ON TWO ELLIPTIC CURVES ASSOCIATED WITH PERFECT CUBOIDS.

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Abstract. A rational perfect cuboid is a rectangular parallelepiped whose edges and face diagonals are given by rational numbers and whose space diagonal is equal to unity. Finding such a cuboid is equivalent to finding a perfect cuboid with all integer edges and diagonals, which is an old unsolved problem. Recently, based on a symmetry approach, it was shown that edges and face diagonals of rational perfect cuboid are roots of two cubic equations whose coefficients depend on two rational parameters. Six special cases where these cubic equations are reducible have been already found. Two more possible cases of reducibility for these cubic equations are considered in the present paper. They lead to a pair of elliptic curves.

1. Introduction.

The problem of a perfect cuboid is known since 1719. For the history of cuboid studies the reader is referred to [1–44]. Let x_1, x_2, x_3 be edges of a cuboid and d_1, d_2, d_3 be its face diagonals. Recently, as a result of the series of papers [45–50], two cubic equations for x_1, x_2, x_3 and d_1, d_2, d_3 were derived:

$$x^3 - E_{10}\,x^2 + E_{20}\,x - E_{30} = 0, \tag{1.1}$$

$$d^3 - E_{01}\,d^2 + E_{02}\,d - E_{03} = 0. \tag{1.2}$$

The numbers x_1, x_2, x_3 are roots of the equation (1.1), while d_1, d_2, d_3 are roots of the equation (1.2). Apart from (1.1) and (1.2), the numbers x_1, x_2, x_3 and d_1, d_2, d_3 should obey the following auxiliary equations:

$$\begin{aligned}
&x_1\,x_2\,d_3 + x_2\,x_3\,d_1 + x_3\,x_1\,d_2 = E_{21},\\
&x_1\,d_2 + d_1\,x_2 + x_2\,d_3 + d_2\,x_3 + x_3\,d_1 + d_3\,x_1 = E_{11},\\
&x_1\,d_2\,d_3 + x_2\,d_3\,d_1 + x_3\,d_1\,d_2 = E_{12}.
\end{aligned} \tag{1.3}$$

The left hand sides of the equations (1.3) are three of nine elementary multisymmetric polynomials that correspond to the permutation group S_3 acting upon the numbers x_1, x_2, x_3 and d_1, d_2, d_3 broken into pairs: (x_1, d_1), (x_2, d_2), (x_3, d_3). For the theory of general multisymmetric polynomials the reader is referred to [51–71].

The coefficients of the cubic equations (1.1) and (1.2) and the right hand sides of the equations (1.3) are rational functions of two rational parameters b and c. They

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are given by explicit formulas. Here are the formulas for E_{01}, E_{10}, and E_{11}:

$$E_{11}=-\frac{b\,(c^2+2-4\,c)}{b^2\,c^2+2\,b^2-3\,b^2\,c+c-b\,c^2\,+2\,b},\tag{1.4}$$

$$E_{01}=-\frac{b\,(c^2+2-2\,c)}{b^2\,c^2+2\,b^2-3\,b^2\,c+c-b\,c^2+2\,b},\tag{1.5}$$

$$E_{10}=-\frac{b^2\,c^2+2\,b^2-3\,b^2\,c\,-c}{b^2\,c^2+2\,b^2-3\,b^2\,c+c-b\,c^2+2\,b}.\tag{1.6}$$

The formulas (1.4), (1.5), and (1.6), were derived in [49] by solving the equation[1]

$$(2\,E_{11})^2+(E_{01}^2+L^2-E_{10}^2)^2-8\,E_{01}^2\,L^2=0,\tag{1.7}$$

which was derived in [48]. Below are the formulas for E_{02}, E_{21}, E_{12}:

$$\begin{gathered}
E_{02}=\frac{1}{2}\,(28\,b^2\,c^2-16\,b^2\,c-2\,c^2-4\,b^2-b^2\,c^4+\,4\,b^3\,c^4-12\,b^3\,c^3+\\
+4\,b\,c^3+24\,b^3\,c-8\,b\,c-2\,b^4\,c^4+12\,b^4\,c^3-26\,b^4\,c^2-8\,b^2\,c^3+\\
+24\,b^4\,c-16\,b^3-8\,b^4)\,(b\,c-1-b)^{-2}\,(b\,c-c-2\,b)^{-2},
\end{gathered}\tag{1.8}$$

