ON A PAIR OF CUBIC EQUATIONS ASSOCIATED WITH PERFECT CUBOIDS.

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Abstract. A perfect cuboid is a rectangular parallelepiped with integer edges and integer face diagonals whose space diagonal is also integer. The existence of such cuboids is neither proved, nor disproved. A rational perfect cuboid is a natural companion of a perfect cuboid absolutely equivalent to the latter one. Its edges and face diagonals are rational numbers, while its space diagonal is equal to unity. Recently, based on a symmetry reduction, it was shown that edges of a rational perfect cuboid are roots of a certain cubic equation with rational coefficients depending on two rational parameters. Face diagonals of this cuboid are roots of another cubic equation whose coefficients are rational numbers depending on the same two rational parameters. In the present paper these two cubic equations are studied for reducibility. Six special cases of their reducibility over the field of rational numbers are found.

1. Introduction.

The problem of a perfect cuboid is known since 1719, but is still not resolved. For the history of this problem the reader is referred to [1–44]. Let x_1, x_2, x_3 be edges of a cuboid, d_1, d_2, d_3 be its face diagonals, and L be its space diagonal. Then the cuboid is described by the following four polynomial equations:

$$\begin{aligned} &x_1^2+x_2^2+x_3^2-L^2=0, && x_2^2+x_3^2-d_1^2=0,\\ &x_3^2+x_1^2-d_2^2=0, && x_1^2+x_2^2-d_3^2=0. \end{aligned} \tag{1.1}$$

This paper continues the series of papers [45–50] applying the symmetry approach to the equations (1.1). Indeed, using three numbers x_1, x_2, x_3, one can build the cubic equation $(x-x_1)(x-x_2)(x-x_3)=0$ which expands to

$$x^3-E_{10}\,x^2+E_{20}\,x-E_{30}=0. \tag{1.2}$$

Similarly, the equation $(d-d_1)(d-d_2)(d-d_3)=0$ expands to

$$d^3-E_{01}\,d^2+E_{02}\,d-E_{03}=0. \tag{1.3}$$

The coefficients E_{10}, E_{20}, and E_{30} of the equation (1.2) are elementary symmetric polynomials of three variables x_1, x_2, x_3 (see [51]). Similarly, the coefficients E_{01}, E_{02}, and E_{03} of the equation (1.3) are elementary symmetric polynomials of three

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variables d_1, d_2, d_3. Here are the formulas for these polynomials:

$$\begin{aligned}
&x_1+x_2+x_3=E_{10}, && d_1+d_2+d_3=E_{01},\\
&x_1\,x_2+x_2\,x_3+x_3\,x_1=E_{20}, && d_1\,d_2+d_2\,d_3+d_3\,d_1=E_{02},\\
&x_1\,x_2\,x_3=E_{30}, && d_1\,d_2\,d_3=E_{03}.
\end{aligned}\tag{1.4}$$

Mixing x_1, x_2, x_3 with d_1, d_2, d_3, one can write the following formulas:

$$\begin{aligned}
&x_1\,x_2\,d_3+x_2\,x_3\,d_1+x_3\,x_1\,d_2=E_{21},\\
&x_1\,d_2+d_1\,x_2+x_2\,d_3+d_2\,x_3+x_3\,d_1+d_3\,x_1=E_{11},\\
&x_1\,d_2\,d_3+x_2\,d_3\,d_1+x_3\,d_1\,d_2=E_{12}.
\end{aligned}\tag{1.5}$$

The left hand sides of the formulas (1.4) complemented with the left hand sides of the formulas (1.5) constitute the complete set of so-called elementary multisymmetric polynomials. For the theory of multisymmetric polynomials, either elementary and non-elementary, the reader is referred to [52–72].

The cuboid equations (1.1) imply some equations for E_{10}, E_{20}, E_{30}, E_{01}, E_{02}, E_{03}, E_{21}, E_{11}, and E_{12} in (1.4) and (1.5). These equations are called factor equations. They were studied in [46] and [47] using ideals in polynomial rings and their Gröbner bases (see the general theory in [73]). In [48] the factor equations were reduced to a single biquadratic equation for three variables E_{10}, E_{01}, and E_{11}:

$$(2\,E_{11})^2+(E_{01}^2+L^2-E_{10}^2)^2-8\,E_{01}^2\,L^2=0.\tag{1.6}$$

The other variables E_{20}, E_{30}, E_{02}, E_{03}, E_{21}, and E_{12} are expressed through E_{10}, E_{01}, and E_{11} by means of the following formulas:

$$E_{20}=\frac{1}{2}\,E_{10}^2-\frac{1}{2}\,L^2, \qquad E_{02}=\frac{1}{2}\,E_{01}^2-L^2,\tag{1.7}$$

$$\begin{aligned}
E_{21}=&\ \frac{2\,E_{10}^3\,E_{11}+2\,E_{01}^2\,E_{10}\,E_{11}-E_{01}\,E_{10}^4+E_{01}^5}{8\,(E_{01}^2+E_{10}^2)}\,+\\
&+\,\frac{6\,E_{10}\,E_{11}\,L^2-2\,E_{01}\,E_{10}^2\,L^2-8\,E_{01}^3\,L^2+3\,E_{01}\,L^4}{8\,(E_{01}^2+E_{10}^2)},
\end{aligned}\tag{1.8}$$

$$\begin{aligned}
E_{12}=&\ \frac{E_{01}^4\,E_{10}-2\,E_{01}^3\,E_{11}-2\,E_{01}\,E_{10}^2\,E_{11}-E_{10}^5}{8\,(E_{01}^2+E_{10}^2)}\,+\\
&+\,\frac{6\,E_{10}^3\,L^2-6\,E_{01}\,E_{11}\,L^2+3\,E_{10}\,L^4}{8\,(E_{01}^2+E_{10}^2)},
\end{aligned}\tag{1.9}$$

$$E_{30}=-\frac{1}{3}\,E_{12}-\frac{1}{6}\,E_{10}\,E_{01}^2-\frac{1}{2}\,E_{10}\,L^2+\frac{1}{6}\,E_{10}^3+\frac{1}{3}\,E_{01}\,E_{11},\tag{1.10}$$

$$E_{03}=-\frac{1}{3}\,E_{21}-\frac{1}{6}\,E_{01}\,E_{10}^2-\frac{5}{6}\,E_{01}\,L^2+\frac{1}{6}\,E_{01}^3+\frac{1}{3}\,E_{10}\,E_{11}.\tag{1.11}$$

