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On minors of maximal determinant matrices

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Abstract

By an old result of Cohn (1965), a Hadamard matrix of order n has no proper Hadamard submatrices of order $m > n/2$. We generalise this result to maximal determinant submatrices of Hadamard matrices, and show that an interval of length $\sim n/2$ is excluded from the allowable orders. We make a conjecture regarding a lower bound for sums of squares of minors of maximal determinant matrices, and give evidence to support it. We give tables of the values taken by the minors of all maximal determinant matrices of orders ≤ 21 and make some observations on the data. Finally, we describe the algorithms that were used to compute the tables.

1 Introduction

A $\{+1, -1\}$-matrix (abbreviated "$\{\pm 1\}$-matrix" below) is a matrix whose elements are $+1$ or -1. The *Hadamard maximal determinant problem*, posed by Hadamard [18], is to find the maximal determinant $D(n)$ of an $n \times n$ $\{\pm 1\}$-matrix of given order n. A matrix that attains the maximum is a *maximal determinant matrix* (abbreviated *maxdet matrix*). Such matrices, of which Hadamard matrices are a special case, are of interest in combinatorics and have applications in statistical design [23], coding theory and signal processing [20, 35, 39]. In design theory a maxdet matrix is also known as a *saturated D-optimal design*.

Finding maxdet matrices is in general difficult, with the difficulty depending on the congruence class of $n \pmod 4$. Most is known for $n \equiv 0 \pmod 4$ (the *Hadamard orders*), and least for $n \equiv 3 \pmod 4$.[1] Maxdet matrices are known for orders 1 through 21 inclusive; for $n = 22$ and $n = 23$ there are conjectures (see [35]) but as yet no proofs of maximality. Constructions exist for various infinite families and ad-hoc examples, see [36].

Consider the $k \times k$ submatrices $S_k(A)$ of an $n \times n$ $\{\pm 1\}$-matrix A, where $0 < k \le n$. For $M \in S_k(A)$, we say that $\det(M)$ is a *minor of order k of A*. Usually only the absolute value of $\det(M)$ is of interest.

One method for finding maxdet or large-determinant matrices involves choosing a large-determinant matrix of a slightly smaller (or larger) order than the desired order, possibly perturbing it by a low-rank modification, and adding (or removing) a small number of suitably chosen rows and columns. For example, Solomon [35, 42] found a (conjectured) maxdet matrix of order 33 and determinant 441×2^{74} in this way by starting with an appropriate Hadamard matrix of order 32, and in [8, 27, 31] the method was used to obtain lower bounds on $D(n)$.

This motivates our interest in the minors of maxdet matrices, and in particular the question: *What is the largest order of a maxdet matrix contained as a proper submatrix of a given maxdet matrix?* In this paper we answer this question for the maxdet submatrices of maxdet matrices of orders $n \le 21$ by computing all minors of maxdet matrices of these orders. Our work extends that of earlier researchers who have considered minors of Hadamard and maxdet matrices with other applications in mind, such as the problem of growth in Gaussian elimination [13, 26, 28, 29, 38].

Schmidt [37, pg. 441] says *"The nature of the construction $\cdots$ is in line with the computer assisted observation that binary matrices with maximal determinants may not contain large order submatrices with large determinants"*.

[1] See Đokovic and Kotsireas [15] for a recent summary of what is known about the cases $n \equiv 2 \pmod 4$, and Brent *et al* [7] for the cases $n \equiv 1 \pmod 2$.

Whether this is true in general depends on the precise meaning of "large". Certainly there are exceptions. For example, the maxdet $\{\pm 1\}$-matrix of order 17 contains a maxdet (Hadamard) submatrix of order 16.

In Theorem 1 of §2 we give a new proof of an old result of Cohn [11] that a Hadamard matrix of order n can not have a proper Hadamard submatrix of order $m > n/2$. We then generalize the proof to cover maxdet submatrices of Hadamard matrices, and show, assuming the Hadamard conjecture, that a Hadamard matrix of order n can not have a maxdet submatrix M of order $m \ge n/2 + 5\log n$ unless $m \ge n-2$ (see Theorem 2). In other words, m can not lie in the interval $[n/2 + 5\log n, n-2)$. Without the assumption of the Hadamard conjecture, we can still exclude an interval $[n/2 + o(n), n - o(n))$ of length $\sim n/2$, see Theorem 3 and Remark 2. These results partially confirm the remark of Schmidt quoted above. However, they apply only to sub-matrices of Hadamard matrices. Except in small cases that are amenable to explicit computation, we have not excluded the possibility that a maxdet matrix of order $n \not\equiv 0 \pmod 4$ has a maxdet submatrix of any given order $m < n$.

In §3 we define two sequences related to the sets of minors of maxdet matrices, and give the first 21 terms of each sequence.

In §4 we describe the minors that occur in maxdet matrices of orders 1 through 21, and make some observations on the patterns that occur. We mention a result (Proposition 3), on small minors of Hadamard matrices, which was suggested by the data before a proof was found.

Motivated by Turán's result [44] that the expected value of $\det(A)^2$ is $m!$ for random $\{\pm 1\}$ matrices of order m, we consider the mean value of $\det(M)^2$ over all $m \times m$ submatrices of maxdet matrices of order $n \ge m$, and conjecture that it is bounded below by $m!$ (see Conjecture 1). The conjecture is consistent with our computations for $1 \le m \le n \le 21$.

Finally, in §5 we describe the algorithms that we used to compute minors of square $\{\pm 1\}$-matrices, as well as some related algorithms that were considered but rejected for various reasons.

Hadamard equivalence and HT-equivalence[2]

Two $\{\pm 1\}$-matrices are *Hadamard equivalent* if one can be obtained from the other by negating rows or columns, and/or by interchanging rows or columns. Clearly the answers to the questions posed above about minors are the same for all matrices in a Hadamard equivalence class, and also for any matrix A and its transpose (dual) A^T. Hence, it is useful to define the notion of *HT-equivalence* by saying that two matrices A and B are *HT-equivalent* if A is Hadamard-equivalent to B *or* A is Hadamard-equivalent to B^T. For example, there are 5 distinct Hadamard equivalence classes of Hadamard matrices of order 16, but two of these classes contain matrices that are dual to those in the other class, so there are only 4 distinct HT-equivalence classes.

2 Excluded minors of Hadamard matrices

In this section we consider the possible orders of submatrices M of a Hadamard matrix H, satisfying the condition that M is Hadamard (see Theorem 1) or, more generally, that M is a maxdet matrix (see Theorems 2–3).

Recently Szöllősi [43, Proposition 5.5] established an elegant correspondence between the minors of order m and of order $n-m$ of a Hadamard matrix of order n. His result applies to complex Hadamard matrices, of which $\{\pm 1\}$ Hadamard matrices are a special case. More precisely, if $d+m=n$, $0 < d < n$, then corresponding to each $m \times m$ submatrix with determinant μ there is a $d \times d$ submatrix with determinant $\pm n^{n/2-d}\mu$. Only special cases (for small d or m) were known before Szöllősi (see for example [13, 28, 38, 40]). This is perhaps surprising as Szöllősi's critical Lemma 5.7 follows in a straightforward manner from Jacobi's determinant identity [21]. We use the following corollary of Szöllősi's theorem.[3]

Corollary 1. *Suppose that a Hadamard matrix H of order n has an $m \times m$ submatrix M, where $n/2 \le m \le n$. Then*

$$|\det(M)| \le n^{m-n/2} D(n-m)\,.$$

[2] There seems to be no widely-accepted name for this concept. In [5] HT-equivalence is called “extended Hadamard equivalence”. Wanless [45] calls an HT-equivalence class a “resemblance class”.

[3] Corollary 1 is essentially the same as Theorem 3 of Cohn [11], the difference being that Cohn replaces $D(n-m)$ by the Hadamard bound. However, Cohn's proof is quite different from ours.

