An Equivalence Relation on the Symmetric Group and Multiplicity-free Flag h-Vectors

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Abstract

We consider the equivalence relation $\sim$ on the symmetric group $\mathfrak{S}_n$ generated by the interchange of two adjacent elements a_i and a_{i+1} of $w = a_1 \cdots a_n \in \mathfrak{S}_n$ such that $|a_i - a_{i+1}| = 1$. We count the number of equivalence classes and the sizes of the equivalence classes. The results are generalized to permutations of multisets. In the original problem, the equivalence class containing the identity permutation is the set of linear extensions of a certain poset. Further investigation yields a characterization of all finite graded posets whose flag h-vector takes on only the values $0, \pm 1$.

1 Introduction.

Let $k \geq 2$. Define two permutations u and v (regarded as words $a_1 a_2 \cdots a_n$) in the symmetric group $\mathfrak{S}_n$ to be *equivalent* if v can be obtained from u by a sequence of interchanges of adjacent terms that differ by at least j. It is a nice exercise to show that the number $f_j(n)$ of equivalence classes of this relation (an obvious equivalence relation) is given by

$$f_j(n) = \begin{cases} n!, & n \leq j \\ j! \cdot j^{n-j}, & n > j. \end{cases}$$

Namely, one can show that every equivalence class contains a unique permutation $w = b_1 b_2 \cdots b_n$ for which we never have $b_i \geq b_{i+1} + j$. To count these permutations w for $n > j$, we first have $j!$ ways of ordering $1, 2, \ldots, j$ within w. Then insert $j+1$ in j ways,

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i.e., at the end or preceding any $i \neq 1$. Next insert $j+2$ in j ways, etc. The case $j=3$ of this argument appears in [7, A025192] and is attributed to Joel Lewis, November 14, 2006. Some equivalence relations on $\mathfrak{S}_n$ of a similar nature are pursued by Linton et al. [3].

The above result suggests looking at some similar equivalence relations on $\mathfrak{S}_n$. The one we will consider here is the following: define u and v to be *equivalent*, denoted $u \sim v$, if v can be obtained from u by interchanging adjacent terms that differ by exactly one. For instance, when $n=3$ we have the two equivalence classes $\{123, 213, 312\}$ and $\{321, 231, 312\}$. We will determine the number of classes, the number of one-element classes, and the sizes of the equivalence classes (always a product of Fibonacci numbers). It turns out that the class containing the identity permutation $12\cdots n$ may be regarded as the set of linear extensions of a certain n-element poset P_n. Moreover, P_n has the most number of linear extensions of any n-element poset on the vertex set $[n]=\{1,2,\ldots,n\}$ such that $i<j$ in P implies $i<j$ in $\mathbb{Z}$, and such that all linear extensions of P (regarded as permutations of $[n]$) have a different descent set. This result leads to the complete classification and enumeration of finite graded posets of rank n whose flag h-vector is "multiplicity-free," i.e., assumes only the values 0 and ± 1.

2 The number of equivalence classes.

To obtain the number of equivalence classes, we first define a canonical element in each class. We then count these canonical elements by the Principle of Inclusion-Exclusion. We call a permutation $w = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ *salient* if we never have $a_i = a_{i+1}+1$ $(1 \leq i \leq n-1)$ or $a_i = a_{i+1}+2 = a_{i+2}+1$ $(1 \leq i \leq n-2)$. For instance, there are eight salient permutations in $\mathfrak{S}_4$: 1234, 1342, 2314, 2341, 2413, 3142, 3412, 4123.

Lemma 2.1. *Every equivalence class with respect to the equivalence relation $\sim$ contains exactly one salient permutation.*

Proof. Let E be an equivalence class, and let $w = a_1 a_2 \cdots a_n$ be the lexicographically least element of E. Then we cannot have $a_i = a_{i+1}+1$ for some i; otherwise we could interchange a_i and a_{i+1} to obtain a lexicographically smaller permutation in E. Similarly if $a_i = a_{i+1}+2 = a_{i+2}+1$ then we can replace $a_i a_{i+1} a_{i+2}$ with $a_{i+2} a_i a_{i+1}$. Hence w is salient.

It remains to show that a class E cannot contain a salient permutation $w = b_1 b_2 \cdots b_n$ that is not the lexicographically least element $v = a_1 a_2 \cdots a_n$ of E. Let i be the least index for which $a_i \neq b_i$. Since $v \sim w$, there must be some b_j satisfying $j > i$ and $b_j < b_i$ that is interchanged with b_i in the transformation of v to w by adjacent transpositions of consecutive integers. Hence $b_j = b_i + 1$. If $j = i+1$ then v is not salient. If $j = i+2$ then we must have $b_{i+1} = b_i + 2$ in order to move b_{j+2} past b_{j+1}, so again v is not salient. If $j > i+2$ then some element b_k between b_i and b_j in v satisfies $|b_j - b_k| > 1$, so we cannot move b_j past b_k unless we first interchange b_i and b_k (which must therefore equal

$b_i + 1$). But then after b_i and b_j are interchanged, we cannot move b_k back to the right of b_i. Hence v and w cannot be equivalent, a contradiction completing the proof. ∎

Theorem 2.2. *Let $f(n)$ be the number of equivalence classes of the relation $\sim$ on $\mathfrak{S}_n$, with $f(0) = 1$. Then*

$$f(n) = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j (n-j)! \binom{n-j}{j}. \tag{2.1}$$

Equivalently,

$$\sum_{n \geq 0} f(n) x^n = \sum_{m \geq 0} m! (x(1-x))^m.$$

Proof. By Lemma 2.1, we need to count the number of salient permutations $w \in \mathfrak{S}_n$. The proof is by an inclusion-exclusion argument. Let A_i, $1 \leq i \leq n-1$, be the set of permutations $v \in \mathfrak{S}_n$ that contain the factor (i.e., consecutive terms) $i+1, i$. Let B_i, $1 \leq i \leq n-2$, be the set of $v \in \mathfrak{S}_n$ that contain the factor $i+2, i, i+1$. Let $C_1, \dots, C_{2n-3}$ be some indexing of the A_i's and B_i's. By the Principle of Inclusion-Exclusion, we have

$$f(n) = \sum_{S \subseteq [2n-3]} (-1)^{\#S} \# \bigcap_{i \in S} C_i, \tag{2.2}$$

