

CUTTING AND SHUFFLING A LINE SEGMENT: MIXING BY INTERVAL EXCHANGE TRANSFORMATIONS

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We present a computational study of finite-time mixing of a line segment by cutting and shuffling. A family of one-dimensional interval exchange transformations is constructed as a model system in which to study these types of mixing processes. Illustrative examples of the mixing behaviors, including pathological cases that violate the assumptions of the known governing theorems and lead to poor mixing, are shown. Since the mathematical theory applies as the number of iterations of the map goes to infinity, we introduce practical measures of mixing (the percent unmixed and the number of intermaterial interfaces) that can be computed over given (finite) numbers of iterations. We find that good mixing can be achieved after a finite number of iterations of a one-dimensional cutting and shuffling map, even though such a map cannot be considered chaotic in the usual sense and/or it may not fulfill the conditions of the ergodic theorems for interval exchange transformations. Specifically, good shuffling can occur with only six or seven intervals of roughly the same length, as long as the rearrangement order is an irreducible permutation. This study has implications for a number of mixing processes in which discontinuities arise either by construction or due to the underlying physics.

Keywords: Cutting and shuffling; piecewise isometries; granular mixing; lamellar structures.

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1. Introduction

The precise “symptoms” expressed by a chaotic dynamical system have been the subject of much discussion over the years [Brown & Chua, 1996, 1998; Martelli *et al.*, 1998; Brown & Chua, 1999]. A positive leading Lyapunov exponent, computed analytically or from a time series, is one common requirement [Baker & Gollub, 1996, Chap. 5]. Recently, piecewise-continuous and discontinuous dynamical systems have provided examples and counterexamples to the “typical” notions associated with chaos, often leading to the development of new types of bifurcation phenomena [di Bernardo *et al.*, 2008], and even the possibility of *nondeterministic chaos* [Jeffrey, 2011]. An overview of the role of discontinuities in mixing, illustrated by recent examples from the literature, is provided by Sturman [2012].

Piecewise isometries (PWIs) [Goetz, 2000, 2002] are one of the most recent examples of a discontinuous dynamical system exhibiting nontrivial dynamics. Under some specific conditions, it can be shown that the topological entropy [Buzzi, 2001] and all of the Lyapunov exponents [Fu & Duan, 2008] of a PWI are equal to zero (wherever and whenever they exist). A smooth distance-preserving map does not lead to complex dynamics in any sense of the word, but “gluing” together different isometries in a piecewise manner across a line of discontinuity leads to highly nontrivial behavior of orbits. The resulting complex dynamics include, but are not limited to, attractors and quasi-periodicity [Goetz, 1998; Scott *et al.*, 2001; Kahng, 2002; Mendes & Nicol, 2004; Ashwin & Goetz, 2005; Lowenstein & Vivaldi, 2010]. In fact, Kahng [2009a,b] has shown that certain PWIs satisfy a modification of Devaney’s criterion for chaos [Devaney, 2003, §1.8]. A dynamical system is Devaney chaotic if it is (i) sensitive to initial conditions, (ii) topologically transitive and (iii) has a set of periodic orbits that is dense in the domain; planar PWIs satisfy criteria (i) and (ii).

PWIs also arise in the study of digital filters [Chua & Lin, 1988; Ogorzałek, 1992]. If one allows for the more general class of piecewise *contraction* (non-area-preserving) maps, there are further applications to electrical engineering [Deane, 2006]. More recently, PWI maps have been proposed as the skeleton of the kinematics of the flow of granular materials in tumblers [Christov *et al.*, 2010b,a; Juarez *et al.*, 2010, 2012]. The PWI dynamics in granular mixing are termed *cutting and shuffling* to emphasize their fundamental difference from the well-known *stretching and folding* mechanism of chaotic fluid mixing [Ottino, 1989]. Some illustrated examples of these differences are provided by Christov *et al.* [2011].

Laboratory experiments have shown [Juarez *et al.*, 2010, 2012] that the mixing properties of granular flows can be predicted based on the underlying PWI framework. However, it is also worthwhile to consider simpler PWI-type mixing problems. The one-dimensional (1D) version of a PWI is an *interval exchange transformation* (IET) [Goetz, 1998]. In a 1D context, cutting and shuffling takes a particularly intuitive form, well known to card players [Golomb, 1961; Aldous & Diaconis, 1986; Trefethen & Trefethen, 2000]. Recent work by Liang & West [2008] showing that 2D chaotic fluid mixing can exhibit *cutoffs* (i.e., abrupt transitions from partially-mixed to well-mixed states) usually associated with card shuffling [Aldous & Diaconis, 1986; Trefethen & Trefethen, 2000] provides further evidence for the relevance of 1D cutting and shuffling. IETs, which replace the deck of cards with a set of subsegments of variable lengths, are one of the simplest dynamical systems that can be successfully studied analytically in some detail [Katok & Hasselblatt, 1995, §14.5]. More striking, however, is that IETs describe the dynamics along lines of discontinuity in some planar PWIs [Kahng, 2009b].

To illustrate mixing by cutting and shuffling in the context of granular flow, consider a spherical device partially-filled with two initially separate colors of granular material in which the mixing dynamics due to short rotations about two axes can be observed from below [Meier *et al.*, 2007; Sturman *et al.*, 2008; Juarez *et al.*, 2010]. Figure 1(a) shows a bottom view of a PWI simulation of such a setup after several iterations of these rotations. As the sphere rotates about an axis, material moves from one side of the surface in Fig. 1(a) to the other, and re-enters the side it came from upon reaching the dashed edge. For the PWI framework to be valid, we must first assume that the re-entry occurs instantaneously [Christov *et al.*, 2010b; Juarez *et al.*, 2010], i.e., there is a discontinuous mapping from one side of the dotted circle in Fig. 1(a) to the other across the axis of rotation (not shown and arbitrary). IET-like dynamics are evident if we “unwrap” the boundary of the surface of discontinuity shown approximately by the dotted curve encircling the mixing pattern in Fig. 1(a). The dynamics occurring on this interface are the source of complexity in

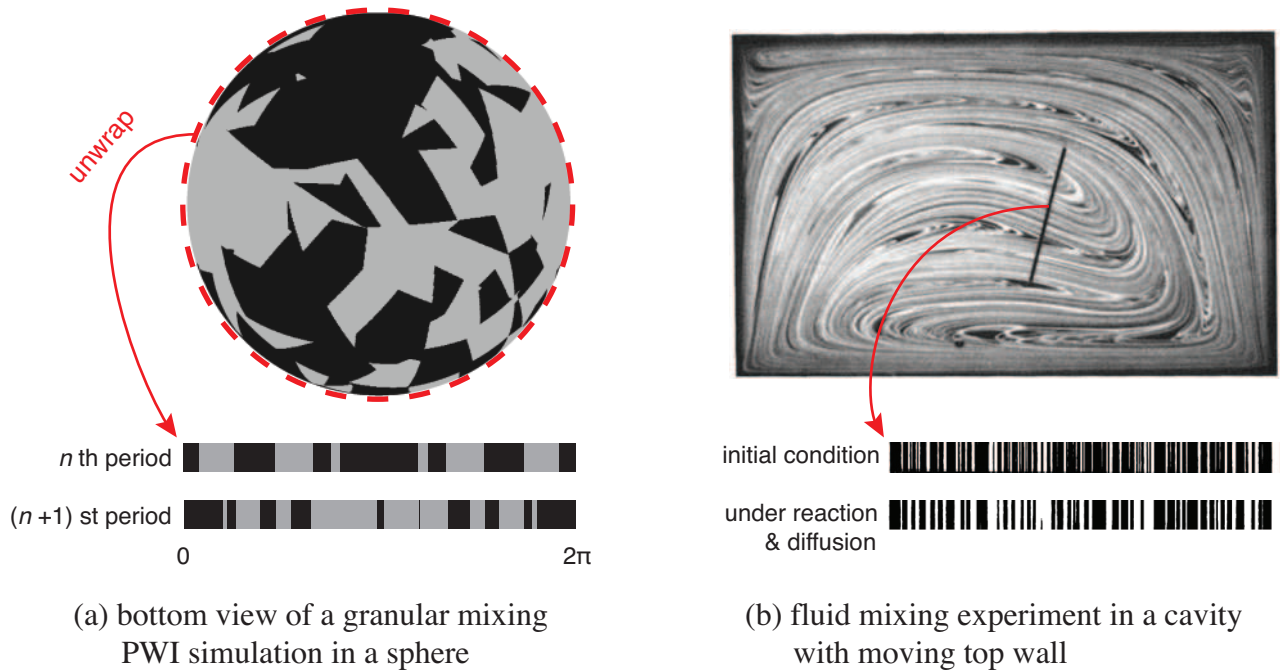


Fig. 1. (a) View from below of a PWI granular mixing simulation in a “blinking spherical tumbler flow” [Juarez *et al.*, 2010, 2012]. Initially, the black and gray materials occupy the left and right quarter-spheres, respectively. Inset shows the cutting and shuffling dynamics occurring at the discontinuity interface (dotted curve). Shuffling occurs upon rotation about an axis, while cutting results from changing the axis of rotation. (b) Photograph showing typical structures produced by chaotic mixing in a cavity flow experiment using two fluids of similar viscosity and negligible interfacial tension. A lamellar structure with striations of distributed thickness is generated by the chaotic flow. Inset shows a cross-section, which is the starting point for a 1D model of the evolution of the lamellar structure. The strips in the insets in panels (a) and (b) have a finite vertical extent for clarity of presentation. Images in panel (b) reproduced, with permission, from F.J. Muzzio & J.M. Ottino, *Phys. Rev. Lett.*, **63**, 1989, 47–50. © 1989 American Physical Society.

the PWI model of granular mixing in a spherical tumbler [Christov, 2011, §8.9]. Thus, an understanding of the behavior of such IETs can shed light on the more complicated PWI dynamics in a plane or on a sphere. Similar discontinuous changes in the flow also occur in a number of chaotic fluid mixing problems such as the periodic reorientation of streamlines in a circular Hele-Shaw cell, which has been used as a model for injection wells in porous geological formations [Metcalfe *et al.*, 2010]. A detailed discussion of how such flows fit into the PWI framework is given by Sturman [2012].