Proof. By Szöllősi's theorem, $|\det(M)| = n^{n/2-d}|\det(M')|$, where $d = n - m$ and M' is some $d \times d$ submatrix of H. Since $n/2 - d = m - n/2$ and $|\det(M')| \le D(d)$ by the definition of D, the corollary follows. □

The following lower bound on $D(m)$ is given in [8, Corollary 3].

Proposition 1. *Assume the Hadamard conjecture.*[4] *Then, for $m \ge 4$, we have $D(m) \ge 4m^{m/2-1}$.*

Suppose that a Hadamard matrix H of order n has a maxdet submatrix M of order m. Corollary 1 gives an upper bound on $|\det(M)|$, and Proposition 1 gives a lower bound. Theorems 1–2 show that these bounds are incompatible for certain values of m. Theorem 1 considers the case that M is a Hadamard matrix, and Theorem 2 considers the more general case that M is a maxdet matrix. We have stated Theorem 2 under the assumption of the Hadamard conjecture, but a weaker result is provable without this assumption – see Theorem 3.

Theorem 1 was proved by Cohn [11, Theorem 2], but we give a different proof which generalizes to give proofs of Theorems 2–3.

Theorem 1 (Cohn). *Let H be a Hadamard matrix of order n having a Hadamard submatrix M of order $m < n$. Then $m \le n/2$.*

Proof. The theorem is trivial if $m \le n/2$, so assume that $n > m > n/2$. Let $d = n - m$. Since M exists and $|\det(M)| = m^{m/2}$, Corollary 1 and Hadamard's bound for $D(d)$ give

$$m^{m/2} \le n^{m-n/2} d^{d/2}\,. \tag{1}$$

Squaring both sides of (1), we have $(m/n)^m \le (d/n)^d$. Taking n-th roots and defining $x := m/n \in (0,1)$, we see that

$$x^x \le (1-x)^{1-x}.$$

This inequality is equivalent to $f(x) \le 0$, where $f : [0,1] \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x \ln x - (1-x)\ln(1-x) & \text{if } x \in (0,1), \\ 0 \text{ otherwise.} \end{cases}$$

[4] It is sufficient to assume that Hadamard matrices of order $4k$ exist for all positive integers $k \le (m+2)/4$. This is known to be true for $4k < 668$, see [24, 39].

It is easy to verify that $f(1/2) = 0$, $f'(x) = 2 + \ln x + \ln(1-x)$, and

$$f''(x) = \frac{1-2x}{x(1-x)} < 0 \ \text{ in } (1/2, 1).$$

Thus, $f(x) > 0$ in $(1/2, 1)$, so we must have $x \le 1/2$ or $x = 1$. The case $x = 1$ is ruled out because it implies that $m = n$, contrary to the assumption that $m < n$. Thus $x \le 1/2$, which implies that $m \le n/2$. □

Theorem 2. *Assume the Hadamard conjecture. Let H be a Hadamard matrix of order n having a maxdet submatrix M of order $m < n$. Then $m < (n/2 + 5\ln n)$ or $m \ge n-2$.*

Proof. The result is trivial if $m \le n/2$, and $n/2 + 5\ln n > n - 3$ for $n \le 28$, so we can assume that $m > n/2 > 14$. By Proposition 1 and Corollary 1, we have

$$m^{m/2}\left(\frac{4}{m}\right) \le n^{m-n/2} d^{d/2}, \tag{2}$$

where $d = n - m$. We prefer to use the slightly weaker inequality

$$m^{m/2}\left(\frac{4}{n}\right) \le n^{m-n/2} d^{d/2}. \tag{3}$$

Taking logarithms, and defining $x := m/n$ and f as in the proof of Theorem 1, we obtain

$$f(x) \le \frac{2\ln(n/4)}{n}. \tag{4}$$

The right side of this inequality is positive (since $n > 4$) and independent of x (this is why we used (3) instead of (2)). We showed in the proof of Theorem 1 that $f(1/2) = f(1) = 0$ and $f''(x) < 0$ on $(1/2, 1)$. Let

$$x_{\max} = (1 + \sqrt{1 - 4/e^2})/2 \approx 0.84$$

be the (unique) point in $(1/2, 1)$ at which $f'(x)$ vanishes. Thus $f(x)$ attains its maximum value at $x = x_{\max}$. Since $2\ln(n/4)/n < 0.14 < f(x_{\max}) \approx 0.15$ for $n > 28$, the inequality (4) is not satisfied for all $x \in (1/2, 1)$, and there is a unique interval $(x_0, x_1) \subseteq (1/2, 1)$ on which $f(x) > 2\ln(n/4)/n$, with $x_{\max} \in (x_0, x_1)$. (Here x_0 and x_1 depend on n but not on m.) It follows that there can not exist a maxdet submatrix of order m with $x_0 < m/n < x_1$.

To locate x_1 we consider the case $d = n - m = 3$. The inequality (3) gives

$$\left(\frac{n-3}{n}\right)^{n-3} \le \frac{27}{16n}\,, \tag{5}$$

but the left side of this inequality is bounded away from zero as $n \to \infty$, whereas the right side tends to zero. Thus, the inequality can not hold for large n. In fact, a computation shows that (5) can only hold for $n < 29$. Thus, for $n \ge 29$ an interval (x_0, x_1) as above exists, with $x_1 > 1 - 3/n$, so $m = n - 3$ is not a possible order of a maxdet submatrix M.

We now show that $nx_0 < n/2 + 5\ln n$. Define $\nu := n/2$, $\delta := m - \nu$. Thus $m = \nu + \delta$, $d = \nu - \delta$, and squaring the inequality (3) gives

$$(\nu+\delta)^{\nu+\delta}\left(\frac{2}{\nu}\right)^2 \le (2\nu)^{2\delta}(\nu-\delta)^{\nu-\delta}\,,$$

or equivalently

$$\left(\frac{\nu+\delta}{\nu-\delta}\right)^{\nu} \le \left(\frac{\nu}{2}\right)^2\left(\frac{4\nu^2}{\nu^2-\delta^2}\right)^{\delta}\,. \tag{6}$$

Write $z := \delta/\nu = 2x - 1$. We can assume that

$$0 < z < z_{max} = 2x_{max} - 1 = \sqrt{1 - 4/e^2} \approx 0.68\,.$$

Taking logarithms in (6) gives

$$\frac{\delta}{z}\ln\left(\frac{1+z}{1-z}\right) \le \delta(\ln 4 - \ln(1-z^2)) + 2\ln(\nu/2). \tag{7}$$

Collecting the terms involving δ gives

$$\delta(2 - \ln 4 - \varepsilon(z)) \le 2\ln(\nu/2),$$

where

$$\varepsilon(z) = 2 - \frac{1}{z}\ln\left(\frac{1+z}{1-z}\right) - \ln(1-z^2) = \sum_{k=1}^{\infty}\frac{z^{2k}}{k(2k+1)}$$

is monotonic increasing on $[0, z_{\max}]$, so

$$\varepsilon(z) \le \varepsilon(z_{\max}) < 0.1803,$$

and thus $2 - \ln 4 - \varepsilon(z) > 0.4334$ for $z \in (0, z_{\max})$. It follows that

$$\delta < \frac{2\ln(\nu/2)}{0.4334} < 5\ln n.$$

□

Remark 1. We do not expect the values $m = n - 1$ or $m = n - 2$ to occur for $n > 4$. They have to be included as possibilities simply because the lower bound on $D(n)$ given by Proposition 1 is too weak to exclude them. It is possible to have $m > n/2$, for example a maxdet submatrix of order $m = n/2 + 1$ occurs in Hadamard matrices of orders $n = 4$ and $n = 12$.