where the empty intersection of the C_i's is $\mathfrak{S}_n$. A little thought shows that any intersection of the C_i's consists of permutations that contain some set of nonoverlapping factors $j, j-1, \dots, i+1, i$ and $j, j-1, \dots, i+3, i+2, i, i+1$. Now permutations containing the factor $j, j-1, \dots, i+1, i$ are those in $A_{j-1} \cap A_{j-2} \cap \cdots \cap A_i$ (an intersection of $j-i$ sets), while permutations containing $j, j-1, \dots, i+3, i+2, i, i+1$ are those in $A_{j-1} \cap A_{j-2} \cap \cdots \cap A_{i+2} \cap B_i$ (an intersection of $j-i-1$ sets). Since $(-1)^{j-i} + (-1)^{j-i-1} = 0$, it follows that all terms on the right-hand side of equation (2.2) involving such intersections will cancel out. The only surviving terms will be the intersections $A_{i_1} \cap \cdots \cap A_{i_j}$ where the numbers $i_1, i_1+1, i_2, i_2+1, \dots, i_j, i_j+1$ are all *distinct*. The number of ways to choose such terms for a given j is the number of sequences of j 2's and $n-2j$ 1's, i.e., $\binom{n-j}{2j}$. The number of permutations of the j factors i_r, i_{r+1}, $1 \leq r \leq j$, and the remaining $n-2j$ elements of $[n]$ is $(n-j)!$. Hence equation (2.2) reduces to equation (2.1), completing the proof. ∎

Note. There is an alternative proof based on the Cartier-Foata theory of partially commutative monoids [1]. Let M be the monoid with generators $g_1, \dots, g_n$ subject only to relations of the form $g_i g_j = g_j g_i$ for certain i and j. Let $x_1, \dots, x_n$ be *commuting* variables. If $w = g_{i_1} \cdots g_{i_m} \in M$, then set $x^w = x_{i_1} \cdots x_{i_m}$. Define

$$F_M(x) = \sum_{w \in M} x^w.$$

Then a fundamental result (equivalent to [1, Thm. 2.4]) of the theory asserts that

$$F_M(x) = \frac{1}{\sum_S (-1)^{\#S} \prod_{g_i \in S} x_i}, \tag{2.3}$$

where S ranges over all subsets of $\{g_1, \ldots, g_n\}$ (including the empty set) whose elements pairwise commute. Consider now the case where the relations are given by $g_i g_{i+1} = g_{i+1} g_i$, $1 \leq i \leq n-1$. Thus $f(n)$ is the coefficient of $x_1 x_2 \cdots x_n$ in $F(x)$. Writing $[x^\alpha]G(x)$ for the coefficient of $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ in the power series $G(x)$, it follows from equation (2.3) that

$$\begin{aligned} f(n) &= [x_1 \cdots x_n] \frac{1}{1 - \sum_{i=1}^n x_i + \sum_{i=1}^{n-1} x_i x_{i+1}} \\ &= [x_1 \cdots x_n] \sum_{j \geq 0} \left(\sum_{i=1}^n x_i - \sum_{i=1}^{n-1} x_i x_{i+1} \right)^j. \end{aligned}$$

A straightforward argument shows that

$$[x_1 \cdots x_n] \left(\sum_{i=1}^n x_i - \sum_{i=1}^{n-1} x_i x_{i+1} \right)^{n-j} = (-1)^j (n-j)! \binom{n-j}{j},$$

and the proof follows.

Note. The numbers $f(n)$ for $n \geq 0$ begin 1, 1, 1, 2, 8, 42, 258, 1824, 14664, ... This sequence appears in [7, A013999] but without a combinatorial interpretation before the present paper.

Various generalizations of Theorem 2.2 suggest themselves. Here we will say a few words about the situation where $\mathfrak{S}_n$ is replaced by all permutations of a *multiset* on the set $[n]$ with the same definition of equivalence as before. For instance, for the multiset $M = \{1^2, 2, 3^2\}$ (short for $\{1, 1, 2, 3, 3\}$), there are six equivalence classes, each with five elements, obtained by fixing a word in $1, 1, 3, 3$ and inserting 2 in five different ways. Suppose that the multiset is given by $M = \{1^{r_1}, \ldots, n^{r_n}\}$. According to equation (2.3), the number f_M of equivalence classes of permutations of M is the coefficient of $x_1^{r_1} \cdots x_n^{r_n}$ in the generating function

$$F_n(x) = \frac{1}{1 - \sum_{i=1}^n x_i + \sum_{i=1}^{n-1} x_i x_{i+1}}.$$

For the case $n = 4$ (and hence $n \leq 4$) we can give an explicit formula for the coefficients of $F_n(x)$, or in fact (as suggested by I. Gessel) for $F_n(x)^t$ where t is an indeterminate. We use the falling factorial notation $(y)_r = y(y-1) \cdots (y-r+1)$.

Theorem 2.3. *We have*

$$F_4(x)^t = \sum_{h,i,j,k \geq 0} \frac{(t+h+j-1)_j (t+h+k-1)_h (t+i+k-1)_{i+k}}{h!\, i!\, j!\, k!} x_1^h x_2^i x_3^k x_4^k. \quad (2.4)$$

In particular,

$$F_4(x) = \sum_{h,i,j,k \geq 0} \binom{h+j}{j} \binom{h+k}{k} \binom{i+k}{i} x_1^h x_2^i x_3^j x_4^k. \quad (2.5)$$

Proof. Since the coefficient of $x_1^h x_2^i x_3^j x_4^k$ in $F_4(x)^t$ is a polynomial in t, it suffices to assume that t is a nonnegative integer. The result can then be proved straightforwardly by induction on t. Namely, the case $t = 0$ is trivial. Assume for $t - 1$ and let $G(x)$ be the right-hand side of equation (2.4). Check that

$$(1 - x_1 - x_2 - x_3 - x_4 + x_1x_2 + x_2x_3 + x_3x_4)G(x) = F_4(x)^{t-1},$$

and verify suitable initial conditions.

Ira Gessel points out (private communication) that we can prove the theorem without guessing the answer in advance by writing

$$\frac{1}{1 - x_1 - x_2 - x_3 - x_4 + x_1x_2 + x_2x_3 + x_3x_4} =$$

$$\frac{1}{(1-x_2)(1-x_3)\left(1 - \frac{x_1}{1-x_3}\right)\left(1 - \frac{x_4}{(1-x_2)\left(1-\frac{x_1}{1-x_3}\right)}\right)},$$

and then expanding one variable at a time in the order x_4, x_1, x_2, x_3. ∎

For multisets supported on sets with more than four elements there are no longer simple explicit formulas for the number of equivalence classes. However, we can still say something about the multisets $\{1^k, 2^k, \ldots, n^k\}$ for k fixed or n fixed. The simplest situation is when n is fixed.

