}\mathbb{Q}[M,L]$. For the general theory of multisymmetric polynomials the reader is referred to [48–68]. For our purposes we need the following theorem from this theory.

Theorem 1.2. *Each multisymmetric polynomial* $p\in\operatorname{Sym}\mathbb{Q}[x_1,x_2,x_3,d_1,d_2,d_3,L]$ *can be expressed through the following elementary multisymmetric polynomials:*

$$\begin{aligned} &e_{[1,0]}=x_1+x_2+x_3, && e_{[0,1]}=d_1+d_2+d_3,\\ &e_{[2,0]}=x_1\,x_2+x_2\,x_3+x_3\,x_1, && e_{[0,2]}=d_1\,d_2+d_2\,d_3+d_3\,d_1,\\ &e_{[3,0]}=x_1\,x_2\,x_3, && e_{[0,3]}=d_1\,d_2\,d_3, \end{aligned} \tag{1.11}$$

$$\begin{aligned} &e_{[2,1]}=x_1\,x_2\,d_3+x_2\,x_3\,d_1+x_3\,x_1\,d_2,\\ &e_{[1,1]}=x_1\,d_2+d_1\,x_2+x_2\,d_3+d_2\,x_3+x_3\,d_1+d_3\,x_1,\\ &e_{[1,2]}=x_1\,d_2\,d_3+x_2\,d_3\,d_1+x_3\,d_1\,d_2. \end{aligned} \tag{1.12}$$

Theorem 1.2 is known as the Fundamental Theorem on Elementary Multisymmetric Polynomials. Its proof can be found in [54]. If we denote

$$\mathbb{Q}[E_{10}, E_{20}, E_{30}, E_{01}, E_{02}, E_{03}, E_{21}, E_{11}, E_{12}, L] = \mathbb{Q}[E, L],$$

then Theorem 1.2 means that for each multisymmetric polynomial p from the ring $\operatorname{Sym}\mathbb{Q}[M, L]$ there is some polynomial $q \in \mathbb{Q}[E, L]$ of ten independent variables such that p is produced from q by substituting the elementary multisymmetric polynomials (1.11) and (1.12) for E_{10}, E_{20}, E_{30}, E_{01}, E_{02}, E_{03}, E_{21}, E_{11}, and E_{12} into its arguments. The substitution procedure can be understood as a mapping:

$$\varphi\colon\ \mathbb{Q}[E, L] \to \operatorname{Sym}\mathbb{Q}[M, L]. \tag{1.13}$$

The mapping (1.13) is a ring homomorphism. It is surjective, which follows from Theorem 1.2, but it is not bijective, i. e. it has a nonzero kernel $K = \operatorname{Ker}\varphi$. The kernel K is an ideal in the ring $\mathbb{Q}[E, L]$. It was calculated as

$$K = \bigl\langle q_1,\, \ldots\, ,\, q_7\bigr\rangle, \tag{1.14}$$

where $q_1, \ldots , q_7$ are seven polynomials being a basis of K. These polynomials are given by the explicit formulas (2.4) through (2.10) in [46]. The ideal (1.14) has a Gröbner basis comprising fourteen polynomials:

$$K = \bigl\langle \tilde q_1, \tilde q_2, \tilde q_3, \tilde q_4, \tilde q_5, \tilde q_6, \tilde q_7, \tilde q_8, \tilde q_9, \tilde q_{10}, \tilde q_{11}, \tilde q_{12}, \tilde q_{13}, \tilde q_{14}\bigr\rangle. \tag{1.15}$$

The polynomials q_1 through q_{14} from (1.15) were calculated in [46] with the use of the Maxima symbolic computations package, but explicit formulas for them were not given. These formulas were given later in Appendix to [69]. For the theory of Gröbner bases and their applications the reader is referred to [70].

Returning to the factor equations (1.3) through (1.10) and noting that their left hand sides are multisymmetric polynomials, one can apply Theorem 1.2 to them. This was done in [69] and the E-forms[1] of the factor equations were derived in [69]. Here are the transformed factor equations:

$$E_{10}^2 - 2\,E_{20} - L^2 = 0, \tag{1.16}$$

$$2\,E_{02} - 4\,E_{20} - E_{01}^2 + 2\,E_{10}^2 = 0, \tag{1.17}$$

$$E_{10}\,E_{11} - 3\,E_{03} - E_{21} + 3\,E_{01}\,E_{02} - E_{20}\,E_{01} - E_{01}^3 = 0, \tag{1.18}$$

$$E_{01}\,E_{11} - E_{12} - 3\,E_{30} + E_{10}\,E_{02} + E_{20}\,E_{10} - E_{01}^2\,E_{10} = 0, \tag{1.19}$$