$$\begin{gathered}
E_{21}=\frac{b}{2}\,(5\,c^6\,b-2\,c^6\,b^2+52\,c^5\,b^2-16\,c^5\,b-2\,c^7\,b^2+2\,b^4\,c^8+\\
+142\,b^4\,c^6-26\,b^4\,c^7-426\,b^4\,c^5-61\,b^3\,c^6+100\,b^3\,c^5+14\,c^7\,b^3-\\
-c^8\,b^3-20\,b\,c^2-8\,b^2\,c^2-16\,b^2\,c-128\,b^2\,c^4-200\,b^3\,c^3+\\
+244\,b^3\,c^2+32\,b\,c^3-112\,b^3\,c+768\,b^4\,c^4-852\,b^4\,c^3+568\,b^4\,c^2+\\
+104\,b^2\,c^3-208\,b^4\,c+8\,c^4-4\,c^3+16\,b^3+32\,b^4-2\,c^5)\times\\
\times(b^2\,c^4-6\,b^2\,c^3+13\,b^2\,c^2-12\,b^2\,c+4\,b^2+c^2)^{-1}\times\\
\times(b\,c-1-b)^{-2}\,(b\,c-c-2\,b)^{-2},
\end{gathered}\tag{1.9}$$

$$\begin{gathered}
E_{12}=(16\,b^6+32\,b^5-6\,c^5\,b^2+2\,c^5\,b-62\,b^5\,c^6+62\,b^6\,c^6-\\
-180\,b^6\,c^5+18\,b^5\,c^7-12\,b^6\,c^7-2\,b^5\,c^8+b^6\,c^8+248\,b^5\,c^2+\\
+248\,b^6\,c^2-96\,b^6\,c+321\,b^6\,c^4-180\,b^5\,c^3-144\,b^5\,c-360\,b^6\,c^3+\\
+b^4\,c^8+8\,b^4\,c^6-6\,b^4\,c^7+18\,b^4\,c^5+7\,b^3\,c^6+90\,b^5\,c^5-14\,b^3\,c^5-\\
-c^7\,b^3+17\,b^2\,c^4+28\,b^3\,c^3-28\,b^3\,c^2-4\,b\,c^3+8\,b^3\,c-57\,b^4\,c^4+\\
+36\,b^4\,c^3+32\,b^4\,c^2-12\,b^2\,c^3-48\,b^4\,c-c^4+16\,b^4)\times\\
\times(b^2\,c^4-6\,b^2\,c^3+13\,b^2\,c^2-12\,b^2\,c+4\,b^2+c^2)^{-1}\times\\
\times(b\,c-1-b)^{-2}\,(b\,c-c-2\,b)^{-2}.
\end{gathered}\tag{1.10}$$

[1] In deriving (1.4), (1.5), (1.6) the parameter L in the equation (1.7) was taken for the unity. However, the corresponding solution for the general case $L\neq 1$ easily follows from (1.4), (1.5), (1.6) by homogeneity.

The formulas (1.8), (1.9), (1.10) were derived in [50] by substituting (1.4), (1.5), (1.6) and the length of the space diagonal $L = 1$ into the appropriate formulas from [48]. The formulas for E_{20}, E_{30}, and E_{03} are similar:

$$\begin{aligned} E_{20} = \frac{b}{2}\,(b\,c^2 - 2\,c - 2\,b)\,(2\,b\,c^2 - c^2 - 6\,b\,c + 2 + 4\,b)\,\times\\ \times\,(b\,c - 1 - b)^{-2}\,(b\,c - c - 2\,b)^{-2}, \end{aligned} \tag{1.11}$$

$$\begin{aligned} E_{30} = c\,b^2\,(1 - c)\,(c - 2)\,(b\,c^2 - 4\,b\,c + 2 + 4\,b)\,\times\\ \times\,(2\,b\,c^2 - c^2 - 4\,b\,c + 2\,b)\,\times\\ \times\,(b^2\,c^4 - 6\,b^2\,c^3 + 13\,b^2\,c^2 - 12\,b^2\,c + 4\,b^2 + c^2)^{-1}\,\times\\ \times\,(b\,c - 1 - b)^{-2}\,(-c + b\,c - 2\,b)^{-2}, \end{aligned} \tag{1.12}$$

$$\begin{aligned} E_{03} = \frac{b}{2}\,(b^2\,c^4 - 5\,b^2\,c^3 + 10\,b^2\,c^2 - 10\,b^2\,c + 4\,b^2 + 2\,b\,c\,+\\ +\,2\,c^2 - b\,c^3)\,(2\,b^2\,c^4 - 12\,b^2\,c^3 + 26\,b^2\,c^2 - 24\,b^2\,c\,+\\ +\,8\,b^2 - c^4\,b + 3\,b\,c^3 - 6\,b\,c + 4\,b + c^3 - 2\,c^2 + 2\,c)\,\times\\ \times\,((b^2\,c^4 - 6\,b^2\,c^3 + 13\,b^2\,c^2 - 12\,b^2\,c + 4\,b^2 + c^2)^{-1}\,\times\\ \times\,(b\,c - 1 - b)^{-2}\,(-c + b\,c - 2\,b)^{-2}. \end{aligned} \tag{1.13}$$

The formulas (1.11), (1.12), (1.13) were also derived in [50] by substituting (1.4), (1.5), (1.6) into the appropriate formulas from [48].