Proposition 2.4. *Let $g(n, k)$ be the number of equivalence classes of permutations of the multiset $\{1^k, \ldots, n^k\}$. For fixed n, $g(n, k)$ is a P-recursive function of k, i.e., for some integer $d \geq 1$ and polynomials $P_0(k), \ldots, P_d(k)$ (depending on n), we have*

$$P_0(k)g(n, k + d) + P_1(k)g(n, k + d - 1) + \cdots + P_d(k)g(n, k) = 0$$

for all $k \geq 0$.

Proof. The proof is an immediate consequence of equation (2.3), the result of Lipshitz [4] that the diagonal of a rational function (or even a D-finite function) is D-finite, and the elementary result [8, Thm. 1.5][10, Prop. 6.4.3] that the coefficients of D-finite series are P-recursive. ∎

To deal with permutations of the multiset $\{1^k, \ldots, n^k\}$ when k is fixed, let $\mathbb{C}\{\{x\}\}$ denote the field of fractional Laurent series $f(x)$ over $\mathbb{C}$ with finitely many terms having a negative exponent, i.e., for some $j_0 \in \mathbb{Z}$ and some $N \geq 1$ we have $f(x) = \sum_{j \geq j_0} a_j x^{j/N}$, $a_j \in \mathbb{C}$. A series $y \in \mathbb{C}\{\{x\}\}$ is *algebraic* if it satisfies a nontrivial polynomial equation whose coefficients are polynomials in x. Any such polynomial equation of degree n has n zeros (including multiplicity) belonging to the field $\mathbb{C}\{\{x\}\}$. In fact, this field is algebraically closed (Puiseux' theorem). For further information, see for instance [10,

§6.1]. Write $\mathbb{C}_{\text{alg}}\{\{x\}\}$ for the field of algebraic fractional (Laurent) series. The next result is a direct generalization of Theorem 2.2. The main point is that the series $z_i(x)$ and $y_j(x)$ are *algebraic.*

Theorem 2.5. *Let k be fixed. Then there exist finitely many algebraic fractional series $y_1, \ldots, y_q,\ z_1, \ldots, z_q \in \mathbb{C}_{\text{alg}}\{\{x\}\}$ and polynomials $P_1, \ldots, P_q \in \mathbb{C}_{\text{alg}}\{\{x\}\}[m]$ (i.e., polynomials in m whose coefficients lie in $\mathbb{C}_{\text{alg}}\{\{x\}\}$) such that*

$$\sum_{n\geq 0} g(n,k)x^n = \sum_{j=1}^{q} z_j(x) \sum_{m\geq 0} m! P_j(m) y_j(x)^m.$$

Proof. Our proof will involve "umbral" methods. By the work of Rota et al. [5][6], this means that we will be dealing with polynomials in t (whose coefficients will be fractional Laurent series in x) and will apply a linear functional $\varphi\colon \mathbb{C}[t]\{\{x\}\} \to \mathbb{C}\{\{x\}\}$. For our situation φ is defined by $\varphi(t^m) = m!$.

NOTE. The ring $\mathbb{C}[t]\{\{x\}\}$ consists of all series of the form $\sum_{j\geq j_0} a_j(t)x^{j/N}$ for some $j_0 \in \mathbb{Z}$ and $N \geq 1$, where $a_j(t) \in \mathbb{C}[t]$. When we replace t^m with $m!$, the coefficient of each $x^{j/N}$ is a well-defined complex number. The function φ is not merely linear; it commutes with *infinite* linear combinations of the form $\sum_{j\geq j_0} a_j(t)x^{j/N}$. In other words, φ is *continuous* is the standard topology on $\mathbb{C}[t]\{\{x\}\}$ defined by $f_n(x,t) \to 0$ if $\deg_t f_n(x,t) \to \infty$ as $n \to \infty$. See [9, p. 7].

By equation (2.3), we have

$$\begin{aligned} g(n,k) &= [x_1^k \cdots x_n^k] \frac{1}{1 - \sum x_i + \sum x_i x_{i+1}} \\ &= [x_1^k \cdots x_n^k] \sum_{r\geq 0} \left(\sum x_i - \sum x_i x_{i+1}\right)^r. \end{aligned}$$

We obtain a term τ in the expansion of $(\sum x_i - \sum x_i x_{i+1})^r$ by picking a term x_i or $-x_i x_{i+1}$ from each factor. Associate with τ the graph G_τ on the vertex set $[n]$ where we put a loop at i every time we choose the term x_i, and we put an edge between i and $i+1$ whenever we choose the term $-x_i x_{i+1}$. Thus G_τ is regular of degree k with r edges, and each connected component has a vertex set which is an interval $\{a, a+1, \ldots, a+b\}$. Let $\mu(i)$ be the number of loops at vertex i and $\mu(i,i+1)$ the number of edges between vertices i and $i+1$. Let $\nu = \sum \mu(i,i+1)$, the total number of nonloop edges. Then

$$[\tau] \left(\sum x_i - \sum x_i x_{i+1}\right)^r = \frac{(-1)^\nu r!}{\prod_{i=1}^r \mu_i! \cdot \prod_{i=1}^{r-1} \mu(i,i+1)!}. \tag{2.6}$$

Define the *umbralized weight* $w(G)$ of $G = G_\tau$ by

$$w(G) = \frac{(-1)^\nu t^r}{\prod_{i=1}^r \mu_i! \cdot \prod_{i=1}^{r-1} \mu(i,i+1)!}.$$

Thus $w(G)$ is just the right-hand side of equation (2.6) with the numerator factor $r!$ replaced by t^r. At the end of the proof we will "deumbralize" by applying the functional φ, thus replacing t^r with $r!$.

Regarding k as fixed, let

$$c(m) = \sum_H w(H), \tag{2.7}$$

summed over all *connected graphs* H on a linearly ordered m-element vertex set, say $[m]$, that are regular of degree k and such that every edge is either a loop or is between two consecutive vertices i and $i+1$. It is easy to see by transfer-matrix arguments (as discussed in [9, §4.7]) that

$$F(x,t) := \sum_{m\geq 1} c(m)x^m$$

is a rational function of x whose coefficients are integer polynomials in t. The point is that we can build up H one vertex at a time in the order $1, 2, \ldots, m$, and the information we need to see what new edges are allowed at vertex i (that is, loops at i and edges between i and $i+1$) depends only on a *bounded* amount of prior information (in fact, the edges at $i-1$). Moreover, the contribution to the umbralized weight $w(G)$ from adjoining vertex i and its incident edges is simply the weight obtained thus far multiplied by $(-1)^{\mu(i,i+1)}(t/2)^{\mu(i)+\mu(i,i+1)}$. Hence we are counting weighted walks on a certain edge-weighted graph with vertex set $[k-1]$ (the possible values of $\mu(i,i+1)$) with certain initial conditions. We have to add a term for one exceptional graph: the graph with two vertices and k edges between them. This extra term does not affect rationality.