A general solution for the equation (1.6) was derived in [49], including a two-parameter solution and several one-parameter solutions. As it was proved in [50], the one-parameter solutions do not lead to perfect cuboids. The two-parameter solution of (1.6) is written in [49] for the case of a rational cuboid with unit space diagonal $L = 1$. This solution with the parameters b and c is given by the formulas

$$E_{11} = -\frac{b\,(c^2 + 2 - 4\,c)}{b^2\,c^2 + 2\,b^2 - 3\,b^2\,c + c - b\,c^2 + 2\,b}, \tag{1.12}$$

$$E_{01} = -\frac{b\,(c^2 + 2 - 2\,c)}{b^2\,c^2 + 2\,b^2 - 3\,b^2\,c + c - b\,c^2 + 2\,b}, \tag{1.13}$$

$$E_{10} = -\frac{b^2\,c^2 + 2\,b^2 - 3\,b^2\,c - c}{b^2\,c^2 + 2\,b^2 - 3\,b^2\,c + c - b\,c^2 + 2\,b}. \tag{1.14}$$

Substituting (1.12), (1.13), and (1.14) into the formulas (1.7), (1.8), (1.9), (1.10), (1.11) and taking into account that $L = 1$, one can find that

$$\begin{gathered} E_{20} = \frac{b}{2}\,(b\,c^2 - 2\,c - 2\,b)\,(2\,b\,c^2 - c^2 - 6\,b\,c + 2 + 4\,b)\,\times \\ \times\,(b\,c - 1 - b)^{-2}\,(b\,c - c - 2\,b)^{-2}, \end{gathered} \tag{1.15}$$

$$\begin{gathered} E_{02} = \frac{1}{2}\,(28\,b^2\,c^2 - 16\,b^2\,c - 2\,c^2 - 4\,b^2 - b^2\,c^4 + 4\,b^3\,c^4 - 12\,b^3\,c^3 + \\ +\,4\,b\,c^3 + 24\,b^3\,c - 8\,b\,c - 2\,b^4\,c^4 + 12\,b^4\,c^3 - 26\,b^4\,c^2 - 8\,b^2\,c^3 + \\ +\,24\,b^4\,c - 16\,b^3 - 8\,b^4)\,(b\,c - 1 - b)^{-2}\,(b\,c - c - 2\,b)^{-2}. \end{gathered} \tag{1.16}$$

$$\begin{gathered} E_{21} = \frac{b}{2}\,(5\,c^6\,b - 2\,c^6\,b^2 + 52\,c^5\,b^2 - 16\,c^5\,b - 2\,c^7\,b^2 + 2\,b^4\,c^8 + \\ +\,142\,b^4\,c^6 - 26\,b^4\,c^7 - 426\,b^4\,c^5 - 61\,b^3\,c^6 + 100\,b^3\,c^5 + 14\,c^7\,b^3 - \\ -\,c^8\,b^3 - 20\,b\,c^2 - 8\,b^2\,c^2 - 16\,b^2\,c - 128\,b^2\,c^4 - 200\,b^3\,c^3 + \\ +\,244\,b^3\,c^2 + 32\,b\,c^3 - 112\,b^3\,c + 768\,b^4\,c^4 - 852\,b^4\,c^3 + 568\,b^4\,c^2 + \\ +\,104\,b^2\,c^3 - 208\,b^4\,c + 8\,c^4 - 4\,c^3 + 16\,b^3 + 32\,b^4 - 2\,c^5)\,\times \\ \times\,(b^2\,c^4 - 6\,b^2\,c^3 + 13\,b^2\,c^2 - 12\,b^2\,c + 4\,b^2 + c^2)^{-1}\,\times \\ \times\,(b\,c - 1 - b)^{-2}\,(b\,c - c - 2\,b)^{-2}, \end{gathered} \tag{1.17}$$

$$\begin{gathered} E_{12} = (16\,b^6 + 32\,b^5 - 6\,c^5\,b^2 + 2\,c^5\,b - 62\,b^5\,c^6 + 62\,b^6\,c^6 - \\ -\,180\,b^6\,c^5 + 18\,b^5\,c^7 - 12\,b^6\,c^7 - 2\,b^5\,c^8 + b^6\,c^8 + 248\,b^5\,c^2 + \\ +\,248\,b^6\,c^2 - 96\,b^6\,c + 321\,b^6\,c^4 - 180\,b^5\,c^3 - 144\,b^5\,c - 360\,b^6\,c^3 + \\ +\,b^4\,c^8 + 8\,b^4\,c^6 - 6\,b^4\,c^7 + 18\,b^4\,c^5 + 7\,b^3\,c^6 + 90\,b^5\,c^5 - 14\,b^3\,c^5 - \\ -\,c^7\,b^3 + 17\,b^2\,c^4 + 28\,b^3\,c^3 - 28\,b^3\,c^2 - 4\,b\,c^3 + 8\,b^3\,c - 57\,b^4\,c^4 + \\ +\,36\,b^4\,c^3 + 32\,b^4\,c^2 - 12\,b^2\,c^3 - 48\,b^4\,c - c^4 + 16\,b^4)\,\times \\ \times\,(b^2\,c^4 - 6\,b^2\,c^3 + 13\,b^2\,c^2 - 12\,b^2\,c + 4\,b^2 + c^2)^{-1}\,\times \\ \times\,(b\,c - 1 - b)^{-2}\,(b\,c - c - 2\,b)^{-2}. \end{gathered} \tag{1.18}$$

$$\begin{aligned}E_{03}=\frac{b}{2}\,(b^2\,c^4-5\,b^2\,c^3+10\,b^2\,c^2-10\,b^2\,c+4\,b^2+2\,b\,c+\\+2\,c^2-b\,c^3)\,(2\,b^2\,c^4-12\,b^2\,c^3+26\,b^2\,c^2-24\,b^2\,c+\\+8\,b^2-c^4\,b+3\,b\,c^3-6\,b\,c+4\,b+c^3-2\,c^2+2\,c)\times\\\times((b^2\,c^4-6\,b^2\,c^3+13\,b^2\,c^2-12\,b^2\,c+4\,b^2+c^2)^{-1}\times\\\times(b\,c-1-b)^{-2}\,(-c+b\,c-2\,b)^{-2},\end{aligned}\tag{1.19}$$

$$\begin{aligned}E_{30}=c\,b^2\,(1-c)\,(c-2)\,(b\,c^2-4\,b\,c+2+4\,b)\times\\\times(2\,b\,c^2-c^2-4\,b\,c+2\,b)\times\\\times(b^2\,c^4-6\,b^2\,c^3+13\,b^2\,c^2-12\,b^2\,c+4\,b^2+c^2)^{-1}\times\\\times(b\,c-1-b)^{-2}\,(-c+b\,c-2\,b)^{-2}.\end{aligned}\tag{1.20}$$

The formulas (1.15), (1.16), (1.17), (1.18), (1.19), and (1.20) were derived in [50]. Based on these formulas, two inverse problems were formulated.