We can prove a result similar to, but weaker than, Theorem 2 without assuming the Hadamard conjecture. Let the *prime gap function* $\lambda(n)$ be the maximum gap between consecutive primes (p_i, p_{i+1}) with $p_i \le n$. Then we have:

Theorem 3. *Let H be a Hadamard matrix of order $n \ge 4$ having a maxdet submatrix M of order $m < n$, and let $\lambda(n)$ be the prime gap function. Then there exist positive constants c_1, c_2 such that $m < n/2 + c_1\lambda(n)\ln n$ or $m \ge n - c_2\lambda(n)$.*

Sketch of proof. The proof is similar to that of Theorem 2, but uses Theorem 1 and Corollary 1 of [8] in place of Proposition 1. Thus (3) is replaced by

$$m^{m/2}\left(\frac{4}{ne}\right)^{\lambda(n/2)/2} \le n^{m-n/2}d^{d/2},$$

and (4) by

$$f(x) \le \lambda(n/2)\,\frac{\ln(ne/4)}{n}\,.$$

The remainder of the proof follows that of Theorem 2, except that some of the explicit constants have to be replaced by $O(\lambda(n/2))$ terms, and we have to assume that n is sufficiently large, say $n \ge n_0$. At the end, we can increase c_1 or c_2 if necessary to ensure that $n/2 + c_1\lambda(n)\ln n > n - c_2\lambda(n)$ for $4 \le n < n_0$. $\square$

Remark 2. By a result of Baker, Harman and Pintz [2], $\lambda(n) = O(n^{21/40})$, so the excluded interval in Theorem 3 has length $\sim n/2$.

Remark 3. It should be possible to sharpen Theorems 2–3 by using the (asymptotically sharper) bounds on $D(m)$ given in [10] instead of the bound of Proposition 1 (this is work in progress).

3 Sequences related to minors

In this section we define two sequences related to the minors of maxdet matrices, and give the first 21 terms in each sequence [34]. In the following definitions, $\mathbb{N}$ denotes the positive integers.

For the convenience of the reader, Tables 1–2 give some data taken from Orrick and Solomon [35], where references to the original sources may be found. Table 1 gives the *spectrum* of possible (absolute values of) determinants of $\{\pm 1\}$-matrices of order $n \le 11$, normalised by the usual factor 2^{n-1}. In this and other tables, the notation "$a..b$" is a shorthand for "$\{x \in \mathbb{N} : a \le x \le b\}$". Table 2 gives $\Delta(n) := D(n)/2^{n-1}$ for $n \le 21$.

n	Spectrum $\{\lvert\det(A)\rvert/2^{n-1}\}$
1	$\{1\}$
2	$\{0, 1\}$
3	$\{0, 1\}$
4	$\{0..2\}$
5	$\{0..3\}$
6	$\{0..5\}$
7	$\{0..9\}$
8	$\{0..18, 20, 24, 32\}$
9	$\{0..40, 42, 44, 45, 48, 56\}$
10	$\{0..102, 104, 105, 108, 110, 112,$ $116, 117, 120, 125, 128, 144\}$
11	$\{0..268, 270..276, 278..280, 282..286,$ $288, 291, 294..297, 304, 312, 315, 320\}$

Table 1: Spectrum of $\{\pm 1\}$-matrices of order $n \le 11$, from Orrick and Solomon [35]; for $n = 13$ see [7].

We are interested in when the full spectrum of possible minor values occurs in the minors of maxdet matrices of given order n.

Definition 1. *The* full-spectrum threshold *of an $n \times n$ $\{\pm 1\}$ matrix A is the maximum $m_f \le n$ such that the full spectrum of possible values occurs for the minors of order m_f of A.*

Definition 2. *The* full-spectrum threshold $m_f : \mathbb{N} \to \mathbb{N}$ *is the maximum of the full-spectrum threshold of A over all maxdet matrices A of order n.*

n	$\Delta(n)$	n	$\Delta(n)$	n	$\Delta(n)$	n	$\Delta(n)$
–	–	1	1	2	1	3	1
4	2	5	3	6	5	7	9
8	4×2^3	9	7×2^3	10	18×2^3	11	40×2^3
12	6×3^5	13	15×3^5	14	39×3^5	15	105×3^5
16	8×4^7	17	20×4^7	18	68×4^7	19	833×4^6
20	10×5^9	21	29×5^9	–	–	–	–

Table 2: Maximal determinants $\Delta(n) = D(n)/2^{n-1}$, $n \le 21$, from Orrick and Solomon [35].

We write $m_f(A)$ or $m_f(n)$ to denote the full-spectrum thresholds of Definitions 1 or 2 respectively; which is meant should be clear from the context. The values of $m_f(n)$ for $1 \le n \le 21$ are given in Table 3. We note that the full-spectrum threshold $m_f(A)$ does depend on the HT-equivalence class of A. For example, the four HT-equivalence classes for order 16 give four different values $m_f \in \{5, 6, 7, 8\}$, see Tables 23–26.

For the reasons mentioned in §1, we are also interested in the largest order of a maxdet matrix contained as a proper submatrix of a given maxdet matrix. We make some definitions analogous to Definitions 1–2.

Definition 3. *The* complementary depth *of an $n \times n$ $\{\pm 1\}$ matrix A is the maximum $m_d < n$ such that a maxdet matrix of order m_d occurs as a proper submatrix of A, or* 0 *if $n = 1$. The* depth *of A is $d(A) := n - m_d(A)$.*

Definition 4. *The* complementary depth *$m_d : \mathbb{N} \to \mathbb{Z}$ is the maximum of the complementary depth of A over all maxdet matrices A of order n. The* depth *$d : \mathbb{N} \to \mathbb{Z}$ is defined by $d(n) := n - m_d(n)$.*

We write $d(A)$ or $d(n)$ for the depths of Definitions 3 or 4 respectively; similarly for $m_d(A)$ and $m_d(n)$. Clearly $d(A)$ depends on the HT-equivalence class of A – for example, see Tables 30–32 for the three HT-equivalence classes of order 18 with depths 7, 7 and 10. From Definition 4, $d(n)$ is the minimum of $d(A)$ over all maxdet matrices A of order n, so $d(18) = 7$. Computed values of $d(n)$, $m_d(n)$ and $m_f(n)$ for $1 \le n \le 21$ are given in Table 3. It is clear from the definitions that $m_f(n) \le m_d(n)$ for $n > 1$.

If $n \equiv 0 \pmod 8$ then $d(n) = n/2$ in the range of our computations. If $n \equiv 4 \pmod 8$ then both $d(n) = n/2$ (for $n = 20$) and $d(n) = n/2 - 1$ (for

n	1	2	3	4	5	6	7	8	9	10	11	12
d	1	1	1	1	1	1	1	4	1	2	3	5
m_d	0	1	2	3	4	5	6	4	8	8	8	7
m_f	1	1	2	2	3	4	6	4	6	6	7	6

n	13	14	15	16	17	18	19	20	21
d	6	7	8	8	1	7	10	10	10
m_d	7	7	7	8	16	11	9	10	11
m_f	7	7	7	8	8	8	9	8	10

Table 3: Depths $d(n)$ of largest maxdet proper submatrices, complementary depth $m_d(n) = n - d(n)$, and full-spectrum threshold $m_f(n)$; see Definitions 2–4.

$n = 4, 12$) are possible. We see from Table 3 that the computed values all satisfy $d(n) \le (n+1)/2$. It is interesting that the value $d = 1$ occurs for $n \le 7$, $n = 9$ and $n = 17$ (contrary to the remark of Schmidt quoted in §1).

4 Further results and observations on minors

In Tables 6–43, which may be found in the Appendix at the end of the paper, we give computational results on the minors of maxdet matrices of order $n \le 21$. Here we make some empirical observations on the results and mention a fact (Proposition 3) that was suggested by them.

Let $k = \lfloor n/4 \rfloor$. Factors of the form $k^{2k}, k^{2k-1}, \ldots, k^2, k$ are present as we descend through the minors of maxdet matrices of even order n. For Hadamard matrices this is an easy consequence of the Hadamard bound $n^{n/2}$ and Szöllősi's theorem, but for $n \equiv 2 \pmod 4$ we do not have a simple explanation. Factors of the form $k^{2k-1}, k^{2k-2}, \ldots, k$ are present in the minors of maxdet matrices with order $n \equiv 1 \pmod 4$, and factors of the form $k^{2k-2}, k^{2k-3}, \ldots, k$ are present if $n \equiv 3 \pmod 4$. It is an open question whether this behaviour persists for $n > 21$. The observed divisibility properties are related to the structure of the Gram matrices $A^T A$ of maxdet matrices A, but in general this structure is unknown. For a summary of what is currently known, see [35].