Based on the formulas (1.4) through (1.6) and (1.8) through (1.13), the following two problems were formulated.

Problem 1.1. *Find all pairs of rational numbers b and c for which the cubic equations (1.1) and (1.2) with the coefficients given by the formulas (1.6), (1.11), (1.12) and (1.5), (1.8), (1.13) possess positive rational roots x_1, x_2, x_3, d_1, d_2, d_3 obeying the auxiliary polynomial equations (1.3) whose right hand sides are given by the formulas (1.9), (1.4), (1.10).*

Problem 1.2. *Find at least one pair of rational numbers b and c for which the cubic equations (1.1) and (1.2) with the coefficients given by the formulas (1.6), (1.11), (1.12) and (1.5), (1.8), (1.13) possess positive rational roots x_1, x_2, x_3, d_1, d_2, d_3 obeying the auxiliary polynomial equations (1.3) whose right hand sides are given by the formulas (1.9), (1.4), (1.10).*

The problems 1.1 and 1.2 are equivalent to the appropriate problems for perfect cuboids. Therefore they are equally difficult. However, now we can consider more simple problems, e. g. we can search for the cases where the polynomials in the left hand sides of the equations (1.1) and (1.2) are reducible.

Definition 1.1. A polynomial with rational coefficients is called reducible over $\mathbb{Q}$ if it splits into a product of two or more polynomials with rational coefficients.

The polynomials (1.1) and (1.2) can be reducible simultaneously or in a separate way. Six simple cases of reducibility were found in [72]. In each of these six cases the polynomials (1.1) and (1.2) are reducible simultaneously. Unfortunately, or maybe fortunately, since otherwise the problem would be closed, none of them leads to a perfect cuboid. Here is the list of reducibility conditions for all of these cases:

$$
\begin{array}{ll}
1)\ \ b=0 \text{ and } c\neq 0; & 4)\ \ c=2 \text{ and } b\neq 1;\\
2)\ \ c=0 \text{ and } b\,(1+b)\neq 0; & 5)\ \ b\,(c-2)^2=-2 \text{ and } c\neq 2;\\
3)\ \ c=1 \text{ and } b\neq -1; & 6)\ \ 2\,b\,(c-1)^2=c^2 \text{ and } c\,(c-1)\neq 0.
\end{array}
\tag{1.14}
$$

In addition to (1.14), there are two more options that could lead to reducibility of the second polynomial (1.2). They are considered in this paper. As for the first polynomial (1.1), until the converse is proved, the reducibility of (1.2) does not imply the reducibility of (1.1) in general.

2. Elliptic reducibility curves.

Let's consider the formula (1.13) for the last term E_{03} in the of the polynomial (1.2). If $E_{03}=0$, then the polynomial (1.2) has the rational root $d=0$, i.e. it is reducible over $\mathbb{Q}$. Looking at (1.13), we see that the numerator in this formula is the product of three terms. One of them is b. The vanishing condition $b=0$ for E_{03} is already listed in (1.14). Let's consider the other two vanishing conditions:

$$
b^2\,c^4-5\,b^2\,c^3+10\,b^2\,c^2-10\,b^2\,c+4\,b^2-b\,c^3+2\,b\,c+2\,c^2=0,
\tag{2.1}
$$

$$
\begin{gathered}
2\,b^2\,c^4-12\,b^2\,c^3+26\,b^2\,c^2-24\,b^2\,c\,+\\
+\,8\,b^2-b\,c^4+3\,b\,c^3-6\,b\,c+4\,b+c^3-2\,c^2+2\,c=0.
\end{gathered}
\tag{2.2}
$$

The equations (2.1) and (2.2) are quadratic with respect to b. Their discriminants with respect to b are given by the following formulas:

$$
D_7=-(7\,c^4-40\,c^3+84\,c^2-80\,c+28)\,c^2,
\tag{2.3}
$$

$$
D_8=(c^4-8\,c^3+12\,c^2-16\,c+4)\,(c-1)^2\,(c-2)^2.
\tag{2.4}
$$

The indices 7 and 8 in (2.3) and (2.4) mean that we consider the seventh and the eighth reducibility cases continuing the list (1.14).

