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The gauge structure of generalised diffeomorphisms

David S. Berman[1], **Martin Cederwall**[2],
Axel Kleinschmidt[3,4] **and Daniel C. Thompson**[4,5]

[1] *Centre for Research in String Theory, School of Physics, Queen Mary College, University of London, Mile End Road, London, E1 4NS, England*

[2] *Dept. of Fundamental Physics, Chalmers University of Technology, SE 412 96 Gothenburg, Sweden*

[3] *Max-Planck-Institut für Gravitationsphysik (Albert Einstein Institut), Am Mühlenberg 1, 14476 Golm, Germany*

[4] *International Solvay Institutes, Campus Plaine C.P. 231, Boulevard du Triomphe, 1050 Bruxelles, Belgium*

[5] *Theoretische Natuurkunde, Vrije Universiteit Brussel, International Solvay Institutes, Pleinlaan 2, B-1050, Brussels, Belgium*

Abstract: We investigate the generalised diffeomorphisms in M-theory, which are gauge transformations unifying diffeomorphisms and tensor gauge transformations. After giving an $E_{n(n)}$-covariant description of the gauge transformations and their commutators, we show that the gauge algebra is infinitely reducible, *i.e.*, the tower of ghosts for ghosts is infinite. The Jacobiator of generalised diffeomorphisms gives such a reducibility transformation. We give a concrete description of the ghost structure, and demonstrate that the infinite sums give the correct (regularised) number of degrees of freedom. The ghost towers belong to the sequences of representations previously observed appearing in tensor hierarchies and Borcherds algebras. All calculations rely on the section condition, which we reformulate as a linear condition on the cotangent directions. The analysis holds for $n < 8$. At $n = 8$, where the dual gravity field becomes relevant, the natural guess for the gauge parameter and its reducibility still yields the correct counting of gauge parameters.

email: d.s.berman@qmul.ac.uk, martin.cederwall@chalmers.se, Axel.Kleinschmidt@aei.mpg.de, dthompson@tena4.vub.ac.be

1. Introduction

It has been known for a long time that compactification on an n-dimensional torus of $D = 11$ supergravity, and of M-theory, enjoys a U-duality symmetry, namely a discrete version of $E_{n(n)}$ (see for example refs. [1,2,3]). Such a symmetry mixes momentum states with winding states of branes. The series can even be continued to the infinite-dimensional algebras E_9 [4,5], E_{10} [6-10] and E_{11} [11-15], although the interpretation is somewhat different in the last two cases. It has later become clear that it should be possible to give the theory a formulation which is manifest under the (continuous) exceptional group [16-21], which plays roughly speaking the same rôle as $GL(n)$ does in gravity.

Such ideas, where space-time is enlarged to accommodate the extra "momenta", found its geometric formulation in the work of Hull, first for T-duality [22] and later for U-duality [23,24]. The doubled field theory relevant for T-duality is closely connected to the generalised geometry of Hitchin [25], and has been thoroughly investigated [26-36], although some geometric understanding still seems to be missing.

Concerning U-duality, much of the structure is similar, but there are some fundamental differences in the structure of the generalised diffeomorphisms. Part of the purpose of the present paper is to clarify these. From investigations of the dynamics of supersymmetric membranes, it has been clear from different arguments that U-duality can be made manifest [37,38,39]. Properties of generalised diffeomorphisms, and of some objects transforming linearly under them (that can serve as equations of motion) have been investigated in refs. [40-45]. A generic geometric picture and a tensor formalism are lacking, as is a full geometric treatment of fermions (though progress has been made in the case of double field theory/ten-dimensional supergravity [46-50]).

An interesting structure arising in the context of U-duality is the concept of tensor hierarchies, connected to the possible gaugings of supergravity [51,52,53]. They can also be understood in terms of Borcherds algebras [54,55,56], and seem to have an origin in the decomposition of the adjoint representation of E_{11} [14,57]. In the present paper we will encounter the same structures in a seemingly different context, namely in the ghost structure of the algebra of generalised diffeomorphisms. Some of the representations are given in Table 1. The representations R_k given there are possible representations of k-form fields in the uncompactified dimensions.

n	R_1	R_2	R_3	R_4
3	$(\mathbf{3},\mathbf{2})$	$(\overline{\mathbf{3}},\mathbf{1})$	$(\mathbf{1},\mathbf{2})$	$(\mathbf{3},\mathbf{1})$
4	$\mathbf{10}$	$\overline{\mathbf{5}}$	$\mathbf{5}$	$\overline{\mathbf{10}}$
5	$\mathbf{16}$	$\mathbf{10}$	$\overline{\mathbf{16}}$	$\mathbf{45}$
6	$\mathbf{27}$	$\overline{\mathbf{27}}$	$\mathbf{78}$	$\overline{\mathbf{351}'}$
7	$\mathbf{56}$	$\mathbf{133}$	$\mathbf{912}$	$\mathbf{8645}\oplus\mathbf{133}$
8	$\mathbf{248}$	$\mathbf{3875}\oplus\mathbf{1}$	$\mathbf{147250}\oplus\mathbf{3875}\oplus\mathbf{248}$	$\mathbf{6696000}\oplus\mathbf{779247}\oplus\mathbf{147250}$ $\oplus\,2\cdot\mathbf{30380}\oplus\mathbf{3875}\oplus 2\cdot\mathbf{248}$

Table 1: Some relevant representations

The section condition, restricting the dependence of fields on the coordinates of the extended manifold, is central in the analysis. It is essential for the equivalence of a model formulated within the framework of generalised geometry with an ordinary supergravity. If it could be meaningfully continued to higher n, it would be a key ingredient in demonstrating the validity of the E_{11} conjecture. A first step in this direction may be found in ref. [57].

The present paper is structured as follows: Section 2 contains basic facts about generalised (exceptional) diffeomorphisms and their algebras. This analysis is not new [45], but its covariant formulation has to our knowledge not been given previously. In Section 3, we investigate the reducibility of generalised diffeomorphisms, which, when formulated covariantly, turns out to be infinite. The version of the representations presented in Table 1 is part of the prediction of Section 3. We show that the counting of gauge parameters arising from the infinite sums is correct, even for $n = 8$. In Section 4, finally, we come back to the section condition and show how it can be cast in a linear form, mimicking the construction of isotropic subspaces from pure spinors in doubled geometry [25]. Throughout the paper we will only be concerned with what is normally viewed as the compactified directions, and completely ignore the rest.

A note on terminology: The term "generalised geometry", when used in this paper, will typically refer to the exceptional $E_{n(n)}$ geometry. When we want to refer to the $O(d, d)$ situation, we use the term "doubled geometry". We have no need to make a terminological distinction between the doubled formalism of Hull [22] and the mathematical setting of Hitchin [25].