$$\begin{aligned} &-E_{10}\,E_{21} - E_{01}\,E_{12} - E_{01}\,E_{30} - E_{01}^3\,E_{10} + E_{01}^2\,E_{11} -\\ &\qquad -\ E_{02}\,E_{11} + E_{11}\,E_{20} - E_{10}\,E_{03} + 2\,E_{10}\,E_{01}\,E_{02} = 0. \end{aligned} \tag{1.20}$$

$$\begin{aligned} &4\,E_{01}\,E_{10}\,E_{11} - 3\,E_{01}^2\,E_{10}^2 + 2\,E_{10}^2\,E_{02} + 2\,E_{20}\,E_{01}^2 - 2\,E_{10}\,E_{12} -\\ &\qquad -2\,E_{02}\,E_{20} - 2\,E_{01}\,E_{21} - E_{11}^2 - 12\,E_{10}\,E_{30} + 6\,E_{20}^2 = 0, \end{aligned} \tag{1.21}$$

[1] The E-form of a polynomial is its preimage under the mapping (1.13).

$$
\begin{aligned}
&4\,E_{01}\,E_{10}\,E_{11}-4\,E_{10}^2\,E_{02}-4\,E_{20}\,E_{01}^2-2\,E_{10}\,E_{12}+10\,E_{02}\,E_{20}-\\
&-2\,E_{01}\,E_{21}-E_{11}^2-12\,E_{01}\,E_{03}-3\,E_{01}^4-6\,E_{02}^2+12\,E_{01}^2\,E_{02}=0,
\end{aligned}
\tag{1.22}
$$

$$
\begin{gathered}
9\,E_{01}\,E_{03}\,E_{20}-7\,E_{01}^2\,E_{02}\,E_{20}+2\,E_{02}\,E_{10}\,E_{12}-2\,E_{01}^2\,E_{10}\,E_{12}+\\
+3\,E_{03}\,E_{10}\,E_{11}+4\,E_{01}^3\,E_{10}\,E_{11}-7\,E_{01}\,E_{02}\,E_{10}\,E_{11}-6\,E_{01}\,E_{03}\,E_{10}^2+\\
+8\,E_{01}^2\,E_{02}\,E_{10}^2+3\,E_{01}\,E_{11}\,E_{30}-2\,E_{01}\,E_{20}\,E_{21}+E_{10}\,E_{12}\,E_{20}-\\
-E_{02}\,E_{10}^2\,E_{20}+E_{01}\,E_{10}\,E_{11}\,E_{20}+9\,E_{02}\,E_{10}\,E_{30}-2\,E_{02}\,E_{20}^2+\\
+2\,E_{01}^2\,E_{20}^2-E_{11}^2\,E_{20}-3\,E_{12}\,E_{30}+E_{02}\,E_{11}^2-E_{01}^2\,E_{11}^2-\\
-2\,E_{02}^2\,E_{10}^2+2\,E_{01}^4\,E_{20}+2\,E_{02}^2\,E_{20}-3\,E_{03}\,E_{21}-\\
-2\,E_{01}^3\,E_{21}+5\,E_{01}\,E_{02}\,E_{21}-6\,E_{01}^2\,E_{10}\,E_{30}-3\,E_{01}^4\,E_{10}^2=0.
\end{gathered}
\tag{1.23}
$$

In [69] the factor equations (1.16), (1.17), (1.18), (1.19), (1.20), (1.21), (1.22), and (1.23) were complemented with fourteen kernel equations

$$
\begin{aligned}
&\tilde q_1=0, &&\tilde q_2=0, &&\tilde q_3=0,\\
&\tilde q_4=0, &&\tilde q_5=0, &&\tilde q_6=0,\\
&\tilde q_7=0, &&\tilde q_8=0, &&\tilde q_9=0,\\
&\tilde q_{10}=0, &&\tilde q_{11}=0, &&\tilde q_{12}=0,\\
&\tilde q_{13}=0, &&\tilde q_{14}=0.
\end{aligned}
\tag{1.24}
$$

As a result a huge system of twenty two polynomial equations with respect to ten variables was obtained. In [69] this system was analyzed and luckily was reduced to a single equation. Here is this equation:

$$
(2\,E_{11})^2+(E_{01}^2+L^2-E_{10}^2)^2-8\,E_{01}^2\,L^2=0.
\tag{1.25}
$$

Equation (1.25) turns out to be explicitly solvable, which is also a lucky event. Its general solution was discovered in [71]. This general solution of (1.25), upon eliminating L by homogeneity, consists of one two-parameter solution and a series of one-parameter solutions. The main goal of the present paper is to propagate these explicit solutions back to the equations (1.16) through (1.24) and formulate polynomial inverse problems relating these solutions with perfect cuboids.