Let's denote through P_7 and P_8 the square free factors of the discriminants D_7 and D_8 respectively. Both of them are fourth order polynomials of c:

$$
P_7(c)=-7\,c^4+40\,c^3-84\,c^2+80\,c-28,
\tag{2.5}
$$

$$
P_8(c)=c^4-8\,c^3+12\,c^2-16\,c+4.
\tag{2.6}
$$

Using the polynomials (2.5) and (2.6), we can write the polynomial equations

$$
y^2=-7\,c^4+40\,c^3-84\,c^2+80\,c-28,
\tag{2.7}
$$

$$
y^2=c^4-8\,c^3+12\,c^2-16\,c+4,
\tag{2.8}
$$

which are more simple than the equations (2.1) and (2.2). The equations (2.5) and (2.6) define two elliptic curves on the (y,c) plane (see [73]). The discriminants of the quartic polynomials $P_7(c)$ and $P_8(c)$ do coincide and are nonzero:

$$
D(P_7)=D(P_8)=-1048576=-2^{20}\neq 0.
\tag{2.9}
$$

Due to (2.9) the elliptic curves defined by the equations (2.7) and (2.8) both are non-degenerate.

In Number Theory an elliptic curve is understood as a genus 1 curve with at least one rational point either finite or at infinity (see [73]). In the case of the curve (2.8) such a rational point is obvious:

$$y = 2, \qquad\qquad c = 0. \tag{2.10}$$

In the case of the curve (2.7) it is not obvious, but it does exist:

$$y = 1, \qquad\qquad c = 1. \tag{2.11}$$

Theorem 2.1. *Each rational point (y, c) of the curve (2.7) with $c \neq 1$ and $c \neq 2$ produces a rational solution (b, c) for the equation (2.1), where*

$$b = \frac{c\,(c^2 + y - 2)}{2\,(c-1)\,(c-2)\,((c-1)^2 + 1)}. \tag{2.12}$$

Theorem 2.2. *Each rational point (y, c) of the curve (2.8) with $c \neq 1$ and $c \neq 2$ produces a rational solution (b, c) for the equation (2.2), where*

$$b = \frac{c^2 + y - 2}{4\,(c-2)\,(c-1)}. \tag{2.13}$$

The proof for both theorems 2.1 and 2.2 is pure calculations. Note that the equalities (2.12) and (2.13) in them are linear with respect to y. Resolving these equalities for y, we can formulate the following converse theorems.

Theorem 2.3. *Each rational solution (b, c) of the equation (2.1) with $c \neq 0$ produces a rational point (y, c) for the elliptic curve (2.7), where*

$$y = \frac{(2\,c^4 - 10\,c^3 + 20\,c^2 - 20\,c + 8)\,b - c^3 + 2\,c}{c}. \tag{2.14}$$

Theorem 2.4. *Each rational solution (b, c) of the equation (2.2) produces a rational point (y, c) for the elliptic curve (2.8), where*

$$y = (4\,c^2 - 12\,c + 8)\,b - c^2 + 2. \tag{2.15}$$

The formulas (2.12), (2.13), (2.14), and (2.15) mean that the curve (2.1) is birationally equivalent to the elliptic curve (2.7), while the curve (2.2) is birationally equivalent to the elliptic curve (2.8).

3. Exceptional solutions and points.

Let's return to the cubic equations (1.1) and (1.2) and to the auxiliary equations (1.3). We can call them cuboid equations since they were derived from the original cuboid equations through the symmetry factorization procedure as a result of the series of papers [45–50]. The simultaneous non-vanishing condition for all denominators in the formulas (1.4) through (1.6) and in the formulas (1.8) through (1.13) is written as the following inequality:

$$\begin{gathered}(b^2\,c^4-6\,b^2\,c^3+13\,b^2\,c^2-12\,b^2\,c+4\,b^2+c^2)\,(b\,c-1-b)\times\\ \times(b\,c-c-2\,b)\,(b^2\,c^2+2\,b^2-3\,b^2\,c+c-b\,c^2+2\,b)\neq 0.\end{gathered}\tag{3.1}$$

Combining the inequality (3.1) with the equation (2.1), we find the only rational exceptional point on the curve (2.1). It is the origin:

$$b=0, \qquad\qquad c=0.\tag{3.2}$$

Similarly, combining (3.1) with the equation (2.2), we find the only rational exceptional point on the curve (2.2). It coincides with the point (3.2).

If $c=0$, the equation (2.1) has the only rational solution with $b=0$, i.e. this solution coincides with (3.2). It contradicts the inequality (3.1).

For $c=1$ and $c=2$ the equation (2.1) has the following rational solutions:

$$b=-2, \qquad\qquad c=1,\tag{3.3}$$

$$b=2, \qquad\qquad c=2.\tag{3.4}$$

The solutions (3.3) and (3.4) do not contradict the inequality (3.1). But they are covered by the cases 3 and 4 listed in (1.14). Therefore we call them exceptional solutions of the equation (2.1) or exceptional points of the curve (2.1).