2. Generalised diffeomorphisms

Since gravitational and tensorial degrees of freedom mix under U-duality, so do their respective gauge transformations, and the concept of diffeomorphisms has to be generalised. In ordinary geometry, the infinitesimal transformation of any tensor is given by the Lie derivative in the direction of the diffeomorphism parameter u^m, which acting on a vector v^m reads

$$\delta_u v^m = L_u v^m = [u,v]^m = u^n \partial_n v^m - \partial_n u^m v^n \ . \tag{2.1}$$

The interpretation of this transformation that best lends itself to generalisations is to view the first term as a transport term, and the second one as a $\mathfrak{gl}(n)$ transformation with the matrix $\partial_n u^m$ valued in the fundamental representation of the Lie algebra $\mathfrak{gl}(n)$. The transformation of any tensor is given by replacing the Lie algebra action by the appropriate representation. Of course, this transformation is already antisymmetric in u and v, and the commutator of two diffeomorphisms is given by the algebra of vector fields:

$$[L_u, L_v] = L_{[u,v]} = L_{L_u v} \ . \tag{2.2}$$

In the context of U-duality, the rôle played by $\mathfrak{gl}(n)$ is assumed by the Lie algebra $\mathfrak{e}_{n(n)}$, together with a real scaling, corresponding to the (on-shell) trombone symmetry [58]. The tensors should now be tensors under $E_{n(n)} \times \mathbb{R}$, and a generalised diffeomorphism should be of the form

$$\delta_U V^M = \mathscr{L}_U V^M = U^N \partial_N V^M - \alpha P_{(adj)}{}^M{}_{N,}{}^P{}_Q \partial_P U^Q V^N + \beta \partial_N U^N V^M \tag{2.3}$$

for some constants α and β. An upper index $M, N, \ldots$ denotes an object in the representation R_1 (see Table 1), the "coordinate representation" of $E_{n(n)}$, and $P_{(adj)}$ is the projection on the adjoint representation that is contained in the tensor product $R_1 \otimes \bar{R}_1$ of the coordinate representation R_1 and its conjugate. This lends itself to a natural generalisation to other representations, and respects composition of tensors since both terms obey the Leibniz rule. These transformations has been been defined, and their algebra examined, in ref. [45], in a formalism where $GL(n)$ is manifest. We will give a covariant analysis. The operation $\mathscr{L}_U$ is referred to as a generalised Lie derivative, or an "exceptional Dorfman bracket".

The Ansatz we will use is not of the form (2.3), but of the general form

$$\mathscr{L}_U V^M = L_U V^M + Y^{MN}{}_{PQ} \partial_N U^P V^Q \tag{2.4}$$

for some $E_{n(n)}$-invariant tensor Y. If the "tensor-friendly" version (2.3) is to hold, there must be an identity

$$Y^{MN}{}_{PQ} = \delta^M_P \delta^N_Q - \alpha P_{(adj)}{}^M{}_{Q,}{}^N{}_P + \beta \delta^M_Q \delta^N_P \ . \tag{2.5}$$

2.1. The section condition

When one starts commuting two generalised diffeomorphisms given by eq. (2.4), one immediately encounters the condition

$$Y^{MN}{}_{PQ} \partial_M \ldots \partial_N \ldots = 0 \ , \tag{2.6}$$

where the ellipses indicate that the derivatives act on different objects. This will be solved by the *section condition*. Often, one makes the difference between a *strong* and a *weak* section condition. The weak version of the section condition reads $P_{(R_2)}^{MN}{}_{PQ} \partial_M \partial_N \Phi = 0$ for any field or variable Φ, while the strong one states that $P_{(R_2)}^{MN}{}_{PQ} \partial_M \Phi \partial_N \Phi' = 0$ for any pair of fields or variables, which we write in shorthand as

$$(\partial \otimes \partial)|_{\bar{R}_2} = 0 \ . \tag{2.7}$$

(The representation R_2 is listed for the various values of n in Table 1.) This latter version is the one that will be needed throughout our analysis, and when we refer to the section condition in the following, we will mean its strong version, unless explicitly stated otherwise.

Section 4 will be devoted to a detailed analysis of the section condition, its reformulation and solution. For now, we just make a brief comment. The interpretation of eq. (2.7) is not as a non-linear condition on the momenta (which would be strange, considering they are cotangent vectors). Instead, eq. (2.7) should be read: Find a largest possible linear subspace of cotangent space such that any pair of vectors A and B belonging to the subspace fulfill $P_{(R_2)}^{MN}{}_{PQ} A_M B_N = 0$. This insinuates that there should be a more direct way of writing the section condition as a linear condition on momenta. We will do this in Section 4. Generally, any such solution to the section condition will pick out an n-dimensional subspace conserved by $GL(n)$.

In doubled geometry, where the manifest duality group is $O(d,d)$, the generalised Lie derivative (Dorfman bracket) is

$$\mathscr{L}_U V^M = U^N \partial_N V^M - (\delta^M_Q \delta^P_N - \eta^{MP} \eta_{NQ}) \partial_P U^Q V^N = L_U V^M + \eta^{MP} \eta_{NQ} \partial_P U^Q V^N \ , \tag{2.8}$$

where η is the $O(d,d)$-invariant metric. Comparing to the different forms for the expression in the U-duality setting, we see that it has analogous properties — the expression $\delta_Q^M\delta_N^P - \eta^{MP}\eta_{NQ}$ projects on the adjoint, and the section condition is simply $\eta^{MN}\partial_M\otimes\partial_N = 0$. The section condition is fulfilled by any pairs of covectors in an isotropic (null) subspace of dimension d.

2.2. The algebra of generalised diffeomorphisms

The tensor Y in the Ansatz for the generalised diffeomorphisms is completely determined by demanding that the transformations form an algebra. We need

$$[\mathscr{L}_U,\mathscr{L}_V] = \mathscr{L}_{[\![U,V]\!]}\ , \tag{2.9}$$

where

$$[\![U,V]\!] = [U,V]^M + \tfrac{1}{2}Y^{MN}{}_{PQ}(\partial_N U^P V^Q - \partial_N V^P U^Q)\ . \tag{2.10}$$

This bracket is analogous to and shares many properties with the Courant bracket in doubled geometry, and may be called an "exceptional Courant bracket". A direct calculation shows that the algebra closes according to eq. (2.9) iff the following conditions are fulfilled:

$$Y^{MN}{}_{PQ}\partial_M\otimes\partial_N = 0\ , \tag{2.11a}$$

$$\big(Y^{MN}{}_{TQ}Y^{TP}{}_{RS} - Y^{MN}{}_{RS}\delta_Q^P\big)\,\partial_{(N}\otimes\partial_{P)} = 0\ , \tag{2.11b}$$

$$\begin{aligned}\big(Y^{MN}{}_{TQ}Y^{TP}{}_{[SR]} &+ 2Y^{MN}{}_{[R|T|}Y^{TP}{}_{S]Q}\\ &-Y^{MN}{}_{[RS]}\delta_Q^P - 2Y^{MN}{}_{[S|Q|}\delta_{R]}^P\big)\,\partial_{(N}\otimes\partial_{P)} = 0\ ,\end{aligned} \tag{2.11c}$$

$$\begin{aligned}\big(Y^{MN}{}_{TQ}Y^{TP}{}_{(SR)} &+ 2Y^{MN}{}_{(R|T|}Y^{TP}{}_{S)Q}\\ &-Y^{MN}{}_{(RS)}\delta_Q^P - 2Y^{MN}{}_{(S|Q|}\delta_{R)}^P\big)\,\partial_{[N}\otimes\partial_{P]} = 0\ .\end{aligned} \tag{2.11d}$$