Problem 1.1. *Find all pairs of rational numbers b and c for which the cubic equations (1.2) and (1.3) with the coefficients (1.14), (1.15), (1.20), (1.13), (1.16), (1.19) possess positive rational roots x_1, x_2, x_3, d_1, d_2, d_3 obeying the auxiliary equations (1.5) with the right hand sides (1.17), (1.18), (1.12).*

Problem 1.2. *Find at least one pair of rational numbers b and c for which the cubic equations (1.2) and (1.3) with the coefficients (1.14), (1.15), (1.20), (1.13), (1.16), (1.19) possess positive rational roots x_1, x_2, x_3, d_1, d_2, d_3 obeying the auxiliary equations (1.5) with the right hand sides (1.17), (1.18), (1.12).*

The problems 1.1 and 1.2 are equivalent to finding all perfect cuboids and to finding at least one perfect cuboid respectively. In the present paper we study the cubic equations (1.2) and (1.3) for reducibility using the methods of [41], which were applied to a twelfth order Diophantine equation in that paper.

2. The first reducibility case $b = 0$.

Note that the formulas (1.12) through (1.20) for the coefficients of the cubic equations (1.2) and (1.3) and for the right hand sides of the auxiliary equations (1.5) possess denominators. The simultaneous non-vanishing condition for all of their denominators is written as follows:

$$\begin{aligned}(b^2\,c^4-6\,b^2\,c^3+13\,b^2\,c^2-12\,b^2\,c+4\,b^2+c^2)\times\\\times(b\,c-1-b)\,(b\,c-c-2\,b)\neq 0.\end{aligned}\tag{2.1}$$

The case $b = 0$ is very simple. The non-vanishing condition (2.1) in this case is written as $c \neq 0$. Substituting $b = 0$ into (1.14), (1.15), (1.20), (1.13), (1.16), (1.19), we find that the cubic equations (1.2) and (1.3) reduce to

$$x^2\,(x-1)=0,\qquad\qquad d\,(d-1)\,(d+1)=0.\tag{2.2}$$

Substituting $b = 0$ into (1.17), (1.18), (1.12), we obtain

$$E_{21}=0,\qquad\qquad E_{11}=0,\qquad\qquad E_{12}=-1.\tag{2.3}$$

The equations (2.2) are already factored. We can choose their roots as follows:

$$\begin{aligned} &x_1 = 1, && x_2 = 0, && x_3 = 0,\\ &d_1 = 0, && d_1 = 1, && d_3 = -1. \end{aligned} \tag{2.4}$$

Substituting (2.3) and (2.4) into (1.5), we find that the auxiliary equations (1.5) are fulfilled. Substituting (2.4) along with $L = 1$ into (1.1), we find that the cuboid equations (1.1) are also fulfilled. However, the formulas (2.4) do not provide a perfect cuboid since its edges and face diagonals cannot be zero or negative.

Theorem 2.1. *If $b = 0$ and $c \neq 0$, then the cubic polynomials in (1.2) and (1.3) are reducible and provide three integer roots for each of the equations (1.2) and (1.3) satisfying the auxiliary equations (1.5) but not resolving the problem 1.2.*

3. The second reducibility case $c = 0$.

The case $c = 0$ is also simple. The non-vanishing condition (2.1) in this case turns to $b\,(1 + b) \neq 0$. The cubic equations (1.2) and (1.3) turn to

$$\begin{aligned} x\,\bigl(2\,(1+b)^2\,x^2 + 2\,b\,(1+b)\,x - (1+2\,b)\bigr) = 0,\\ (d+1)\,\bigl(2\,(1+b)^2\,d^2 - 2\,b\,(1+b)\,d - (1+2\,b)\bigr) = 0. \end{aligned} \tag{3.1}$$

Substituting $c = 0$ into the formulas (1.17), (1.18), and (1.12), we obtain

$$E_{21} = \frac{1+2\,b}{2\,(1+b)^2}, \qquad E_{11} = \frac{-1}{1+b}, \qquad E_{12} = 1. \tag{3.2}$$

The equations (3.1) are already factored. Upon splitting off the linear terms they turn to quadratic equations very similar to each other:

$$\begin{aligned} 2\,(1+b)^2\,x^2 + 2\,b\,(1+b)\,x - (1+2\,b) = 0,\\ 2\,(1+b)^2\,d^2 - 2\,b\,(1+b)\,d - (1+2\,b) = 0. \end{aligned} \tag{3.3}$$

The discriminants of the quadratic equations (3.3) do coincide:

$$D = 4\,(b^2 + 4\,b + 2)\,(1+b)^2. \tag{3.4}$$

The formula (3.4) means that in order to solve the equations (3.3) in rational numbers one should resolve the following quadratic equation in rational numbers:

$$b^2 + 4\,b + 2 = \beta^2. \tag{3.5}$$

The equation (3.5) can be written as $(b+2)^2 - \beta^2 = 2$. The lemma 2.2 from [49] is applicable to this equation. Applying this lemma, we get

$$b = \frac{t^2 - 4\,t + 2}{2\,t}, \qquad \beta = \frac{t^2 - 2}{2\,t} \tag{3.6}$$

for some rational $t \neq 0$. Note that the condition $b\,(1+b) \neq 0$ derived from (2.1) for $c = 0$ implies no restrictions for t since the equations

$$\frac{t^2-4\,t+2}{2\,t}=0, \qquad\qquad \frac{t^2-4\,t+2}{2\,t}=-1$$

have no rational roots. Now, substituting b from (3.6) into the quadratic equations (3.3), we can find their roots. Since the roots $x = 0$ and $d = -1$ of the cubic equations (3.1) are already known, we can write formulas for all their roots:

$$\begin{aligned} &x_1=0, && x_2=\frac{-t\,(t-2)}{(t-1)^2+1}, && x_3=\frac{2\,(t-1)}{(t-1)^2+1},\\ &d_1=-1, && d_2=\frac{-2\,(t-1)}{(t-1)^2+1}, && d_3=\frac{t\,(t-2)}{(t-1)^2+1}. \end{aligned} \tag{3.7}$$

Taking into account the formula for b in (3.6), we transform (3.2) to

$$E_{21}=\frac{2\,t\,(t^2-3\,t+2)}{(t-1)^2+1}, \qquad E_{11}=\frac{-2\,t}{(t-1)^2+1}, \qquad E_{12}=1. \tag{3.8}$$

Now, if we substitute the formulas (3.7) and (3.8) into (1.5), we find that the auxiliary equations (1.5) are fulfilled. Similarly, substituting (3.7) along with $L = 1$ into (1.1), we find that the cuboid equations (1.1) are also fulfilled. However, the formulas (3.7) do not provide a perfect cuboid since x_1 is zero and d_1 is negative.