The formula (2.14) maps the solutions (3.3) and (3.4) of the equation (2.1) to the following rational points of the elliptic curve (2.7):

$$y=1, \qquad\qquad c=1,\tag{3.5}$$

$$y=-2, \qquad\qquad c=2.\tag{3.6}$$

The point (3.5) coincides with (2.11). Along with (3.5) and (3.6), the elliptic curve (2.7) has the following two solutions being mirror images of the previous two:

$$y=-1, \qquad\qquad c=1,\tag{3.7}$$

$$y=2, \qquad\qquad c=2.\tag{3.8}$$

Note that the rational points (3.5), (3.6), (3.7), and (3.8) are exceptional in the sense of the theorem 2.1. And finally, note that the elliptic curve (2.7) has no rational points with $c=0$. Then we can formulate the following result.

Theorem 3.1. *Non-exceptional rational solutions of the equation* (2.1)*, if they exist, are in one-to-one correspondence with non-exceptional rational points of the curve* (2.7)*. The correspondence is established by the formulas* (2.14) *and* (2.12)*.*

Let's proceed to the equation (2.2) associated with the curve (2.8). Along with the solution (3.2), it has the following rational solution with $c=0$:

$$b=-\frac{1}{2}, \qquad\qquad c=0.\tag{3.9}$$

The solution (3.9) does not contradict the inequality (3.1). But it is covered by the case 2 listed in (1.14). Therefore we call it an exceptional solution of the equation (2.2) or an exceptional point of the curve (2.2).

The formula (2.15) maps the solution (3.9) of the equation (2.2) to the following rational point of the elliptic curve (2.8):

$$y = -2, \qquad\qquad c = 0, \tag{3.10}$$

Along with (3.10), the elliptic curve (2.8) has the following solution being a mirror image of (3.10) and coinciding with (2.10):

$$y = 2, \qquad\qquad c = 0. \tag{3.11}$$

The points (3.10) and (3.11) of the curve (2.8) are not exceptional in the sense of the theorem 2.2. The formula (2.13) maps them to the solutions (3.9) and (3.2) of the equation (2.2) respectively. But the latter ones are exceptional. Therefore we call the points (3.10) and (3.11) exceptional by convention. Note also that the the equations (2.2) has no solutions with $c = 1$ or $c = 2$ and the curve (2.8) has no rational points with $c = 1$ or $c = 2$ as well. Therefore we can formulate the following theorem similar to the theorem 3.1.

Theorem 3.2. *Non-exceptional rational solutions of the equation* (2.2)*, if they exist, are in one-to-one correspondence with non-exceptional rational points of the curve* (2.8)*. The correspondence is established by the formulas* (2.15) *and* (2.13)*.*

4. The seventh reducibility case.

The seventh and the eighth reducibility cases considered below are based on non-exceptional rational solutions of the equations (2.1) and (2.2) or, which is equivalent, on non-exceptional rational points of the elliptic curves (2.7) and (2.8). We do not study the problem of existence for such solutions and/or such points in the present paper. Therefore the results below are conditional provided these point and these solutions do exist.

Like in [72], let's denote through $P(x)$ and $Q(d)$ the cubic polynomials in the left hand sides of the equations (1.1) and (1.2) respectively. Then due to (1.6), (1.11), and (1.12) the coefficients of the polynomial $P(x)$ are functions of b and c. Due to (1.5), (1.8), and (1.13) the coefficients of $Q(d)$ are also functions of b and c.

The seventh reducibility case occurs if (b, c) is a non-exceptional rational solution of the equation (2.1). This solution is produced from some non-exceptional rational point (y, c) on the elliptic curve (2.7) by means of the formula (2.12). Let's substitute (2.12) into the coefficients of the polynomials $P(x)$ and $Q(d)$ and take into account the equation (2.7) in calculating y^2, y^3, y^4, y^5, etc. As a result we get two very huge expressions for $P(x)$ and $Q(d)$, but they turn out to be factorable in x and d so that we can formulate the following theorem.

Theorem 4.1. *If the rational numbers b and c are produced from some non-exceptional rational point (y, c) of the elliptic curve* (2.7) *by means of the formula* (2.12)*, then the cubic polynomials in* (1.1) *and* (1.2) *are reducible over $\mathbb{Q}$ and the equations* (1.1) *and* (1.2) *are factored as*

$$P_{7.2}(x)\,(x + 1) = 0, \qquad\qquad Q_{7.2}(d)\; d = 0. \tag{4.1}$$

The second formula (4.1) is not surprising since the equation (2.1), and hence the equation (2.7), were derived from the condition $E_{03} = 0$. As for the first formula (4.1), it is proved by substituting $x = -1$ into the formula for $P(x)$ transformed by means of the formula (2.12) as described above. Selecting those rational points of the curve (2.7) where the polynomials $P(x) = P_{7.2}(x)\,(x+1)$ and $Q(d) = Q_{7.2}(d)\;d$ split into three linear factors is a separate problem. We do not consider this problem in the present paper.