Eq. (2.11a), which roughly speaking is the section condition of the previous subsection, comes from a single remaining term in the commutator containing a derivative. Equation (2.11b) comes from terms with $\partial^2 UV$ and $U\partial^2 V$, while eqs. (2.11c) and (2.11d) multiply $\partial U\partial V$ (hence the opposite symmetrisations). No specific symmetry properties have been assumed for Y.

The easiest possible expression for $Y^{MN}{}_{PQ}$, satisfying the section condition, would be that it is proportional to the projector on R_2 itself,

$$Y^{MN}{}_{PQ} = kP_{(R_2)}^{MN}{}_{PQ}\ . \tag{2.12}$$

This is indeed the case for $n \leq 6$, where it then immediately follows that Y is symmetric in pairs of indices. For higher n, Y also contains some other term that separately vanishes with the help of the section condition when contracted with derivatives as in eq. (2.11a).

In all cases up to $n = 6$, with Y given by eq. (2.12), the equations (2.11) simplify. Terms with coefficients 1 and 2 in the third and fourth equation combine into symmetrisations in three indices. Note that this is also precisely what is needed for the second equation to hold; if

$$Y^{(MN}{}_{TQ}Y^{P)T}{}_{RS} - Y^{(MN}{}_{RS}\delta_Q^{P)} = 0 \ , \tag{2.13}$$

which turns out to be true for $n \leq 5$, but needs a little modification for $n = 6, 7$, the indices on the derivative can be cycled to Y, and the equations are then satisfied thanks to the section condition.

Let $Y^{MN}{}_{PQ} = kP^{MN}_{(R_2)}{}_{PQ}$, where $P_{(R_2)}$ is the projector on R_2 in the symmetric product of two R_1's (this will be modified for $n = 7$). Eq. (2.13) has the structure $R_1 \otimes R_2 \otimes (\otimes_s^3 \bar{R}_1)$. The number of possible invariant tensors is the number of singlets in this tensor product, *i.e.*, the number of irreducible modules in $T = (R_1 \otimes R_2) \cap (\otimes_s^3 R_1)$.

For $n \leq 5$, this number is 1, which means that the two terms in eq. (2.13) are proportional to each other. Then the constant k can be determined simply by taking some trace of the equation. One gets $k = 2(n-1)$. The explicit expressions are given below.

For $n = 6, 7$, the number of irreducible modules in T is 2. In E_6, there is an invariant symmetric tensor c^{MNP}. If one normalises it so that $c^{MNP}c_{MNP} = 27$ (in which case $P^{MN}_{(\mathbf{27})}{}_{PQ} = c^{MNR}c_{PQR}$), the relevant identity reads

$$10P^{(MN}_{(\mathbf{27})}{}_{QT}P^{P)T}_{(\mathbf{27})}{}_{RS} - P^{(MN}_{(\mathbf{27})}{}_{RS}\delta_Q^{P)} - \tfrac{1}{3}c^{MNP}c_{QRS} = 0 \ . \tag{2.14}$$

The last term is of course **27**- or $\overline{\mathbf{27}}$-projected on any pair of indices. The tensor Y is given by the same expression as for lower n.

When $n = 7$, R_1 is the fundamental 56-dimensional module. It is symplectic, so there is an invariant tensor ε_{MN}. We choose conventions where ε^{MN} is the inverse to ε_{MN}. R_2 is **133**, the adjoint. There is a completely symmetric 4-index tensor c^{MNPQ}, which we choose to normalise so that $c^{MNPQ} = P^{(MNPQ)}_{(\mathbf{133})}$. Then, the projector on **133** can be written

$$P^{MN}_{(\mathbf{133})}{}_{PQ} = c^{MN}{}_{PQ} + \tfrac{1}{12}\delta_P^{(M}\delta_Q^{N)} \ , \tag{2.15}$$

which is a practical expression when one wants to move indices on P. The relevant identity generalising eqs. (2.13,2.14) is

$$12P_{(\mathbf{133})QT}^{(MN}P_{(\mathbf{133})RS}^{P)T} - 4c^{MNPT}P_{(\mathbf{133})TQRS} - P_{(\mathbf{133})RS}^{(MN}\delta_Q^{P)} = 0 \ . \tag{2.16}$$

The tensor Y now necessarily contains an antisymmetric part, and takes the form

$$Y^{MN}{}_{PQ} = 12P_{(\mathbf{133})PQ}^{MN} + \tfrac{1}{2}\varepsilon^{MN}\varepsilon_{PQ} \ . \tag{2.17}$$

It is clear (even more so from the argument in Section 4.5) that if an $SL(7)$ vector is picked out by the section condition, one will also have $\varepsilon^{MN}\partial_M \otimes \partial_N = 0$.

In Section 2.4, we will study the case $n = 8$, for which the generalised Lie derivatives fail to form an ordinary Lie algebra.

To summarise, the forms of the tensor Y in the different cases are:

$$\begin{array}{ll} n=3: & Y^{i\alpha,j\beta}{}_{k\gamma,l\delta} = 4\delta_{kl}^{ij}\delta_{\gamma\delta}^{\alpha\beta} \ , \\ n=4: & Y^{mn,pq}{}_{rs,tu} = 6\delta_{rstu}^{mnpq} \ , \\ n=5: & Y^{\alpha\beta}{}_{\gamma\delta} = \tfrac{1}{2}\gamma_a^{\alpha\beta}\gamma_{\gamma\delta}^a \ , \\ n=6: & Y^{MN}{}_{PQ} = 10c^{MNR}c_{PQR} \ , \\ n=7: & Y^{MN}{}_{PQ} = 12c^{MN}{}_{PQ} + \delta_P^{(M}\delta_Q^{N)} + \tfrac{1}{2}\varepsilon^{MN}\varepsilon_{PQ} \ , \end{array} \tag{2.18}$$

with index notation that is hopefully self-explanatory.