The presence of high powers of $k = \lfloor n/4 \rfloor$ in the minors of order m, and high powers of a possibly different integer $k' = \lfloor m/4 \rfloor$ in the maximal determinants of order m, gives one explanation of why certain minors can not meet the maximal determinant for that order.[5] For example, Table 19 shows that a Hadamard matrix of order $n = 12$ has minors of order 11 with scaled value 3^5, but $\Delta(11) = 5 \times 2^6$, which contains a high power of 2, not of 3.

Proposition 2 is from [9, Theorem 1]. The upper bound is sharp because it is attained for Hadamard matrices. For the case that A is a Hadamard matrix, the result is due to de Launey and Levin [31, Proposition 2].

Proposition 2. *Let A be a square $\{\pm 1\}$ matrix of order $n \geq m > 1$. Then the mean value $E(\det(M)^2)$ of $\det(M)^2$, taken over all $m \times m$ submatrices M of A, satisfies*

$$E(\det(M)^2) \leq n^m \Big/ \binom{n}{m}. \tag{8}$$

Moreover, equality holds in (8) *iff A is a Hadamard matrix.*

Remark 4. For random $\{\pm 1\}$ matrices M of order m, the expected value of $\det(M)^2$ is $m!$, by a result of Turán [44]. The last sentence of Proposition 2 implies that $E(\det(M)^2) \geq m!$ for the order-m submatrices M of a Hadamard matrix, with strict inequality if $m > 1$.

As a check on the correctness of our programs, we computed the mean value of $\det(M)^2$ for submatrices of order m of maxdet matrices of order $n \leq 21$, and $2 \leq m \leq n$. The results agreed with the predictions of Proposition 2. The following conjecture is consistent with our computations.

Conjecture 1. *Let A be a maxdet matrix. Then the mean value $E(\det(M)^2)$ of $\det(M)^2$ taken over all $m \times m$ submatrices M of A satisfies the inequality*

$$E(\det(M)^2) \geq m! \tag{9}$$

Moreover, the inequality (9) *is strict for $m > 1$.*

Table 4 gives some data for orders $13 \leq n \leq 15$ to support Conjecture 1. In the table, $R_{\mathrm{L}}(m, n)$ is the ratio of $E(\det(M)^2)$, for submatrices M of order m of a maxdet matrix of order n, to the conjectured lower bound $m!$.

[5] Of course, this begs the question of *why* the maximal determinants are divisible by a high power of k' – we do not have a convincing explanation for this unless the order m is such that the Hadamard, Barba [3], or Ehlich-Wojtas [16, 47] bound is achieved, in which case it follows from the form of the relevant bound, see Osborn [36, pp. 98–99].

Similarly, $R_{\mathrm{H}}(m,n)$ is the ratio of $E(\det(M)^2)$ to the upper bound (8). We see that $R_{\mathrm{L}}(m,n) > 1$ for $2 \le m \le n$ (as conjectured), and the lower bound is reasonably good for $m \le 5$, but deteriorates for larger m. The upper bound is within a factor of three of $E((\det M)^2)$ for all m. A similar pattern occurs for all orders $n \le 21$, except that $R_{\mathrm{H}}(m,n) = 1$ if n is a Hadamard order, in accordance with Proposition 2.

m	$R_{\mathrm{L}}(m,13)$	$R_{\mathrm{H}}(m,13)$	$R_{\mathrm{L}}(m,14)$	$R_{\mathrm{H}}(m,14)$	$R_{\mathrm{L}}(m,15)$	$R_{\mathrm{H}}(m,15)$
2	1.077	0.994	1.067	0.991	1.059	0.988
3	1.259	0.983	1.222	0.973	1.195	0.966
4	1.611	0.968	1.516	0.948	1.445	0.935
5	2.283	0.949	2.054	0.917	1.890	0.897
6	3.625	0.928	3.072	0.882	2.694	0.852
7	6.560	0.904	5.139	0.843	4.233	0.804
8	13.81	0.879	9.773	0.802	7.427	0.752
9	34.80	0.851	21.58	0.759	14.79	0.699
10	109.4	0.823	56.93	0.715	34.14	0.645
11	457.4	0.795	187.0	0.671	94.00	0.592
12	2864	0.765	815.6	0.627	321.7	0.540
13	35796	0.736	5318	0.584	1461	0.491
14			69137	0.542	9902	0.444
15					133638	0.399

Table 4: Ratio of $E(\det(M)^2)$ to lower and upper bounds, $13 \le n \le 15$. For definitions of $R_{\mathrm{L}}(m,n)$ and $R_{\mathrm{H}}(m,n)$, see text.

The frequencies of occurrence of small singular submatrices of Hadamard matrices are given in the following Proposition [9, Corollary 4], which was suggested by the computational results before we found a proof. The case $m = 2$ is implicit in a paper of Little and Thuente [32, pg. 254].

Proposition 3. *Let H be a Hadamard matrix of order n, and let $Z(m,H)$ be the number of minors of order m of H that vanish. Then*

$$Z(2,H) = n^2(n-1)(n-2)/8, \quad \textit{and} \tag{10}$$

$$Z(3,H) = n^2(n-1)(n-2)(n-4)(5n-4)/288. \tag{11}$$

Remark 5. There is no analogue of Proposition 3 for minors of order $m > 3$, because the value of $Z(m, H)$ can depend on the HT-equivalence class of H, so is not given by a polynomial in m (unless $m \le 3$). For example, the four HT-equivalence classes of Hadamard matrices of order 16 have 1717520, 1712912, 1710608, and 1709456 vanishing minors of order 4. For maxdet matrices of order $n \equiv 3 \pmod 4$, we sometimes find different numbers of vanishing minors of order 2. For example, if $n = 11$, we get 1391, 1389, and 1401 vanishing minors for the three HT-equivalence classes. The right-hand-side of eqn. (10), which by [9, Corollary 3.1] is a lower bound on the number of vanishing minors in this non-Hadamard case, gives 1362 (rounded up).

Finally, we briefly consider the frequencies (or *multiplicities*) with which the different values of $|\det(M)/2^{n-1}|$ occur for minors of order m of a maxdet matrix of order n. In Table 5 we give the results of computations for $m = 7$, $n = 15$, which gives the typical behaviour that we have observed.[6] The second column gives the observed multiplicity of a minor with $|\det(M)|/2^{n-1}$ equal to the integer in the first column. The third column gives the multiplicities observed when taking a random sample of $\binom{15}{7}^2 = 41409225$ uniformly distributed $\{\pm 1\}$-matrices of order m (we call this the *random model*). It is clear from the table that the actual distribution is nothing like the distribution for the random model. A χ^2 test gives an absurdly small probability $< 10^{-10^{10}}$ that the two samples were drawn from the same distribution. Similar behaviour occurs for other values of $m \ge 2$. For example, when $m = 2$ we find 5187 zero minors and 5838 nonzero minors, but for random matrices of order 2 we expect zero and nonzero values to occur with equal probability.

Table 5 shows that the normalised minors are biased towards even values. For the random model, this bias can be explained by reducing to the $\{0, 1\}$ case and considering the evaluation of the determinant in $\mathbb{Z}/2\mathbb{Z}$. Then, for large n, we expect even values to occur about 71% of the time.[7] This prediction is in agreement with the data for the random model (the third column). For the second column we find that even values occur about 69% of the time, which is close to the prediction for the random model. Thus, although the actual distribution differs considerably from that of the random model, the bias towards even values persists. In comparison with the random model, the extreme bias in favour of even values in the tail of the distribution compensates for a bias against zero minors.

[6] Data on multiplicities for other values of m and n may be found at [4].

[7] The precise constant in the limit as $n \to \infty$ is $1 - \prod_{k \ge 1}(1 - 2^{-k})$, see [6, 17, 30].