The concept of intermaterial area is central in the theory of chaotic mixing [Ottino, 1989]. A cross-section of the mixing process reveals a pattern of pieces of material of different types and varying thickness. Stretching and folding motions create fine lamellar structures, such as those seen in Fig. 1(b), across which reaction and diffusion processes act. The distribution of the thickness of lamella along a 1D slice of the domain (e.g., across the black line in Fig. 1(b)) has been studied, showing universal properties of the evolution of the distribution toward a steady state [Muzzio & Ottino, 1989]. One-dimensional models of the lamellar dynamics are useful, because precise numerical tracking of the intermaterial area in a chaotic flow, even in 2D, is a difficult (if not impossible) task [Franjione & Ottino, 1987]. What is more, the effective 1D dynamics across a cross-section (as in Fig. 1(b)) can be thought of as an IET with molecular diffusion blurring the boundaries between subsegments before they are permuted. Different models of “cutting” can lead to significant enhancement over a purely diffusive mixing process [Ottino *et al.*, 1992; Ashwin *et al.*, 2002]. In an industrial setting, various designs of mixers have been proposed that create such lamellar structures in a controlled and reproducible fashion [Hobbs & Muzzio, 1997; Neerincx *et al.*, 2011]. In such a device, the speed of a two-step chemical reaction depends sensitively on the arrangement and thickness of the lamella [Clifford *et al.*, 1999, 2000]. More importantly, in multiple reactions, the final product distribution depends on the dynamics of mixing [Chella & Ottino, 1984]. Therefore, a better

understanding of the practical details of IETs can lead to improvements in the design of devices that must mix materials or allow them to react completely in a precise manner.

While the mathematical theory of IETs is mature (see, e.g., the recent review of Viana [2006]), few computational examples and parameter space studies can be found in the literature. Moreover, theorems focus on *infinite-time* concepts such as ergodicity, weak mixing and Lyapunov exponents. It is apparent, however, that infinite-time concepts, while very useful, do not fully address the questions that arise in practice. Mixing experiments and chemical reactions take place over a *finite* period of time. For mixing of granular materials, the diameter d of a typical particle defines a “cut off” length scale beyond which further mixing is impossible because smaller-scale structures cannot be produced. Similarly, in fluid mixing the diffusion length scale $\sim \sqrt{Dt}$, where D is the molecular diffusivity and t is time, sets a lower-bound beyond which it is not worthwhile to continue creating finer lamellar structures [Toussaint & Carrière, 1999]. Therefore, questions such as “How many iterations does it take to reduce longest continuous segment in an IET to a given fraction of the interval?” cannot be answered by current theory on IETs.

The only previous detailed numerical study of mixing of a line segment is that of Ashwin *et al.* [2002], who only consider protocols based on exchanges of equal-length subsegments, which do not necessarily mix. Ashwin *et al.* [2002] noted that “mixing properties of interval exchange maps are very subtle and relatively poorly understood and depend on parameters in a sensitive way.” Thus, the goal of the present work is to extend the latter study to mixing by cutting and shuffling within the framework of the ergodic theory developed for interval exchange transformations (IETs). Specifically, we aim to develop illustrative examples of the behavior of 1D cutting and shuffling maps in order to clarify how the pathological cases that lead to poor mixing (e.g., those that violate the known IET theorems). We shall do so in the context of finite-time transport, characterizing the mixing properties of those maps over a given (fixed) number of iterations. The proofs of the ergodic and mixing theorems for IETs are difficult, and similar questions for PWIs remain open. As a result, computational studies have recently proven to be an important part of the mathematical discovery process in this field [Bressaud & Poggiaspalla, 2007].

This paper is organized as follows. Section 2 introduces our IET construction and defines the parameter space to be explored. Section 3 summarizes the relevant previous mathematical results regarding IETs, their complexity and their mathematical mixing properties. In Sec. 4, we present practical numerical results on mixing by 1D cutting and shuffling, and connect the behaviors to the mathematical theory from Sec. 3. Section 5 summarizes the most important results of our study and presents our conclusions.

2. IET construction and simulation methodology

To study 1D cutting and shuffling, i.e., the “mixing” of a line segment \mathcal{I} , we construct a specific type of IET, which we denote as $T_{S,\Pi}: \mathcal{I} \rightarrow \mathcal{I}$, with a well-defined parameter space that we can systematically explore. To this end, without loss of generality, we take the line segment to be an interval of the real line, specifically $\mathcal{I} = [0, 1]$. Then, cutting and shuffling proceeds by dividing \mathcal{I} into a collection of N disjoint subsegments given by the set $S = \{\mathcal{I}_1, \dots, \mathcal{I}_N\}$. Specifically, the interval is divided into N subsegments of length $|\mathcal{I}_i|$ and the ratio $r_i = |\mathcal{I}_i|/|\mathcal{I}_{i-1}|$ between the lengths of consecutive subsegments is taken to be constant (i.e., $r_i = r \geq 1 \forall i$), as shown in Fig. 2. Now, we specify r and require the length of the interval \mathcal{I} to be 1, which gives

$$\sum_{i=1}^N r^{i-1} x = 1 \quad \Rightarrow \quad x = \frac{r-1}{r^N-1}, \quad (1)$$

where $x = |\mathcal{I}_1|$ is the primary interval length.

To visualize the process of cutting and shuffling, the initial subsegments (after \mathcal{I} is first subdivided into the set S) are each assigned a different color, as shown in Fig. 2. One application of the IET $T_{S,\Pi}$ shuffles the subsegments according to the prescribed rearrangement order given by the permutation Π . The subsegments after the initial subdivision of \mathcal{I} are consecutively numbered from left to right, as shown in the top row of Fig. 3. Hence, the rearrangement order defines a one-to-one mapping from the set $\{1, 2, \dots, N\}$ into itself. The image of $\{1, 2, \dots, N\}$ under Π is the order of the subsegments after one iteration of the cutting and shuffling process, i.e., after one application of the IET. For example, in Fig. 3, the rearrangement

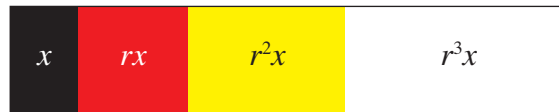


Fig. 2. (Color online.) Sketch of how an interval \mathcal{I} is subdivided in our IET construction with $N = 4$ and $r = 1.5$.

order is given by $\Pi([1234]) = [3142]$, where the bracket notation (rather than set notation) is used for the permutation to emphasize that the order of both the input and output integers is important and fixed.

After the initial subdivision and shuffling of \mathcal{I} , the line segment is reassembled, then it is cut once again at the same locations, and the process continues. The result of a second application of the IET is shown in the bottom row of Fig. 3. The cutting and shuffling process continues until either a certain number of iterations have been completed, or the line segment is “mixed” to a predetermined level, pending a proper definition of “mixed.” Details of the numerical procedure used to simulate cutting and shuffling of a line segment are provided in Appendix A.

To quantify the degree of mixing after n iterations, we use two measures: the percent unmixed U_n and number of distinct cuts (i.e., interfaces) C_n . The percent unmixed is the longest continuous subsegment of like color, even if that subsegment comes about as a result of two like-color subsegments having been recombined. Since the length of \mathcal{I} was normalized to 1, the length of longest piece of material of like color corresponds to the largest percent of the material that is “unmixed.” Ideally, $U_n \rightarrow 0$ as $n \rightarrow \infty$ would imply complete mixing. For example, in Fig. 3, the central unmixed portion after $n = 2$ iterations is the longer white interval, giving $U_2 = 23\%$.

The number of interfaces C_n , on the other hand, is the number of locations where subsegments of different initial color are in contact. Locations where pieces of subsegments of like color reassemble are not included in the calculation of C_n , even though a cut was formally made there at some previous iteration. In contrast to U_n , a larger value of C_n corresponds to better mixing because many pieces of differing color are in contact. In Fig. 3, after 2 iterations, we have $C_2 = 6$.

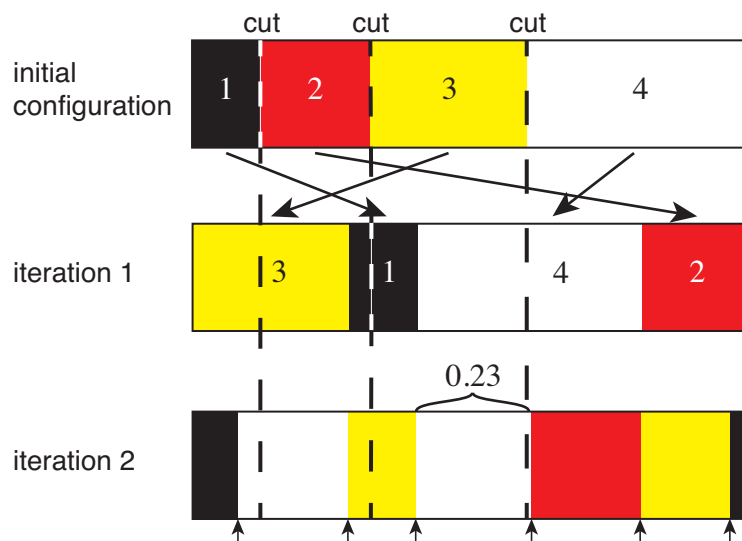


Fig. 3. (Color online.) Illustration of how the first two iterations of our IET construction with $N = 4$, $r = 1.5$ and $\Pi([1234]) = [3142]$ proceed. After the second iteration, the maximum percent unmixed (corresponding to the subsegment delineated by the curly brace in the bottom row) is $U_2 = 23\%$, and number the number of interfaces (denoted by arrows at the bottom) is $C_2 = 6$.