In all cases, it can be checked that if one makes a Fierz-like rearrangement and rewrites $Y^{MN}{}_{PQ}$ in a basis where $R_1 \otimes \bar{R}_1$ represented by the indices N and P (which are the ones contracting $\partial_N U^P$) is expanded in irreducible modules, one gets

$$Y^{MN}{}_{PQ} = -\alpha_n P_{(adj)}{}^M{}_{Q,}{}^N{}_P + \beta_n\delta_Q^M\delta_P^N + \delta_P^M\delta_Q^N \ . \tag{2.19}$$

Here, the last term cancels the second term in the ordinary Lie derivative, when inserted in eq. (2.4). The projector on the adjoint is defined so that $P_{(adj)}{}^M{}_{N,}{}^R{}_S P_{(adj)}{}^S{}_{R,}{}^P{}_Q = P_{(adj)}{}^M{}_{N,}{}^P{}_Q$ and $P_{(adj)}{}^M{}_{N,}{}^N{}_M = \dim(adj)$ (unfortunately, the convention for raising and lowering indices leads to $P_{(adj)} = -P_{\mathbf{133}}$ for $n = 7$). The constants α_n and β_n take the numerical values $(\alpha_4, \beta_4) = (3, \frac{1}{5})$, $(\alpha_5, \beta_5) = (4, \frac{1}{4})$, $(\alpha_6, \beta_6) = (6, \frac{1}{3})$, $(\alpha_7, \beta_7) = (12, \frac{1}{2})$. For $n = 3$, the U-duality group $SL(3) \times SL(2)$ is not semisimple. There one has

$$Y^{i\alpha,j\beta}{}_{k\gamma,l\delta} = \delta_k^i\delta_l^j\delta_\gamma^\alpha\delta_\delta^\beta - (2P_{(\mathbf{8},\mathbf{1})} + 3P_{(\mathbf{1},\mathbf{3})})^{i\alpha}{}_{l\delta,}{}^{j\beta}{}_{k\gamma} + \tfrac{1}{6}\delta_l^i\delta_k^j\delta_\delta^\alpha\delta_\gamma^\beta \ . \tag{2.20}$$

This provides the precise relations with the expressions of ref. [45]. There seems to be a pattern: $\beta_n = \frac{1}{9-n}$. The coefficient in front of $P_{(R_2)}$ is always $2(n-1)$.

2.3. The Jacobi identity

For an ordinary Lie derivative, $L_U V$ is already antisymmetric in U and V, so $[U,V]=L_U V$. This is now no longer the case. Define the symmetric part of $\mathscr{L}_U V$ as $(\!(U,V)\!) = \frac{1}{2}(\mathscr{L}_U V + \mathscr{L}_V U)$. Then

$$\mathscr{L}_{(\!(U,V)\!)} W^M = -\left(Y^{M[N}{}_{PQ} Y^{|P|R]}{}_{[ST]} + Y^{M[N}{}_{[ST]} \delta^{R]}_Q \right) \partial_N U^S \partial_R V^T W^Q \ , \tag{2.21}$$

after eq. (2.11b) has been used for the part symmetric in indices on derivatives. This vanishes trivially for $n \leq 6$ and equals $-\frac{1}{4}\varepsilon^{NR}\varepsilon_{PQ}\partial_{[N}U^P \partial_{R]} V^Q W^M = 0$ for $n=7$ by the section condition. Hence a generalised diffeomorphism generated by $(\!(U,V)\!)$ gives a zero transformation.

The Jacobiator $[\![\cdot,\cdot,\cdot]\!]$ can be calculated using the same method as in ref. [25]. Let $[\![U,V,W]\!] = [\![[\![U,V]\!],W]\!] + \text{cycl.}$. Using

$$\begin{aligned} [\![[\![U,V]\!],W]\!] &= \tfrac{1}{2}(\mathscr{L}_{[\![U,V]\!]} W - \mathscr{L}_W [\![U,V]\!]) \\ &= \tfrac{1}{2}(\mathscr{L}_U \mathscr{L}_V W - \mathscr{L}_V \mathscr{L}_U W) - \tfrac{1}{4}(\mathscr{L}_W \mathscr{L}_U V - \mathscr{L}_W \mathscr{L}_V U) \ , \end{aligned} \tag{2.22}$$

we see that the Jacobiator can be written using either the first or the second term of the first line of eq. (2.22):

$$[\![U,V,W]\!] = \begin{cases} \frac{1}{4}\mathscr{L}_{[\![U,V]\!]} W + \text{cycl.} \\ \frac{1}{2}\mathscr{L}_W [\![U,V]\!] + \text{cycl.} \end{cases} \tag{2.23}$$

and thus also as

$$[\![U,V,W]\!] = \tfrac{1}{6}(\mathscr{L}_{[\![U,V]\!]} W + \mathscr{L}_W [\![U,V]\!]) + \text{cycl.} = \tfrac{1}{3}(\!([\![U,V]\!],W)\!) + \text{cycl.} \ . \tag{2.24}$$

So, the Jacobiator generates a zero transformation for all $n \leq 7$. We will give a more careful interpretation of this statement in the following section.

2.4. $n = 8$

For $n=8$, the algebra does not work. This is more or less expected, since this is where the dual (in the 11-dimensional sense) gravity field becomes relevant. As we will see in the following section, the counting of gauge parameters nevertheless matches the ones in component form, including the "dual diffeomorphisms". For this reason, we would still like to say a few words about the failure of the algebra.

The section condition will necessarily be in $\bar{R}_2 = \mathbf{1} \oplus \mathbf{3875}$, as can be seen at an early stage in the calculation (this is vindicated by the entry in Table 1, as it will be calculated in Section 3). This leaves only **27000** in the symmetric product of two derivatives, and it can be deduced that it also implies that the projections of two derivatives on **248** vanish. The projection operators are [59]

$$\begin{aligned}
&P_{(\mathbf{1})}^{MN}{}_{PQ} = \tfrac{1}{248}\eta^{MN}\eta_{PQ} \ , \\
&P_{(\mathbf{248})}^{MN}{}_{PQ} = -\tfrac{1}{60}f_A{}^{MN}f^A{}_{PQ} \ , \\
&P_{(\mathbf{3875})}^{MN}{}_{PQ} = -\tfrac{1}{14}f^{A(M}{}_P f_A{}^{N)}{}_Q + \tfrac{1}{7}\delta_P^{(M}\delta_Q^{N)} - \tfrac{1}{56}\eta^{MN}\eta_{PQ} \ ,
\end{aligned} \tag{2.25}$$

where the structure constants are normalised so that $f^{MAB}f_{NAB} = -60\delta_N^M$.