We may describe the vertex sets of the connected components of G_τ by a composition $(\alpha_1, \ldots, \alpha_s)$ of n, i.e., a sequence of positive integers summing to n. Thus the vertex sets are

$$\{1, \ldots, \alpha_1\}, \{\alpha_1+1, \ldots, \alpha_1+\alpha_2\}, \ldots, \{\alpha_1+\cdots+\alpha_{s-1}+1, \ldots, n\}.$$

Now set

$$f(n) = \sum_G w(G),$$

summed over *all* graphs G on a linearly ordered n-element vertex set, say $[n]$, that are regular of degree k and such that every edge is either a loop or is between two consecutive vertices i and $i+1$. Thus

$$f(n) = \sum_{(\alpha_1,\ldots,\alpha_r)} g(\alpha_1)\cdots g(\alpha_s),$$

where the sum ranges over all compositions of n. Note the crucial fact that if G has r edges and the connected components have $r_1, \ldots, r_s$ edges, then $t^r = t^{r_1+\cdots+r_s}$. It therefore follows that if

$$G(x,t) = \sum_{n\geq 0} f(n)x^n,$$

then

$$G(x,t) = \frac{1}{1-F(x,t)}.$$

It follows from equation (2.6) that

$$\sum_{n\geq 0} g(n,k)x^n = \varphi G(x,t), \tag{2.8}$$

where φ is the linear functional mentioned above which is defined by $\varphi(t^r) = r!$.

By Puiseux' theorem we can write

$$G(x,t) = \frac{1}{(1-y_1t)^{d_1}\cdots(1-y_qt)^{d_q}}$$

for certain distinct algebraic fractional Laurent series $y_i \in \mathbb{C}_{\mathrm{alg}}\{\{x\}\}$ and integers $d_j \geq 1$. By partial fractions we have

$$\begin{aligned} G(x,t) &= \sum_{j=1}^{q}\sum_{i=1}^{d_j} \frac{u_{ij}}{(1-y_jt)^i} \\ &= \sum_{j=1}^{q}\sum_{i=1}^{d_j} u_{ij} \sum_{m\geq 0} \binom{i+m-1}{i} y_j^m t^m \\ &= \sum_{j=1}^{q}\sum_{m\geq 0}\sum_{j=1}^{q}\sum_{m\geq 0} P_j(m) z_j y_j^m t^m, \end{aligned} \tag{2.9}$$

where $u_{ij}, z_i \in \mathbb{C}_{\mathrm{alg}}\{\{x\}\}$ and $P_j(m) \in \mathbb{C}_{\mathrm{alg}}\{\{x\}\}[m]$. Apply the functional φ to complete the proof. ∎

Example 2.6. Consider the case $k = 2$. Let G be a connected regular graph of degree 2 with vertex set $[m]$ and edges that are either loops or are between vertices i and $i+1$ for some $1 \leq i \leq m-1$. Then G is either a path with vertices $1, 2, \ldots, m$ (in that order) with a loop at both ends (allowing a double loop when $m = 1$), or a double edge when $m = 2$. Hence

$$\begin{aligned} F(x,t) &= \frac{1}{2}t^2x + (\frac{1}{2}t^2 - t^3)x^2 + t^4x^3 - t^5x^4 + \cdots \\ &= \frac{1}{2}t^2x - (t^3 - \frac{1}{2}t^2)x^2 + \frac{t^4x^3}{1+tx}, \end{aligned}$$

and

$$\begin{aligned} G(x,t) &= \frac{1}{1-F(x,t)} \\ &= \frac{1+tx}{1+xt-\frac{1}{2}(x+x^2)t^2+\frac{1}{2}(x^2-x^3)t^3}. \end{aligned}$$

The denominator of $G(x,t)$ factors as $(1-y_1t)(1-y_2t)(1-y_3t)$, where

$$\begin{aligned}
y_1 &= x+2x^2+18x^3+194x^4+2338x^5+30274x^6+411698x^7 \\
&\quad +5800066x^8+\cdots \\
y_2 &= \frac{1}{2}\sqrt{2}x^{1/2}-x-\frac{1}{4}\sqrt{2}x^{3/2}-x^2-\frac{33}{16}\sqrt{2}x^{5/2}-9x^3 \\
&\quad -\frac{657}{32}\sqrt{2}x^{7/2}-97x^4-\cdots \\
y_3 &= -\frac{1}{2}\sqrt{2}x^{1/2}-x+\frac{1}{4}\sqrt{2}x^{3/2}-x^2+\frac{33}{16}\sqrt{2}x^{5/2}-9x^3 \\
&\quad +\frac{657}{32}\sqrt{2}x^{7/2}-97x^4+\cdots.
\end{aligned}$$

Since the y_i's are distinct we can take each $P_i(m)=1$. The coefficients z_1,z_2,z_3 are given by

$$\begin{aligned}
z_1 &= -4x-48x^2-676x^3-10176x^4-158564x^5-2523696x^6+\cdots \\
z_2 &= \frac{1}{2}+\frac{1}{2}\sqrt{2}x^{1/2}+2x+\frac{19}{4}\sqrt{2}x^{3/2}+24x^4+\frac{1007}{16}\sqrt{2}x^{5/2}+338x^3 \\
&\quad +\frac{29507}{32}\sqrt{2}x^{7/2}+\cdots \\
z_3 &= \frac{1}{2}-\frac{1}{2}\sqrt{2}x^{1/2}+2x-\frac{19}{4}\sqrt{2}x^{3/2}+24x^4-\frac{1007}{16}\sqrt{2}x^{5/2}+338x^3 \\
&\quad -\frac{29507}{32}\sqrt{2}x^{7/2}+\cdots.
\end{aligned}$$

Finally we obtain

$$\sum_{n\geq 0} g(n,2)x^n = \sum_{m\geq 0} m!(z_1y_1^m+z_2y_2^m+z_3y_3^m).$$

In Theorem 2.2 we determined the number of equivalence classes of the equivalence relation $\sim$ on $\mathfrak{S}_n$. Let us now turn to the structure of the individual equivalence classes. Given $w\in\mathfrak{S}_n$, write $\langle w\rangle$ for the class containing w. First we consider the case where w is the identity permutation $\mathrm{id}_n = 12\cdots n$ or its reverse $\overline{\mathrm{id}}_n = n\cdots 21$. (Clearly for any w and its reverse $\bar{w}$ we have $\#\langle w\rangle = \#\langle\bar{w}\rangle$.) Let F_n denote the nth Fibonacci number, i.e., $F_1=F_2=1$, $F_n=F_{n-1}+F_{n-2}$.

