ON SINGULARITIES OF THE INVERSE PROBLEMS ASSOCIATED WITH PERFECT CUBOIDS.

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Abstract. Two cubic equations and three auxiliary equations for edges and face diagonals of a rational perfect cuboid have been recently derived. They constitute a background for two inverse problems. The coefficients of cubic equations and the right hand sides of auxiliary equations are rational functions of two rational parameters, i. e. they have denominators. Hence the inverse problems have singular points. These singular points are studied in the present paper.

1. Introduction.

A rational perfect cuboid is a rectangular parallelepiped whose edges and face diagonals are rational numbers and whose space diagonal is equal to unity: $L = 1$. Finding such a cuboid is equivalent to finding a cuboid with all integer edges and diagonals, which is an unsolved problem for many years (see [1–44]).

Let x_1, x_2, x_3 be edges of a cuboid and let d_1, d_2, d_3 be its face diagonals. Then x_1, x_2, x_3 are roots of the cubic equation

$$x^3 - E_{10}\,x^2 + E_{20}\,x - E_{30} = 0. \tag{1.1}$$

Similarly, d_1, d_2, d_3 are roots of the other cubic equation

$$d^3 - E_{01}\,d^2 + E_{02}\,d - E_{03} = 0. \tag{1.2}$$

Apart from (1.1) and (1.2), the rational numbers x_1, x_2, x_3 and d_1, d_2, d_3 should obey the following three auxiliary equations:

$$\begin{aligned}
&x_1\,x_2\,d_3 + x_2\,x_3\,d_1 + x_3\,x_1\,d_2 = E_{21},\\
&x_1\,d_2 + d_1\,x_2 + x_2\,d_3 + d_2\,x_3 + x_3\,d_1 + d_3\,x_1 = E_{11},\\
&x_1\,d_2\,d_3 + x_2\,d_3\,d_1 + x_3\,d_1\,d_2 = E_{12}.
\end{aligned} \tag{1.3}$$

The cubic equations (1.1), (1.2) and the auxiliary equations (1.3) were obtained as a result of the series of papers [45–50]). The coefficients E_{10}, E_{20}, E_{30}, E_{01}, E_{02}, E_{03} in (1.1) and (1.2) as well as the right hand sides E_{21}, E_{11}, E_{12} in (1.3) are given by explicit formulas. Here is the formula for E_{11}:

$$E_{11} = -\frac{b\,(c^2 + 2 - 4\,c)}{b^2\,c^2 + 2\,b^2 - 3\,b^2\,c + c - b\,c^2 + 2\,b}. \tag{1.4}$$

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The formulas for E_{10}, E_{01} are similar to the formula (1.4) for E_{11}:

$$E_{10}=-\frac{b^2\,c^2+2\,b^2-3\,b^2\,c\,-c}{b^2\,c^2+2\,b^2-3\,b^2\,c+c-b\,c^2+2\,b}, \tag{1.5}$$

$$E_{01}=-\frac{b\,(c^2+2-2\,c)}{b^2\,c^2+2\,b^2-3\,b^2\,c+c-b\,c^2+2\,b}. \tag{1.6}$$

Below are the formulas for E_{20}, E_{02}, E_{30}, E_{03}, E_{21}, E_{12} in (1.1), (1.2), and (1.3):

$$\begin{aligned}E_{20}=\frac{b}{2}\,(b\,c^2-2\,c-2\,b)\,(2\,b\,c^2-c^2-6\,b\,c+2+4\,b)\,\times\\ \times\,(b\,c-1-b)^{-2}\,(b\,c-c-2\,b)^{-2},\end{aligned} \tag{1.7}$$

$$\begin{aligned}E_{02}=\frac{1}{2}\,(28\,b^2\,c^2-16\,b^2\,c-2\,c^2-4\,b^2-b^2\,c^4+4\,b^3\,c^4-12\,b^3\,c^3+\\ +4\,b\,c^3+24\,b^3\,c-8\,b\,c-2\,b^4\,c^4+12\,b^4\,c^3-26\,b^4\,c^2-8\,b^2\,c^3+\\ +24\,b^4\,c-16\,b^3-8\,b^4)\,(b\,c-1-b)^{-2}\,(b\,c-c-2\,b)^{-2},\end{aligned} \tag{1.8}$$

$$\begin{aligned}E_{30}=c\,b^2\,(1-c)\,(c-2)\,(b\,c^2-4\,b\,c+2+4\,b)\,(2\,b\,c^2-c^2-4\,b\,c+\\ +2\,b)\,(b^2\,c^4-6\,b^2\,c^3+13\,b^2\,c^2-12\,b^2\,c+4\,b^2+c^2)^{-1}\,\times\\ \times\,(b\,c-1-b)^{-2}\,(-c+b\,c-2\,b)^{-2},\end{aligned} \tag{1.9}$$