There is a certain combination of these projectors that combine in the same way as for lower n, and this is a reasonable candidate:

$$\begin{aligned}
\mathscr{L}_U V^M &= L_U V^M + (14P_{(\mathbf{3875})} - 30P_{(\mathbf{248})} + 62P_{(\mathbf{1})})^{MN}{}_{PQ}\partial_N U^P V^Q \\
&= U^N\partial_N V^M + \partial_N U^N V^M - f^{AM}{}_Q f_A{}^N{}_P \partial_N U^P V^Q \ .
\end{aligned} \tag{2.26}$$

Note that the coefficients follow the pattern deduced from lower n. It certainly looks a lot easier to try this than to make a general Ansatz with the P's. The section condition now implies

$$\begin{aligned}
&\eta^{MN}\partial_M \otimes \partial_N = 0 \ , \\
&f^{AMN}\partial_M \otimes \partial_N = 0 \ , \\
&(f^{AM}{}_P f_A{}^N{}_Q - 2\delta_P^{(M}\delta_Q^{N)})\partial_M \otimes \partial_N = 0
\end{aligned} \tag{2.27}$$

(note that symmetrisation is not needed in the first term in the last equation).

The terms in $([\mathscr{L}_U, \mathscr{L}_V] - \mathscr{L}_{[\![U,V]\!]})W$ with a derivative on W vanish due to the section condition. It turns out that also the terms of the form "$\partial U \partial V W$" all cancel. A long calculation leads to a single remaining obstruction with the structure $f^{MN}{}_Q f^P{}_{ST}\partial_N\partial_P U^S V^T W^Q - (U \leftrightarrow V)$. Even if our guess (2.26) was dictated by the tensor-friendly form (2.3), it can be checked explicitly that other combinations fail even more seriously to fulfill an algebra.

3. The ghost tower

The tensor gauge transformations are reducible. A 2-form transformation has a 1-form reducibility and a 0-form second order reducibility, so that the effective number of gauge parameters in n dimensions is $\binom{n}{2} - n + 1 = \binom{n-1}{2}$, and analogously for a a 5-form parameter $\binom{n-1}{5}$. Including diffeomorphisms, the effective number of generalised diffeomorphisms should be $n + \binom{n-1}{2} + \binom{n-1}{5}$, as long as dual gravity does not enter. This number will be checked by examining the reducibility of the generalised diffeomorphisms in their covariant form.

In doubled geometry, the gauge transformations contained in the generalised diffeomorphisms are diffeomorphisms and the gauge transformation of the 2-form B, $\delta B = d\Lambda$. The latter is reducible, since $\Lambda = d\phi$ gives rise to no transformation on the fields. This reducibility is directly reflected in the reducibility of the generalised diffeomorphisms: a parameter $U^M = \eta^{MN}\partial_N\xi$ does not enter the transformation (2.8) due to the section condition.

Analogously, the reducibility in the exceptional setting is also associated with the section condition. A parameter constructed as $U^M[\xi] = \partial_N\xi^{MN}$, where ξ is in the representation R_2 conjugate to the section condition, will generate a zero transformation through $\mathscr{L}_{U[\xi]}$. This is easily seen from the form (2.3) of the generalised diffeomorphisms. In the transport term, $\partial_P\xi^{NP}\partial_P = 0$ due to the section condition, and in the $E_{n(n)} \times \mathbb{R}$ transformation terms, $\partial^2\xi$ contains neither the singlet nor the adjoint. This is the first order reducibility. The relation for $U[\xi]$ will in turn be reducible, in the sense that for an $\xi^{MN} = \partial_P\xi'^{MNP}$, with ξ' in a certain representation R_3, $U[\xi[\xi']] = 0$, and so on.

We will now examine the representation content of the reducibility, and show that it is directly connected to the properties of the weak section condition. Namely, consider an object λ_M in $\bar{R}_1$, satisfying the weak section condition, $T \equiv \lambda^2|_{\bar{R}_2} = 0$. This constraint is reducible, there is always some representation $\bar{R}_3$ such that $T' \equiv (\lambda T)|_{\bar{R}_3} = 0$. Again, given this form of T' it will satisfy $T'' \equiv (\lambda T')|_{\bar{R}_4} = 0$, and so on. The representations in question can be determined by examining the partition function for the object λ, as will be done below. A typical example is provided by pure spinors in $D = 10$, which will actually be one of the cases.

Now, consider a momentum in R_1 conjugate to λ, and call it $w^M = \frac{\partial}{\partial\lambda_M}$. A naked w is not invariant with respect to the constraint on λ, $\lambda^2|_{\bar{R}_2} = 0$. This constraint, with parameter ξ^{MN} in R_2, generates a transformation of w, $\delta_\xi w^M = [-\frac{1}{2}\xi^{NP}\lambda_N\lambda_P, w^M] = \lambda_N\xi^{MN}$. Here we note that this transformation is formally equivalent to the reducibility of the parameter of generalised diffeomorphisms, if we replace λ_M by ∂_M and w^M by U^M. Once this is established, it is clear that the parameters of higher order reducibility are identical to those of the reducibility of the weak section condition on ∂. The first order reducibility is precisely such that it leaves both the scalar and adjoint parts of $\partial_M U^N$ invariant. This point of view gives yet another, algebraic, reason why only the scalar and adjoint of ∂U can appear in the generalised Lie derivative (2.3).

Such towers of ghosts have been examined in the cases of pure spinors in various dimensions. The details can be derived as follows. Write a partition function by counting the homogeneous functions of degree i of the constrained object λ:

$$Z(t)=\sum_{i=0}^{\infty}\dim(r_i)t^i\ . \tag{3.1}$$

(This, and everything below, can of course in principle be refined by not only counting dimensions, but also actual representations. For low R_k's, the actual representations can however be deduced safely by just observing the dimension.) In all the cases under consideration, the weak section condition is such that the representation r_i contained in the i'th power of λ is the irreducible representation with highest weight i times the one of the coordinate representation $\bar{R}_1$. This is a direct consequence of the fact that all smaller representations than the largest one are absent in r_2 due to the bilinear constraint. In r_i, any smaller representation would have to be formed by tensor some pair of λ's into such a smaller representation.

These partition functions Z_n are given for different values of n below. They can either be calculated with the help of explicit expressions[†] for $\dim(r_i)$, or alternatively in a pragmatic way: by forming the series in eq. (3.1) from dimensions calculated with LiE, up to some high power where we safely can conclude that it coincides with $(1-t)^{-(\text{some number})}\times F(t)$, where $F(t)$ is a polynomial with $F(1)\neq 0$.

$$\begin{aligned}
Z_3(t)&=\sum_{i=0}^{\infty}(i+1)\binom{i+2}{2}t^i=(1-t)^{-4}(1+2t)\ ,\\
Z_4(t)&=\sum_{i=0}^{\infty}\frac{1}{3}\binom{i+4}{4}\binom{i+3}{2}t^i=(1-t)^{-7}(1+3t+t^2)\ ,\\
Z_5(t)&=\sum_{i=0}^{\infty}\frac{1}{10}\binom{i+7}{7}\binom{i+5}{3}t^i=(1-t)^{-11}(1+t)(1+4t+t^2)\ ,\\
Z_6(t)&=\sum_{i=0}^{\infty}\frac{1}{56}\binom{i+11}{11}\binom{i+8}{5}t^i=(1-t)^{-17}(1+t)(1+9t+19t^2+9t^3+t^4)\ ,\qquad (3.2)\\
Z_7(t)&=\sum_{i=0}^{\infty}\frac{1}{3^2\cdot 5\cdot 11\cdot 13}(i+9)\binom{i+17}{17}\binom{i+13}{9}t^i
\end{aligned}$$