Proposition 2.7. *We have* $\#\langle\mathrm{id}_n\rangle = \#\langle\overline{\mathrm{id}}_n\rangle = F_{n+1}$.

Proof. Let $g(n)=\#\langle\mathrm{id}_n\rangle=\#\langle\overline{\mathrm{id}}_n\rangle$, so $g(1)=1=F_2$ and $g(2)=2=F_3$. If $w=a_1a_2\cdots a_n\sim\mathrm{id}$, then either $a_n=n$ with $g(n-1)$ possibilities for $a_1a_2\cdots a_{n-1}$, or else $a_{n-1}=n$ and $a_n=n-1$ with $g(n-2)$ possibilities for $a_1a_2\cdots a_{n-2}$. Hence $g(n)=g(n-1)+g(n-2)$, and the proof follows. ∎

We now consider an arbitrary equivalence class. Proposition 2.9 below is due to Joel Lewis (private communication).

Lemma 2.8. *Each equivalence class $\langle w\rangle$ of permutations of any finite subset S of $\{1,2,\dots\}$ contains a permutation $v = v_1v_2\cdots v_k$ (concatenation of words) such that (a) each v_i is an increasing or decreasing sequence of consecutive integers, and (b) every $u\sim w$ has the form $u = v_1'v_2'\cdots v_k'$, where $v_i'\sim v_i$. Moreover, the permutation v is unique up to reversing v_i's of length two.*

Proof. Let j be the largest integer for which some $v\sim w$ has the property that $v_1v_2\cdots v_j$ is either an increasing or decreasing sequence of consecutive integers. It is easy to see that v_k for $k>j$ can never be interchanged with some v_i for $1\leq i\leq j$ in a sequence of transpositions of adjacent consecutive integers. Moreover, $v_1v_2\cdots v_j$ cannot be converted to the reverse $v_j\cdots v_2v_1$ unless $j\leq 2$. The result follows by induction. ∎

Proposition 2.9. *Let m_i be the length of v_i in Lemma 2.8. Then*

$$\#\langle w\rangle = F_{m_1+1}\cdots F_{m_k+1}.$$

Proof. Immediate from Proposition 2.7 and Lemma 2.8. ∎

We can also ask for the number of equivalence classes of a given size r. Here we consider $r=1$. Let $N(n)$ denote the number of one-element equivalence classes of permutations in $\mathfrak{S}_n$. Thus $N(n)$ is also the number of permutations $a_1a_2\cdots a_n\in\mathfrak{S}_n$ for which $|a_i-a_{i+1}|\geq 2$ for $1\leq i\leq n-1$. This problem is discussed in OEIS [7, A002464]. In particular, we have the generating function

$$\begin{aligned}\sum_{n\geq 0} N(n)x^n &= \sum_{m\geq 0} m!\left(\frac{x(1-x)}{1+x}\right)^m\\ &= 1+x+2x^4+14x^5+90x^6+646x^7+5242x^8+\cdots.\end{aligned}$$

3 Multiplicity-free flag h-vectors of distributive lattices.

Let P be a finite graded poset of rank n with $\hat{0}$ and $\hat{1}$, and let ρ be the rank function of P. (Unexplained poset terminology may be found in [9, Ch. 3].) Write $2^{[n-1]}$ for the set of all subsets of $[n-1]$. The *flag f-vector* of P is the function $\alpha_P\colon 2^{[n-1]}\to\mathbb{Z}$ defined as follows: if $S\subseteq[n-1]$, then $\alpha_P(S)$ is the number of chains C of P such that $S=\{\rho(t)\colon t\in C\}$. For instance, $\alpha_P(\emptyset)=1$, $\alpha_P(i)$ (short for $\alpha_P\{i\}$) is the number of elements of P of rank i, and $\alpha_P([n-1])$ is the number of maximal chains of P. Define the *flag h-vector* $\beta_P\colon 2^{[n-1]}\to\mathbb{Z}$ by

$$\beta_P(S)=\sum_{T\subseteq S}(-1)^{\#(S-T)}\alpha_P(T).$$

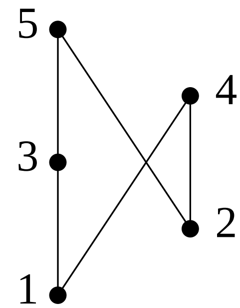

Figure 1: The poset Q_{12345}

Equivalently,

$$\alpha_P(S) = \sum_{T \subseteq S} \beta_P(T).$$

We say that β_P is *multiplicity-free* if $\beta_P(S) = 0, \pm 1$ for all $S \subseteq [n-1]$.

In this section we will classify and enumerate all P for which β_P is multiplicity-free. First we consider the case when P is a distributive lattice, so $P = J(Q)$ (the lattice of order ideals of Q) for some n-element poset Q (see [9, Thm. 3.4.1]). Suppose that Q is a natural partial ordering of $[n]$, i.e., if $i < j$ in Q, then $i < j$ in $\mathbb{Z}$. We may regard a *linear extension* of Q as a permutation $w = a_1 \cdots a_n \in \mathfrak{S}_n$ for which i precedes j in w if $i < j$ in Q. Write $\mathcal{L}(Q)$ for the set of linear extensions of Q, and let

$$D(w) = \{i \colon a_i > a_{i+1}\} \subseteq [n-1],$$

the *descent set* of w. A basic result in the theory of P-partitions [9, Thm. 3.13.1] asserts that

$$\beta_P(S) = \#\{w \in \mathcal{L}(Q) \colon D(w) = S\}.$$

It will follow from our results that the equivalence class containing $12 \cdots n$ of the equivalence relation $\sim$ on $\mathfrak{S}_n$ is the set $\mathcal{L}(Q)$ for a certain natural partial ordering Q of $[n]$ for which $\beta_{J(Q)}$ is multiplicity-free, and that Q has the most number of linear extensions of any n-element poset for which $\beta_{J(Q)}$ is multiplicity-free.