A number of alternative mathematical definitions of the quality of mixing can found in the literature (see, e.g., [Ashwin *et al.*, 2002, §1.2]). Recently, mathematically-rigorous measures of mixing capable of quantifying the “multiscale” nature of the self-similar lamellar structures developed by stretching and

folding in chaotic flows [Muzzio *et al.*, 1992] have been developed using the tools of dynamical systems [Mathew *et al.*, 2005] and partial differential equations [Thiffeault, 2012]. For our purposes, however, U_n and C_n are easy to compute, require few assumptions, are conceptually simple, and have concrete physical meaning. Thus, we use them to quantify the complexity generated by cutting and shuffling a line segment.

3. Summary of mathematical results on IETs

In this section, we review the pertinent previous mathematical results that characterise IETs to aid in the interpretation of our computational results. Specifically, we introduce the necessary terminology and the key theorems applicable to our study.

An in-depth discussion of the *ergodic hierarchy* that defines “levels” of mixing using mathematically-precise concepts from dynamical systems theory can be found in [Sturman *et al.*, 2006, §3.7] and [Berkovitz *et al.*, 2006]. An elementary discussion in relation to IETs and PWIs can be found in [Christov *et al.*, 2011, §V]. Keane’s minimality condition [Keane, 1975, §2] is an important result in this respect as it guarantees (unique) ergodicity, the “weakest” form of mixing, of (almost all) IETs [Masur, 1982; Veech, 1982]. The “almost all” part the latter result is strict as non-trivial IETs that are minimal but not (uniquely) ergodic have been constructed [Keane, 1977]. We give a short summary of the background on these results through some equivalent and easy-to-state theorems due to Viana [2006], after providing some necessary definitions.

Definition 3.1. A permutation Π is said to be *irreducible* if applying Π to any of the subsets $\{1\}$, $\{1, 2\}$, $\{1, 2, 3\}$ up to $\{1, 2, \dots, N - 1\}$ does not yield a permutation of just the elements of *that* subset.

By $\Pi(i)$ we denote the value that the integer i is mapped to by the permutation, while the compacted notation $\Pi([123\dots])$ is used to show (all at once) the image under the action of Π of the integers $\{1, 2, 3, \dots\}$. For example, the permutation $\Pi([12345]) = [31254]$ is reducible (i.e., *not irreducible*) because the first three elements (and the last two elements) are a permutation of only themselves (neither 1, 2, nor 3 maps to 4 or 5). In contrast, $\Pi([12345]) = [31524]$ is *irreducible*. See Appendix B for further details.

Definition 3.2. A permutation Π of $\{1, 2, \dots, N\}$ is said to be a rotation if $\Pi(i + 1) \equiv \Pi(i) + 1 \pmod{N}$.

For example, it easy to show that $\Pi([12345]) = [34512]$ satisfies this definition, hence it is a rotation. On the other hand, $\Pi([12345]) = [54312]$ violates the definition for $i = 1$ because 2 maps to $\Pi(2) = 4$ while 1 maps to $\Pi(1) = 5$ and $4 \not\equiv 5 + 1 \pmod{5} = 1$, hence $\Pi([12345]) = [54312]$ is not a rotation.

Theorem 1 [Viana, 2006, Proposition 3.2]. *If the lengths of the subsegments $\mathcal{I}_i \in S$ are rationally independent and Π is irreducible, then S and Π satisfy the Keane minimality condition.*

For example, to ensure that the lengths of subsegments $\mathcal{I}_i \in S$ are rationally independent, we could require that the lengths of adjacent intervals $r_i = |\mathcal{I}_i|/|\mathcal{I}_{i-1}|$ be irrational numbers. Here, we will not state Keane’s condition in its original form [Keane, 1975, §2] because Theorem 1 is more accessible. Furthermore, Theorems 2 and 3 below characterize all IETs satisfying Keane’s condition, which allows us to bypass its cumbersome mathematical definition.

Theorem 2 [Viana, 2006, Proposition 4.1]. *If S and Π satisfy the Keane condition, then every orbit of $T_{S,\Pi}$ is dense in the whole domain; such an IET is termed minimal.*

Here, by “orbit” we mean the set of points $\{y, T_{S,\Pi}(y), T_{S,\Pi}^2(y), \dots, T_{S,\Pi}^n(y), \dots\}$ for a given $y \in \mathcal{I}$. An orbit is dense in the domain if, for some n , it comes arbitrarily close to any given point in the domain.

Theorem 3 [Viana, 2006, Lemma 4.4]. *If S and Π satisfy the Keane condition, then $T_{S,\Pi}$ has no periodic points.*

Thus, if an IET satisfies the conditions of Theorem 1, it satisfies the Keane condition, meaning (by Theorems 2 and 3) that “all its orbits visit all of the domain” and, specifically, none are periodic. This would suggest good mixing because material is distributed throughout all of the interval. However, we could be more precise if we consider the map’s behavior in the limit as the number of iterations goes to infinity.

Let $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{I}$ be *any* two subintervals of \mathcal{I} , where $\mathcal{I} = [0, 1]$ in our construction. (These subintervals are arbitrary and do not have to coincide with the subsegments \mathcal{I}_i .) Denoting the cutting and shuffling process (i.e., the IET $T_{S,\Pi}$) simply by T , the condition for *strong* mixing can be stated mathematically as

$$\lim_{n \rightarrow \infty} \mu(T^n(\mathcal{A}_1) \cap \mathcal{A}_2) = \mu(\mathcal{A}_1)\mu(\mathcal{A}_2), \quad (2)$$

where $\mu(\cdot)$ is the appropriate *invariant measure*, e.g., length in 1D, and $T^n(\mathcal{A}_1) \cap \mathcal{A}_2$ denotes the material in common between $T^n(\mathcal{A}_1)$ and \mathcal{A}_2 . In other words, as any subinterval \mathcal{A}_1 is cut and shuffled throughout the domain \mathcal{I} , we would expect to find the same amount of material from \mathcal{A}_1 in any other subinterval \mathcal{A}_2 .

A cutting and shuffling map based on an IET cannot be strongly mixing [Katok, 1980]. However, the following can be shown.

Theorem 4 [Avila & Forni, 2007, Theorem A]. *Let Π be an irreducible permutation that is not a rotation. Then, for almost every S , $T_{S,\Pi}$ is weakly mixing, i.e.,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} |\mu(T^i(\mathcal{A}_1) \cap \mathcal{A}_2) - \mu(\mathcal{A}_1)\mu(\mathcal{A}_2)| = 0. \quad (3)$$

Put simply, Theorem 4 relaxes the requirement that the same amount of material from the subinterval \mathcal{A}_1 be found in any other subinterval \mathcal{A}_2 , to the requirement that Eq. (2) hold except for a countable number of $n \in \mathbb{N}$ [Walters, 1982, p. 45]. Still, weak mixing is a stronger result than the ergodicity property mentioned previously that an IET acquires by satisfying Keane’s minimality condition. Again, the “almost all” requirement is strict as there are non-trivial minimal IETs that are (uniquely) ergodic but not weakly mixing [Hmili, 2010]. The situation becomes much more complicated in 2D, where only a few results are known for *rectangle exchange transformations* [Haller, 1981] and almost none for PWIs. However, there is at least one example, due to Kahng [2002], of a PWI that is *not* weakly-mixing.

The last mathematical result we review concerns the growth of the number of interfaces C_n with n . It is characterized by the following theorem due to Novak [2009].

Theorem 5 [Novak, 2009, Theorem 1.1]. *For any interval exchange transformation $T_{S,\Pi}$, either $C_n(T)$ exhibits linear growth in n or $C_n(T)$ is bounded above independently of n .*

In the case when C_n exhibits linear growth, the slope is given by the number of *nonresolving fundamental discontinuities* [Novak, 2009, Proposition 2.3], i.e., those cuts at which two like colors are never “glued back together,” as $n \rightarrow \infty$. Unlike chaotic dynamical systems in which any analog of C_n grows exponentially with n , quickly leading to an intractable numerical problem in 2D or 3D [Franjione & Ottino, 1987], Theorem 5 ensures that keeping track of the interfaces generated by an IET is feasible. Finally, it is important to note that the dichotomy of the growth of C_n does not correspond directly to the dichotomy of whether $T_{S,\Pi}$ satisfies Keane’s condition or not. We explore the relationship between the two through the numerical examples below.