[†] See for example The Online Encyclopedia of Integer Sequences, `http://oeis.org`.

$$
\begin{aligned}
&= (1-t)^{-28}(1+28t+273t^2+1248t^3+3003t^4+4004t^5+3003t^6+1248t^7\\
&\qquad +273t^8+28t^9+t^{10})\ ,\\
Z_8(t) &= \sum_{i=0}^{\infty}\frac{1}{2\cdot 7\cdot 11^2\cdot 13\cdot 17\cdot 19^2\cdot 23\cdot 29}(2i+29)\binom{i+28}{28}\binom{i+23}{18}\binom{i+19}{10}t^i\\
&= (1-t)^{-58}(1+t)(1+189t+14\,080t^2+562\,133t^3+13\,722\,599t^4\\
&\qquad +220\,731\,150t^5+2\,454\,952\,400t^6+19\,517\,762\,786t^7+113\,608\,689\,871t^8\\
&\qquad +492\,718\,282\,457t^9+1\,612\,836\,871\,168t^{10}+4\,022\,154\,098\,447t^{11}\\
&\qquad +7\,692\,605\,013\,883t^{12}+11\,332\,578\,013\,712t^{13}+12\,891\,341\,012\,848t^{14}\\
&\qquad +11\,332\,578\,013\,712t^{15}+7\,692\,605\,013\,883t^{16}+4\,022\,154\,098\,447t^{17}\\
&\qquad +1\,612\,836\,871\,168t^{18}+492\,718\,282\,457t^{19}+113\,608\,689\,871t^{20}\\
&\qquad +19\,517\,762\,786t^{21}+2\,454\,952\,400t^{22}+220\,731\,150t^{23}+13\,722\,599t^{24}\\
&\qquad +562\,133t^{25}+14\,080t^{26}+189t^{27}+t^{28})\ .
\end{aligned}
$$

The effective number of independent gauge parameters is read off as the negative power of the first factor (the number of "bosonic degrees of freedom"). For $n \leq 6$ the corresponding spaces and their dimensions are known earlier. For $n = 4$, the 7-dimensional space is a cône over the Grassmannian $Gr(2,5)$ of 2-planes in 5 dimensions. For $n = 5$, 11 is the dimension of the space of pure spinors of $Spin(5,5)$. For $n = 6$, an object X^M with $c_{MNP}X^NX^P = 0$ lies on a 17-dimensional cône over the 16-dimensional Cayley plane [60].

For $n \leq 7$, the dimension is 1 greater than the dimension of R_1 for the next lower value of n. This observation should be related to the existence of a 3-grading of the algebra corresponding to the subgroup $E_{n+1(n+1)} \supset E_{n(n)} \times \mathbb{R}$, providing a non-linear "conformal" realisation of $E_{n+1(n+1)}$ on R_1 of $E_{n(n)}$ [61,62]. In fact, the present construction provides an infinite-dimensional linear representation of $E_{n+1(n+1)}$ on polynomials of the constrained objects in R_1 of $E_{n(n)}$, *i.e.*, on $\oplus_{i=0}^{\infty}(i0\ldots 0)$ (the Dynkin index for R_1 is taken to be $(10\ldots 0)$), which can be thought of as a singleton representation. For $n = 5$, this was also observed in ref. [63]. For $n = 7$, the grading corresponding to $E_{8(8)} \supset E_{7(7)} \times SL(2,\mathbb{R})$ is a 5-grading [62]. The dimensions can also be identified as the dimensions of coadjoint nilpotent orbits of $\frac{1}{2}$-BPS instantons [64].

For $n \leq 7$, the number of gauge parameters thus calculated match the number of diffeomorphisms, 2-form and 5-form (for $n \geq 6$) transformations calculated above. For $n = 8$, strikingly enough, the counting also matches if one includes also $n\binom{n-1}{7} = 8$ gauge parameters for the vector-valued 7-form transformations of the dual gravity field. The counting, and the comparison with the results from reducibility are summarised in Table 2.

n	diffeo	2-form	5-form	dual diffeo	total
3	3	1			4
4	4	3			7
5	5	6	0		11
6	6	10	1		17
7	7	15	6	0	28
8	8	21	21	8	58

Table 2: The counting of gauge parameters.

More information on the structure of the reducibility can be extracted by rewriting the partition functions as products of ghost partitions,

$$Z(t)=\prod_{k=1}^{\infty}(1-t^k)^{-A_k}\ , \tag{3.3}$$

where the power A_k is $(-1)^{k-1}$ times the dimension of the $(k-1)$'th reducibility ghost representation. To get the number of effective gauge transformations, we want to calculate a regulated sum $\sum_{k=1}^{\infty}A_k$. Taking the logarithm,

$$\log Z(t)=-\sum_{k=1}^{\infty}A_k\log(1-t^k)=-\sum_{k=1}^{\infty}A_k\left(\log(1-t)+\log\sum_{i=0}^{k-1}t^i\right)\ . \tag{3.4}$$

The second logarithm is regular at $t=1$, so the sum is obtained as the coefficient of the singular behaviour $-\log(1-t)$ at $t=1$, as argued above. This result is also what one obtains from regulating the sum with analytic continuation. A more refined treatment (see for example ref. [65]) is required if one wants to calculate other moments like ghost number $\sum_{k=1}^{\infty}kA_k$.

A completely refined partition function gives information about the exact representations R_k. It requires rewriting the known partition function, including complete information about representations,

$$\mathscr{Z}(t)=\bigoplus_{i=0}^{\infty}(i0\ldots0)t^i\ . \tag{3.5}$$

on a product form

$$\mathscr{Z}(t)=\left(\prod_{k\in 2\mathbb{N}+1}\bigoplus_{j=0}^{\infty}t^{jk}\otimes_s^j R_k\right)\otimes\left(\prod_{k\in 2\mathbb{N}+2}\bigoplus_{j=0}^{\dim(R_k)}(-1)^j t^{jk}\wedge^j R_k\right)\ , \tag{3.6}$$

where the first factor contains partitions for bosons in R_{odd} and the second one partitions for fermions in R_{even}. This can be done recursively to find arbitrary R_k.

Unlike tensor gauge transformations and generalised diffeomorphisms in doubled field theory, the tower of ghosts (reducibility) is infinite in all cases. Such a statement can of course change if one is allowed to break E_n invariance to some smaller covariance. Note that the representations R_k, listed in Table 1, coincide with the representations of "form fields", listed in various tables (see *e.g.* refs. [51,52,56,40])*. The representations R_k are possible representations for k-form fields in the uncompactified $11-n$ dimensions. The sequences continues beyond those of the form fields, and do not halt at any finite k.

