Bessel polynomials, double factorials and context-free grammars*

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Abstract

The purpose of this paper is to show that Bessel polynomials, factorials and Catalan triangle can be generated by using context-free grammars.

Keywords: Bessel polynomials; Double factorials; Catalan triangle; Context-free grammars

1 Introduction

The grammatical method was introduced by Chen [2] in the study of exponential structures in combinatorics. Let A be an alphabet whose letters are regarded as independent commutative indeterminates. Following Chen [2], a *context-free grammar* G over A is defined as a set of substitution rules replacing a letter in A by a formal function over A. The formal derivative D is a linear operator defined with respect to a context-free grammar G. For example, if $G = \{x \to xy, y \to y\}$, then

$$D(x) = xy, D(y) = y, D^2(x) = x(y + y^2), D^3(x) = x(y + 3y^2 + y^3).$$

For any formal functions u and v, we have

$$D(u + v) = D(u) + D(v), \quad D(uv) = D(u)v + uD(v) \quad \textit{and} \quad D(f(u)) = \frac{\partial f(u)}{\partial u} D(u),$$

where $f(x)$ is a analytic function. Using Leibniz's formula, we obtain

$$D^n(uv) = \sum_{k=0}^{n} \binom{n}{k} D^k(u) D^{n-k}(v). \tag{1}$$

Let $[n] = \{1, 2, \ldots, n\}$. *The Stirling number of the second kind* $\left\{ {n \atop k} \right\}$ is the number of ways to partition $[n]$ into k blocks. Let $\mathcal{S}_n$ denote the symmetric group of all permutations of $[n]$.

*This work is supported by NSFC (11126217) and the Fundamental Research Funds for the Central Universities (N100323013).

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The *Eulerian number* $\left\langle{n \atop k}\right\rangle$ enumerates the number of permutations in $\mathcal{S}_n$ with k descents (i.e., $i < n, \pi(i) > \pi(i+1)$). The numbers $\left\langle{n \atop k}\right\rangle$ satisfy the recurrence relation

$$\left\langle{n \atop k}\right\rangle = (k+1)\left\langle{n-1 \atop k}\right\rangle + (n-k)\left\langle{n-1 \atop k-1}\right\rangle,$$

with initial condition $\left\langle{0 \atop 0}\right\rangle = 1$ and boundary conditions $\left\langle{0 \atop k}\right\rangle = 0$ for $k \geq 1$. There is a close relationship between context-free grammars and combinatorics. The reader is referred to [3, 7] for recent results on this subject. Let us now recall two classical results.

Proposition 1 ([2, Eq. 4.8])**.** *If* $G = \{x \to xy, y \to y\}$, *then*

$$D^n(x) = x\sum_{k=1}^{n}\left\{{n \atop k}\right\}y^k.$$

Proposition 2 ([4, Section 2.1])**.** *If* $G = \{x \to xy, y \to xy\}$, *then*

$$D^n(x) = x\sum_{k=0}^{n-1}\left\langle{n \atop k}\right\rangle x^k y^{n-k}.$$

The purpose of this paper is to show that Bessel polynomials, factorials and Catalan triangle can be generated by using context-free grammars.

2 Bessel polynomials

The well known *Bessel polynomials* $y_n(x)$ were introduced by Krall and Frink [5] as the polynomial solutions of the second-order differential equation

$$x^2\frac{d^2y_n(x)}{dx^2} + (2x+2)\frac{dy_n(x)}{dx} = n(n+1)y_n(x).$$

The Bessel polynomials $y_n(x)$ are a family of orthogonal polynomials and they have been extensively studied and applied (see [9, A001498]). The polynomials $y_n(x)$ can be generated by using the Rodrigues formula

$$y_n(x) = \frac{1}{2^n}e^{\frac{2}{x}}\frac{d^n}{dx^n}\left(x^{2n}e^{-\frac{2}{x}}\right).$$

Explicitly, we have

$$y_n(x) = \sum_{k=0}^{n}\frac{(n+k)!}{(n-k)!k!}\left(\frac{x}{2}\right)^k.$$

These polynomials satisfy the recurrence relation

$$y_{n+1}(x) = (2n+1)xy_n(x) + y_{n-1}(x) \quad \text{for} \quad n \geq 0,$$

with initial conditions $y_{-1}(x) = y_0(x) = 1$. The first few of the polynomials $y_n(x)$ are

$$y_1(x) = 1 + x,$$
$$y_2(x) = 1 + 3x + 3x^2,$$
$$y_3(x) = 1 + 6x + 15x^2 + 15x^3.$$

We present here a grammatical characterization of the Bessel polynomials $y_n(x)$.