$\lvert\det(M)\rvert/2^{n-1}$	multiplicity of minors	multiplicity in random model	ratio
0	12857784	24030613	0.54
1	8402100	11140444	0.75
2	10831128	4662108	2.32
3	3483909	924336	3.77
4	3935280	504938	7.79
5	622842	76496	8.14
6	927162	55811	16.61
7	129576	7769	16.68
8	201900	6102	33.09
9	17544	608	28.86
total	41409225	41409225	

Table 5: Comparison of observed multiplicities of minors of order 7 in a maxdet matrix of order 15 with a random model

5 Algorithms for computing the set of minors

Recall that $S_k(A)$ is the set of $k \times k$ submatrices of an $n \times n$ $\{\pm 1\}$-matrix A, where $0 < k \le n$. In this section we describe algorithms for enumerating all the minors of A. Our application is to maxdet matrices, but the algorithms apply to all square $\{\pm 1\}$-matrices A, and with trivial changes they would also apply to rectangular $\{\pm 1\}$-matrices.

In our enumeration we consider only the absolute values of minors, normalised by the factor 2^{n-1} which always divides the determinant of an $n \times n$ $\{\pm 1\}$-matrix, so the *set of minors* of A is defined to be the set

$$\mathcal{S}(A) = \cup_{k=1}^{n} \{\lvert\det(M)\rvert/2^{n-1} : M \in S_k(A)\}.$$

There are $\binom{n}{k}$ possible choices of rows and $\binom{n}{k}$ possible choices of columns for a minor of order k, so altogether a total of

$$T_n := \sum_{k=1}^{n} \binom{n}{k}^2 = \binom{2n}{n} - 1 \sim \frac{4^n}{\sqrt{\pi n}}$$

possibilities to consider when finding the set $\mathcal{S}(A)$ for an $n \times n$ matrix A. In this section we consider four possible algorithms (of which we used two) for finding $\mathcal{S}(A)$. Their running times all involve the factor 4^n, so none of them is practical for n much larger than 20, but they differ significantly in the factor multiplying 4^n and in their space requirements.

Each algorithm has two variants: the first just determines the set $\mathcal{S}(A)$ of minors; the second also counts the multiplicity of each minor, that is, how often a given value $d \in \mathcal{S}(A)$ occurs. We describe the first variant of each algorithm, and briefly mention the changes required for the second variant.

In the descriptions of the algorithms we explain how to compute the complete set $\mathcal{S}(A)$; it should be clear how to compute the subset of $\mathcal{S}(A)$ corresponding to the minors of a given order k.

Algorithm A

Algorithm A simply considers, for each k in $\{1, 2, \ldots, n\}$, the set of all $k \times k$ submatrices of A, and evaluates the determinant of each such submatrix M by Gaussian elimination with partial pivoting, using floating-point arithmetic. The computed determinant is scaled by division by 2^{n-1} and rounded to the nearest integer.

Clearly there is a danger that rounding errors during the process of Gaussian elimination could lead to an incorrectly rounded integer result. However, our experiments, using IEEE standard 64-bit floating-point arithmetic [1], showed that this is not a problem for the values of n that we considered ($n \le 25$). Gaussian elimination with partial pivoting is numerically stable [19, 46], and the maximum scaled determinant of a 25×25 $\{\pm 1\}$-matrix is $42 \times 6^{11} = 15237476352$, meeting the Barba bound [3, 35], so only requires 34 bits of precision, significantly fewer than the 53 bits provided by IEEE standard arithmetic. As a precaution, our implementation prints a warning and halts if the fractional part of a scaled determinant exceeds 1/8; this never occurred for $n \le 25$.

Gaussian elimination requires $O(k^3)$ arithmetic operations to evaluate the determinant of a $k \times k$ matrix. As is traditional in numerical analysis, we count multiplications/divisions but ignore additions/subtractions. With this convention, Gaussian elimination requires $k^3/3 + O(k^2)$ operations. Thus the total cost is

$$W_A \sim \sum_{k=1}^{n} \binom{n}{k}^2 \frac{k^3}{3} \sim \frac{4^n n^{5/2}}{24\sqrt{\pi}} \,.$$

The storage requirements of Algorithm A are minimal, apart from the space required to store the results, that is the set $\mathcal{S}(A)$ of minors and (if required) their multiplicities. This is common to all the algorithms considered – they all need space to store their results.

The set $\mathcal{S}(A)$ can be represented using one bit for each possible value $|\det|/2^{n-1}$. From the Hadamard bound, this requires at most $2^{1-n}n^{n/2}+1$ bits. For example, if $n = 20$, it requires 19531251 bits (2.33 MB). For $n = 24$ it requires 519 MB, which is still feasible. For the variant that counts multiplicities, each bit needs to be replaced by an integer word (say 32 bits or 4 bytes), so the storage required would be acceptable for $n = 20$ (75 MB) but excessive for $n = 24$ (16 GB), since the computers available to us typically have memories of 1 to 4 GB.

Fortunately, a much more economical representation of $\mathcal{S}(A)$ is usually possible, because not all minors in the range $[0, \lfloor 2^{1-n}n^{n/2}\rfloor]$ actually occur. The set $\mathcal{S}(A)$ is usually quite sparse, especially when the order n of A is divisible by 4. For example, with Hadamard matrices of order 16, we have $\#\mathcal{S}(A) < 100$ (see §4). Thus, instead of using 131073 bits to represent $\mathcal{S}(A)$, we can use a hash table with say 200 words [25]. With such an implementation, the storage requirements are moderate for $n \le 25$.

Algorithm B

Algorithm B is similar to Algorithm A, but uses a rank-1 updating formula to update the inverse and determinant of each $k \times k$ submatrix B of A if we already know the inverse and determinant of a submatrix that differs from B in only one row or column. The inverse updating formula

$$(B + uv^T)^{-1} = B^{-1} - (1 + v^T B^{-1} u)^{-1} B^{-1} uv^T B^{-1}$$

is known at the *Sherman-Morrison* formula [41] – the determinant updating formula

$$\det(B + uv^T) = \det(B)(1 + v^T B^{-1} u)$$

seems to be known only as a "matrix determinant lemma".

Since the updating steps require $\sim k^2$ operations, the complexity is

$$W_B \sim 4^n n^{3/2}/\sqrt{\pi}\,,$$

so $W_A/W_B \sim n/24$.

We did not use Algorithm B because the constant factors involved make it slower than Algorithm A for $n \le 20$, and because it is difficult to guarantee a correctly rounded integer result due to possible numerical instability. Also, even with exact arithmetic, we would have to use a different method whenever B is singular.

Algorithm C

Algorithm C uses integer arithmetic and evaluates each $k \times k$ determinant using the method (attributed to Laplace) of expansion by minors, see for example [33, Chapter 4]. To compute a $k \times k$ minor $\det(B)$, we need to know the $(k-1) \times (k-1)$ minors formed by deleting the first row and an arbitrary column of B. If these minors have been saved from a previous computation, then the work involved in computing one minor $|\det(B)|$ is only k multiplications (by ± 1) and $k-1$ integer additions, plus any overheads involved in retrieving the previously stored values. If we assume, for purposes of comparison with Algorithms A–B, that the work involved amounts to k operations, then the total cost is

$$W_C = \sum_{k=1}^{n} \binom{n}{k}^2 k \sim \frac{4^n n^{1/2}}{2\sqrt{\pi}}\,,$$

giving $W_A/W_C \sim n^2/12$.

Unfortunately, this algorithm has a potentially large memory requirement. If we compute the minors of order k in increasing order $k = 1, 2, \ldots, n$, then to compute the minors of order k we need all $\binom{n}{k-1}^2$ minors of order $k-1$. In the worst case, when $k \approx n/2$, the memory required to store the minors of order $k-1$ and k is about $4^{n+1}/(\pi n)$ words, which is too large to be practical for the values of n that we wish to consider. More memory-efficient implementations are possible, but complicated. For this reason we discarded Algorithm C and implemented a slightly slower, but much simpler algorithm, Algorithm D.