2. Rational perfect cuboids.

Let $x_1,\ x_2,\ x_3,\ d_1,\ d_2,\ d_3$ be edges and face diagonals of some perfect cuboid and let L be its space diagonal. Then, dividing these numbers by L, we get a cuboid whose edges and face diagonals are given by rational numbers, while the space diagonal is equal to unity. Such a cuboid is called a rational perfect cuboid. Conversely, if we have a rational perfect cuboid with unit space diagonal, we can take the common denominator of the rational numbers $x_1,\ x_2,\ x_3,\ d_1,\ d_2,\ d_3$ for L and then, multiplying these numbers by L, we get a perfect cuboid with integer edges and face diagonals whose space diagonal is equal to the integer number L. Thus, integer perfect cuboids and rational perfect cuboids with unit space diagonal are equivalent to each other.

The equivalence of perfect cuboids and rational perfect cuboids was already used in [41] for deriving three cuboid conjectures. In the present paper we use this fact by setting $L = 1$ and thus reducing the number of variables in the equations (1.16) through (1.23) and in the equation (1.25).

3. The two-parameter case.

Let's substitute $L = 1$ into (1.25). As a result, using the notation $E_{11} = x$, $E_{01} = y$, $E_{10} = z$, we get the equation coinciding with the equation (1.1) in [71]:

$$(2\,E_{11})^2 + (E_{01}^2 + 1 - E_{10}^2)^2 = 8\,E_{01}^2. \tag{3.1}$$

This equation was solved in [71]. Theorem 2.1 from [71] yields the following two-parameter solution of the equation (3.1):

$$E_{11} = -\frac{b\,(c^2 + 2 - 4\,c)}{b^2\,c^2 + 2\,b^2 - 3\,b^2\,c + c - b\,c^2 + 2\,b}, \tag{3.2}$$

$$E_{01} = -\frac{b(c^2 + 2 - 2\,c)}{b^2\,c^2 + 2\,b^2 - 3\,b^2\,c + c - b\,c^2 + 2\,b}, \tag{3.3}$$

$$E_{10} = -\frac{b^2\,c^2 + 2\,b^2 - 3\,b^2\,c - c}{b^2\,c^2 + 2\,b^2 - 3\,b^2\,c + c - b\,c^2 + 2\,b}. \tag{3.4}$$

The numbers b and c are two rational parameters in (3.2), (3.3), and (3.4).

Apart from E_{11}, E_{10}, and E_{01} there are six other variables E_{20}, E_{30}, E_{02}, E_{03}, E_{21}, E_{12} in the equations (1.16) through (1.23). These variables are expressed through E_{11}, E_{10}, E_{01}, and L by means of the formulas (4.1), (4.3), (4.6), (4.7), (5.1), and (5.2) from [69]. Here are the formulas for E_{20} and E_{02}:

$$E_{20} = \frac{1}{2}\,E_{10}^2 - \frac{1}{2}\,L^2, \qquad\qquad E_{02} = \frac{1}{2}\,E_{01}^2 - L^2. \tag{3.5}$$

Substituting $L = 1$, (3.3), and (3.4) into (3.5), we derive

$$\begin{gathered} E_{20} = \frac{b}{2}\,(b\,c^2 - 2\,c - 2\,b)\,(2\,b\,c^2 - c^2 - 6\,b\,c + 2 + 4\,b)\,\times \\ \times\,(b\,c - 1 - b)^{-2}\,(b\,c - c - 2\,b)^{-2}, \end{gathered} \tag{3.6}$$

$$\begin{gathered} E_{02} = \frac{1}{2}\,(28\,b^2\,c^2 - 16\,b^2\,c - 2\,c^2 - 4\,b^2 - b^2\,c^4 + \\ +\,4\,b^3\,c^4 - 12\,b^3\,c^3 + 4\,b\,c^3 + 24\,b^3\,c - 8\,b\,c - 2\,b^4\,c^4 + \\ +\,12\,b^4\,c^3 - 26\,b^4\,c^2 - 8\,b^2\,c^3 + 24\,b^4\,c - 16\,b^3 - 8\,b^4)\,\times \\ \times\,(b\,c - 1 - b)^{-2}\,(b\,c - c - 2\,b)^{-2}. \end{gathered} \tag{3.7}$$

The formulas for E_{21} and E_{12} are taken from (5.1) and (5.2) in [69] respectively:

$$\begin{aligned} E_{21} = {} & \frac{2\,E_{10}^3\,E_{11} + 2\,E_{01}^2\,E_{10}\,E_{11} - E_{01}\,E_{10}^4 + E_{01}^5}{8\,(E_{01}^2 + E_{10}^2)} + \\ & + \frac{6\,E_{10}\,E_{11}\,L^2 - 2\,E_{01}\,E_{10}^2\,L^2 - 8\,E_{01}^3\,L^2 + 3\,E_{01}\,L^4}{8\,(E_{01}^2 + E_{10}^2)}, \end{aligned} \tag{3.8}$$

$$E_{12} = \frac{E_{01}^4\,E_{10} - 2\,E_{01}^3\,E_{11} - 2\,E_{01}\,E_{10}^2\,E_{11} - E_{10}^5}{8\,(E_{01}^2 + E_{10}^2)} + \\ + \frac{6\,E_{10}^3\,L^2 - 6\,E_{01}\,E_{11}\,L^2 + 3\,E_{10}\,L^4}{8\,(E_{01}^2 + E_{10}^2)}. \tag{3.9}$$