4. Numerical results

Given the simple system of an interval divided into subsegments that we constructed in Section 2, we can set about to explore the relationship between the mathematical results on IETs summarized in Section 3 and mixing by cutting and shuffling. For example, a natural question to ask is what happens when Keane’s condition is violated? Finite-precision arithmetic ensures that, on a computer, there are no “truly irrational” choices for ratio r between adjacent subsegment lengths. We would like to determine how this affects the mixing properties of IETs. In this section, we investigate several examples of IETs constructed as described in Sec. 2 with the goal of understanding the impact of the theorems from Sec. 3 on finite-time mixing by cutting and shuffling. The latter is an aspect of IETs that is not covered by the mathematical theory. To make the computational analysis easier, we restrict ourselves to a relatively small number of subsegments $N \in \{4, 5, 6, 7\}$ and a ratio between subsegment lengths $r \in [1, 2.5]$.

To visually analyze the impact of the various parameters on the mixing properties of our IET, we construct space-time plots of the cut and shuffled line segment \mathcal{I} as shown in Fig. 4. In such space-time plots, the initial subdivision of \mathcal{I} into N differently-colored subsegments is at the top. The cut and shuffled versions of \mathcal{I} resulting from subsequent applications of the IET, i.e., $T^n(\mathcal{I})$, are stacked below the initial condition. A small number of pieces of different colors at the bottom of a space-time plot indicates poor mixing. This is one of the most useful aspects of the space-time plot: repeating patterns in the graph correlate with poor mixing. For instance, in Fig. 4(a), periodic dynamics, which leads to poor mixing, are immediately evident. An IET that can produce significant mixing, on the other hand, leads to the bottom row of the space-time plot having many subsegments of different colors, as shown in Fig. 4(b).

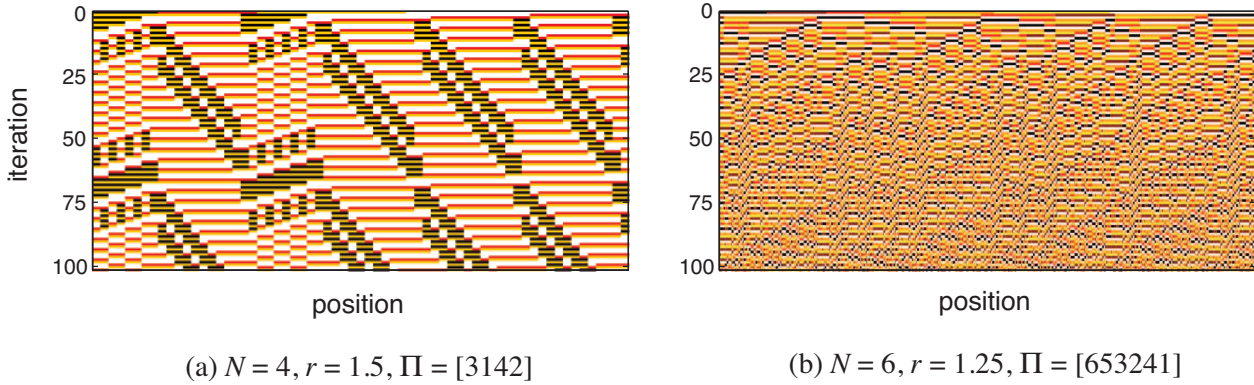


Fig. 4. (Color online.) Space-time plots over 100 iterations of our IET-based cutting and shuffling map for two different combinations of parameters. (a) Periodic dynamics and poor mixing result from choosing $N = 4$, $r = 1.5$ and $\Pi([1234]) = [3142]$. (b) Substantial mixing is observed for $N = 6$, $r = 1.25$ and $\Pi([123456]) = [653241]$.

It is worthwhile comparing space-time diagrams to *Poincaré sections*, a common diagnostic of mixing. A Poincaré section consists of the collection of locations of “tracers” after each application of a map representing the action of, say, one period of a flow. When each tracer is assigned a different color, the Poincaré section takes a particularly visually appealing form (see, e.g., [Ottino, 1989, Chap. 6]). In this one-dimensional context, however, the “tracers” can be thought of as points distributed along a line segment, filling it densely in the case of good mixing, or leaving gaps (depending on their seed locations) in the case of poor mixing. Constructing a space-time diagram as in Fig. 4 is a clearer way of visualizing the same information: any row represents a mixing pattern that can be compared to a previous or later iteration showing where colors, which correspond to different initial locations, are distributed under the action of cutting and shuffling.

4.1. Effect of rearrangement order

The choice of the subsegment rearrangement order, represented by the permutation Π , obviously affects the quality of mixing as made precise by the theorems in Sec. 3. Permutations that are reducible, as described in Definition 3.1, result in poor mixing and, often, periodic dynamics. Figure 5 illustrates this behavior for three choices of Π with N and r fixed.

The simplest reducible permutations are those for which either $\Pi(1) = 1$ (meaning the first subsegment remains in place throughout the cutting and shuffling process) or $\Pi(N) = N$ (meaning the last subsegment remains in place, as shown in Fig. 5(a)). In either case, the percent unmixed U_n is equal (or becomes equal at some $n > 1$) to the length of this fixed subsegment; no further reduction in U_n can be achieved. Figure 5(b), on the other hand, displays a different type of reducible rearrangement order in which subsegments 1, 2 and 3, 4 are exchanged pairwise. Again, U_n does not tend to zero during the cutting and shuffling process. Additionally, the number of interfaces C_n is easily seen to remain between 3 and 5. This is, of course, indicative of poor mixing. Permutations that are rotations, as described in Definition 3.2, also result in poor mixing, though they do not explicitly violate Keane’s condition (our choice of r , however, does). Figure 4(c)

shows such a case, with the space-time plot clearly illustrating why this Π is termed a “rotation.” In each iteration the last two subsegments are shifted to the beginning of the line segment, resulting in periodic dynamics and poor mixing.

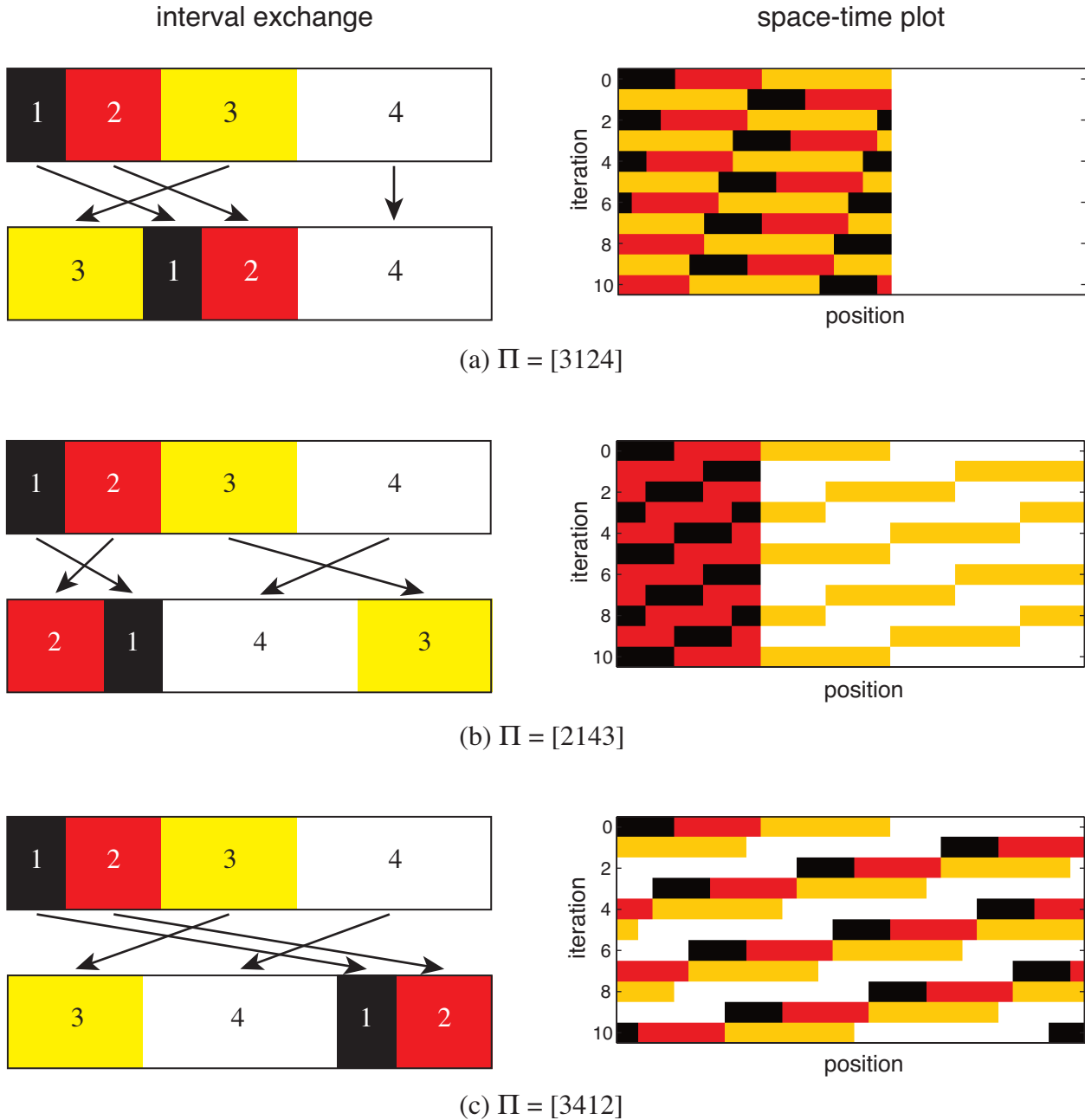


Fig. 5. (Color online.) Cutting and shuffling of a line segment with rearrangement order given by reducible permutations: (a) $\Pi([1234]) = [3124]$ for which the last subsegment remains in place, (b) $\Pi([1234]) = [2143]$ in which the first and last pair of subsegments are exchanged pairwise and (c) $\Pi([1234]) = [3412]$ which is a rotation. In all three cases $N = 4$ and $r = 1.5$. On the right are space-time plots of the first 10 iterations of the cutting and shuffling process.