For example, when $n=6$,

$$Z_6(t)=(1-t)^{-27}(1-t^2)^{27}(1-t^3)^{-78}(1-t^4)^{351}(1-t^5)^{-1755}(1-t^6)^{8983}(1-t^7)^{-47034}\times\ldots \tag{3.7}$$

Here, we recognise the $\mathbf{27}$, $\overline{\mathbf{27}}$, $\mathbf{78}$, $\overline{\mathbf{351'}}$ and $\overline{\mathbf{1728}}\oplus\overline{\mathbf{27}}$ from the table of fields†.

The Dynkin indices of R_k for the first few k are depicted in Figure 1. For $n=7,8$, this gives the leading (biggest) representations. For $n=3$, $R_1=(10)(1)$ (*i.e.*, 1's at the nodes marked R_4 and R_3 in the figure), but $R_{2,3,4}$ are given accurately by the figure.

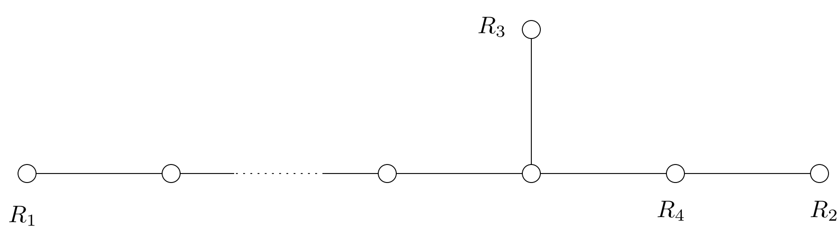


Figure 1: Dynkin indices of some reducibility representations.

* The representations we derive coincide exactly with those appearing in Borcherds algebras. For $n=8$, we have verified this up to R_4. The reason for this is probably that Serre relations for the Borcherds algebra is effectively encoded in an algebraic constraint (the section condition). We may come back to this in a future publication.

† We write $\mathbf{351'}$ for a tensor $A^{[MN]}$. There are four 351-dimensional representations of E_6: this one, the symmetric $\mathbf{351}$ of a tensor $S^{(MN)}$ with $c_{MNP}S^{NP}=0$, and their conjugates.

Let us spell out an example in some detail. For $n=5$, the parameter is a spinor, and we have

$$\begin{aligned} Y^{\alpha\beta}{}_{\gamma\delta} &= 8P_{\mathbf{10}}^{\alpha\beta}{}_{\gamma\delta} = \tfrac{1}{2}\gamma_a^{\alpha\beta}\gamma^a_{\gamma\delta} \\ &= \delta_\gamma^\alpha\delta_\delta^\beta + \tfrac{1}{8}(\gamma^{ab})^\alpha{}_\delta(\gamma_{ab})^\beta{}_\gamma + \tfrac{1}{4}\delta_\delta^\alpha\delta_\gamma^\beta \ , \end{aligned} \tag{3.8}$$

so that generalised diffeomorphisms are generated by $\mathscr{L}_U$, where

$$\mathscr{L}_U V^\alpha = (U\partial)V^\alpha + \tfrac{1}{8}(\partial\gamma^{ab}U)(\gamma_{ab}V)^\alpha + \tfrac{1}{4}(\partial U)V^\alpha \ . \tag{3.9}$$

Now, consider a parameter U which is constructed as a derivative of a $Spin(5,5)$ vector as

$$U^\alpha[\xi] = \gamma_a^{\alpha\beta}\partial_\beta\xi^a \ . \tag{3.10}$$

Substituting this parameter in the transformation gives

$$\mathscr{L}_{U[\xi]}V^\alpha = \gamma_a^{\beta\gamma}\partial_\gamma\xi^a\partial_\beta V^\alpha + \tfrac{1}{8}(\partial\gamma^{ab}\gamma_c\partial)\xi^c(\gamma_{ab}V)^\alpha + \tfrac{1}{4}(\partial\gamma_c\partial)\xi^c V^\alpha \ . \tag{3.11}$$

All three terms vanish, since the section condition implies $\gamma_a^{\alpha\beta}\partial_\alpha\otimes\partial_\beta = 0$. So, the reducibility of the parameter U^α lies at least in ξ^a (it is not difficult to show that this the complete reducibility). Then ξ in turn has a second order reducibility, $\xi^a = (\partial\gamma^a\xi')$, which gives zero in eq. (3.10) by virtue of the Fierz identity $\gamma^{a\alpha(\beta}\gamma_a{}^{\gamma\delta)} = 0$ and the section condition. Next, $\xi'_\alpha = (\gamma_{ab}\partial)_\alpha\xi''^{ab}$, etc. The structure is reflected in the product form

$$Z_5(t) = (1-t)^{-16}(1-t^2)^{10}(1-t^3)^{-16}(1-t^4)^{45}(1-t^5)^{-144}(1-t^6)^{456}(1-t^7)^{-1440}\times\ldots \tag{3.12}$$

The parameter U in $R_1 = \mathbf{16}$ branches into the diffeomorphism vector and a 2-form and a 5-form gauge transformation. The vector ξ in $R_2 = \mathbf{10}$ branches into a 1-form and a 4-form when $Spin(5,5)\to GL(5)$. This is of course the reducibility one wants for a 2-form and a 5-form gauge transformation. At the next level, something has to change, however. The second order reducibility in the $GL(n)$ language contains a scalar and a 3-form (in total 11 second order ghosts), while the smallest representation of ξ' such that $\xi = \partial\xi'$ is $R_3 = \overline{\mathbf{16}}$. There is an excess of a 1-form. This does not imply any extra reducibility as compared to the $GL(n)$-covariant considerations, but has to be compensated for by higher reducibilities. The reducibility becomes infinite, with ever growing representations R_k, but in a way that makes the resulting infinite alternating sum meaningful.

4. The section condition as a linear constraint

In the generalised space-time the coordinates form the representation R_1 of the duality group $E_{n(n)}$. The section condition is given by projecting out a particular representation $\bar{R}_2$:

$$(\partial \otimes \partial)|_{\bar{R}_2} = 0 \ . \tag{4.1}$$

We propose instead to introduce an auxiliary object Λ in a representation T which will play the analogous role to the pure spinor in the $O(d,d)$ case and pick out a subspace akin to the way the pure spinor identifies a maximal isotropic subspace. The "pure spinor" constraints in this case transform in a representation P, *i.e.*, has the form

$$\Lambda^2|_P = 0 \ . \tag{4.2}$$

The linear section equation will have the form

$$(\Lambda \otimes \partial)|_N = 0 \tag{4.3}$$

for some representation N, and will imply the section condition.

We will make this reformulation for $n \leq 7$. In all cases, the representation T of Λ is $\bar{R}_3$, and the representation N for the linear constraint of eq. (4.3) is $\bar{R}_4$. The constraint on Λ is absent for $n \leq 4$. We are not yet able to deduce a pattern for the representation P of this constraint.