5. The eighth reducibility case.

The eighth reducibility case occurs if (b, c) is a non-exceptional rational solution of the equation (2.2). This solution is produced from some non-exceptional rational point (y, c) on the elliptic curve (2.8) by means of the formula (2.13). Let's substitute (2.13) into the coefficients of the polynomials $P(x)$ and $Q(d)$ and take into account the equation (2.8) in calculating y^2, y^3, y^4, y^5, etc. As a result we get two very huge expressions for $P(x)$ and $Q(d)$, but they turn out to be factorable in x and d so that we can formulate the following theorem.

Theorem 5.1. *If the rational numbers b and c are produced from some non-exceptional rational point (y, c) of the elliptic curve (2.8) by means of the formula (2.13), then the cubic polynomials in (1.1) and (1.2) are reducible over $\mathbb{Q}$ and the equations (1.1) and (1.2) are factored as*

$$P_{8.2}(x)\,(x-1) = 0, \qquad\qquad Q_{8.2}(d)\;d = 0. \tag{5.1}$$

The second formula (5.1) is not surprising since the equation (2.2), and hence the equation (2.8), were derived from the condition $E_{03} = 0$. As for the first formula (5.1), it is proved by substituting $x = 1$ into the formula for $P(x)$ transformed by means of the formula (2.13) as described above. Selecting those non-exceptional rational points of the curve (2.8) where the polynomials $P(x) = P_{8.2}(x)\,(x-1)$ and $Q(d) = Q_{8.2}(d)\;d$ split into three linear factors is a separate problem. We do not consider this problem in the present paper.

9. Concluding remarks.

The theory of rational points on elliptic curves is a very advanced and still developing area in modern mathematics. It comprises some intriguing open questions, e. g. the Birch and Swinnerton-Dyer conjecture, which is one of the seven Millennium Prize Problems (see [74] and [75]). The author expects that the observations and results of the present paper will make the perfect cuboid problem closer to this fascinating area of mathematics.

References

1. *Euler brick*, Wikipedia, Wikimedia Foundation Inc., San Francisco, USA.
2. Halcke P., *Deliciae mathematicae oder mathematisches Sinnen-Confect*, N. Sauer, Hamburg, Germany, 1719.
3. Saunderson N., *Elements of algebra,* Vol. 2, Cambridge Univ. Press, Cambridge, 1740.
4. Euler L., *Vollständige Anleitung zur Algebra,* 3 Theile, Kaiserliche Akademie der Wissenschaften, St. Petersburg, 1770-1771.
5. Pocklington H. C., *Some Diophantine impossibilities*, Proc. Cambridge Phil. Soc. **17** (1912), 108–121.