So far, as expected, we have shown that choices of Π and r that violate Keane’s condition lead to trivial dynamics and poor mixing. It is worthwhile to also consider cases when the permutation is *irreducible* (as required by Theorem 1) but r is still chosen to violate Keane’s condition. Now, as shown in Fig. 6, we find that a slight change in the rearrangement order can have a significant impact on the quality of mixing by

cutting and shuffling.

Having performed a number of numerical experiments, we find that five typical outcomes can be observed. First, as shown in Fig. 6(a), there can be significant mixing. The percent unmixed U_n decays to a small value quickly, and the number of interfaces C_n grows to a large value. However, there is a clear bound (lower for U_n , upper for C_n) on the quality of mixing that can be achieved because $r = 3/2$ is a rational number. By switching the location of only two subsegments in the rearrangement order, we observe a second kind of mixing behavior, shown in Fig. 6(b). Now, the quality of mixing remains poor

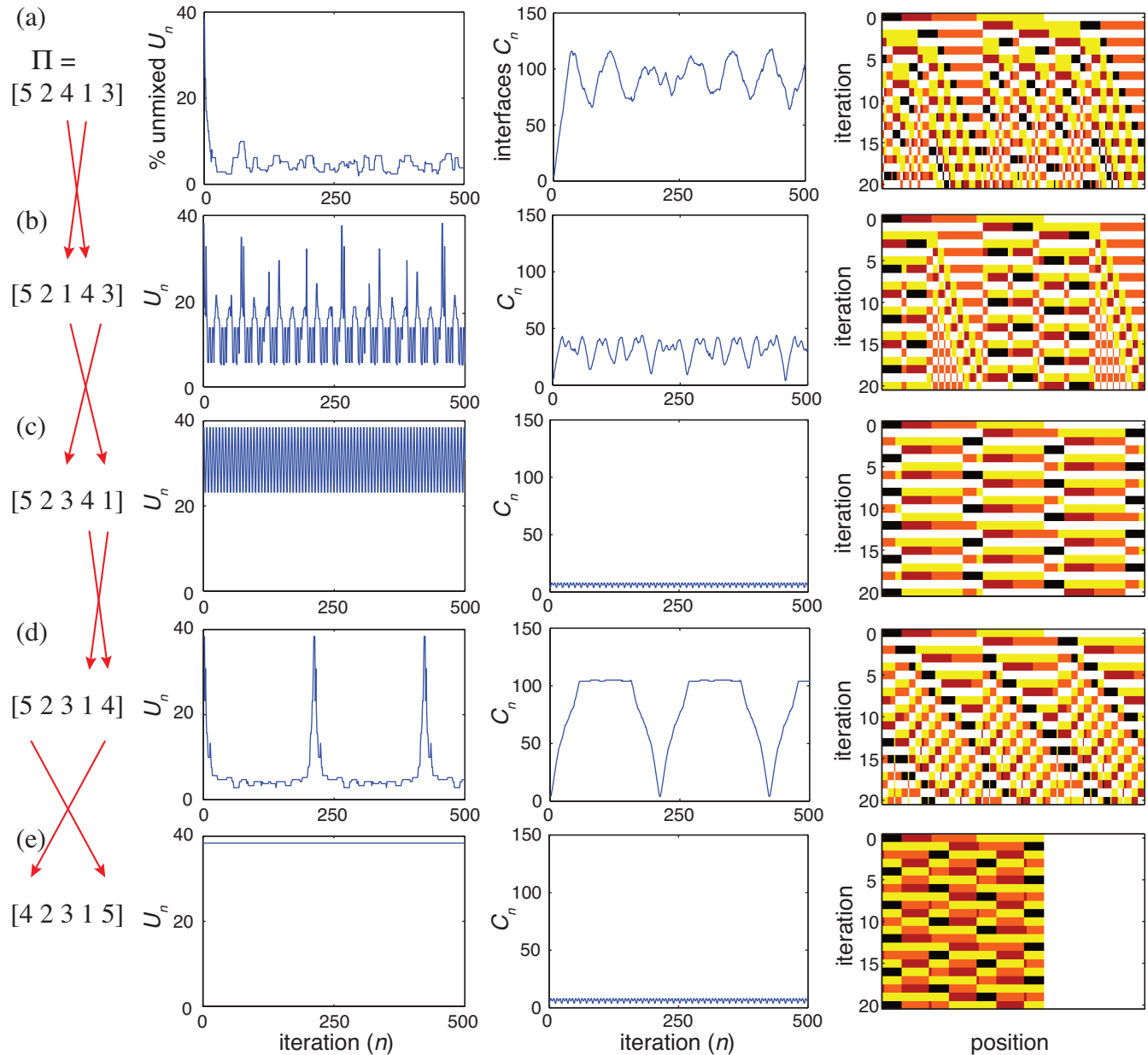


Fig. 6. (Color online.) When Keane's condition is violated by choosing $r = 1.5$ for $N = 5$, switching the locations of only 2 elements in the rearrangement order can have a significant impact on the mixing dynamics. There are five typical outcomes. (a) Significant mixing for $\Pi([12345]) = [52413]$. (b) U_n and C_n oscillate under the rearrangement order $= \Pi([1234]) = [52143]$. (c) Periodic dynamics result for $\Pi([1234]) = [52341]$ with U_n and C_n oscillating at a high frequency. (d) Significant mixing can be achieved but the dynamics are ultimately periodic (jumps in U_n , dips in C_n always occur after a fixed number of iterations) for $\Pi([12345]) = [52314]$. (e) U_n remains constant and C_n is bounded above by a small number under the reducible rearrangement order $\Pi([12345]) = [42315]$.

with large-amplitude oscillations present in U_n and C_n . This occurs because subsegments that are cut and shuffled can become reassembled at a later iteration. Upon switching another two elements in the rearrangement order, a third type of behavior is observed in Fig. 6(c): U_n and C_n now oscillate at a much higher frequency; U_n remains high and C_n remains low, indicating poor mixing. Again, this results because the subsegments are reassembling into their initial configuration every few iterations due to the periodic dynamics of this IET. The fourth type of behavior, produced by another small change in the rearrangement order, results in the dynamics shown in Fig. 6(d), where U_n and C_n now exhibit two distinct time scales of oscillation. One is at high frequency and with low amplitude (“local” changes every few iterations), the other at low frequency and with large amplitude (“global” reassemblies of the initial condition). At the iteration corresponding to a peak in U_n , the interval has reassembled into its original state. Though there is, apparently, significant mixing at certain n , periodic dynamics still exist because $r = 3/2$ is a rational number, which violates Keane’s condition. Finally, Fig. 6(e) presents the fifth type of behavior observed. In this case, the rearrangement order is a reducible permutation with $\Pi(N) = N$. As expected, there is very poor mixing; U_n remains constant (because the last, and longest, subsegment is never cut and shuffled) and C_n is bounded above by a small integer for all n .

4.2. Effect of interval length ratio

Both the interval length ratio r and the number of subsegments N significantly affect the quality of mixing by cutting and shuffling. In Fig. 7, the percent unmixed U_n (averaged over all irreducible permutations for a given N) is plotted as a function of the number of iterations n for several choices of r and N . As expected for such irreducible permutations, U_n decreases with n ; the greatest decrease occurs within the first 20 or so iterations. For larger N , the curves in Fig. 7 are systematically lower. This indicates that starting with more subsegments leads to faster mixing. Furthermore, the change in U_n resulting from adding an additional segment is greater when going from $N = 4$ to $N = 5$ than when going from $N = 5$ to $N = 6$ or from $N = 6$ to $N = 7$. This illustrates that there is a point of diminishing returns. The errors bars are larger for smaller N because there are more protocols that mix poorly in those cases. Thus, in practice, at least 6 or 7 subsegments are necessary to obtain significant mixing across a range of protocols, but starting with more subsegments than that has minimal impact on the overall mixing. With a larger number of subsegments, the precise rearrangement order chosen in the protocol has less impact than for smaller N , as long as the permutation is irreducible.

It is also clear from Fig. 7 that reducing r results in better mixing. To further analyze the impact of r , we plot U_n —averaged over all irreducible rearrangement orders for a given N and then averaged from iteration 50 to 100—as a function of r in Fig. 8(a). Clearly this average U_n increases as r increases, with a larger number of subsegments effecting better mixing. The upward trend in Fig. 8(a) can be understood by recalling that larger r leads to an increase in the length difference between adjacent initial subsegments. In particular, it is easy to show that the longest initial subsegment, which takes the most “effort” to cut and shuffle, has length

$$U_{n,\max}(r, N) = r^{N-1} \left(\frac{r-1}{r^N-1} \right). \quad (4)$$

We use the notation $U_{n,\max}$ because, by construction, the length of the longest initial subsegment of a given color is the upper bound on the percent unmixed for all n . Note that $U_{n,\max}$ is well behaved at $r = 1$, specifically $\lim_{r \rightarrow 1} U_{n,\max}(r, N) = 1/N$. Thus, for a fixed number of iterations (as in Fig. 8, where all averages are calculated from iteration 50 to 100), increasing r should increase the average percent unmixed. To account for this effect, in Fig. 8(b), we divide the average percent unmixed by $U_{n,\max}(r, N)$. Clearly, this normalization eliminates part of the upward trend. It is natural to interpret the normalized average U_n in Fig. 8(b) as a *mixing efficiency*, i.e., a measure of the average shortest segment length produced by cutting and shuffling compared to the initial longest segment length. On the other hand, from a practical standpoint, we want the unmixed portion to be as short as possible, leading to a preference for r being close, but not exactly equal, to one, as indicated in Fig. 8(a) and discussed shortly.