Algorithm D

The idea of Algorithm D is the same as that of Algorithm C. However, when computing the minors of order k of an $n \times n$ matrix A, the outer loop runs over all $\binom{n}{k}$ combinations of k rows of A. Having selected these k rows, forming a $k \times n$ matrix B, we now compute all minors of order k of B. At the j-th step we compute all minors of order j in the last j rows of B. Thus the number of operations (counting as for Algorithm C) to compute the minors of order

k of B is

$$\sum_{j=1}^{k} \binom{n}{j} j$$

and the space requirement is at most $2\binom{n}{\lfloor n/2 \rfloor} \sim 2^{n+\frac{3}{2}}/\sqrt{\pi n}$ words, much less than for Algorithm C. The overall operation count is

$$\sum_{k=1}^{n} \binom{n}{k} \sum_{j=1}^{k} \binom{n}{j} j = 4^{n-1} n.$$

This is larger than the operation count for Algorithm C by a factor $\sim \sqrt{\pi n}/2$, but smaller than the operation count for Algorithm A by a factor $\sim n^{3/2}/(6\sqrt{\pi})$.

If a parallel implementation is desired, then it is easy to parallelise over the outer loop – different processors can work on different combinations of k rows of A in parallel.

We ran both Algorithms A and D on small cases to check the correctness of our implementations. For the large cases we used mainly Algorithm D, which is much faster than Algorithm A for the most time-consuming cases ($k \approx n/2$), as expected from the operation counts given above.

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Appendix: Tables of minors for orders ≤ 21

In Tables 6–43, $k = \lfloor n/4 \rfloor$, where n is the order of the $\{\pm 1\}$-matrix. The first column gives the order m of the minor, $1 \le m \le n$. The second column gives the set of absolute values of the minors of order m, divided by the known factor 2^{m-1}. In this column the notation "$\{a, b, \ldots\} \times k^{\alpha}$" is a shorthand for "$\{ak^{\alpha}, bk^{\alpha}, \ldots\}$", etc. For $n \in \{19, 21\}$ we have abbreviated the entry in the second column by giving only the minimum and maximum rather than the complete set, using "$(\min, \max)$" instead of "$\{a, b, \ldots\}$". In such cases we write "$(a, b) \times k^{\alpha}$" instead of "$(ak^{\alpha}, bk^{\alpha})$".

In the third column we give the scaled maximum determinant $\Delta(m) = D(m)/2^{m-1}$ (redundant, but included for easy comparison with the entries in the second column). The fourth column answers whether some minor meets the maximum possible determinant for its order (see Table 2), and the last column answers whether the full spectrum of possible values of minors (as given in Table 1) occurs.

If there is more than one HT-equivalence class for an order n, the classes are listed in the same order as they are given in [35]. The information given in Tables 6–43 is sufficient to uniquely identify each HT-equivalence class.

To avoid making this Appendix excessively long, we have omitted details such as the frequency of occurrence of each minor value. Further information is available on our website [4].

Orders 1–6

m	{minors}	$\Delta(m)$	max?	full?
1	$\{1\}$	1	yes	yes

Table 6: $n = 1$, $k = 0$, $m_f = 1$

m	{minors}	$\Delta(m)$	max?	full?
2	$\{1\}$	1	yes	no
1	$\{1\}$	1	yes	yes

Table 7: $n = 2$, $k = 0$, $d = 1$, $m_f = 1$

m	{minors}	$\Delta(m)$	max?	full?
3	$\{1\}$	1	yes	no
1–2	full spectrum	1	yes	yes

Table 8: $n = 3$, $k = 0$, $d = 1$, $m_f = 2$

m	{minors}	$\Delta(m)$	max?	full?
4	$\{2\}$	2	yes	no
3	$\{1\}$	1	yes	no
1–2	full spectrum	1	yes	yes

Table 9: $n = 4$, $k = 1$, $d = 1$, $m_f = 2$

m	{minors}	$\Delta(m)$	max?	full?
5	$\{3\}$	3	yes	no
4	$\{1, 2\}$	2	yes	no
1–3	full spectrum	1	yes	yes

Table 10: $n = 5$, $k = 1$, $d = 1$, $m_f = 3$

m	{minors}	$\Delta(m)$	max?	full?
6	$\{5\}$	5	yes	no
5	$\{1..3\}$	3	yes	no
1–4	full spectrum	≤ 2	yes	yes

Table 11: $n = 6$, $k = 1$, $d = 1$, $m_f = 4$

Orders 7–11(a)

m	{minors}	$\Delta(m)$	max?	full?
7	$\{9\}$	9	yes	no
1–6	full spectrum	≤ 5	yes	yes

Table 12: $n = 7$, $k = 1$, $d = 1$, $m_f = 6$

m	{minors}	$\Delta(m)$	max?	full?
8	$\{2\} \times 2^4$	2×2^4	yes	no
7	$\{1\} \times 2^3$	9	no	no
6	$\{0, 1\} \times 2^2$	5	no	no
5	$\{0, 1\} \times 2^1$	3	no	no
1–4	full spectrum	≤ 2	yes	yes

Table 13: $n = 8$, $k = 2$, $d = 4$, $m_f = 4$

m	{minors}	$\Delta(m)$	max?	full?
9	$\{7\} \times 2^3$	7×2^3	yes	no
8	$\{2, 3, 4, 6, 8\} \times 2^2$	8×2^2	yes	no
7	$\{0..4\} \times 2^1$	9	no	no
1–6	full spectrum	≤ 5	yes	yes

Table 14: $n = 9$, $k = 2$, $d = 1$, $m_f = 6$

m	{minors}	$\Delta(m)$	max?	full?
10	$\{9\} \times 2^4$	9×2^4	yes	no
9	$\{3, 6\} \times 2^3$	7×2^3	no	no
8	$\{0..5, 8\} \times 2^2$	8×2^2	yes	no
7	$\{0..4\} \times 2^1$	9	no	no
1–6	full spectrum	≤ 5	yes	yes

Table 15: $n = 10$, $k = 2$, $d = 2$, $m_f = 6$

m	{minors}	$\Delta(m)$	max?	full?
11	$\{20\} \times 2^4$	20×2^4	yes	no
10	$\{0, 2..8, 10, 12, 16\} \times 2^3$	18×2^3	no	no
9	$\{0, 2..4, 6, 8..10, 12..24,$ $26..33, 36, 40, 48\}$	56	no	no
8	$\{0..18, 20, 24\}$	32	no	no
1–7	full spectrum	≤ 9	yes	yes

Table 16: $n = 11(\text{a})$, $k = 2$, $d = 4$, $m_f = 7$

Orders 11(b)–12

m	{minors}	$\Delta(m)$	max?	full?
11	$\{20\} \times 2^4$	20×2^4	yes	no
10	$\{0, 1, 4..6, 8..11, 14, 16\} \times 2^3$	18×2^3	no	no
9	$\{0, 2..4, 6, 8..24, 26..28, 30..32, 36, 40, 44, 48\}$	56	no	no
8	$\{0..18, 20, 24\}$	32	no	no
1–7	full spectrum	≤ 9	yes	yes

Table 17: $n = 11(b)$, $k = 2$, $d = 4$, $m_f = 7$

m	{minors}	$\Delta(m)$	max?	full?
11	$\{20\} \times 2^4$	20×2^4	yes	no
10	$\{4, 6, 8, 12, 16\} \times 2^3$	18×2^3	no	no
9	$\{0..4, 6, 8, 12\} \times 2^2$	14×2^2	no	no
8	$\{0..8, 12, 16\} \times 2^1$	16×2^1	yes	no
7	$\{0..8\}$	9	no	no
1–6	full spectrum	≤ 5	yes	yes

Table 18: $n = 11(c)$, $k = 2$, $d = 3$, $m_f = 6$

m	{minors}	$\Delta(m)$	max?	full?
12	$\{2\} \times 3^6$	2×3^6	yes	no
11	$\{1\} \times 3^5$	5×2^6	no	no
10	$\{0, 1\} \times 3^4$	$3^2 \times 2^4$	no	no
9	$\{0, 1\} \times 3^3$	7×2^3	no	no
8	$\{0..2\} \times 3^2$	2^5	no	no
7	$\{0..3\} \times 3^1$	3^2	yes	no
1–6	full spectrum	≤ 5	yes	yes