Theorem 3.1. *If $c = 0$ and $b\,(1 + b) \neq 0$, then the cubic polynomials in (1.2) and (1.3) are reducible over $\mathbb{Q}$. Moreover, if b is given by the first formula (3.6) for some rational $t \neq 0$, then each of the cubic equations (1.2) and (1.3) has three rational roots satisfying the auxiliary equations (1.5) but not resolving the problem 1.2.*

4. The third reducibility case $c = 1$.

The case $c = 1$ is similar to the previous case $c = 0$. The non-vanishing condition (2.1) in this case turns to $b \neq -1$. The cubic equations (1.2) and (1.3) turn to

$$\begin{gathered} x\,\bigl(2\,(1+b)^2\,x^2-(2\,b+2)\,x-b\,(b+2)\bigr)=0,\\ (d+1)\,\bigl(2\,(1+b)^2\,d^2-(2\,b+2)\,d-b\,(b+2)\bigr)=0. \end{gathered} \tag{4.1}$$

Substituting $c = 1$ into the formulas (1.17), (1.18), and (1.12), we obtain

$$E_{21}=\frac{b\,(b+2)}{2\,(1+b)^2}, \qquad E_{11}=\frac{b}{1+b}, \qquad E_{12}=-1. \tag{4.2}$$

The equations (4.1) are already factored. Upon splitting out linear terms they turn to quadratic equations. These two quadratic equations do coincide with each other up to the change of x for d and vice versa:

$$\begin{gathered} 2\,(1+b)^2\,x^2-(2\,b+2)\,x-b\,(b+2)=0,\\ 2\,(1+b)^2\,d^2-(2\,b+2)\,d-b\,(b+2)=0. \end{gathered} \tag{4.3}$$

Certainly, the discriminants of the coinciding equations (4.3) do also coincide:

$$D=4\,(2\,b^2+4\,b+1)\,(b+1)^2. \tag{4.4}$$

The formula (4.4) means that in order to solve the equations (4.3) in rational numbers one should resolve the following quadratic equation in rational numbers:

$$2\,b^2+4\,b+1=\beta^2. \tag{4.5}$$

If $b=0$, then (4.5) implies $\beta=\pm 1$, which yields two trivial rational solutions for the equation (4.5). If $b\neq 0$, we can write the equation (4.5) as

$$\frac{1}{b^2}+\frac{4}{b}+2=\frac{\beta^2}{b^2}. \tag{4.6}$$

The equation (4.6) is very similar to (3.5). Therefore the formulas (3.6) yield

$$\frac{1}{b}=\frac{t^2-4\,t+2}{2\,t}, \qquad\qquad \frac{\beta}{b}=\frac{t^2-2}{2\,t} \tag{4.7}$$

for some rational $t\neq 0$. Then the formulas (4.7) can be transformed to

$$b=\frac{2\,t}{(t-2)^2-2}, \qquad\qquad \beta=\frac{t^2-2}{(t-2)^2-2}. \tag{4.8}$$

Note that the formulas (4.8) are consistent since their denominators cannot vanish for any rational t. Note also that one of the above trivial solutions with $b=0$ and $\beta=-1$ can be obtained from (4.8) for $t=0$. In order to cover the other trivial solution with $b=0$ and $\beta=1$ we need to add the following sign option to (4.8):

$$b=\frac{2\,t}{(t-2)^2-2}, \qquad\qquad \beta=\pm\,\frac{t^2-2}{(t-2)^2-2}. \tag{4.9}$$

Due to these observations the restriction $t\neq 0$ is removed and the formulas (4.9) cover all rational solutions of the equation (4.5). The condition $b\neq -1$ derived from (2.1) for $c=1$ implies no restrictions for t since the equation

$$\frac{2\,t}{(t-2)^2-2}=-1$$

has no rational roots. Now we can substitute b from (4.9) into the quadratic equations (4.3) and find their roots. Since the roots $x=0$ and $d=-1$ of the cubic equations (4.1) are already known, we can write formulas for all their roots:

$$\begin{aligned} &x_1=0, && x_2=\frac{t\,(t-2)}{(t-1)^2+1}, && x_3=\frac{-2\,(t-1)}{(t-1)^2+1},\\ &d_1=-1, && d_2=\frac{-2\,(t-1)}{(t-1)^2+1}, && d_3=\frac{t\,(t-2)}{(t-1)^2+1}. \end{aligned} \tag{4.10}$$

Taking into account the formula for b in (4.9), we transform (4.2) to

$$E_{21}=\frac{2\,(t^2-3\,t+2)\,t}{((t-1)^2+1)^2},\qquad E_{11}=\frac{2\,t}{(t-1)^2+1},\qquad E_{12}=-1. \tag{4.11}$$

Now, if we substitute the formulas (4.10) and (4.11) into (1.5), we find that the auxiliary equations (1.5) are fulfilled. Similarly, substituting (4.10) along with $L=1$ into (1.1), we find that the cuboid equations are also fulfilled. But again, the formulas (4.10) do not provide a perfect cuboid since x_1 is zero and d_1 is negative.