In addition, it is evident that certain ratios, e.g., $r = 3/2$, $2/1$, $5/2$ or $3/1$, lead to particularly poor mixing, resulting in peaks in Fig. 8, similar to the resonances leading to *Arnold tongues* in a chaotic

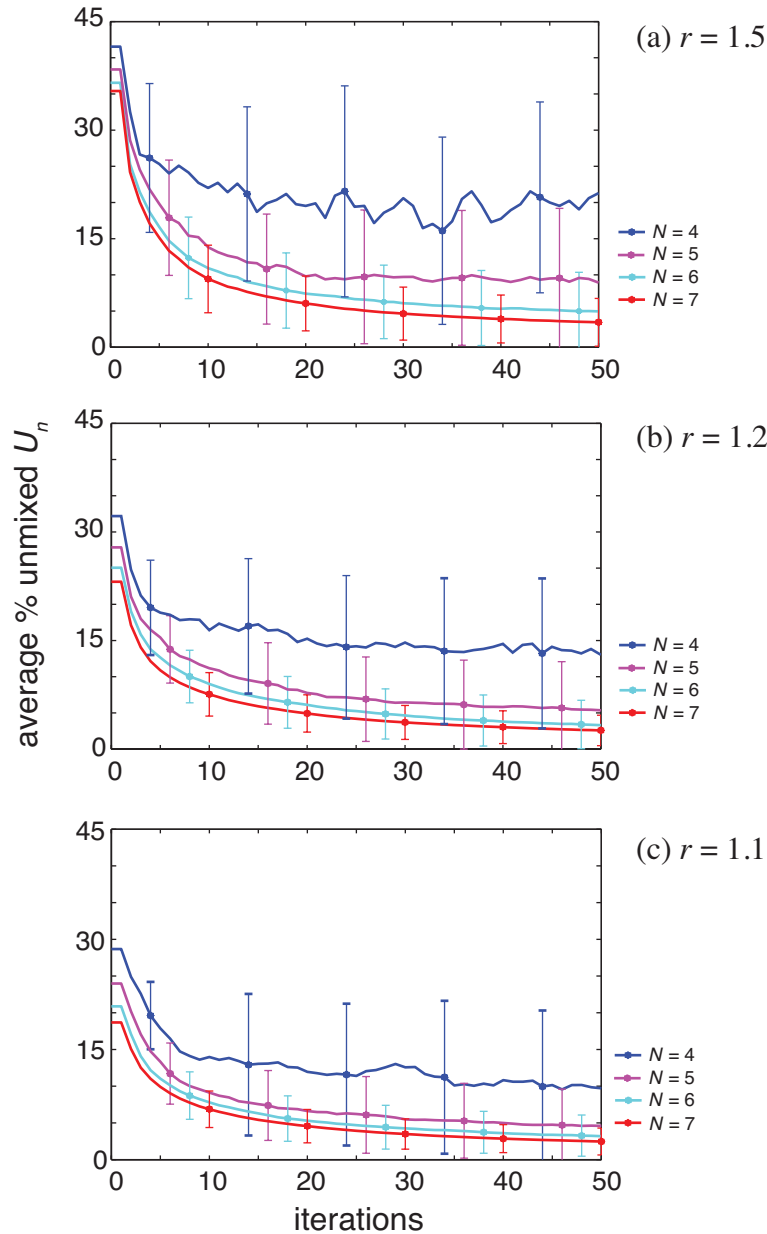


Fig. 7. (Color online.) Average (across all irreducible rearrangement orders) percent unmixed U_n for (a) $r = 1.5$ and $N = 4$ to 7, (b) $r = 1.2$ and $N = 4$ to 7, (c) $r = 1.1$ and $N = 4$ to 6. Error bars represent one standard deviation about the mean. $N = 6$ appears to be a point of diminishing returns, and r closer to 1 (a more uniform distribution of initial subsegment lengths) leads to better mixing on average.

dynamical system [Pikovsky & Rosenblum, 2007]. This corroborates the theoretical results from Sec. 3 that in order to get the best mixing possible (specifically, to eliminate periodic orbits and ensure ergodicity), r should be an irrational number. Furthermore, for the present IET construction, Fig. 8(a) suggests that an irrational number close to unity is preferable. The reason for this is the manner in which we define r . For r close to unity, the initial distribution of subsegment lengths is almost uniform, meaning no subsegment is very long compared to any other. The opposite case ($r \gg 1$) is undesirable because long subsegments can take many iterations to be cut and shuffled. After normalizing by $U_{n,\max}$ in Fig. 8(b), we observe that the mixing efficiency is still better for ratios closer to one. However, there is little difference in the normalized unmixed portion for $r \in [1, 2]$. It is quite evident that as r increases beyond 2, the second longest subsegment is only a small fraction of the longest subsegment. Hence, it takes many more iterations to

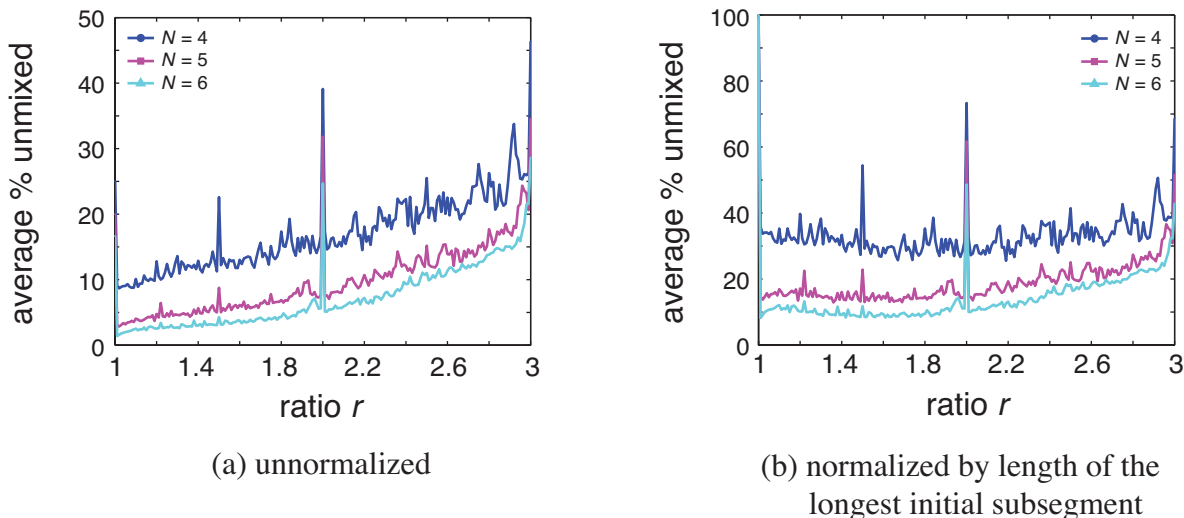


Fig. 8. (Color online.) Average (across iterations 50 to 100 *and* across all irreducible rearrangement orders) percent unmixed U_n versus the subsegment ratio r for $N = 4, 5$ and 6 . Arnold tongue-like phenomena are seen at the resonances $2/1, 3/1, 3/2, 5/2$, etc. The best mixing (on average) occurs for r close, but not equal, to 1 because the distribution of lengths of the initial subsegments is close to uniform. The fact that the length of the longest initial subsegment increases with r , meaning that (on average) more iterations are needed to reach the same percent unmixed (upward trend in (a)), can be accounted by normalizing with $U_{n,\max}$, as shown in (b).

significantly reduce the length of the longest subsegment. In the opposite extreme, as $r \rightarrow 1$, the difference in length between the longest subsegment and the shortest subsegment becomes small, which makes it possible to accomplish good mixing after an adequate number of iterations. Overall, we can conclude that the mixing quality and efficiency are improved (on average) by starting with a more uniform distribution of subsegment lengths.

4.3. Growth of the number of interfaces

As discussed earlier, an important aspect of the mixing properties of a cutting and shuffling process is the number of interfaces it generates between materials of different color. Such interfaces arise from the discontinuities in the cutting and shuffling map, specifically where the cuts are placed in the interval exchange transformation. For PWIs, the higher-dimensional analogue of IETs that arise in the context of granular mixing in a spherical container, it was shown that the discontinuities in the map are the sole source of intermaterial interfaces in the absence of the “usual” chaotic dynamics [Christov, 2011, §8.9]. The recent work by Novak [2009] on the growth of the number of discontinuities as $n \rightarrow \infty$, which we summarized in Sec. 3, is the only mathematical characterization of the set of interfaces. Beyond that, determining the growth of C_n is, in general, an open problem.

In this section, we aim to better understand the behavior of C_n by supplementing the mathematical theorems with numerical results. Specifically, we wish to determine what effect violating Keane’s condition has on the growth of C_n , and whether violating it to different “degrees” matters. To this end, we take N fixed and consider the two cases of a given reducible or a given irreducible rearrangement order Π . In each case, we vary the value of r . Ideally, r should be irrational to observe the asymptotic results of Novak [2009]. To approach this limit, we take r to be increasingly “more irrational” and compute the evolution of C_n for the given N and Π .