In general, if we have a partially commutative monoid M with generators $g_1, \ldots, g_n$ and if $w = g_{i_1} g_{i_2} \cdots g_{i_r} \in M$, then the set of all words in the g_i's that are equal to w correspond to the linear extensions of a poset Q_w with elements $1, \ldots, r$ [9, Exer. 3.123]. Namely, if $1 \leq a < b \leq r$ in $\mathbb{Z}$, then let $a < b$ in Q_w if $g_{i_a} = g_{i_b}$ or if $g_{i_a} g_{i_b} \neq g_{i_b} g_{i_a}$. In the case $g_i = i$ and $w = 12 \cdots n$, then the set of all words equal to w are themselves the linear extensions of Q_w. For instance, if $n = 5$, $g_i = i$, and the commuting relations are $12 = 21$, $23 = 32$, $34 = 43$, and $45 = 54$, then the poset Q_{12345} is shown in Figure 1. The linear extensions are the words equivalent to 12345 under $\sim$, namely 12345, 12354, 12435, 21345, 21354, 21435, 13245, 13254. Write Q_n for the poset $Q_{12 \cdots n}$. Define a subset S of $\mathbb{Z}$ to be *sparse* if it does not contain two consecutive integers.

Lemma 3.10. *For each sparse $S \subset [n-1]$, there is exactly one $w \in \langle 12 \cdots n \rangle$ (the equivalence class containing $12 \cdots n$ of the equivalence relation $\sim$) satisfying $D(w) = S$. Conversely, if $w \in \langle 12 \cdots n \rangle$ then $D(w)$ is sparse.*

Proof. The permutations $w \in \langle 12 \cdots n \rangle$ are obtained by taking the identity permutation $\mathrm{id} = 12 \cdots n$, choosing a sparse subset $S \subset [n-1]$, and transposing i and $i+1$ in id when $i \in S$. The proof follows. ∎

Proposition 3.11. *Let Q be an n-element poset for which the flag h-vector of $J(Q)$ is multiplicity-free. Then $e(Q) \leq F_{n+1}$ (a Fibonacci number). Moreover, the unique such poset (up to isomorphism) for which equality holds is Q_n.*

We will prove Proposition 3.11 as a consequence of a stronger result: the complete classification of all posets Q for which the flag h-vector of $J(Q)$ is multiplicity-free. The key observation is the following trivial result.

Lemma 3.12. *Let P be any graded poset whose flag h-vector β_P is multiplicity-free. Then P has at most two elements of each rank.*

Proof. Let P have rank n and $1 \leq i \leq n-1$. Then $\beta_P(i) = \alpha_P(i) - 1$, i.e., one less than the number of elements of rank i. The proof follows. ∎

Thus we need to consider only distributive lattices $J(Q)$ of rank n with at most two elements of each rank. A poset Q is said to be $(\mathbf{2}+\mathbf{2})$-*free* if it does not have an induced subposet isomorphic to the disjoint union of two 2-element chains. Such a poset is also an *interval order* [2][9, Exer. 3.15][11]. Similarly a poset is $(\mathbf{1}+\mathbf{1}+\mathbf{1})$-*free* or of *width* at most two if it does not have a 3-element antichain.

Theorem 3.13. *Let Q be an n-element poset. The following conditions are equivalent.*

- *(a) The flag h-vector $\beta_{J(Q)}$ of $J(Q)$ is multiplicity-free (in which case if $\beta_{J(Q)}(S) \neq 0$, then S is sparse).*
- *(b) For $0 \leq i \leq n$, $J(Q)$ has at most two elements of rank i. Equivalently, Q has at most two i-element order ideals.*
- *(c) Q is $(\mathbf{2}+\mathbf{2})$-free and $(\mathbf{1}+\mathbf{1}+\mathbf{1})$-free.*

Moreover, if $f(n)$ is the number of nonisomorphic n-element posets satisfying the above conditions, then

$$\sum_{n \geq 0} f(n) x^n = \frac{1-2x}{(1-x)(1-2x-x^2)}.$$

If $g(n)$ is the number of such posets which are not a nontrivial ordinal sum (equivalently, $J(Q)$ has exactly two elements of each rank $1 \leq i \leq n-1$), then

$$g(1) = g(2) = 1 \text{ and } g(n) = 2^{n-3}, \; n \geq 3. \tag{3.1}$$

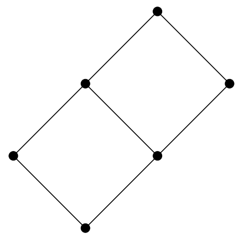

Figure 2: A distributive lattice of rank three

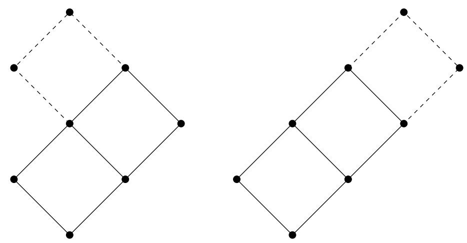

Figure 3: Extending a lattice of rank three

Proof. Consider first a distributive lattices $J(Q)$ of rank n with exactly two elements of rank i for $1 \leq i \leq n-1$. Such lattices are described in [9, Exer. 3.35(a)]. There is one for $n \leq 3$. The unique such lattice of rank three is shown in Figure 2. Once we have such a lattice L of rank $n \geq 3$, we can obtain two of rank $n+1$ by adjoining an element covering the left or right coatom (element covered by $\hat{1}$) of L and a new maximal element, as illustrated in Figure 3 for $n = 3$. When $n = 1$ (so L is a 2-element chain), we say by convention that the $\hat{0}$ of L is a *left* coatom. When $n = 2$ (so L is the boolean algebra B_2) we obtain isomorphic posets by adjoining an element covering either the left or right coatom, so again by convention we always choose the *right* coatom. Thus every such L of rank $n \geq 1$ can be described by a word $\gamma = \gamma_1\gamma_2\cdots\gamma_{n-1}$, where $\gamma_1 = 0$ (when $n \geq 2$), $\gamma_2 = 1$ (when $n \geq 3$), and $\gamma_i = 0$ or 1 for $i \geq 3$. If $\gamma_i = 0$, then at rank i we adjoin an element on the left, otherwise on the right. Write $L(\gamma)$ for this lattice. Figure 4 shows $L(01001)$.