Substituting $L = 1$, (3.2), (3.3), and (3.4) into (3.8) and (3.9), we derive

$$\begin{aligned} E_{21} = \frac{b}{2}\,&(5\,c^6\,b - 2\,c^6\,b^2 + 52\,c^5\,b^2 - 16\,c^5\,b - 2\,c^7\,b^2 + 2\,b^4\,c^8 + \\ &+ 142\,b^4\,c^6 - 26\,b^4\,c^7 - 426\,b^4\,c^5 - 61\,b^3\,c^6 + 100\,b^3\,c^5 + 14\,c^7\,b^3 - \\ &- c^8\,b^3 - 20\,b\,c^2 - 8\,b^2\,c^2 - 16\,b^2\,c - 128\,b^2\,c^4 - 200\,b^3\,c^3 + \\ &+ 244\,b^3\,c^2 + 32\,b\,c^3 - 112\,b^3\,c + 768\,b^4\,c^4 - 852\,b^4\,c^3 + 568\,b^4\,c^2 + \\ &+ 104\,b^2\,c^3 - 208\,b^4\,c + 8\,c^4 - 4\,c^3 + 16\,b^3 + 32\,b^4 - 2\,c^5) \times \\ &\times (b^2\,c^4 - 6\,b^2\,c^3 + 13\,b^2\,c^2 - 12\,b^2\,c + 4\,b^2 + c^2)^{-1} \times \\ &\times (b\,c - 1 - b)^{-2}\,(b\,c - c - 2\,b)^{-2}, \end{aligned} \tag{3.10}$$

$$\begin{aligned} E_{12} = (&16\,b^6 + 32\,b^5 - 6\,c^5\,b^2 + 2\,c^5\,b - 62\,b^5\,c^6 + 62\,b^6\,c^6 - \\ &- 180\,b^6\,c^5 + 18\,b^5\,c^7 - 12\,b^6\,c^7 - 2\,b^5\,c^8 + b^6\,c^8 + 248\,b^5\,c^2 + \\ &+ 248\,b^6\,c^2 - 96\,b^6\,c + 321\,b^6\,c^4 - 180\,b^5\,c^3 - 144\,b^5\,c - 360\,b^6\,c^3 + \\ &+ b^4\,c^8 + 8\,b^4\,c^6 - 6\,b^4\,c^7 + 18\,b^4\,c^5 + 7\,b^3\,c^6 + 90\,b^5\,c^5 - 14\,b^3\,c^5 - \\ &- c^7\,b^3 + 17\,b^2\,c^4 + 28\,b^3\,c^3 - 28\,b^3\,c^2 - 4\,b\,c^3 + 8\,b^3\,c - 57\,b^4\,c^4 + \\ &+ 36\,b^4\,c^3 + 32\,b^4\,c^2 - 12\,b^2\,c^3 - 48\,b^4\,c - c^4 + 16\,b^4) \times \\ &\times (b^2\,c^4 - 6\,b^2\,c^3 + 13\,b^2\,c^2 - 12\,b^2\,c + 4\,b^2 + c^2)^{-1} \times \\ &\times (b\,c - 1 - b)^{-2}\,(b\,c - c - 2\,b)^{-2}. \end{aligned} \tag{3.11}$$

The formulas for E_{30} and E_{03} are taken from (4.6) and (4.7) in [69]:

$$E_{30} = -\frac{1}{3}\,E_{12} - \frac{1}{6}\,E_{10}\,E_{01}^2 - \frac{1}{2}\,E_{10}\,L^2 + \frac{1}{6}\,E_{10}^3 + \frac{1}{3}\,E_{01}\,E_{11}, \tag{3.12}$$

$$E_{03} = -\frac{1}{3}\,E_{21} - \frac{1}{6}\,E_{01}\,E_{10}^2 - \frac{5}{6}\,E_{01}\,L^2 + \frac{1}{6}\,E_{01}^3 + \frac{1}{3}\,E_{10}\,E_{11}. \tag{3.13}$$