To make the idea of “more irrational” precise, we apply the concept of a *finite continued fraction* of k

partial denominators, which is defined as

$$[a_0; a_1, \dots, a_k] := a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_k}}}}. \quad (5)$$

The continued fraction is *simple* if $\{a_0, \dots, a_k\}$ are all integers. Every real number has a unique simple continued fraction representation; rational numbers correspond to finite continued fractions, while irrational numbers correspond to infinite continued fractions [Burton, 2007, Chap. 15]. We call one rational number “more irrational” than another if it has a longer continued fraction expansion.

Thus, in Fig. 9, we consider the following three choices for the ratio of subsegment lengths: $r = 1.25 = 5/4 = [1; 4]$ as the “least irrational,” $r = 1.3 = 13/10 = [1; 3, 3]$ as the “more irrational” and $r = 1.35 = 27/20 = [1; 2, 1, 6]$ as the “most irrational.” Figure 9(a) shows the growth of C_n for the protocol generated by the reducible permutation $\Pi([12345]) = [13524]$. In this case, the dynamics is eventually periodic for all three choices of r with the period of C_n becoming longer as r becomes “more irrational.”

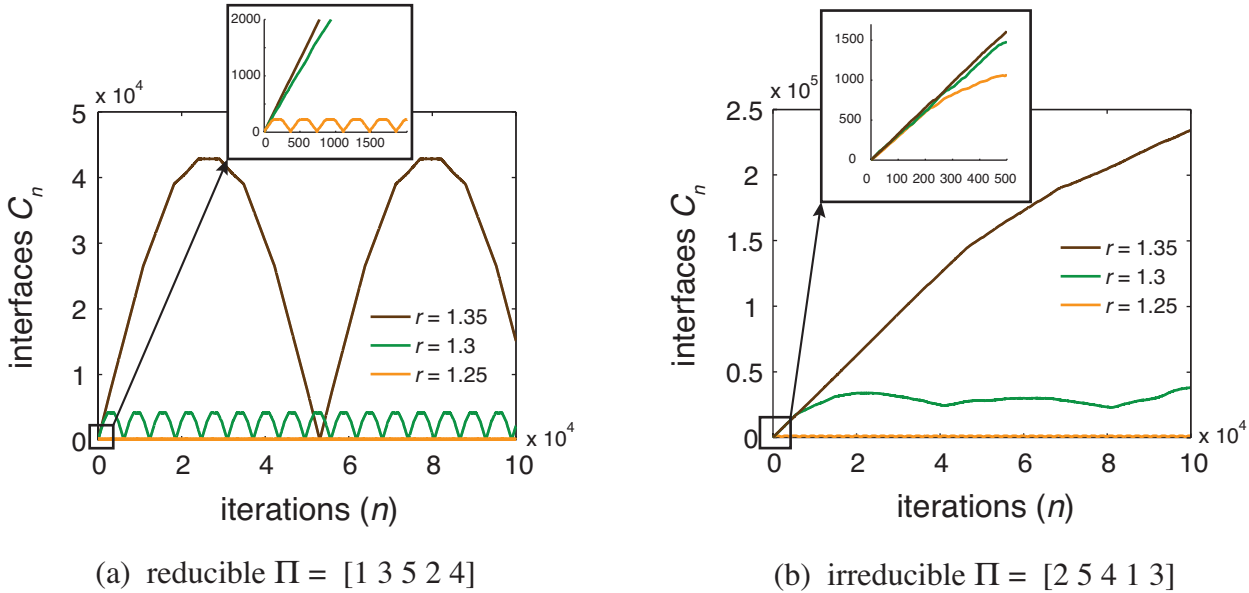


Fig. 9. (Color online.) Number of interfaces C_n plotted as a function of iterations for $N = 5$, ratios $r = 1.25$ (orange), 1.3 (green), 1.35 (brown) and (a) the reducible rearrangement order $\Pi([12345]) = [13524]$ and (b) the irreducible rearrangement order $\Pi([12345]) = [25413]$. Note that, for clarity, different vertical scales are used in panels (a) and (b).

Meanwhile, for the irreducible permutation $\Pi([12345]) = [25413]$ in Fig. 9(b), we do not observe periodic dynamics for $r = 1.3$ and $r = 1.35$ over the 10^5 iterations considered. For $r = 1.3$, however, C_n is bounded above independently of n , which leads us to conjecture that this case of Theorem 5 corresponds to IETs that violate Keane’s condition. For $r = 1.35$, we expect that there is also an upper bound for C_n that is independent of n , but it is not achieved over the 10^5 iterations considered. Thus, this last protocol can be considered, for all practical purposes, to behave as if Keane’s condition were satisfied. As a consequence, this is strong evidence that the dichotomy of grow–no growth in Theorem 5 corresponds to the dichotomy of satisfying–violating the assumptions of Theorem 1 (equivalently, Keane’s minimality condition).

The inset in Fig. 9(b) shows that the slope of all three C_n curves for small n is essentially independent of r . This suggests the interesting possibility that the slope is fixed by the number of *fundamental discontinuities* (in the language of Novak [2009]), which is encoded by the chosen irreducible permutation Π alone. All three protocols in Fig. 9(b) violate Keane’s condition to various “degrees,” and thus the number of

fundamental discontinuities is, technically speaking, zero because these protocols must be eventually periodic. Indeed, the linear growth of C_n is sustained over a shorter range of n as r is made “more rational.” Nevertheless, we observe a similar slope initially, which suggests that if r were irrational, then this is the growth rate of C_n and is thus the number of fundamental (non-resolvable) discontinuities permitted for this choice of N and Π .

4.4. Statistics on finite-time mixing

As mentioned in the Introduction, the mathematical theory (Sec. 3) does not provide an answer to questions such as “How many iterations does it take to reduce the longest continuous-color segment in an IET to a given fraction of the interval?”. In this section, we address this question through numerical simulations. Following [Juarez *et al.*, 2012], we evaluate the quality of mixing of *all* the cutting and shuffling protocols that can be constructed for given N and r by plotting the percentage of protocols (i.e., percentage of permutations Π for fixed N and r) that achieve a given percent unmixed value U_n after $n = 50$ iterations.

Figure 10 shows the finite-time mixing statistics for the protocols with $r = 1.5$ (a “bad” choice, corresponding to a local maximum in Fig. 8) and $r = 1.35$ (a “better” choice) with $N = 4, 5$ and 6 . Both of these ratios are rational numbers, and we have thus violated the assumptions of Theorems 1 and 4. Consequently, these IETs are neither ergodic nor weakly mixing in the mathematical sense. Nevertheless, the majority of these protocols achieve a substantial amount of mixing (as quantified by U_n) over 50 iterations. Furthermore, there is a clear benefit to picking r to be “more irrational” (as discussed in Sec. 4.3) because the curves for $r = 1.35$ (solid) consistently lie below those for $r = 1.5$ (dashed), indicating a higher degree of mixing across all protocols after 50 iterations.

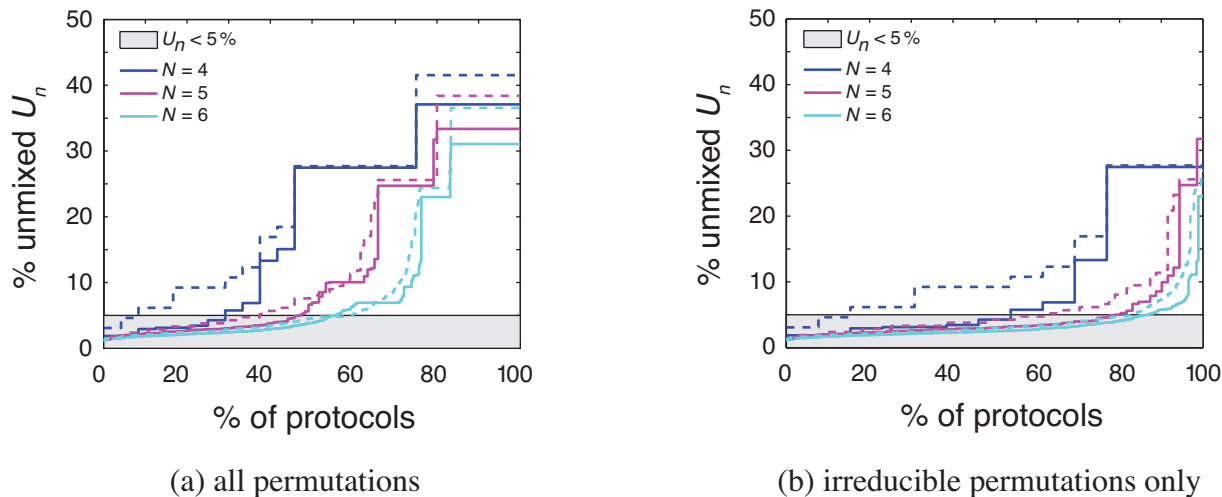


Fig. 10. (Color online.) Statistics on finite-time mixing by cutting and shuffling showing the percent unmixed achieved by a given percentage of protocols after 50 iterations, where the ratio r and number of initial subsegments N is given and (a) all permutations or (b) only irreducible permutations are considered for the rearrangement order. Solid and dashed curves correspond to $r = 1.35$ and $r = 1.5$, respectively.