6. Dickson L. E, *History of the theory of numbers,* Vol. 2*: Diophantine analysis*, Dover, New York, 2005.
7. Kraitchik M., *On certain rational cuboids*, Scripta Math. **11** (1945), 317–326.
8. Kraitchik M., *Théorie des Nombres,* Tome 3*, Analyse Diophantine et application aux cuboides rationelles*, Gauthier-Villars, Paris, 1947.
9. Kraitchik M., *Sur les cuboides rationelles*, Proc. Int. Congr. Math. **2** (1954), Amsterdam, 33–34.
10. Bromhead T. B., *On square sums of squares*, Math. Gazette **44** (1960), no. 349, 219–220.
11. Lal M., Blundon W. J., *Solutions of the Diophantine equations* $x^2 + y^2 = l^2$, $y^2 + z^2 = m^2$, $z^2 + x^2 = n^2$, Math. Comp. **20** (1966), 144–147.
12. Spohn W. G., *On the integral cuboid*, Amer. Math. Monthly **79** (1972), no. 1, 57-59.
13. Spohn W. G., *On the derived cuboid*, Canad. Math. Bull. **17** (1974), no. 4, 575-577.
14. Chein E. Z., *On the derived cuboid of an Eulerian triple*, Canad. Math. Bull. **20** (1977), no. 4, 509–510.
15. Leech J., *The rational cuboid revisited*, Amer. Math. Monthly **84** (1977), no. 7, 518–533; see also Erratum, Amer. Math. Monthly **85** (1978), 472.
16. Leech J., *Five tables relating to rational cuboids*, Math. Comp. **32** (1978), 657–659.
17. Spohn W. G., *Table of integral cuboids and their generators*, Math. Comp. **33** (1979), 428–429.
18. Lagrange J., *Sur le dérivé du cuboide Eulérien*, Canad. Math. Bull. **22** (1979), no. 2, 239–241.
19. Leech J., *A remark on rational cuboids*, Canad. Math. Bull. **24** (1981), no. 3, 377–378.
20. Korec I., *Nonexistence of small perfect rational cuboid*, Acta Math. Univ. Comen. **42/43** (1983), 73–86.
21. Korec I., *Nonexistence of small perfect rational cuboid II*, Acta Math. Univ. Comen. **44/45** (1984), 39–48.
22. Wells D. G., *The Penguin dictionary of curious and interesting numbers*, Penguin publishers, London, 1986.
23. Bremner A., Guy R. K., *A dozen difficult Diophantine dilemmas*, Amer. Math. Monthly **95** (1988), no. 1, 31–36.
24. Bremner A., *The rational cuboid and a quartic surface*, Rocky Mountain J. Math. **18** (1988), no. 1, 105–121.
25. Colman W. J. A., *On certain semiperfect cuboids*, Fibonacci Quart. **26** (1988), no. 1, 54–57; see also *Some observations on the classical cuboid and its parametric solutions*, Fibonacci Quart. **26** (1988), no. 4, 338–343.
26. Korec I., *Lower bounds for perfect rational cuboids*, Math. Slovaca **42** (1992), no. 5, 565–582.
27. Guy R. K., *Is there a perfect cuboid? Four squares whose sums in pairs are square. Four squares whose differences are square*, Unsolved Problems in Number Theory, 2nd ed., Springer-Verlag, New York, 1994, pp. 173–181.
28. Rathbun R. L., Granlund T., *The integer cuboid table with body, edge, and face type of solutions*, Math. Comp. **62** (1994), 441–442.
29. Van Luijk R., *On perfect cuboids,* Doctoraalscriptie, Mathematisch Instituut, Universiteit Utrecht, Utrecht, 2000.
30. Rathbun R. L., Granlund T., *The classical rational cuboid table of Maurice Kraitchik*, Math. Comp. **62** (1994), 442–443.
31. Peterson B. E., Jordan J. H., *Integer hexahedra equivalent to perfect boxes*, Amer. Math. Monthly **102** (1995), no. 1, 41–45.
32. Rathbun R. L., *The rational cuboid table of Maurice Kraitchik*, e-print math.HO/0111229 in Electronic Archive http://arXiv.org.
33. Hartshorne R., Van Luijk R., *Non-Euclidean Pythagorean triples, a problem of Euler, and rational points on K3 surfaces*, e-print math.NT/0606700 in Electronic Archive http://arXiv.org.
34. Waldschmidt M., *Open diophantine problems*, e-print math.NT/0312440 in Electronic Archive http://arXiv.org.
35. Ionascu E. J., Luca F., Stanica P., *Heron triangles with two fixed sides*, e-print math.NT/0608185 in Electronic Archive http://arXiv.org.
36. Ortan A., Quenneville-Belair V., *Euler's brick*, Delta Epsilon, McGill Undergraduate Mathematics Journal **1** (2006), 30-33.
37. Knill O., *Hunting for Perfect Euler Bricks*, Harvard College Math. Review **2** (2008), no. 2, 102; see also http://www.math.harvard.edu/~knill/various/eulercuboid/index.html.