Theorem 3. *If* $G = \{a \to ab, b \to b^2c, c \to bc^2\}$, *then*

$$D^n(ab) = ab^{n+1}y_n(c) \quad \textit{for} \quad n \geq 0.$$

Proof. Let

$$a(n,k) = \frac{(n+k)!}{2^k(n-k)!k!}.$$

Then $y_n(c) = \sum_{k=0}^{n} a(n,k)c^k$. It is easy to verify that

$$a(n+1,k) = a(n,k) + (n+k)a(n,k-1). \tag{2}$$

For $n \geq 0$, we define

$$D^n(ab) = ab^{n+1}\sum_{k=0}^{n} E(n,k)c^k. \tag{3}$$

Note that $D(ab) = ab^2(1+c)$. Hence $E(1,0) = a(1,0), E(1,1) = a(1,1)$. It follows from (3) that

$$D^{n+1}(ab) = D(D^n(ab)) = ab^{n+2}\sum_{k=0}^{n} E(n,k)c^k + ab^{n+2}\sum_{k=0}^{n}(n+k+1)E(n,k)c^{k+1}.$$

Therefore,

$$E(n+1,k) = E(n,k) + (n+k)E(n,k-1).$$

Comparing with (2), we see that the coefficients $E(n,k)$ satisfy the same recurrence relation and initial conditions as $a(n,k)$, so they agree. □

For the context-free grammar

$$G = \{a \to ab, b \to b^2c, c \to bc^2\},$$

in the same way as above we find that

$$D^n(a^2b) = 2^n a^2 b^{n+1} y_n\left(\frac{c}{2}\right) \quad \text{for} \quad n \geq 0.$$

By Theorem 3, we obtain $D^k(a) = ab^k y_{k-1}(c)$ for $k \geq 0$. The *double factorial* of odd numbers are defined by

$$(2n-1)!! = 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1),$$

and for even numbers

$$(2n)!! = 2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n).$$

As usual, set $(-1)!! = 0!! = 1$. It is clear that

$$D^n(b) = (2n-1)!!b^{n+1}c^n \quad \textit{for} \quad n \geq 0.$$

By (1), the following corollary is immediate.

Corollary 4. *For* $n \geq 0$, *we have*

$$y_n(x) = \sum_{k=0}^{n}(2n-2k-1)!!\binom{n}{k}y_{k-1}(x)x^{n-k}.$$

3 Polynomials associated with diagonal Padé approximation to the exponential function

The Padé approximations arise naturally in many branches of mathematics and have been extensively investigated (see [6] for instance). The *diagonal Padé approximation* to the exponential function e^x is the unique rational function

$$R_n(x) = \frac{P_n(x)}{P_n(-x)},$$

where

$$P_n(x) = \sum_{k=0}^{n} M(n,k)x^{n-k} \quad \text{and} \quad M(n,k) = \frac{(n+k)!}{(n-k)!k!}.$$

Clearly, $P_n(1) = y_n(2)$, where $y_n(x)$ is the Bessel polynomials. It is easy to verify that the numbers $M(n,k)$ satisfy the recurrence relation

$$M(n+1,k) = M(n,k) + (2n+2k)M(n,k-1). \tag{4}$$

The first few of the polynomials $P_n(x)$ are given as follows (see [9, A113025]):

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x+2, \\ P_2(x) &= x^2+6x+12, \\ P_3(x) &= x^3+12x^2+60x+120. \end{aligned}$$

We present here a grammatical characterization of the polynomials $P_n(x)$.

Theorem 5. *If $G = \{a \to ab^2, b \to b^3c^2, c \to b^2c^3\}$, then*

$$D^n(ab^2) = ab^{2n+2}c^{2n}P_n\left(\frac{1}{c^2}\right).$$

Proof. For $n \geq 0$, we define

$$D^n(ab^2) = ab^{2n+2}\sum_{k=0}^{n} N(n,k)c^{2k}. \tag{5}$$

Note that $D(ab^2) = ab^4(1+2c^2)$. Hence $N(1,0) = M(1,0), N(1,1) = M(1,1)$. It follows from (5) that

$$D^{n+1}(ab^2) = ab^{2n+4}\sum_{k=0}^{n} N(n,k)c^{2k} + ab^{2n+4}\sum_{k=0}^{n}(2n+2k+2)N(n,k)c^{2k+2}.$$

Therefore,

$$N(n+1,k) = N(n,k) + (2n+2k)N(n,k-1).$$

Comparing with (4), we see that the coefficients $N(n,k)$ satisfy the same recurrence relation and initial conditions as $M(n,k)$, so they agree. □

Along the same lines, we immediately deduce the following corollary.