Theorem 4.1. *If $c=1$ and $b\neq-1$, then the cubic polynomials in (1.2) and (1.3) are reducible over $\mathbb{Q}$. Moreover, if b is given by the first formula (4.9) for some rational t, then each of the cubic equations (1.2) and (1.3) has three rational roots satisfying the auxiliary equations (1.5) but not resolving the problem 1.2.*

5. The fourth reducibility case $c=2$.

The case $c=2$ is similar to both of the previous cases $c=0$ and $c=1$. The non-vanishing condition (2.1) in this case turns to $b\neq 1$. The cubic equations (1.2) and (1.3) in this case turn to the following ones:

$$\begin{gathered}x\,\bigl(2\,(b-1)^2\,x^2+2\,(b-1)\,x-b\,(b-2)\bigr)=0,\\(d+1)\,\bigl(2\,(b-1)^2\,d^2-2\,(b-1)\,d-b\,(b-2)\bigr)=0.\end{gathered} \tag{5.1}$$

Substituting $c=2$ into the formulas (1.17), (1.18), and (1.12), we obtain

$$E_{21}=-\frac{b\,(b-2)}{2\,(b-1)^2},\qquad E_{11}=-\frac{b}{b-1},\qquad E_{12}=-1. \tag{5.2}$$

The equations (5.1) are already factored. If we split off the linear terms, they turn to quadratic equations. These two quadratic equations are very similar:

$$\begin{gathered}2\,(b-1)^2\,x^2+2\,(b-1)\,x-b\,(b-2)=0,\\2\,(b-1)^2\,d^2-2\,(b-1)\,d-b\,(b-2)=0.\end{gathered} \tag{5.3}$$

The discriminants of the quadratic equations (5.3) do coincide:

$$D=4\,(2\,b^2-4\,b+1)\,(b-1)^2. \tag{5.4}$$

The formula (5.4) means that in order to solve the equations (5.3) in rational numbers one should resolve the following quadratic equation in rational numbers:

$$2\,b^2-4\,b+1=\beta^2. \tag{5.5}$$

The equation (5.5) is similar to the equation (4.5). If $b=0$, then it has two trivial solutions with $\beta=\pm 1$. If $b\neq 0$, we can write the equation (5.5) as

$$\Bigl(\frac{1}{b}-2\Bigr)^2-\frac{\beta^2}{b^2}=2. \tag{5.6}$$

The lemma 2.2 from [49] is applicable to the equation (5.6). This lemma yields

$$\frac{1}{b}=\frac{t^2+4\,t+2}{2\,t},\qquad\qquad \frac{\beta}{b}=\frac{t^2-2}{2\,t}\tag{5.7}$$

for some rational $t\neq 0$. Now the formulas (5.7) can be transformed to

$$b=\frac{2\,t}{(t+2)^2-2},\qquad\qquad \beta=\frac{t^2-2}{(t+2)^2-2}.\tag{5.8}$$

Like (4.8), the formulas (5.8) are consistent since their denominators cannot vanish for any rational t. One of the above trivial solutions with $b=0$ and $\beta=-1$ can be obtained from (5.8) for $t=0$. In order to cover the other trivial solution with $b=0$ and $\beta=1$ we need to add the following sign option to (5.8):

$$b=\frac{2\,t}{(t+2)^2-2},\qquad\qquad \beta=\pm\frac{t^2-2}{(t+2)^2-2}.\tag{5.9}$$

Due to the above observations the restriction $t\neq 0$ is removed and the formulas (5.9) cover all rational solutions of the equation (5.5). The condition $b-1\neq 0$ derived from (2.1) for $c=2$ implies no restrictions for t since the equation

$$\frac{2\,t}{(t+2)^2-2}=1$$

has no rational roots. Now we can substitute b from (5.9) into the quadratic equations (5.3) and find their roots. Since the roots $x=0$ and $d=1$ of the cubic equations (5.1) are already known, we can write formulas for all their roots:

$$\begin{aligned}
&x_1=0, &&x_2=\frac{2\,(t+1)}{(t+1)^2+1}, &&x_3=\frac{t\,(t+2)}{(t+1)^2+1},\\
&d_1=1, &&d_2=\frac{-t\,(t+2)}{(t+1)^2+1}, &&d_3=\frac{-2\,(t+1)}{(t+1)^2+1}.
\end{aligned}\tag{5.10}$$

Taking into account the formula for b in (5.9), the formula (5.2) is transformed to

$$E_{21}=\frac{2\,(t^2+3\,t+2)\,t}{((t+1)^2+1)^2},\qquad E_{11}=\frac{2\,t}{(t+1)^2+1},\qquad E_{12}=-1.\tag{5.11}$$

If we substitute the formulas (5.10) and (5.11) into (1.5), we find that the auxiliary equations (1.5) are fulfilled. Similarly, substituting (5.10) into (1.1) along with $L=1$, we find that the cuboid equations are also fulfilled. But, like in the previous cases, the formulas (5.10) do not provide a perfect cuboid since x_1 is zero.

Theorem 5.1. *If $c=2$ and $b\neq 1$, then the cubic polynomials in* (1.2) *and* (1.3) *are reducible over $\mathbb{Q}$. Moreover, if b is given by the first formula* (5.9) *for some rational t, then each of the cubic equations* (1.2) *and* (1.3) *has three rational roots satisfying the auxiliary equations* (1.5) *but not resolving the problem 1.2.*

6. Some other reducibility cases.

Note that in each of the previous four cases the cubic equation (1.2) has the root $x=0$. The necessary and sufficient condition for that is written as

$$P(x)=E_{30}=0, \tag{6.1}$$

where $P(x)$ is the cubic polynomial in the left hand side of the equation (1.2). Looking at the formulas (6.1) and (1.20), we see that along with the conditions $b=0$, $c=0$, $c=1$, $c=2$, which were already considered in the previous cases, there are the following two conditions for vanishing E_{30}:

$$b\,c^2-4\,b\,c+4\,b+2=0, \tag{6.2}$$

$$2\,b\,c^2-4\,b\,c+2\,b-c^2=0. \tag{6.3}$$

Let's denoter through $Q(d)$ the cubic polynomial in the left hand side of the equation (1.3). Then one can easily derive the following formulas:

$$\begin{gathered}Q(-1)=-(c-1)^2\,(b\,c^2-4\,b\,c+4\,b+2)^2\,b^2\,c^2\times\\ \times(b^2\,c^4-6\,b^2\,c^3+13\,b^2\,c^2-12\,b^2\,c+4\,b^2+c^2)^{-1}\times\\ \times(b\,c-1-b)^{-2}\,(b\,c-c-2\,b)^{-2},\end{gathered} \tag{6.4}$$

$$\begin{gathered}Q(1)=(c-2)^2\,(2\,b\,c^2-4\,b\,c+2\,b-c^2)^2\,b^2\times\\ \times(b^2\,c^4-6\,b^2\,c^3+13\,b^2\,c^2-12\,b^2\,c+4\,b^2+c^2)^{-1}\times\\ \times(b\,c-1-b)^{-2}\,(b\,c-c-2\,b)^{-2}.\end{gathered} \tag{6.5}$$

Comparing (6.2) with (6.4), we see that the condition (6.2) implies $Q(-1)=0$. Similarly, comparing (6.3) with (6.5), we find that the condition (6.3) implies $Q(1)=0$. These observations yield the following theorems.