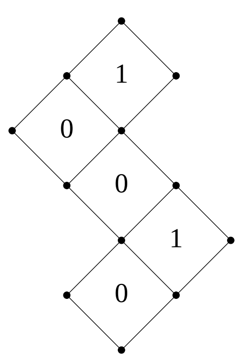

Figure 4: The distributive lattice $L(01001)$

Suppose that $n \geq 3$ and γ has the form $\delta j i^r$, where $r \geq 1$, $i = 0$ or 1, and $j = 1 - i$. For example, if $\gamma = 0111$ then $\delta = \emptyset$ (the empty word) and $r = 3$. If $\gamma = 0101100$, then $\delta = 0101$ and $r = 2$. Consider the lattice $L = L(\delta j i^r)$. Then for one coatom t of L we have $[\hat{0}, t] \cong L(\delta j i^{r-1})$. For the other coatom u of L there is an element $v < u$ such that $[v, u]$ is a chain and $[\hat{0}, v] \cong L(\delta)$. It follows easily that for $S \subseteq [n-1]$ we have the recurrence

$$\beta_{L(\delta i j^r)}(S) = \begin{cases} \beta_{L(\delta i j^{r-1})}(S), & n-1 \notin S \\ \beta_{L(\delta)}(S), & n-1 \in S. \end{cases} \tag{3.2}$$

Hence by induction β_L is multiplicity-free. Morever, if $\beta_L(S) \neq 0$, then S is sparse.

Suppose now that $J(Q)$ has exactly one element t of some rank $1 \leq i \leq n-1$. Let $[\hat{0}, t] \cong J(Q_q)$ and $[t, \hat{1}] \cong J(Q_2)$. Then $Q = Q_1 \oplus Q_2$ (ordinal sum), and

$$\beta_{J(Q)}(S) = \beta_{J(Q_1)}(S \cap [i-1]) \cdot \beta_{J(Q_1)}(S' \cap [n-i-1]), \tag{3.3}$$

where $S' = \{j \colon i + j \in S\}$. In particular, $\beta_{J(Q)}(S) = 0$ if $i \in S$. It follows from ?? Lemma 3.12 and equations (3.2) and (3.3) that (a) and (b) are equivalent.

Let us now consider condition (c). One can easily check that if $L(\gamma) = J(Q(\gamma))$, then $Q(\gamma)$ is $(\mathbf{2}+\mathbf{2})$-free and $(\mathbf{1}+\mathbf{1}+\mathbf{1})$-free. (Alternatively, if a poset Q contains an induced $\mathbf{2}+\mathbf{2}$ or $\mathbf{1}+\mathbf{1}+\mathbf{1}$ then it contains them as a convex subset, i.e., as a subset $I - I'$ where $I \leq I'$ in $J(Q)$. By considering linear extensions of Q that first use the elements of I' and then those of $I' - I$, one sees that at least two linear extensions have the same descent set.) Thus (b)$\Rightarrow$(c).

Conversely, suppose that $J(Q)$ has three elements of the same rank i. It is easy to see that the restriction of $J(Q)$ to ranks i and $i+1$ is a connected bipartite graph. If an element t of rank $i+1$ covers at least three elements of rank i, then Q contains an induced $\mathbf{1}+\mathbf{1}+\mathbf{1}$. Otherwise there must be elements s, t of rank $i+1$ and u, v, w of rank i for which $u, v < s$ and $v, w < t$. The interval $[u \wedge v \wedge w, u \vee v \vee w]$ is either isomorphic to a boolean algebra B_3, in which case Q contains an induced $\mathbf{1}+\mathbf{1}+\mathbf{1}$, or to $\mathbf{3} \times \mathbf{3}$, in which case P contains an induced $\mathbf{2}+\mathbf{2}$. Hence (c)$\Rightarrow$(b), completing the proof of the equivalence of (a), (b), and (c).

We have already observed that $g(n)$ is given by equation (3.1). Thus

$$A(x) := \sum_{n \geq 1} g(n) x^n = x + x^2 + \frac{x^3}{1-2x}.$$

Elementary combinatorial reasoning shows that

$$\begin{aligned} \sum_{n \geq 0} f(n) x^n &= \frac{1}{1 - A(x)} \\ &= \frac{1-2x}{(1-x)(1-2x-x^2)}, \end{aligned}$$

completing the proof. ∎

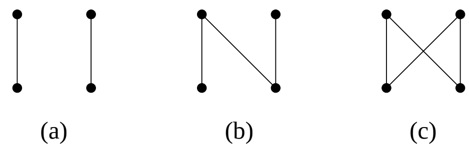

Figure 5: Two consecutive ranks

Corollary 3.14. *Let Q be an n-element poset for which $\beta_{J(Q)}$ is multiplicity-free. Then $e(Q) \leq F_{n+1}$ (a Fibonacci number), with equality if and only if $Q = Q(0101\cdots)$ where $0101\cdots$ is an alternating sequence of $n-1$ 0's and 1's.*

Proof. There are F_{n+1} sparse subsets of $[n-1]$. It follows from the parenthetical comment in Theorem 3.13(a) that $e(Q) \leq F_{n+1}$. Moreover, equations (3.2) and (3.3) make it clear that $\beta_{J(Q)}(S) = 1$ for all sparse $S \subset [n-1]$ if and only if $Q = Q(0101\cdots)$, so the proof follows. ∎

Proposition 3.15. *The n-element poset $Q(0101\cdots)$ is isomorphic to $Q_{12\cdots n}$. Hence The set $\mathcal{L}(Q(0101\cdots)$ of linear extensions of $Q(0101\cdots)$ is equal to the equivalence class $\langle 12\cdots n\rangle$.*

Proof. Immediate from Lemma 3.10. ∎

4 Multiplicity-free flag h-vectors of graded posets.

We now consider *any* graded poset P of rank n for which β_P is multiplicity-free. By Lemma 3.12 there are at most two elements of each rank. If there is just one element t of some rank $1 \leq i \leq n-1$, then let $P_1 = [\hat{0}, t]$ and $P_2 = [t, \hat{1}]$. Equation (3.3) generalizes easily to

$$\beta_P(S) = \begin{cases} \hfill 0, \quad i \in S \\ \beta_{P_1}(S \cap [i-1]) \cdot \beta_{P_2}(S' \cap [n-i-1], i \notin S, \end{cases}$$

where $S_1 = \{j \colon i+j \in S\}$. Hence β_P is multiplicity-free if and only if both β_{P_1} and β_{P_2} are multiplicity-free.

By the previous paragraph we may assume that P has exactly two elements of each rank $1 \leq i \leq n-1$, i.e., of each *interior rank*. There are up to isomorphism three possibilities for the restriction of P to two consecutive interior ranks i and $i+1$ ($1 \leq i \leq n-2$). See Figure 5. If only type (b) occurs, then we obtain the distributive lattice $L(\gamma)$ for some γ. Hence all graded posets with two elements of each interior rank can be obtained from some $L(\gamma)$ by a sequence of replacing the two elements of some interior rank with one of the posets in Figure 5(a) or (c).