Table 19: $n = 12$, $k = 3$, $d = 5$, $m_f = 6$

Orders 13–15

m	{minors}	$\Delta(m)$	max?	full?
13	$\{5\} \times 3^6$	5×3^6	yes	no
12	$\{2, 3\} \times 3^5$	6×3^5	no	no
11	$\{0..3\} \times 3^4$	5×2^6	no	no
10	$\{0..4\} \times 3^3$	$3^2 \times 2^4$	no	no
9	$\{0..5\} \times 3^2$	7×2^3	no	no
8	$\{0..6\} \times 3^1$	2^5	no	no
1–7	full spectrum	≤ 9	yes	yes

Table 20: $n = 13$, $k = 3$, $d = 6$, $m_f = 7$

m	{minors}	$\Delta(m)$	max?	full?
14	$\{13\} \times 3^6$	13×3^6	yes	no
13	$\{4, 6, 7, 9\} \times 3^5$	15×3^5	no	no
12	$\{0..7, 9, 10\} \times 3^4$	18×3^4	no	no
11	$\{0..9, 11\} \times 3^3$	5×2^6	no	no
10	$\{0..13\} \times 3^2$	$3^2 \times 2^4$	no	no
9	$\{0..15\} \times 3^1$	7×2^3	no	no
8	$\{0..18, 20\}$	2^5	no	no
1–7	full spectrum	≤ 9	yes	yes

Table 21: $n = 14$, $k = 3$, $d = 7$, $m_f = 7$

m	{minors}	$\Delta(m)$	max?	full?
15	$\{35\} \times 3^6$	35×3^6	yes	no
14	$\{7, 8, 12, 14, 17, 18, 21, 23, 27\} \times 3^5$	39×3^5	no	no
13	$\{0..21, 23, 24, 26, 27\} \times 3^4$	45×3^4	no	no
12	$\{0..22, 24..27\} \times 3^3$	54×3^3	no	no
11	$\{0..29, 31, 35\} \times 3^2$	5×2^6	no	no
10	$\{0..36, 39, 40\} \times 3^1$	$3^2 \times 2^4$	no	no
9	$\{0..36, 38..40, 42, 44, 45\}$	56	no	no
8	$\{0..18, 20, 24\}$	32	no	no
1–7	full spectrum	≤ 9	yes	yes

Table 22: $n = 15$, $k = 3$, $d = 8$, $m_f = 7$

Order 16

m	{minors}	$\Delta(m)$	max?	full?
16	$\{2\} \times 4^8$	2×4^8	yes	no
15	$\{1\} \times 4^7$	35×3^6	no	no
14	$\{0,1\} \times 4^6$	13×3^6	no	no
13	$\{0,1\} \times 4^5$	5×3^6	no	no
12	$\{0..2\} \times 4^4$	2×3^6	no	no
11	$\{0..3\} \times 4^3$	5×2^6	no	no
10	$\{0..2\} \times 2^5$	9×2^4	no	no
9	$\{0..2\} \times 2^4$	7×2^3	no	no
8	$\{0..4\} \times 2^3$	32	yes	no
7	$\{0..2\} \times 2^2$	9	no	no
6	$\{0..2\} \times 2^1$	5	no	no
1–5	full spectrum	≤ 3	yes	yes

Table 23: $n = 16$(a), $k = 4$, $d = 8$, $m_f = 5$

m	{minors}	$\Delta(m)$	max?	full?
11–16	as for 16(a)	–	–	no
10	$\{0..5\} \times 4^2$	9×4^2	no	no
9	$\{0..4\} \times 2^3$	7×2^3	no	no
8	$\{0..6, 8\} \times 2^2$	32	yes	no
7	$\{0..4\} \times 2^1$	9	no	no
1–6	full spectrum	≤ 5	yes	yes

Table 24: $n = 16$(b), $k = 4$, $d = 8$, $m_f = 6$

m	{minors}	$\Delta(m)$	max?	full?
10–16	as for 16(b)	–	–	no
9	$\{0..9\} \times 4^1$	14×4^1	no	no
8	$\{0..10, 12, 16\} \times 2^1$	16×2^1	yes	no
1–7	full spectrum	≤ 9	yes	yes

Table 25: $n = 16$(c), $k = 4$, $d = 8$, $m_f = 7$

m	{minors}	$\Delta(m)$	max?	full?
9–16	as for 16(c)	–	–	no
1–8	full spectrum	≤ 32	yes	yes

Table 26: $n = 16$(d), $k = 4$, $d = 8$, $m_f = 8$

Order 17

m	{minors}	$\Delta(m)$	max?	full?
17	$\{5\} \times 4^8$	5×4^8	yes	no
16	$\{2,3,8\} \times 4^7$	8×4^7	yes	no
15	$\{0..4\} \times 4^6$	35×3^6	no	no
14	$\{0..4\} \times 4^5$	13×3^6	no	no
13	$\{0..7\} \times 4^4$	5×3^6	no	no
12	$\{0..9\} \times 4^3$	2×3^6	no	no
11	$\{0..13, 15\} \times 4^2$	20×4^2	no	no
10	$\{0..21, 24, 27\} \times 4^1$	36×4^1	no	no
9	$\{0..40, 42, 44, 45, 48\}$	56	no	no
1–8	full spectrum	≤ 32	yes	yes

Table 27: $n = 17$(a), $k = 4$, $d = 1$, $m_f = 8$

m	{minors}	$\Delta(m)$	max?	full?
10–17	as for 17(a)	–	–	no
9	$\{0..22, 24\} \times 2^1$	28×2^1	no	no
1–8	full spectrum	≤ 32	yes	yes

Table 28: $n = 17$(b), $k = 4$, $d = 1$, $m_f = 8$

m	{minors}	$\Delta(m)$	max?	full?
11–17	as for 17(a)	–	–	no
10	$\{0..10, 12\} \times 2^3$	18×2^3	no	no
9	$\{0..12\} \times 2^2$	14×2^2	no	no
8	$\{0..10, 12, 16\} \times 2^1$	16×2^1	yes	no
1–7	full spectrum	≤ 9	yes	yes

Table 29: $n = 17$(c), $k = 4$, $d = 1$, $m_f = 7$

Order 18

m	$\{\text{minors}\}$	$\Delta(m)$	max?	full?
18	$\{17\} \times 4^8$	17×4^8	yes	no
17	$\{6, 7, 10, 11\} \times 4^7$	20×4^7	no	no
16	$\{0..8, 10, 11, 13\} \times 4^6$	32×4^6	no	no
15	$\{0..16\} \times 4^5$	35×3^6	no	no
14	$\{0..20\} \times 4^4$	13×3^6	no	no
13	$\{0..24, 26, 28\} \times 4^3$	5×3^6	no	no
12	$\{0..38, 40, 41, 44, 52\} \times 4^2$	2×3^6	no	no
11	$\{0..62, 64, 68, 80\} \times 4^1$	80×4^1	yes	no
10	$\{0..94, 96..98, 100..102, 104, 108, 112, 128\}$	144	no	no
9	$\{0..40, 42, 44, 45, 48\}$	56	no	no
1–8	full spectrum	≤ 32	yes	yes

Table 30: $n = 18$(a), $k = 4$, $d = 7$, $m_f = 8$

m	$\{\text{minors}\}$	$\Delta(m)$	max?	full?
14–18	as for 18(a)	–	–	no
13	$\{0..24, 26, 28, 32\} \times 4^3$	5×3^6	no	no
12	$\{0..38, 40, 41, 44, 64\} \times 4^2$	2×3^6	no	no
11	$\{0..62, 64, 68, 80\} \times 4^1$	80×4^1	yes	no
10	$\{0..52, 54, 56, 64\} \times 2^1$	72×2^1	no	no
9	$\{0..40, 42, 44, 45, 48\}$	56	no	no
1–8	full spectrum	≤ 32	yes	yes