Theorem 6.1. *If the condition (6.2) is fulfilled, then both cubic equations (1.2) and (1.3) are reducible. In this case the first of them has the root $x=0$, while the second has the root $d=-1$.*

Theorem 6.2. *If the condition (6.3) is fulfilled, then both cubic equations (1.2) and (1.3) are reducible. In this case the first of them has the root $x=0$, while the second has the root $d=1$.*

7. The fifth reducibility case.

The fifth reducibility case is defined by the condition (6.2). The equality (6.2) is linear with respect to b. It can be resolved as follows:

$$b=\frac{-2}{(c-2)^2}. \tag{7.1}$$

Substituting (7.1) into the formulas (1.14), (1.15), and (1.20), we find that the first cubic equation (1.2) reduces to the following one:

$$x\,\bigl((c^2-2\,c+2)\,x+2\,(c-1)\bigr)\bigl((c^2-2\,c+2)\,x-c\,(c-2)\bigr)=0. \tag{7.2}$$

It is easy to see that the equation (7.2) has three rational roots

$$x_1=0,\qquad x_2=\frac{-2\,(c-1)}{(c-1)^2+1},\qquad x_3=\frac{c\,(c-2)}{(c-1)^2+1}.\tag{7.3}$$

Now let's substitute (7.1) into the formulas (1.13), (1.16), and (1.19). As a result we find that the second cubic equation (1.3) reduces to the following one:

$$(d+1)\,\bigl((c^2-2\,c+2)\,d-2\,(c-1)\bigr)\bigl((c^2-2\,c+2)\,d-c\,(c-2)\bigr)=0.\tag{7.4}$$

Again, it is easy to see that the equation (7.4) has three rational roots

$$d_1=-1,\qquad d_2=\frac{c\,(c-2)}{(c-1)^2+1},\qquad d_3=\frac{2\,(c-1)}{(c-1)^2+1}.\tag{7.5}$$

Now let's substitute (7.1) into the formulas (1.17), (1.18), and (1.12). This yields

$$\begin{aligned}E_{21}&=\frac{2\,(c-2)\,(c-1)\,c}{((c-1)^2+1)^2},\\ E_{11}&=\frac{2\,((c-2)^2-2)\,(c-2)}{((c-1)^2+1)^2},\\ E_{12}&=\frac{-((c-2)^2-2)\,(c^2-2)}{((c-1)^2+1)^2}.\end{aligned}\tag{7.6}$$

If we substitute the formulas (7.3), (7.5), and (7.6) into (1.5), we find that the auxiliary equations (1.5) are fulfilled. Similarly, substituting (7.3) and (7.5) along with $L=1$ into the equations (1.1), we find that the cuboid equations are also fulfilled. But, like in the previous cases, the formulas (7.3) and (7.5) do not provide a perfect cuboid since x_1 is zero.

Note that (7.1) provides the restriction $c\neq 2$. Substituting (7.1) into (2.1), we get the condition $(c-1)^2+1\neq 0$ which is fulfilled for all rational c. Summarizing the results of this section, we can formulate the following theorem.

Theorem 7.1. *If $c\neq 2$ and $b\,(c-2)^2=-2$, then the cubic polynomials in (1.2) and (1.3) are reducible over the field of rational numbers $\mathbb{Q}$. Moreover, each of the cubic equations (1.2) and (1.3) has three rational roots satisfying the auxiliary equations (1.5) but not resolving the problem 1.2.*

8. The sixth reducibility case.

The sixth reducibility case is defined by the condition (6.3). The equality (6.3) is linear with respect to b. It can be resolved as follows:

$$b=\frac{c^2}{2\,(c-1)^2}.\tag{8.1}$$

Substituting (8.1) into the formulas (1.14), (1.15), and (1.20), we find that the first cubic equation (1.2) reduces to the following one:

$$x\,\bigl((c^2-2\,c+2)\,x+2\,(c-1)\bigr)\bigl((c^2-2\,c+2)\,x-c\,(c-2)\bigr)=0.\tag{8.2}$$

It is easy to see that the equation (8.2) has three rational roots

$$x_1 = 0, \qquad x_2 = \frac{-2\,(c-1)}{(c-1)^2+1)} \qquad x_3 = \frac{c\,(c-2)}{(c-1)^2+1}. \tag{8.3}$$

Now let's substitute (8.1) into the formulas (1.13), (1.16), and (1.19). As a result we find that the second cubic equation (1.3) reduces to the following one:

$$(d-1)\,\bigl((c^2-2\,c+2)\,d-2\,(c-1)\bigr)\bigl((c^2-2\,c+2)\,d-c\,(c-2)\bigr)=0. \tag{8.4}$$

Again, it is easy to see that the equation (8.4) has three rational roots

$$d_1 = 1, \qquad d_2 = \frac{c\,(c-2)}{(c-1)^2+1}, \qquad d_3 = \frac{2\,(c-1)}{(c-1)^2+1}. \tag{8.5}$$

The formulas (8.3) and (8.5) are almost identical to the formulas (7.3) and (7.5). The only difference is the sign of d_1.

Now let's substitute (7.1) into the formulas (1.17), (1.18), and (1.12). This yields

$$\begin{aligned}
&E_{21} = -\frac{2\,(c-2)\,(c-1)\,c}{((c-1)^2+1)^2},\\
&E_{11} = \frac{2\,c\,((c-2)^2-2)\,(c-1)}{((c-1)^2+1)^2},\\
&E_{12} = \frac{((c-2)^2-2)\,(c^2-2)}{((c-1)^2+1)^2}.
\end{aligned} \tag{8.6}$$

If we substitute the formulas (8.3), (8.5), and (8.6) into (1.5), we find that the auxiliary equations (1.5) are fulfilled. Similarly, substituting (8.3) and (8.5) along with $L = 1$ into the equations (1.1), we find that the cuboid equations are also fulfilled. But like in all previous cases, the formulas (8.3) and (8.5) do not provide a perfect cuboid since x_1 is zero.

Note that (8.1) provides the restriction $c \neq 1$. Substituting (7.1) into (2.1), we get the condition $c\,((c-2)^2+1) \neq 0$, which reduces to $c \neq 0$ since $(c-2)^2+1$ is always positive. Summarizing the results of this section, we can formulate the following theorem.