Corollary 6. *Let $y_n(x)$ be the Bessel polynomials. If $G = \{a \to ab^2, b \to b^3c^2, c \to b^2c^3\}$, then*

$$D^n(a^2b^2) = 2^n a^2 b^{2n+2} y_n(c^2).$$

4 Double factorials

The following identity was studied systematically by Callan [1, Section 4.8]:

$$\sum_{k=1}^{n} k!\binom{2n-k-1}{k-1}(2n-2k-1)!! = (2n-1)!!. \tag{6}$$

As pointed out by Callan [1], the identity (6) counts different combinatorial structures, such as *increasing ordered trees* of n edges by outdegree k of the root and the sum of the weights of all vertices labeled k at depth $n-1$ in the *Catalan tree* (see [9, A102625]).

Let

$$R(n,k) = k!\binom{2n-k-1}{k-1}(2n-2k-1)!!.$$

Thus, $\sum_{k=1}^{n} R(n,k) = (2n-1)!!$. It is easy to verify that

$$R(n+1,k) = (2n-k)R(n,k) + kR(n,k-1), \tag{7}$$

with initial conditions $R(0,0) = 1$ and $R(0,k) = 0$ for $k \geq 1$ or $k < 0$. For $n \geq 1$, let $R_n(x) = \sum_{k=1}^{n} R(n,k)x^k$. The first few of the polynomials $R_n(x)$ are

$$\begin{aligned}
R_1(x) &= x, \\
R_2(x) &= x + 2x^2, \\
R_3(x) &= 3x + 6x^2 + 6x^3, \\
R_4(x) &= 15x + 30x^2 + 36x^3 + 24x^4.
\end{aligned}$$

Theorem 7. *If $G = \{a \to a^2b, b \to b^2c, c \to bc^2\}$, then*

$$D^n(a) = ab^n \sum_{k=1}^{n} R(n,k)a^k c^{n-k} \quad \textit{for} \quad n \geq 0. \tag{8}$$

Proof. Note that $D(a) = a^2b$ and $D^2(a) = ab^2(ac + 2a^2)$. For $n \geq 1$, we define

$$D^n(a) = ab^n \sum_{k=1}^{n} r(n,k)a^k c^{n-k}. \tag{9}$$

Hence $r(1,1) = R(1,1), r(2,1) = R(2,1)$ and $r(2,2) = R(2,2)$. It follows from (9) that

$$D(D^n(a)) = ab^{n+1}\sum_{k=1}^{n}(2n-k)r(n,k)a^k c^{n-k+1} + ab^{n+1}\sum_{k=0}^{n}(k+1)r(n,k)a^{k+1}c^{n-k}.$$

Therefore,

$$r(n+1,k) = (2n-k)r(n,k) + kr(n,k-1).$$

Comparing with (7), we see that the coefficients $r(n,k)$ satisfy the same recurrence relation and initial conditions as $R(n,k)$, so they agree. $\square$

In the following discussion, we also consider the context-free grammar

$$G = \{a \to a^2b, b \to b^2c, c \to bc^2\}.$$

Note that

$$D(ab) = a^2b^2 + ab^2c, D^2(ab) = ab^3(3c^2 + 3ac + 2a^2).$$

For $n \geq 0$, we define

$$D^n(ab) = ab^{n+1}\sum_{k=0}^{n} H(n,k)a^k c^{n-k}. \tag{10}$$

It follows that

$$D(D^n(ab)) = ab^{n+2}\sum_{k=0}^{n}(2n-k+1)H(n,k)a^k c^{n-k+1} + ab^{n+2}\sum_{k=0}^{n}(k+1)H(n,k)a^{k+1}c^{n-k}.$$

Hence, the numbers $H(n,k)$ satisfy the recurrence relation

$$H(n+1,k) = (2n-k+1)H(n,k) + kH(n,k-1), \tag{11}$$

with initial conditions $H(1,0) = H(1,1) = 1$ and $H(1,k) = 0$ for $k \geq 2$ or $k < 0$. Using (11), it is easy to verify that

$$H(n,k) = \frac{(2n-k)!}{2^{n-k}(n-k)!}.$$

It should be noted that the numbers $H(n,k)$ are entries in a *double factorial triangle* (see [9, A193229]). In particular, we have $H(n,0) = (2n-1)!!, H(n,n) = n!$ and $\sum_{k=0}^{n} H(n,k) = (2n)!!$. Moreover, combining (1), (8) and (10), we obtain

$$H(n,k) = \sum_{m=k}^{n} \binom{n}{m}(2n-2m-1)!!R(m,k)$$

for $n \geq 1$ and $0 \leq k \leq n$.