The “worst case scenario” corresponds to the intersection of a curve with the vertical line at 100% of protocols. No protocols can mix any less than the percent of the domain corresponding to the longest initial subsegment. Based on the IET construction from Sec. 2, we know that the length of longest subsegment at $n = 0$ is $U_n = U_{n,\max}$ as defined in Eq. (4). Thus, 100% of protocols must achieve *at least* this value of U_n . From Eq. (4) we find that $U_{n,\max}(r = 1.35, N = 4, 5, 6) \approx 37, 33, 31\%$ and $U_{n,\max}(r = 1.5, N = 4, 5, 6) \approx 41.5, 38, 36.5\%$. Indeed, these are the highest values (i.e., U_n at 100% of protocols) observed in Fig. 10(a). However, it is clear that most protocols do much better, and the percentage of protocols that effect significant mixing, say those falling below the 5% unmixed line in the shaded area, is much higher when only irreducible permutations are considered in Fig. 10(b). No protocols achieve complete mixing over

the 50 iterations considered, therefore all the curves have a non-zero limiting value on the left. However, it is evident that in all cases there exists at least one protocol that achieves $U_n < 5\%$ because limiting value as % of protocols $\rightarrow 0$ is always in the shaded portion of the graph.

This last observation allows us to determine bounds on the *95% mixing time*, which is defined as the first iteration n^* for which $U_{n^*} < 5\%$ [Ashwin *et al.*, 2002]. The shaded portions of Fig. 10 correspond to $U_n < 5\%$, and it is evident that for $N = 5$ and 6 (and both $r = 1.5$ and 1.35), the majority (i.e., over 50%) of *all* protocols achieve this value of U_n in at most 50 iterations. Therefore, the 95% mixing time is, on average, $\lesssim 50$ iterations. When only irreducible permutations are considered for the rearrangement order (Fig. 10(b)), we see that nearly 80% of the protocols for $N = 5$ and 6 (and both $r = 1.5$ and 1.35) reduce U_n below 5% over 50 iterations. Consequently, if a cutting and shuffling map can achieve the 95% mixing time, then, in practice, we expect the presence of diffusion (or other kind of irreversibility) to result in complete and thorough mixing, even if the cutting and shuffling protocol may be eventually periodic (see, e.g., the relevant discussion and examples in [Ashwin *et al.*, 2002; Sturman, 2012]).

5. Conclusion

Although the problem of mixing a line segment by cutting and shuffling seems simple at a first glance, this first comprehensive numerical study of cutting and shuffling protocols based on interval exchange transformations (IETs) demonstrates a breadth of possible dynamical behaviors. Some of these behaviors can be predicted by the mathematical theory of IETs, while others could only be found through numerical experimentation. In addition to observing and classifying the dynamics of this cutting and shuffling process, we quantified its finite-time mixing properties within a well-defined parameter space. This is not only of practical interest in determining how cutting and shuffling leads to complex dynamics but is also of abstract interest because it could stimulate new theoretical results in the area of IETs.

Based on our study, the following design rules for mixing a line segment by cutting and shuffling can be drawn:

- (i) The rearrangement order should be an irreducible permutation (see Appendix B for more technical details on this topic).
- (ii) An IET with $N = 6$ subsegments is the most practical number of subsegments. The improvement in mixing diminishes with larger N .
- (iii) The ratio of adjacent subsegment lengths should not be large, specifically it should be an irrational number slightly larger than 1. (Continued fraction expansions can be used to systematically make r “more irrational.”)

Thus, an important result of the present work is that even if a cutting and shuffling map cannot be proven to be mixing in the mathematical sense, it can generate significant complexity over a finite number of iterations. Specifically, for all practical purposes it can behave (over a finite number of iterations) as if satisfied the mathematical requirements for mixing (see Sec. 4.3). Thus, in physical systems where cutting and shuffling is one of several underlying mixing mechanisms (e.g., in granular flows [Christov *et al.*, 2010b,a; Juarez *et al.*, 2010, 2012] as motivated in the Introduction), optimizing for cutting and shuffling is beneficial and, indeed, desirable. If the cutting and shuffling process is in some sense “optimal,” then the additional mixing mechanisms that become relevant at later times can achieve greater efficiency. The design rules listed above can have an impact in performing such optimization of physical mixing processes, once the underlying cutting and shuffling skeleton is uncovered.

At the same time, the present study suggests several mathematical questions regarding IETs that have not been answered in the pure mathematics literature:

- (i) Is there a rigorous connection between resonances in chaotic dynamical systems and the spikes of the average of U_n in Fig. 8 at values of r that are simple fractions?
- (ii) Is the number of nonresolving fundamental discontinuities of an IET solely a function of Π ? If so, what kind?
- (iii) Are there conditions that can be imposed on subsegment length ratios r_i and the permutation Π so that U_n is less than a given percentage after n iterations?

Finally, a possible avenue of future research is to extend the present methodology to study mixing by cutting and shuffling of the unit square $\mathcal{I} \times \mathcal{I} = [0, 1]^2$. This could be accomplished by, for example, applying an interval exchange transformation in the y -direction and extending it in the x -direction by making each subinterval \mathcal{I}_i into a rectangle of unit horizontal length [Christov *et al.*, 2011]. More generally, such an interval exchange in the y -direction can be sequentially composed with another in the x -direction to produce a special case of the more general class of *rectangle exchange transformations* (RETs) [Haller, 1981]. In comparison to IETs, for which there are a number of definitive theoretical results (recall Sec. 3), RETs are poorly understood. Thus, providing concrete numerical results could be quite useful. In addition, adding irreversibility to the cutting and shuffling process, e.g., through diffusion as in [Ashwin *et al.*, 2002; Sturman, 2012], opens further possibilities. It is conceivable that the competition between cutting and shuffling and diffusion would lead to *one-dimensional strange eigenmodes*, which have been found to be important in understanding mixing in open chaotic flows [Gouillart *et al.*, 2009].

Acknowledgments

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Appendices

Appendix A

In this Appendix, we outline how a numerical simulation of the cutting and shuffling of a line segment by an interval exchange transformation (IET) as constructed in Sec. 2 can be performed. The code takes the following inputs: the adjacent subsegment ratio r (a finite-precision decimal), the number of subsegments N (an integer > 1), the rearrangement order Π (a permutation of the integers $\{1, \dots, N\}$) and the desired number of iterations n_{\max} of the process (an integer > 1). Then, it proceeds through the steps below.

- (1) Convert r to a fraction r_n/r_d and compute the *least common denominator* (LCD) of the fractions $\{r^{i-1}x\}_{i=1}^N$ (x is given by Eq. (1)), which correspond to the lengths of the subsegments in the IET construction. The LCD can be found to be $\text{lcd} = (r_n^N - r_d^N)/(r_n - r_d)$. (Note that $r_n - r_d$ is always a factor of $r_n^N - r_d^N$ for integer N , so that lcd is an integer.) The shortest piece that can be created by cutting and shuffling is thus 1 unit out of lcd .
- (2) Create two vectors of length lcd , which we call **segment** and **newsegment**, to represent the line segment before and after shuffling, respectively. Accordingly, the initial subsegment divisions are located between cells $r^{i-1} \cdot x \cdot \text{lcd}$ and $r^{i-1} \cdot x \cdot \text{lcd} + 1$ ($i = 1, \dots, N - 1$) in the array.
- (3) ‘‘Color’’ the interval to be shuffled by assigning all elements between 1 and $x \cdot \text{lcd}$ (the first subsegment) of the array **segment** the value of 1, all elements between $x \cdot \text{lcd} + 1$ and $r \cdot x \cdot \text{lcd}$ (the second subsegment) the value of 2, and so on.
- (4) Using the rearrangement order Π , copy all elements corresponding to the first ($i = 1$) subsegment portion of the array **segment** into the appropriate parts of **newsegment**; for example, for $\Pi(1) \neq 1$,

$$\text{newsegment}\left(\left(r^{\Pi(1)-1} \cdot x \cdot \text{lcd} + 1\right) : r^{\Pi(1)} \cdot x \cdot \text{lcd}\right) = \text{segment}(1 : x \cdot \text{lcd}). \quad (\text{A.1})$$

(Here, a colon is used in the sense of the MATLAB array subscripting operator.) Repeat for all $i = 1, \dots, N$ to complete one iteration of the shuffling process.

- (5) Determine the longest number of consecutive elements of `newsegment` containing the same value and divide this number by the total number of elements (`lcD`) to obtain the percent unmixed U_n for this (say, the n th) iteration.
- (6) Calculate the number of adjacent elements of `newsegment` that have different values to obtain the number of distinct cuts C_n at this iteration.
- (7) Set `segment = newsegment` and repeat Steps (4) through (7) until the desired number of iterations n_{\max} is reached.

Appendix B

In this Appendix, we give some remarks on irreducible permutations. The following MATHEMATICA code determines whether `p` is an irreducible permutation of $\{1, 2, \dots, N\}$ (i.e., it returns 0 for false or 1 for true):

```
<< Combinatorica`
IsReducibleQ[p_] := Module[{j, ans = 0},
  For[j = 1, j <= Length[p] - 1, j++,
    If[PermutationQ[Take[p, j]], ans = 1; Break[]];
  ];
  ans
];
```

Table B.1 lists the number of total and irreducible permutations as a function of N . Klazar [2003] gives a discussion of the integer sequence corresponding to the number of irreducible permutations of length N and its properties (see also entry A003319 in the On-Line Encyclopedia of Integer Sequences [OEIS Foundation Inc., 2012]).

Table B.1. Number of permutations of the integers $\{1, 2, \dots, N\}$.

N	1	2	3	4	5	6	7	8	9	10
total = $N!$	1	2	6	24	120	720	5 040	40 320	362 880	3 628 800
irreducible	1	1	3	13	71	461	3 447	29 093	27 3343	2 829 325

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