The formulas (3.12) and (3.13) comprise E_{21} and E_{12} from (3.10) and (3.11). Applying $L = 1$, (3.10), (3.11), (3.2), (3.3), and (3.4) to (3.12) and (3.13), we get

$$\begin{aligned} E_{03} = \frac{b}{2}\,&(b^2\,c^4 - 5\,b^2\,c^3 + 10\,b^2\,c^2 - 10\,b^2\,c + 4\,b^2 + 2\,b\,c + \\ &+ 2\,c^2 - b\,c^3)\,(2\,b^2\,c^4 - 12\,b^2\,c^3 + 26\,b^2\,c^2 - 24\,b^2\,c + \\ &+ 8\,b^2 - c^4\,b + 3\,b\,c^3 - 6\,b\,c + 4\,b + c^3 - 2\,c^2 + 2\,c) \times \\ &\times ((b^2\,c^4 - 6\,b^2\,c^3 + 13\,b^2\,c^2 - 12\,b^2\,c + 4\,b^2 + c^2)^{-1} \times \\ &\times (b\,c - 1 - b)^{-2}\,(-c + b\,c - 2\,b)^{-2}, \end{aligned} \tag{3.14}$$

$$\begin{gathered} E_{30}=c\,b^2\,(1-c)\,(c-2)\,(b\,c^2-4\,b\,c+2+4\,b)\,\times\\ \times\,(2\,b\,c^2-c^2-4\,b\,c+2\,b)\,\times\\ \times\,(b^2\,c^4-6\,b^2\,c^3+13\,b^2\,c^2-12\,b^2\,c+4\,b^2+c^2)^{-1}\,\times\\ \times\,(b\,c-1-b)^{-2}\,(-c+b\,c-2\,b)^{-2}. \end{gathered} \tag{3.15}$$

Note that the quantities E_{10}, E_{20}, and E_{30} given by the formulas (3.4), (3.6), (3.15) are the values of three elementary multisymmetric polynomials in the left column of (1.11). One can easily see that these polynomials coincide with the regular symmetric polynomials of three variables x_1, x_2, and x_3 (see [72]). Therefore, if we write the cubic equation $(x-x_1)(x-x_2)(x-x_3)=0$, it expands to

$$x^3-E_{10}\,x^2+E_{20}\,x-E_{30}=0. \tag{3.16}$$

Similarly, the quantities E_{01}, E_{02}, and E_{03} given by the formulas (3.3), (3.7), and (3.14) are the values of the elementary multisymmetric polynomials in the right column of (1.11). These three polynomials coincide with regular symmetric polynomials of three variables d_1, d_2, and d_3. For this reason, if we write the cubic equation $(d-d_1)(d-d_2)(d-d_3)=0$, this equation expands to

$$d^3-E_{01}\,d^2+E_{02}\,d-E_{03}=0. \tag{3.17}$$

The remaining three quantities E_{11}, E_{21}, and E_{12} given by the formulas (3.2), (3.10), (3.11) are the values of three elementary multisymmetric polynomials in (1.12). They lead to the following auxiliary polynomial equations:

$$\begin{aligned} &x_1\,x_2\,d_3+x_2\,x_3\,d_1+x_3\,x_1\,d_2=E_{21},\\ &x_1\,d_2+d_1\,x_2+x_2\,d_3+d_2\,x_3+x_3\,d_1+d_3\,x_1=E_{11},\\ &x_1\,d_2\,d_3+x_2\,d_3\,d_1+x_3\,d_1\,d_2=E_{12}. \end{aligned} \tag{3.18}$$

Now we can formulate the following two inverse cuboid problems.

Problem 3.1. *Find all pairs of rational numbers b and c for which the cubic equations (3.16) and (3.17) with the coefficients given by the formulas (3.4), (3.6), (3.15), (3.3), (3.7), (3.14) possess positive rational roots x_1, x_2, x_3, d_1, d_2, d_3 obeying the auxiliary polynomial equations (3.18) whose right hand sides are given by the formulas (3.2), (3.10), (3.11).*

Problem 3.2. *Find at least one pair of rational numbers b and c for which the cubic equations (3.16) and (3.17) with the coefficients given by the formulas (3.4), (3.6), (3.15), (3.3), (3.7), (3.14) possess positive rational roots x_1, x_2, x_3, d_1, d_2, d_3 obeying the auxiliary polynomial equations (3.18) whose right hand sides are given by the formulas (3.2), (3.10), (3.11).*

4. The first one-parameter case.

The four one-parameter cases considered below correspond to the special solutions of Theorem 2.1 in [71]. The first one-parameter case corresponds to the following choice of sign options in this theorem:

$$x=y, \qquad\qquad y=z+1. \tag{4.1}$$

Equations (4.1) are easily resolved in the one-parameter form:

$$x=c, \qquad y=c, \qquad z=c-1. \tag{4.2}$$

Taking into account the notations $E_{11}=x$, $E_{01}=y$, $E_{10}=z$, we derive the following one-parameter solution of the equation (3.1) from (4.2):

$$E_{11}=c, \qquad E_{01}=c, \qquad E_{10}=c-1. \tag{4.3}$$

Substituting $L=1$ and (4.3) into (3.5), (3.8), (3.9), (3.12), and (3.13), we get

$$\begin{aligned} &E_{20}=\frac{c^2-2\,c}{2}, && E_{02}=\frac{c^2-2}{2},\\ &E_{21}=\frac{c^2-2\,c}{2}, && E_{12}=1,\\ &E_{30}=0, && E_{03}=\frac{c^2-2\,c}{2}. \end{aligned} \tag{4.4}$$

In the formulas (4.4) we find $E_{30}=0$, i.e. the last term E_{30} of the cubic equation (3.16) is zero. Then one of its roots x_1, x_2, or x_3 is zero. But a non-degenerate cuboid cannot have a zero edge. As a result we have the following theorem.