Theorem 8.1. *If $c \neq 0$, $c \neq 1$, and $2\,b\,(c-1)^2 = c^2$, then the cubic polynomials in* (1.2) *and* (1.3) *are reducible over the field of rational numbers $\mathbb{Q}$. Moreover, each of the cubic equations* (1.2) *and* (1.3) *has three rational roots satisfying the auxiliary equations* (1.5) *but not resolving the problem 1.2.*

9. Concluding remarks.

Six special cases of reducibility considered in the above sections do not exhaust all of the cases where the cubic equations (1.2) and (1.3) are reducible over $\mathbb{Q}$. There is one very special case with the following values of b and c:

$$b = \frac{14}{5}, \qquad c = -\frac{7}{2}.$$

In this very special case the equations (1.2) and (1.3) are factored as follows:

$$\begin{aligned}&(17\,x+15)\,(9248\,x^2+3128\,x-495)=0,\\&(17\,d+8)\,(9248\,d^2-952\,d-8175)=0.\end{aligned}\tag{9.1}$$

The search for such special cases of reducibility like (9.1) is planned for the future.

References

1. *Euler brick*, Wikipedia, Wikimedia Foundation Inc., San Francisco, USA.
2. Halcke P., *Deliciae mathematicae oder mathematisches Sinnen-Confect*, N. Sauer, Hamburg, Germany, 1719.
3. Saunderson N., *Elements of algebra,* Vol. 2, Cambridge Univ. Press, Cambridge, 1740.
4. Euler L., *Vollständige Anleitung zur Algebra,* 3 Theile, Kaiserliche Akademie der Wissenschaften, St. Petersburg, 1770-1771.
5. Pocklington H. C., *Some Diophantine impossibilities*, Proc. Cambridge Phil. Soc. **17** (1912), 108–121.
6. Dickson L. E, *History of the theory of numbers,* Vol. 2*: Diophantine analysis*, Dover, New York, 2005.
7. Kraitchik M., *On certain rational cuboids*, Scripta Math. **11** (1945), 317–326.
8. Kraitchik M., *Théorie des Nombres,* Tome 3*, Analyse Diophantine et application aux cuboides rationelles*, Gauthier-Villars, Paris, 1947.
9. Kraitchik M., *Sur les cuboides rationelles*, Proc. Int. Congr. Math. **2** (1954), Amsterdam, 33–34.
10. Bromhead T. B., *On square sums of squares*, Math. Gazette **44** (1960), no. 349, 219–220.
11. Lal M., Blundon W. J., *Solutions of the Diophantine equations* $x^2+y^2=l^2$, $y^2+z^2=m^2$, $z^2+x^2=n^2$, Math. Comp. **20** (1966), 144–147.
12. Spohn W. G., *On the integral cuboid*, Amer. Math. Monthly **79** (1972), no. 1, 57-59.
13. Spohn W. G., *On the derived cuboid*, Canad. Math. Bull. **17** (1974), no. 4, 575-577.
14. Chein E. Z., *On the derived cuboid of an Eulerian triple*, Canad. Math. Bull. **20** (1977), no. 4, 509–510.
15. Leech J., *The rational cuboid revisited*, Amer. Math. Monthly **84** (1977), no. 7, 518–533; see also Erratum, Amer. Math. Monthly **85** (1978), 472.
16. Leech J., *Five tables relating to rational cuboids*, Math. Comp. **32** (1978), 657–659.
17. Spohn W. G., *Table of integral cuboids and their generators*, Math. Comp. **33** (1979), 428–429.
18. Lagrange J., *Sur le dérivé du cuboide Eulérien*, Canad. Math. Bull. **22** (1979), no. 2, 239–241.
19. Leech J., *A remark on rational cuboids*, Canad. Math. Bull. **24** (1981), no. 3, 377–378.
20. Korec I., *Nonexistence of small perfect rational cuboid*, Acta Math. Univ. Comen. **42/43** (1983), 73–86.
21. Korec I., *Nonexistence of small perfect rational cuboid II*, Acta Math. Univ. Comen. **44/45** (1984), 39–48.
22. Wells D. G., *The Penguin dictionary of curious and interesting numbers*, Penguin publishers, London, 1986.
23. Bremner A., Guy R. K., *A dozen difficult Diophantine dilemmas*, Amer. Math. Monthly **95** (1988), no. 1, 31–36.
24. Bremner A., *The rational cuboid and a quartic surface*, Rocky Mountain J. Math. **18** (1988), no. 1, 105–121.
25. Colman W. J. A., *On certain semiperfect cuboids*, Fibonacci Quart. **26** (1988), no. 1, 54–57; see also *Some observations on the classical cuboid and its parametric solutions*, Fibonacci Quart. **26** (1988), no. 4, 338–343.
26. Korec I., *Lower bounds for perfect rational cuboids*, Math. Slovaca **42** (1992), no. 5, 565–582.
27. Guy R. K., *Is there a perfect cuboid? Four squares whose sums in pairs are square. Four squares whose differences are square*, Unsolved Problems in Number Theory, 2nd ed., Springer-Verlag, New York, 1994, pp. 173–181.
28. Rathbun R. L., Granlund T., *The integer cuboid table with body, edge, and face type of solutions*, Math. Comp. **62** (1994), 441–442.