38. Sloan N. J. A, *Sequences A031173, A031174, and A031175*, On-line encyclopedia of integer sequences, OEIS Foundation Inc., Portland, USA.
39. Stoll M., Testa D., *The surface parametrizing cuboids*, e-print arXiv:1009.0388 in Electronic Archive http://arXiv.org.
40. Sharipov R. A., *A note on a perfect Euler cuboid.*, e-print arXiv:1104.1716 in Electronic Archive http://arXiv.org.
41. Sharipov R. A., *Perfect cuboids and irreducible polynomials*, Ufa Mathematical Journal **4,** (2012), no. 1, 153–160; see also e-print arXiv:1108.5348 in Electronic Archive http://arXiv.org.
42. Sharipov R. A., *A note on the first cuboid conjecture*, e-print arXiv:1109.2534 in Electronic Archive http://arXiv.org.
43. Sharipov R. A., *A note on the second cuboid conjecture. Part I*, e-print arXiv:1201.1229 in Electronic Archive http://arXiv.org.
44. Sharipov R. A., *A note on the third cuboid conjecture. Part I*, e-print arXiv:1203.2567 in Electronic Archive http://arXiv.org.
45. Sharipov R. A., *Perfect cuboids and multisymmetric polynomials*, e-print arXiv:1205.3135 in Electronic Archive http://arXiv.org.
46. Sharipov R. A., *On an ideal of multisymmetric polynomials associated with perfect cuboids*, e-print arXiv:1206.6769 in Electronic Archive http://arXiv.org.
47. Sharipov R. A., *On the equivalence of cuboid equations and their factor equations*, e-print arXiv:1207.2102 in Electronic Archive http://arXiv.org.
48. Sharipov R. A., *A biquadratic Diophantine equation associated with perfect cuboids*, e-print arXiv:1207.4081 in Electronic Archive http://arXiv.org.
49. Ramsden J. R., *A general rational solution of an equation associated with perfect cuboids*, e-print arXiv:1207.5339 in Electronic Archive http://arXiv.org.
50. Ramsden J. R., Sharipov R. A., *Inverse problems associated with perfect cuboids*, e-print arXiv:1207.6764 in Electronic Archive http://arXiv.org.
51. Shläfli L., *Über die Resultante eines systems mehrerer algebraishen Gleihungen*, Denkschr. Kaiserliche Acad. Wiss. Math.-Natur. Kl. **4** (1852); reprinted in «Gesammelte mathematische Abhandlungen», Band II (1953), Birkhäuser Verlag, 9–112.
52. Cayley A., *On the symmetric functions of the roots of certain systems of two equations*, Phil. Trans. Royal Soc. London **147** (1857), 717–726.
53. Junker F., *Über symmetrische Functionen von mehreren Veränderlishen*, Mathematische Annalen **43** (1893), 225–270.
54. McMahon P. A., *Memoir on symmetric functions of the roots of systems of equations*, Phil. Trans. Royal Soc. London **181** (1890), 481–536.
55. McMahon P. A., *Combinatory Analysis.* Vol. I and Vol. II, Cambridge Univ. Press, 1915–1916; see also Third ed., Chelsea Publishing Company, New York, 1984.
56. Noether E., *Der Endlichkeitssats der Invarianten endlicher Gruppen*, Mathematische Annalen **77** (1915), 89–92.
57. Weyl H., *The classical groups*, Princeton Univ. Press, Princeton, 1939.
58. Macdonald I. G., *Symmetric functions and Hall polynomials,* Oxford Mathematical Monographs, Clarendon Press, Oxford, 1979.
59. Pedersen P., *Calculating multidimensional symmetric functions using Jacobi's formula*, Proceedings AAECC 9, volume 539 of Springer Lecture Notes in Computer Science, Springer, 1991, pp. 304–317.
60. Milne P., *On the solutions of a set of polynomial equations*, Symbolic and numerical computation for artificial intelligence. Computational Mathematics and Applications (Donald B. R., Kapur D., Mundy J. L., eds.), Academic Press Ltd., London, 1992, pp. 89–101.
61. Dalbec J., *Geometry and combinatorics of Chow forms*, PhD thesis, Cornell University, 1995.
62. Richman D. R., *Explicit generators of the invariants of finite groups*, Advances in Math. **124** (1996), no. 1, 49–76.
63. Stepanov S. A., *On vector invariants of the symmetric group*, Diskretnaya Matematika **8** (1996), no. 2, 48–62.
64. Gonzalez-Vega L., Trujillo G., *Multivariate Sturm-Habicht sequences: real root counting on n-rectangles and triangles*, Revista Matemática Complutense **10** (1997), 119–130.
65. Stepanov S. A., *On vector invariants of symmetric groups*, Diskretnaya Matematika **11** (1999), no. 3, 4–14.
66. Dalbec J., *Multisymmetric functions*, Beiträge zur Algebra und Geom. **40** (1999), no. 1, 27–51.

67. Rosas M. H., *MacMahon symmetric functions, the partition lattice, and Young subgroups*, Journ. Combin. Theory **96 A** (2001), no. 2, 326–340.
68. Vaccarino F., *The ring of multisymmetric functions*, e-print math.RA/0205233 in Electronic Archive http://arXiv.org.
69. Briand E., *When is the algebra of multisymmetric polynomials generated by the elementary multisymmetric polynomials?*, Beiträge zur Algebra und Geom. **45** (2004), no. 2, 353–368.
70. Rota G.-C., Stein J. A., *A problem of Cayley from 1857 and how he could have solved it*, Linear Algebra and its Applications (special issue on determinants and the legacy of Sir Thomas Muir) **411** (2005), 167–253.
71. Briand E., Rosas M. H., *Milne's volume function and vector symmetric polynomials*, Journ. Symbolic Comput. **44** (2009), no. 5, 583–590.
72. Sharipov R. A., *On a pair of cubic equations associated with perfect cuboids*, e-print arXiv:1208.0308 in Electronic Archive http://arXiv.org.
73. Connel I., *Elliptic curve handbook*, McGill University, Montreal, 1999; see http://www.math.mcgill.ca/connell.
74. *Birch and Swinnerton-Dyer Conjecture*, Millennium Prize Problems (2000), Clay Mathematics Institute, Cambridge, Massachusetts, USA.
75. *Millennium Prize Problems*, Wikipedia, Wikimedia Foundation Inc., San Francisco, USA.

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