Theorem 4.1. *There are no perfect cuboids associated with the one-parameter solution* (4.3) *of the equation* (3.1)*.*

5. The second one-parameter case.

In the second one-parameter case we choose the following sign options in Theorem 2.1 from [71]: $x=y$ and $y=-z-1$. Then instead of (4.3) we derive

$$E_{11}=c, \qquad E_{01}=c, \qquad E_{10}=-c-1. \tag{5.1}$$

Substituting $L=1$ and (5.1) into (3.5), (3.8), (3.9), (3.12), and (3.13), we get

$$\begin{aligned} &E_{20}=\frac{c^2+2\,c}{2}, && E_{02}=\frac{c^2-2}{2},\\ &E_{21}=-\frac{c^2+2\,c}{2}, && E_{12}=1,\\ &E_{30}=0, && E_{03}=-\frac{c^2+2\,c}{2}. \end{aligned} \tag{5.2}$$

In (5.2) we again see $E_{30}=0$. Therefore we can formulate the following theorem.

Theorem 5.1. *There are no perfect cuboids associated with the one-parameter solution* (5.1) *of the equation* (3.1)*.*

6. The third and the fourth one-parameter cases.

These two one-parameter cases correspond to the last two sign options in Theorem 2.1 from [71]. These two sign options are given by the formulas $x=-y$ and

$y = \pm (z+1)$. They yield the following one-parameter solutions for (3.1):

$$E_{11} = c, \qquad E_{01} = -c, \qquad E_{10} = \pm c - 1. \tag{6.1}$$

The formulas $E_{11} = c$ and $E_{01} = -c$ mean that E_{11} and E_{01} are of opposite signs or both of them are zero. On the other hand, we have two equalities

$$\begin{aligned} &x_1\, d_2 + d_1\, x_2 + x_2\, d_3 + d_2\, x_3 + x_3\, d_1 + d_3\, x_1 = E_{11},\\ &d_1 + d_2 + d_3 = E_{01} \end{aligned}$$

derived from (1.11) and (1.12). The left hand sides of these equalities are positive numbers since edges and face diagonals of a perfect cuboid are positive. The contradiction obtained leads to the following theorem.

Theorem 6.1. *There are no perfect cuboids associated with the one-parameter solutions* (6.1) *of the equation* (3.1)*.*

7. Conclusions.

Theorems 4.1, 5.1, and 6.1 show that the inverse problem 3.1 formulated in Section 3 is equivalent to finding all perfect cuboids, while the problem 3.2 is equivalent to finding at least one perfect cuboid.

References

1. *Euler brick*, Wikipedia, Wikimedia Foundation Inc., San Francisco, USA.
2. Halcke P., *Deliciae mathematicae oder mathematisches Sinnen-Confect*, N. Sauer, Hamburg, Germany, 1719.
3. Saunderson N., *Elements of algebra,* Vol. 2, Cambridge Univ. Press, Cambridge, 1740.
4. Euler L., *Vollständige Anleitung zur Algebra,* 3 Theile, Kaiserliche Akademie der Wissenschaften, St. Petersburg, 1770-1771.
5. Pocklington H. C., *Some Diophantine impossibilities*, Proc. Cambridge Phil. Soc. **17** (1912), 108–121.
6. Dickson L. E, *History of the theory of numbers,* Vol. 2*: Diophantine analysis*, Dover, New York, 2005.
7. Kraitchik M., *On certain rational cuboids*, Scripta Math. **11** (1945), 317–326.
8. Kraitchik M., *Théorie des Nombres,* Tome 3*, Analyse Diophantine et application aux cuboides rationelles*, Gauthier-Villars, Paris, 1947.
9. Kraitchik M., *Sur les cuboides rationelles*, Proc. Int. Congr. Math. **2** (1954), Amsterdam, 33–34.
10. Bromhead T. B., *On square sums of squares*, Math. Gazette **44** (1960), no. 349, 219–220.
11. Lal M., Blundon W. J., *Solutions of the Diophantine equations* $x^2 + y^2 = l^2$, $y^2 + z^2 = m^2$, $z^2 + x^2 = n^2$, Math. Comp. **20** (1966), 144–147.
12. Spohn W. G., *On the integral cuboid*, Amer. Math. Monthly **79** (1972), no. 1, 57-59.
13. Spohn W. G., *On the derived cuboid*, Canad. Math. Bull. **17** (1974), no. 4, 575-577.
14. Chein E. Z., *On the derived cuboid of an Eulerian triple*, Canad. Math. Bull. **20** (1977), no. 4, 509–510.
15. Leech J., *The rational cuboid revisited*, Amer. Math. Monthly **84** (1977), no. 7, 518–533; see also Erratum, Amer. Math. Monthly **85** (1978), 472.
16. Leech J., *Five tables relating to rational cuboids*, Math. Comp. **32** (1978), 657–659.
17. Spohn W. G., *Table of integral cuboids and their generators*, Math. Comp. **33** (1979), 428–429.
18. Lagrange J., *Sur le dérivé du cuboide Eulérien*, Canad. Math. Bull. **22** (1979), no. 2, 239–241.
19. Leech J., *A remark on rational cuboids*, Canad. Math. Bull. **24** (1981), no. 3, 377–378.
20. Korec I., *Nonexistence of small perfect rational cuboid*, Acta Math. Univ. Comen. **42/43** (1983), 73–86.