29. Van Luijk R., *On perfect cuboids,* Doctoraalscriptie, Mathematisch Instituut, Universiteit Utrecht, Utrecht, 2000.
30. Rathbun R. L., Granlund T., *The classical rational cuboid table of Maurice Kraitchik*, Math. Comp. **62** (1994), 442–443.
31. Peterson B. E., Jordan J. H., *Integer hexahedra equivalent to perfect boxes*, Amer. Math. Monthly **102** (1995), no. 1, 41–45.
32. Rathbun R. L., *The rational cuboid table of Maurice Kraitchik*, e-print math.HO/0111229 in Electronic Archive http://arXiv.org.
33. Hartshorne R., Van Luijk R., *Non-Euclidean Pythagorean triples, a problem of Euler, and rational points on K3 surfaces*, e-print math.NT/0606700 in Electronic Archive http://arXiv.org.
34. Waldschmidt M., *Open diophantine problems*, e-print math.NT/0312440 in Electronic Archive http://arXiv.org.
35. Ionascu E. J., Luca F., Stanica P., *Heron triangles with two fixed sides*, e-print math.NT/0608 185 in Electronic Archive http://arXiv.org.
36. Ortan A., Quenneville-Belair V., *Euler's brick*, Delta Epsilon, McGill Undergraduate Mathematics Journal **1** (2006), 30-33.
37. Knill O., *Hunting for Perfect Euler Bricks*, Harvard College Math. Review **2** (2008), no. 2, 102; see also http://www.math.harvard.edu/~knill/various/eulercuboid/index.html.
38. Sloan N. J. A, *Sequences A031173, A031174, and A031175*, On-line encyclopedia of integer sequences, OEIS Foundation Inc., Portland, USA.
39. Stoll M., Testa D., *The surface parametrizing cuboids*, e-print arXiv:1009.0388 in Electronic Archive http://arXiv.org.
40. Sharipov R. A., *A note on a perfect Euler cuboid.*, e-print arXiv:1104.1716 in Electronic Archive http://arXiv.org.
41. Sharipov R. A., *Perfect cuboids and irreducible polynomials*, Ufa Mathematical Journal **4,** (2012), no. 1, 153–160; see also e-print arXiv:1108.5348 in Electronic Archive http://arXiv.org.
42. Sharipov R. A., *A note on the first cuboid conjecture*, e-print arXiv:1109.2534 in Electronic Archive http://arXiv.org.
43. Sharipov R. A., *A note on the second cuboid conjecture. Part I*, e-print arXiv:1201.1229 in Electronic Archive http://arXiv.org.
44. Sharipov R. A., *A note on the third cuboid conjecture. Part I*, e-print arXiv:1203.2567 in Electronic Archive http://arXiv.org.
45. Sharipov R. A., *Perfect cuboids and multisymmetric polynomials*, e-print arXiv:1205.3135 in Electronic Archive http://arXiv.org.
46. Sharipov R. A., *On an ideal of multisymmetric polynomials associated with perfect cuboids*, e-print arXiv:1206.6769 in Electronic Archive http://arXiv.org.
47. Sharipov R. A., *On the equivalence of cuboid equations and their factor equations*, e-print arXiv:1207.2102 in Electronic Archive http://arXiv.org.
48. Sharipov R. A., *A biquadratic Diophantine equation associated with perfect cuboids*, e-print arXiv:1207.4081 in Electronic Archive http://arXiv.org.
49. Ramsden J. R., *A general rational solution of an equation associated with perfect cuboids*, e-print arXiv:1207.5339 in Electronic Archive http://arXiv.org.
50. Ramsden J. R., Sharipov R. A., *Inverse problems associated with perfect cuboids*, e-print arXiv:1207.6764 in Electronic Archive http://arXiv.org.
51. *Symmetric polynomial*, Wikipedia, Wikimedia Foundation Inc., San Francisco, USA.
52. Shläfli L., *Über die Resultante eines systems mehrerer algebraishen Gleihungen*, Denkschr. Kaiserliche Acad. Wiss. Math.-Natur. Kl. **4** (1852); reprinted in «Gesammelte mathematische Abhandlungen», Band II (1953), Birkhäuser Verlag, 9–112.
53. Cayley A., *On the symmetric functions of the roots of certain systems of two equations*, Phil. Trans. Royal Soc. London **147** (1857), 717–726.
54. Junker F., *Über symmetrische Functionen von mehreren Veränderlishen*, Mathematische Annalen **43** (1893), 225–270.
55. McMahon P. A., *Memoir on symmetric functions of the roots of systems of equations*, Phil. Trans. Royal Soc. London **181** (1890), 481–536.
56. McMahon P. A., *Combinatory Analysis.* Vol. I and Vol. II, Cambridge Univ. Press, 1915–1916; see also Third ed., Chelsea Publishing Company, New York, 1984.
57. Noether E., *Der Endlichkeitssats der Invarianten endlicher Gruppen*, Mathematische Annalen **77** (1915), 89–92.

58. Weyl H., *The classical groups*, Princeton Univ. Press, Princeton, 1939.
59. Macdonald I. G., *Symmetric functions and Hall polynomials,* Oxford Mathematical Monographs, Clarendon Press, Oxford, 1979.
60. Pedersen P., *Calculating multidimensional symmetric functions using Jacobi's formula*, Proceedings AAECC 9, volume 539 of Springer Lecture Notes in Computer Science, Springer, 1991, pp. 304–317.
61. Milne P., *On the solutions of a set of polynomial equations*, Symbolic and numerical computation for artificial intelligence. Computational Mathematics and Applications (Donald B. R., Kapur D., Mundy J. L., eds.), Academic Press Ltd., London, 1992, pp. 89–101.
62. Dalbec J., *Geometry and combinatorics of Chow forms*, PhD thesis, Cornell University, 1995.
63. Richman D. R., *Explicit generators of the invariants of finite groups*, Advances in Math. **124** (1996), no. 1, 49–76.
64. Stepanov S. A., *On vector invariants of the symmetric group*, Diskretnaya Matematika **8** (1996), no. 2, 48–62.
65. Gonzalez-Vega L., Trujillo G., *Multivariate Sturm-Habicht sequences: real root counting on n-rectangles and triangles*, Revista Matemática Complutense **10** (1997), 119–130.
66. Stepanov S. A., *On vector invariants of symmetric groups*, Diskretnaya Matematika **11** (1999), no. 3, 4–14.
67. Dalbec J., *Multisymmetric functions*, Beiträge zur Algebra und Geom. **40** (1999), no. 1, 27–51.
68. Rosas M. H., *MacMahon symmetric functions, the partition lattice, and Young subgroups*, Journ. Combin. Theory **96 A** (2001), no. 2, 326–340.
69. Vaccarino F., *The ring of multisymmetric functions*, e-print math.RA/0205233 in Electronic Archive http://arXiv.org.
70. Briand E., *When is the algebra of multisymmetric polynomials generated by the elementary multisymmetric polynomials?*, Beiträge zur Algebra und Geom. **45** (2004), no. 2, 353–368.
71. Rota G.-C., Stein J. A., *A problem of Cayley from 1857 and how he could have solved it*, Linear Algebra and its Applications (special issue on determinants and the legacy of Sir Thomas Muir) **411** (2005), 167–253.
72. Briand E., Rosas M. H., *Milne's volume function and vector symmetric polynomials*, Journ. Symbolic Comput. **44** (2009), no. 5, 583–590.
73. Cox D. A., Little J. B., O'Shea D., *Ideals, Varieties, and Algorithms*, Springer Verlag, New York, 1992.

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