Table 31: $n = 18$(b), $k = 4$, $d = 7$, $m_f = 8$

m	$\{\text{minors}\}$	$\Delta(m)$	max?	full?
14–18	as for 18(a)	–	–	no
13	$\{0..24, 32\} \times 4^3$	5×3^6	no	no
12	$\{0..37, 40, 64\} \times 4^2$	2×3^6	no	no
11	$\{0..32\} \times 2^3$	40×2^3	no	no
10	$\{0..28, 32\} \times 2^2$	36×2^2	no	no
9	$\{0..22, 24\} \times 2^1$	28×2^1	no	no
1–8	full spectrum	≤ 32	yes	yes

Table 32: $n = 18$(c), $k = 4$, $d = 10$, $m_f = 8$

Order 19

m	(min, max) |minor|	$\Delta(m)$	max?	full?
19	$(833, 833) \times 4^6$	833×4^6	yes	no
18	$(140, 784) \times 4^5$	1088×4^5	no	no
17	$(0, 672) \times 4^4$	1280×4^4	no	no
16	$(0, 676) \times 4^3$	2048×4^3	no	no
15	$(0, 1050) \times 4^2$	35×3^6	no	no
14	$(0, 1470) \times 4^1$	13×3^6	no	no
13	$(0, 1904)$	3645	no	no
12	$(0, 756)$	1458	no	no
11	$(0, 312)$	320	no	no
10	$(0, 128)$	144	no	no
1–9	full spectrum	≤ 56	yes	yes

Table 33: $n = 19$(a), $k = 4$, $d = 10$, $m_f = 9$

m	(min, max) |minor|	$\Delta(m)$	max?	full?
19	$(833, 833) \times 4^6$	833×4^6	yes	no
18	$(168, 616) \times 4^5$	1088×4^5	no	no
17	$(0, 672) \times 4^4$	1280×4^4	no	no
16	$(0, 740) \times 4^3$	2048×4^3	no	no
15	$(0, 1024) \times 4^2$	35×3^6	no	no
14	$(0, 1536) \times 4^1$	13×3^6	no	no
13	$(0, 2048)$	3645	no	no
12	$(0, 1024)$	1458	no	no
11	$(0, 288)$	320	no	no
10	$(0, 128)$	144	no	no
1–9	full spectrum	≤ 56	yes	yes

Table 34: $n = 19$(b), $k = 4$, $d = 10$, $m_f = 9$

m	(min, max) |minor|	$\Delta(m)$	max?	full?
14–19	as for 19(b)	–	–	–
13	$(0, 2560)$	3645	no	no
1–12	as for 19(b)	–	–	–

Table 35: $n = 19$(c), $k = 4$, $d = 10$, $m_f = 9$

Order 20

m	{minors}	$\Delta(m)$	max?	full?
20	$\{2\} \times 5^{10}$	2×5^{10}	yes	no
19	$\{1\} \times 5^{9}$	833×4^{6}	no	no
18	$\{0, 1\} \times 5^{8}$	17×4^{8}	no	no
17	$\{0, 1\} \times 5^{7}$	5×4^{8}	no	no
16	$\{0..2\} \times 5^{6}$	2×4^{8}	no	no
15	$\{0..3\} \times 5^{5}$	35×3^{6}	no	no
14	$\{0..5\} \times 5^{4}$	13×3^{6}	no	no
13	$\{0..9\} \times 5^{3}$	5×3^{6}	no	no
12	$\{0..18, 20, 24, 32\} \times 5^{2}$	2×3^{6}	no	no
11	$\{0..40, 42, 44, 48\} \times 5^{1}$	64×5^{1}	no	no
10	$\{0..92, 95, 96, 100, 104, 108, 112, 125, 144\}$	144	yes	no
9	$\{0..40, 42, 44, 48\}$	56	no	no
1–8	full spectrum	≤ 32	yes	yes

Table 36: $n = 20(\mathrm{a})$, $k = 5$, $d = 10$, $m_f = 8$

m	{minors}	$\Delta(m)$	max?	full?
12–20	as for 20(a)	–	–	no
11	$\{0..40, 42, 44, 45, 48\} \times 5^{1}$	64×5^{1}	no	no
10	$\{0..90, 92, 93, 95..102, 104, 108, 112, 117, 120, 125, 128, 144\}$	144	yes	no
9	$\{0..40, 42, 44, 45, 48\}$	56	no	no
1–8	full spectrum	≤ 32	yes	yes

Table 37: $n = 20(\mathrm{b})$, $k = 5$, $d = 10$, $m_f = 8$

m	{minors}	$\Delta(m)$	max?	full?
11–20	as for 20(b)	–	–	no
10	$\{0..88, 90, 92, 93, 96, 99, 100, 102, 104, 108, 112, 120, 125, 128, 144\}$	144	yes	no
1–9	as for 20(b)	–	–	–

Table 38: $n = 20(\mathrm{c})$, $k = 5$, $d = 10$, $m_f = 8$

Order 21(a)–(b)

m	(min, max) |minor|	$\Delta(m)$	max?	full?
21	$(29, 29) \times 5^9$	29×5^9	yes	no
20	$(10, 30) \times 5^8$	50×5^8	no	no
19	$(0, 35) \times 5^7$	833×4^6	no	no
18	$(0, 40) \times 5^6$	17×4^8	no	no
17	$(0, 45) \times 5^5$	5×4^8	no	no
16	$(0, 65) \times 5^4$	2×4^8	no	no
15	$(0, 100) \times 5^3$	35×3^6	no	no
14	$(0, 240) \times 5^2$	13×3^6	no	no
13	$(0, 416) \times 5^1$	5×3^6	no	no
12	$(0, 800)$	1458	no	no
11	$(0, 320)$	320	yes	no
10	$(0, 144)$	144	yes	no
1–9	full spectrum	≤ 56	yes	yes

Table 39: $n = 21$(a), $k = 5$, $d = 10$, $m_f = 9$

m	(min, max) |minor|	$\Delta(m)$	max?	full?
15–21	as for 21(a)	–	–	no
14	$(0, 216) \times 5^2$	13×3^6	no	no
13	$(0, 400) \times 5^1$	5×3^6	no	no
12	$(0, 800)$	1458	no	no
11	$(0, 320)$	320	yes	no
1–10	full spectrum	≤ 144	yes	yes

Table 40: $n = 21$(b), $k = 5$, $d = 10$, $m_f = 10$

Order 21(c)–(e)

m	(min, max) |minor|	$\Delta(m)$	max?	full?
15–21	as for 21(a)	–	–	no
14	$(0,200)\times 5^2$	13×3^6	no	no
13	$(0,400)\times 5^1$	5×3^6	no	no
12	$(0,800)$	1458	no	no
11	$(0,304)$	320	no	no
10	$(0,144)$	144	yes	no
1–9	full spectrum	≤ 56	yes	yes

Table 41: $n=21$(c), $k=5$, $d=11$, $m_f=9$

m	(min, max) |minor|	$\Delta(m)$	max?	full?
15–21	as for 21(a)	–	–	no
14	$(0,240)\times 5^2$	13×3^6	no	no
13	$(0,416)\times 5^1$	5×3^6	no	no
12	$(0,800)$	1458	no	no
11	$(0,320)$	320	yes	no
1–10	full spectrum	≤ 144	yes	yes

Table 42: $n=21$(d), $k=5$, $d=10$, $m_f=10$

m	(min, max) |minor|	$\Delta(m)$	max?	full?
15–21	as for 21(a)	–	–	no
14	$(0,212)\times 5^2$	13×3^6	no	no
13	$(0,368)\times 5^1$	5×3^6	no	no
12	$(0,800)$	1458	no	no
11	$(0,288)$	320	no	no
10	$(0,144)$	144	yes	no
1–9	full spectrum	≤ 56	yes	yes

Table 43: $n=21$(e), $k=5$, $d=11$, $m_f=9$