21. Korec I., *Nonexistence of small perfect rational cuboid II*, Acta Math. Univ. Comen. **44/45** (1984), 39–48.
22. Wells D. G., *The Penguin dictionary of curious and interesting numbers*, Penguin publishers, London, 1986.
23. Bremner A., Guy R. K., *A dozen difficult Diophantine dilemmas*, Amer. Math. Monthly **95** (1988), no. 1, 31–36.
24. Bremner A., *The rational cuboid and a quartic surface*, Rocky Mountain J. Math. **18** (1988), no. 1, 105–121.
25. Colman W. J. A., *On certain semiperfect cuboids*, Fibonacci Quart. **26** (1988), no. 1, 54–57; see also *Some observations on the classical cuboid and its parametric solutions*, Fibonacci Quart. **26** (1988), no. 4, 338–343.
26. Korec I., *Lower bounds for perfect rational cuboids*, Math. Slovaca **42** (1992), no. 5, 565–582.
27. Guy R. K., *Is there a perfect cuboid? Four squares whose sums in pairs are square. Four squares whose differences are square*, Unsolved Problems in Number Theory, 2nd ed., Springer-Verlag, New York, 1994, pp. 173–181.
28. Rathbun R. L., Granlund T., *The integer cuboid table with body, edge, and face type of solutions*, Math. Comp. **62** (1994), 441–442.
29. Van Luijk R., *On perfect cuboids,* Doctoraalscriptie, Mathematisch Instituut, Universiteit Utrecht, Utrecht, 2000.
30. Rathbun R. L., Granlund T., *The classical rational cuboid table of Maurice Kraitchik*, Math. Comp. **62** (1994), 442–443.
31. Peterson B. E., Jordan J. H., *Integer hexahedra equivalent to perfect boxes*, Amer. Math. Monthly **102** (1995), no. 1, 41–45.
32. Rathbun R. L., *The rational cuboid table of Maurice Kraitchik*, e-print math.HO/0111229 in Electronic Archive http://arXiv.org.
33. Hartshorne R., Van Luijk R., *Non-Euclidean Pythagorean triples, a problem of Euler, and rational points on K3 surfaces*, e-print math.NT/0606700 in Electronic Archive http://arXiv.org.
34. Waldschmidt M., *Open diophantine problems*, e-print math.NT/0312440 in Electronic Archive http://arXiv.org.
35. Ionascu E. J., Luca F., Stanica P., *Heron triangles with two fixed sides*, e-print math.NT/0608185 in Electronic Archive http://arXiv.org.
36. Ortan A., Quenneville-Belair V., *Euler's brick*, Delta Epsilon, McGill Undergraduate Mathematics Journal **1** (2006), 30-33.
37. Knill O., *Hunting for Perfect Euler Bricks*, Harvard College Math. Review **2** (2008), no. 2, 102; see also http://www.math.harvard.edu/~knill/various/eulercuboid/index.html.
38. Sloan N. J. A, *Sequences A031173, A031174, and A031175*, On-line encyclopedia of integer sequences, OEIS Foundation Inc., Portland, USA.
39. Stoll M., Testa D., *The surface parametrizing cuboids*, e-print arXiv:1009.0388 in Electronic Archive http://arXiv.org.
40. Sharipov R. A., *A note on a perfect Euler cuboid.*, e-print arXiv:1104.1716 in Electronic Archive http://arXiv.org.
41. Sharipov R. A., *Perfect cuboids and irreducible polynomials*, Ufa Mathematical Journal **4,** (2012), no. 1, 153–160; see also e-print arXiv:1108.5348 in Electronic Archive http://arXiv.org.
42. Sharipov R. A., *A note on the first cuboid conjecture*, e-print arXiv:1109.2534 in Electronic Archive http://arXiv.org.
43. Sharipov R. A., *A note on the second cuboid conjecture. Part I*, e-print arXiv:1201.1229 in Electronic Archive http://arXiv.org.
44. Sharipov R. A., *A note on the third cuboid conjecture. Part I*, e-print arXiv:1203.2567 in Electronic Archive http://arXiv.org.
45. Sharipov R. A., *Perfect cuboids and multisymmetric polynomials*, e-print arXiv:1205.3135 in Electronic Archive http://arXiv.org.
46. Sharipov R. A., *On an ideal of multisymmetric polynomials associated with perfect cuboids*, e-print arXiv:1206.6769 in Electronic Archive http://arXiv.org.
47. Sharipov R. A., *On the equivalence of cuboid equations and their factor equations*, e-print arXiv:1207.2102 in Electronic Archive http://arXiv.org.
48. Shläfli L., *Über die Resultante eines systems mehrerer algebraishen Gleihungen*, Denkschr. Kaiserliche Acad. Wiss. Math.-Natur. Kl. **4** (1852); reprinted in «Gesammelte mathematische Abhandlungen», Band II (1953), Birkhäuser Verlag, 9–112.