$$\begin{aligned}E_{03}=\frac{b}{2}\,(b^2\,c^4-5\,b^2\,c^3+10\,b^2\,c^2-10\,b^2\,c+4\,b^2+2\,b\,c+2\,c^2-\\ -\,b\,c^3)\,(2\,b^2\,c^4-12\,b^2\,c^3+26\,b^2\,c^2-24\,b^2\,c+\,8\,b^2-c^4\,b+3\,b\,c^3-\\ -\,6\,b\,c+4\,b+c^3-2\,c^2+2\,c)\,(b^2\,c^4-6\,b^2\,c^3+13\,b^2\,c^2-\\ -12\,b^2\,c+4\,b^2+c^2)^{-1}\,(b\,c-1-b)^{-2}\,(-c+b\,c-2\,b)^{-2},\end{aligned} \tag{1.10}$$

$$\begin{aligned}E_{21}=\frac{b}{2}\,(5\,c^6\,b-2\,c^6\,b^2+52\,c^5\,b^2-16\,c^5\,b-2\,c^7\,b^2+2\,b^4\,c^8-\\ -\,26\,b^4\,c^7-426\,b^4\,c^5-61\,b^3\,c^6+100\,b^3\,c^5+14\,c^7\,b^3-c^8\,b^3-20\,b\,c^2-\\ -\,8\,b^2\,c^2-16\,b^2\,c-128\,b^2\,c^4-200\,b^3\,c^3+244\,b^3\,c^2+32\,b\,c^3+\\ +\,768\,b^4\,c^4-852\,b^4\,c^3+568\,b^4\,c^2+104\,b^2\,c^3-208\,b^4\,c+8\,c^4+\\ +16\,b^3-112\,b^3\,c+142\,b^4\,c^6+32\,b^4-2\,c^5)\,(b^2\,c^4-6\,b^2\,c^3+13\,b^2\,c^2-\\ -12\,b^2\,c-4\,c^3+4\,b^2+c^2)^{-1}\,(b\,c-1-b)^{-2}\,(b\,c-c-2\,b)^{-2},\end{aligned} \tag{1.11}$$

$$\begin{aligned}E_{12}=(16\,b^6+32\,b^5-6\,c^5\,b^2+2\,c^5\,b-62\,b^5\,c^6+62\,b^6\,c^6+16\,b^4-\\ -\,180\,b^6\,c^5-c^7\,b^3+18\,b^5\,c^7-12\,b^6\,c^7-2\,b^5\,c^8+b^6\,c^8+248\,b^5\,c^2+\\ +\,248\,b^6\,c^2-96\,b^6\,c+321\,b^6\,c^4-180\,b^5\,c^3-144\,b^5\,c-360\,b^6\,c^3+\\ +\,b^4\,c^8+8\,b^4\,c^6-6\,b^4\,c^7+18\,b^4\,c^5+7\,b^3\,c^6+90\,b^5\,c^5-14\,b^3\,c^5+\\ +\,17\,b^2\,c^4+32\,b^4\,c^2+28\,b^3\,c^3-28\,b^3\,c^2-4\,b\,c^3+8\,b^3\,c-57\,b^4\,c^4+\\ +\,36\,b^4\,c^3-12\,b^2\,c^3-48\,b^4\,c-c^4)\,(b^2\,c^4-6\,b^2\,c^3+13\,b^2\,c^2-\\ -\,12\,b^2\,c+4\,b^2+c^2)^{-1}\,(b\,c-1-b)^{-2}\,(b\,c-c-2\,b)^{-2}.\end{aligned} \tag{1.12}$$

Based on the equations (1.1), (1.2), (1.3) and on the formulas (1.4), (1.5), (1.6), (1.7), (1.8), (1.9), (1.10), (1.11), (1.12), in [50] the following two inverse problems were formulated.

Problem 1.1. *Find all pairs of rational numbers b and c for which the cubic equations* (1.1) *and* (1.2) *with the coefficients given by the formulas* (1.5), (1.7), (1.9), (1.6), (1.8), (1.10) *possess positive rational roots x_1, x_2, x_3, d_1, d_2, d_3 obeying the auxiliary polynomial equations* (1.3) *whose right hand sides are given by the formulas* (1.4), (1.11), (1.12).

Problem 1.2. *Find at least one pair of rational numbers b and c for which the cubic equations* (1.1) *and* (1.2) *with the coefficients given by the formulas* (1.5), (1.7), (1.9), (1.6), (1.8), (1.10) *possess positive rational roots x_1, x_2, x_3, d_1, d_2, d_3 obeying the auxiliary polynomial equations* (1.3) *whose right hand sides are given by the formulas* (1.4), (1.11), (1.12).

The formulas (1.4) through (1.12) possess denominators. Therefore the inverse problems are singular for some values of b and c. The main goal of the present paper is to study these singularities.