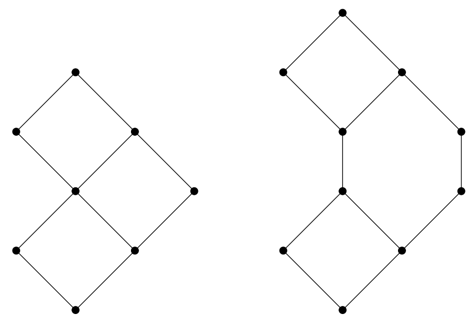

Figure 6: An example of R and its stretching $R[2]$

First consider the situation where we replace the two elements of some interior rank of P with the poset of Figure 5(a). We can work with the following somewhat more general setup. Let R be any graded poset of rank n with $\hat{0}$ and $\hat{1}$. For $1 \leq i \leq n-1$ let $R[i]$ denote the *stretching* of R at rank i, namely, for each element $t \in R$ of rank i, adjoin a new element $t' > t$ such that $t' < u$ whenever $t < u$ (and no additional relations not implied by these conditions). Figure 6 shows an example. Regarding i as fixed, let $S \subset [n]$. If not both $i, i+1 \in S$ then let S° be obtained from S by replacing each element $j \in S$ satisfying $j \geq i+1$ with $j-1$. On the other hand, if both $i, i+1 \in S$ then let S° be obtained from S by removing $i+1$ and replacing each $j \in S$ such that $j > i+1$ with $j-1$. It is easily checked that

$$\beta_{R[i]}(S) = \begin{cases} \beta_R(S^\circ), & \text{if not both } i, i+1 \in S \\ -\beta_R(S^\circ), & \text{if both } i, i+1 \in S. \end{cases}$$

It follows immediately that if β_R is multiplicity-free, then so is $\beta_{R[i]}$.

Now consider the situation where we replace the two elements of some interior rank of P with the poset of Figure 5(c). Again we can work in the generality of any graded poset R of rank n with $\hat{0}$ and $\hat{1}$. If $1 \leq i \leq n-1$, let $R\langle i \rangle$ denote the *proliferation* of R at rank i, namely, for each element $t \in R$ of rank i, adjoin a new element $t' > s$ for every s of rank i such that $t' < u$ whenever $t < u$ (and no additional relations not implied by these conditions). Figure 7 shows an example. Note that if R_1 denotes the restriction of R to ranks $0, 1, \ldots, i$, and if R_2 denotes the restriction of R to ranks $i, i+1, \ldots, n$, then $R\langle i \rangle = R_1 \oplus R_2$ (ordinal sum). Let $\bar{R}_1$ denote R_1 with a $\hat{1}$ adjoined and $\bar{R}_2$ denote R_2 with a $\hat{0}$ adjoined. It is then clear (in fact, a simple variant of equation (3.3)) that

$$\beta_{R\langle i \rangle}(S) = \beta_{R_1}(S \cap [i]) \cdot \beta_{R_2}(S'),$$

where $S' = \{j : i+j \in S\}$. ??

We have therefore proved the following result.

Theorem 4.16. *Let P be a finite graded poset with $\hat{0}$ and $\hat{1}$. The following conditions are equivalent:*

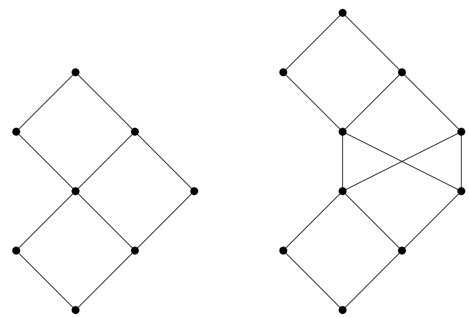

Figure 7: An example of R and its proliferation $R\langle 2\rangle$

(a) The flag h-vector β_P is multiplicity-free.

(b) P has at most two elements of each rank.

The above description of graded posets with multiplicity-free flag h-vectors allows us to enumerate such posets.

Theorem 4.17. *Let $h(n,k)$ denote the number of k-element graded posets P of rank n with $\hat{0}$ and $\hat{1}$ for which β_P is multiplicity-free. Let*

$$U(x,y)=\sum_{n\geq 1}\sum_{k\geq 2}h(n,k)x^k y^n.$$

Then

$$U(x,y)=\frac{xy^2(1-xy^2)(1-3xy^3)}{1-xy-5xy^2+4x^2y^3+5x^2y^4-3x^3y^5}.$$

Proof. The factor xy^2 in the numerator accounts for the $\hat{0}$ and $\hat{1}$ of P. Write $P'=P-\{\hat{0},\hat{1}\}$. We first consider those P' that are not an ordinal sum of smaller nonempty posets. These will be the one-element poset **1** and posets for which every rank has two elements, with the restrictions to two consecutive ranks given by Figure 5(a,b). We obtain P' by first choosing a poset R whose consecutive ranks are given by Figure 5(b) and then doing a sequence of stretches. By equation (3.1), the number of ways to choose R with m levels is 1 for $m=1$ and 2^{m-2} for $m\geq 2$. We can stretch such an R by choosing a sequence $(j_1,\ldots,j_m)$ of nonnegative integers and stretching the ith level of R j_i times. Hence the generating function for the posets P' is given by

$$\begin{aligned} R(x,y) &= xy+\frac{xy^2}{1-xy^2}+\sum_{m\geq 2}\frac{2^{m-2}(xy^2)^m}{(1-xy^2)^m} \\ &= xy+\frac{xy^2}{1-xy^2}+\frac{x^2y^4}{(1-xy^2)(1-3xy^2)}. \end{aligned}$$

All posets being enumerated are unique ordinal sums of posets P', with a $\hat{0}$ and $\hat{1}$ adjoined at the end. Thus

$$\begin{aligned} U(x,y) &= \frac{xy^2}{1-R(x,y)} \\ &= \frac{xy^2(1-xy^2)(1-3xy^2)}{1-xy-5xy^2+4x^2y^3+5x^2y^4-3x^3y^5}. \end{aligned}$$

∎

As special cases, the enumeration by rank of graded posets P with $\hat{0}$ and $\hat{1}$ for which β_P is multiplicity-free is given by

$$\begin{aligned} R(x,1) &= \frac{x(1-x)(1-3x)}{1-6x+9x^2-3x^3} \\ &= x+2x^2+6x^3+21x^4+78x^5+297x^6+1143x^7+4419x^8+\cdots. \end{aligned}$$

Similarly, if we enumerate by number of elements we get

$$\begin{aligned} R(1,y) &= \frac{y^2(1-y^2)(1-3y^2)}{1-y-5y^2+4y^3+5y^4-3y^5} \\ &= y^2+y^3+2y^4+3y^5+7y^6+12y^7+28y^8+51y^9+117y^{10}+\cdots. \end{aligned}$$

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