49. Cayley A., *On the symmetric functions of the roots of certain systems of two equations*, Phil. Trans. Royal Soc. London **147** (1857), 717–726.
50. Junker F., *Über symmetrische Functionen von mehreren Veränderlishen*, Mathematische Annalen **43** (1893), 225–270.
51. McMahon P. A., *Memoir on symmetric functions of the roots of systems of equations*, Phil. Trans. Royal Soc. London **181** (1890), 481–536.
52. McMahon P. A., *Combinatory Analysis.* Vol. I and Vol. II, Cambridge Univ. Press, 1915–1916; see also Third ed., Chelsea Publishing Company, New York, 1984.
53. Noether E., *Der Endlichkeitssats der Invarianten endlicher Gruppen*, Mathematische Annalen **77** (1915), 89–92.
54. Weyl H., *The classical groups*, Princeton Univ. Press, Princeton, 1939.
55. Macdonald I. G., *Symmetric functions and Hall polynomials,* Oxford Mathematical Monographs, Clarendon Press, Oxford, 1979.
56. Pedersen P., *Calculating multidimensional symmetric functions using Jacobi's formula*, Proceedings AAECC 9, volume 539 of Springer Lecture Notes in Computer Science, Springer, 1991, pp. 304–317.
57. Milne P., *On the solutions of a set of polynomial equations*, Symbolic and numerical computation for artificial intelligence. Computational Mathematics and Applications (Donald B. R., Kapur D., Mundy J. L., eds.), Academic Press Ltd., London, 1992, pp. 89–101.
58. Dalbec J., *Geometry and combinatorics of Chow forms*, PhD thesis, Cornell University, 1995.
59. Richman D. R., *Explicit generators of the invariants of finite groups*, Advances in Math. **124** (1996), no. 1, 49–76.
60. Stepanov S. A., *On vector invariants of the symmetric group*, Diskretnaya Matematika **8** (1996), no. 2, 48–62.
61. Gonzalez-Vega L., Trujillo G., *Multivariate Sturm-Habicht sequences: real root counting on n-rectangles and triangles*, Revista Matemática Complutense **10** (1997), 119–130.
62. Stepanov S. A., *On vector invariants of symmetric groups*, Diskretnaya Matematika **11** (1999), no. 3, 4–14.
63. Dalbec J., *Multisymmetric functions*, Beiträge zur Algebra und Geom. **40** (1999), no. 1, 27–51.
64. Rosas M. H., *MacMahon symmetric functions, the partition lattice, and Young subgroups*, Journ. Combin. Theory **96 A** (2001), no. 2, 326–340.
65. Vaccarino F., *The ring of multisymmetric functions*, e-print math.RA/0205233 in Electronic Archive http://arXiv.org.
66. Briand E., *When is the algebra of multisymmetric polynomials generated by the elementary multisymmetric polynomials?*, Beiträge zur Algebra und Geom. **45** (2004), no. 2, 353–368.
67. Rota G.-C., Stein J. A., *A problem of Cayley from 1857 and how he could have solved it*, Linear Algebra and its Applications (special issue on determinants and the legacy of Sir Thomas Muir) **411** (2005), 167–253.
68. Briand E., Rosas M. H., *Milne's volume function and vector symmetric polynomials*, Journ. Symbolic Comput. **44** (2009), no. 5, 583–590.
69. Sharipov R. A., *A biquadratic Diophantine equation associated with perfect cuboids*, e-print arXiv:1207.4081 in Electronic Archive http://arXiv.org.
70. Cox D. A., Little J. B., O'Shea D., *Ideals, Varieties, and Algorithms*, Springer Verlag, New York, 1992.
71. Ramsden J. R., *A general rational solution of an equation associated with perfect cuboids*, e-print arXiv:1207.5339 in Electronic Archive http://arXiv.org.
72. *Symmetric polynomial*, Wikipedia, Wikimedia Foundation Inc., San Francisco, USA.

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