For $n \geq 1$, we define

$$x(x+2)(x+4)\cdots(x+2n-2) = \sum_{k=1}^{n} p(n,k)x^k$$

and

$$(x+1)(x+3)\cdots(x+2n-1) = \sum_{k=0}^{n} q(n,k)x^k.$$

The the triangular arrays $\{p(n,k)\}_{n\geq 1, 1\leq k\leq n}$ and $\{q(n,k)\}_{n\geq 1, 0\leq k\leq n}$ are both *double Pochhammer triangles* (see [9, A039683,A028338]). The following theorem is in a sense "dual" to Theorem 7, and we omit the proof for brevity.

Theorem 8. *If* $G = \{a \to ab^2, b \to b^2c, c \to bc^2\}$*, then we have*

$$D^n(a) = ab^n \sum_{k=1}^{n} p(n,k) b^k c^{n-k}$$

and

$$D^n(ab) = ab^{n+1} \sum_{k=0}^{n} q(n,k) b^k c^{n-k}.$$

Set $p(0,0) = q(0,0) = 1$. By (1), we immediately obtain

$$q(n,k) = \sum_{m=k}^{n} \binom{n}{m} (2n-2m-1)!! p(m,k)$$

for $n \geq 0$ and $0 \leq k \leq n$.

5 Catalan triangle

The classical *Catalan triangle* is defined by the recurrence relation

$$T(n,k) = T(n-1,k) + T(n,k-1),$$

with initial conditions $T(0,0) = 1$ and $T(0,k) = 0$ for $k > 0$ or $k < 0$ (see [9, A009766]). The numbers $T(n,k)$ are often called *ballot numbers*. Explicitly,

$$T(n,k) = \binom{n+k}{k} \frac{n-k+1}{n+1} \quad \text{for} \quad 0 \leq k \leq n. \tag{12}$$

Moreover, $\sum_{k=0}^{n} T(n,k) = T(n+1,n+1) = C(n+1)$, where $C(n)$ is the well known *Catalan number*. Catalan numbers appear in a wide range of problems (see [8] for instance).

It follows from (12) that

$$(n+2)T(n+1,k) = (n-k+2)T(n,k) + (2n+2k)T(n,k-1). \tag{13}$$

This recurrence relation gives rise to the following result.

Theorem 9. *If* $G = \{a \to a^2b^2, b \to b^3c^2, c \to b^2c^3\}$*, then we have*

$$D^n(a^2b^2) = (n+1)! a^2 b^{2n+2} \sum_{k=0}^{n} T(n,k) a^{n-k} c^{2k}.$$

Proof. It is easy to verify that $D(a^2b^2) = 2a^2b^4(a+c^2)$ and $D^2(a^2b^2) = 3!a^2b^6(a^2+2ac^2+2c^4)$. For $n \geq 0$, we define

$$D^n(a^2b^2) = (n+1)! a^2 b^{2n+2} \sum_{k=0}^{n} t(n,k) a^{n-k} c^{2k}.$$

Note that

$$\frac{D^{n+1}(a^2b^2)}{(n+1)!a^2b^{2n+4}} = \sum_{k=0}^{n} (n-k+2) t(n,k) a^{n-k+1} c^{2k} + \sum_{k=0}^{n} (2n+2k+2) t(n,k) a^{n-k} c^{2k+2}.$$

Thus, we get

$$(n+2)t(n+1,k) = (n-k+2)t(n,k) + (2n+2k)t(n,k-1).$$

Comparing with (13), we see that the coefficients $t(n,k)$ satisfy the same recurrence relation and initial conditions as $T(n,k)$, so they agree. □

In the same way as above we find that if $G = \{a \to a^2b^2, b \to b^3c^2, c \to b^2c^3\}$, then

$$D^n(ab^2) = n!ab^{2n+2}\sum_{k=0}^{n}\binom{n+k}{k}a^{n-k}c^{2k}$$

and

$$D^n(b) = \prod_{k=0}^{n-1}(4k+1)b^{2n+1}c^{2n}.$$

It should be noted that $\binom{n+k}{k}$ is the number of lattice paths from $(0,0)$ to (n,k) using steps $(1,0)$ and $(0,1)$ (see [9, A046899]) and $\prod_{k=0}^{n-1}(4k+1)$ is the *quartic factorial number* (see [9, A007696]).

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