2. The common denominator and its reduction.

Let's calculate the common denominator of the fractions (1.4) through (1.12):

$$\begin{gathered}(b^2\,c^4-6\,b^2\,c^3+13\,b^2\,c^2-12\,b^2\,c+4\,b^2+c^2)\,(b\,c-1-b)^2\times\\ \times(b\,c-c-2\,b)^2\,(b^2\,c^2+2\,b^2-3\,b^2\,c+c-b\,c^2+2\,b)=0.\end{gathered}\tag{2.1}$$

The vanishing condition (2.1) determines all singular points of the inverse problems 1.1 and 1.2. The last multiplicand in the left hand side of the formula (2.1) is taken from the denominators of (1.4), (1.5), and (1.6). Studying this multiplicand we find that it is factorable. It factors as follows:

$$b^2\,c^2+2\,b^2-3\,b^2\,c+c-b\,c^2+2\,b=(b\,c-1-b)\,(b\,c-c-2\,b).$$

Applying this formula to (2.1), we reduce the equality (2.1) to

$$\begin{gathered}(b^2\,c^4-6\,b^2\,c^3+13\,b^2\,c^2-12\,b^2\,c+4\,b^2+c^2)\times\\ \times(b\,c-1-b)^3\,(b\,c-c-2\,b)^3=0.\end{gathered}\tag{2.2}$$

Some multiplicands in (2.1) are raised to the third power. But the exponents do not really affect the vanishing condition (2.2). Therefore we reduce it to

$$\begin{gathered}(b^2\,c^4-6\,b^2\,c^3+13\,b^2\,c^2-12\,b^2\,c+4\,b^2+c^2)\times\\ \times(b\,c-1-b)\,(b\,c-c-2\,b)=0.\end{gathered}\tag{2.3}$$

The left hand side of the formula (2.3) is the reduced common denominator of the fractions (1.4) through (1.12). It is broken into the product of three terms. Therefore singular points of the inverse problems 1.1 and 1.2 are subdivided into three singularity subvarieties.

3. The first singularity subvariety.

The first subvariety of singular points is the most simple. It is determined by the most simple factor in (2.2) through the following equation:

$$b\,c-1-b=0. \tag{3.1}$$

The equation (3.1) is linear with respect to both b and c. Resolving it for b, we get

$$b=\frac{1}{c-1},\ \text{ where }\ c\neq 1. \tag{3.2}$$

The formula (3.2) means that the first subvariety of singular points of the inverse problems 1.1 and 1.2 is a rational curve birationally equivalent to a straight line.

4. The second singularity subvariety.

The second subvariety of singular points is similar to the first one and is also very simple. It is determined by the following equation:

$$b\,c-c-2\,b=0. \tag{4.1}$$

Like in (3.2), resolving the equation (4.1) with respect to b, we get

$$b=\frac{c}{c-2},\ \text{ where }\ c\neq 2. \tag{4.2}$$

The formula (4.2) means that the second subvariety of singular points of the inverse problems 1.1 and 1.2 is a rational curve birationally equivalent to a straight line.

5. The third singularity subvariety.

The third subvariety of singular points is different from the first two. It is determined by the following equation which is quadratic with respect to b:

$$b^2\,c^4-6\,b^2\,c^3+13\,b^2\,c^2-12\,b^2\,c+4\,b^2+c^2=0. \tag{5.1}$$

The discriminant of the quadratic equation (5.1) with respect to b is

$$D=-4\,(c-1)^2\,(c-2)^2\,c^2. \tag{5.2}$$

Looking at the discriminant formula (5.2), we see that we can expect to find rational solutions only in one of the following three cases:

$$c=0, \qquad\qquad c=1, \qquad\qquad c=2. \tag{5.3}$$

Substituting $c=0$ into the formula (5.1) we derive $4\,b^2=0$. This yields one rational point at the origin ob the (b,c) plane:

$$b=0, \qquad\qquad c=0. \tag{5.4}$$

Substituting $c = 1$ and $c = 2$ into (5.1), we get two equalities $1 = 0$ and $4 = 0$ which are contradictory. Thus two of the three options in (5.3) do not actually produce any rational solutions for the equation (5.1). This result is not surprising since the equation (5.1) turns out to be reducible to $(c-1)^2\,(c-2)^2\,b^2 + c^2 = 0$.

6. Conclusions.

Despite being a fourth order equation in c, the equation (5.1) does not produce elliptic curves. Therefore the structure of the set of singular points of the inverse problems 1.1 and 1.2 is very simple. It comprises one isolated point (5.4) and two parametric subsets given by the formulas (3.2) and (4.2). This fact can be useful in computerized search for a solution of at least one of the problems 1.1 and 1.2, or in solving both of them if somehow it will be proved that the number of perfect cuboids is finite and they are in a certain range.

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