Let us remind of the situation in doubled generalised geometry. The section condition in its non-linear form reads $\eta^{MN}\partial_M \otimes \partial_N = 0$. The largest linear subspace of the cône of null vectors is an isotropic subspace. Such a space is determined by the choice of a pure $Spin(d,d)$ spinor Λ^α, obeying $(\Lambda\gamma^M\Lambda) = 0$, and the strong section condition is replaced by the linear condition $(\gamma^M\Lambda)_\alpha\partial_M = 0$.

4.1. $n = 3$

The duality group is $G = SL(3) \times SL(2)$. The six coordinates of extended space time are in the $R_1 = (\mathbf{3}, \mathbf{2})$ representation of this duality group. The section condition reads

$$\varepsilon^{\alpha\beta}\partial_{a\alpha}A\partial_{b\beta}B = 0 \ , \quad \alpha = 1,2 \quad a = 1,2,3 \ . \tag{4.4}$$

Let us instead introduce a Λ^α in $\bar{R}_3 = (\mathbf{1}, \mathbf{2})$ and consider the linear section equation

$$\Lambda^\alpha \partial_{a\alpha} = 0 \ . \tag{4.5}$$

This transforms in $\bar{R}_4 = (\overline{\mathbf{3}}, \mathbf{1})$ and it implies the section condition. For instance, choose a frame where $\Lambda^1 \neq 0$ with $\Lambda^2 = 0$, then we see that eq. (4.5) would imply that $\partial_{a1} = 0$ and clearly then eq. (4.4) is satisfied. The linear section condition serves to reduce from the six extended coordinates to three physical coordinates.

4.2. $n = 4$

In this case the duality group is $G = SL(5)$. The ten coordinates of extended space time are in the $R_1 = \overline{\mathbf{10}}$ representation of G. The section condition is in $\bar{R}_2 = \mathbf{5}$:

$$\varepsilon^{abcde} \partial_{ab} A \partial_{cd} B = 0 \ , \quad a = 1 \ldots 5 \ . \tag{4.6}$$

Instead let us introduce an Λ_a in the $\bar{R}_3 = \overline{\mathbf{5}}$ and consider the linear section equation

$$\Lambda_{[a} \partial_{bc]} = 0 \ . \tag{4.7}$$

This transforms in $\bar{R}_4 = \mathbf{10}$ and it implies the section condition. For instance choose $\Lambda_1 \neq 0$ with other components vanishing. Then we see that eq. (4.7) would imply that $\partial_{23} = \partial_{24} = \partial_{25} = \partial_{34} = \partial_{35} = \partial_{45} = 0$ and clearly then eq. (4.6) is satisfied. The linear section condition serves to reduce from the ten extended coordinates to four physical coordinates.

4.3. $n = 5$

The duality group is $G = Spin(5,5)$ and the coordinates form an $R_1 = \mathbf{16}$. The section condition is

$$\partial_\alpha \gamma^{a\alpha\beta} \partial_\beta = 0 \ , \quad a = 1 \ldots 10, \ \alpha = 1 \ldots 16 \ . \tag{4.8}$$

We now introduce a linear section condition

$$0 = \Lambda^\alpha (\gamma^{ab})_\alpha{}^\beta \partial_\beta = 0 \ , \tag{4.9}$$

which transforms in $\bar{R}_4 = \mathbf{45}$, involving a spinor Λ^α in $\bar{R}_3 = \mathbf{16}$. If we further make the restriction that Λ is pure we see that this constraint implies that 11 components of

∂_α are constrained to vanish leaving correctly 5 physical coordinates. To see this consider decomposing $\Lambda = \phi + u_{[ab]} + t_{[abcd]}$ and considering the special choice for which $\phi \neq 0$ is the only non-vanishing. Then in terms of the Mukai pairing we need

$$\begin{aligned}
&(\Lambda, \omega_1 \wedge \omega_2 \partial) = 0 \ , \\
&(\Lambda, \iota_{X_1} \iota_{X_2} \partial) = 0 \ , \\
&(\Lambda, \iota_{X_1}(\omega_1 \wedge \partial) - \omega_1 \wedge (\iota_{X_1} \partial)) = 0 \ .
\end{aligned} \tag{4.10}$$

With this choice for Λ we see that first of these sets the 3-form of ∂ to zero, the second is trivial and the final one sets the five-form to zero. What remains unconstrained are the five coordinates in the direction of the one-form. This is a solution to the section condition.

4.4. $n = 6$

The duality group is the split form of E_6 and the coordinates are in the $\mathbf{27}^\star$. The section conditions must eliminate 21 of these components.

To match with the field content we decompose according to $SL(6) \times SL(2)$ so that the derivatives decompose as $\overline{\mathbf{27}} \to (\overline{\mathbf{15}}, \mathbf{1}) \oplus (\mathbf{6}, \mathbf{2})$:

$$\partial = \partial^{ab} + \partial_a^\alpha \ . \tag{4.11}$$

Then the section conditions are given by a $\mathbf{27} \to (\mathbf{15}, \mathbf{1}) \oplus (\overline{\mathbf{6}}, \mathbf{2})$:

$$\varepsilon_{\alpha\beta} \partial_a^\alpha \partial_b^\beta + \varepsilon_{abcdef} \partial^{cd} \partial^{ef} = 0 \ , \quad \partial^{ab} \partial_b^\beta = 0 \ , \quad a = 1 \ldots 6 \ , \quad \alpha = 1 \ldots 2 \ , \tag{4.12}$$

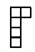
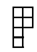
In other words we are projecting the $(\overline{\mathbf{27}} \otimes \overline{\mathbf{27}})|_{\mathbf{27}}$ to zero.

Now to build the linear section condition we start with a Λ in $\bar{R}_3 = \mathbf{78}$ and impose the "purity constraint" $\Lambda^2|_{\mathbf{650}} = 0$. Then for this restricted choice we tensor with a derivative and demand $\Lambda\partial|_{\bar{R}_4 = \mathbf{351}'} = 0$. We decompose the $\mathbf{78}$ according to the $SL(6) \times SL(2)$, $\mathbf{78} \to (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{35}, \mathbf{1}) \oplus (\mathbf{20}, \mathbf{2})$,

$$\Lambda \to \phi^{(\alpha\beta)} + u_a{}^b + w_{[abc]}{}^\alpha \ . \tag{4.13}$$

⋆ It seems that the number of point-like charge in five dimensions gives a counting of $\mathbf{27} \oplus \mathbf{1}$ and provides an extra singlet which is unneeded or irrelevant. Probably it can be accommodated but is rather trivially projected out by the section condition.

The constraint decomposes as $\mathbf{650} \to (\mathbf{1},\mathbf{1}) \oplus (\mathbf{70},\mathbf{2}) \oplus (\overline{\mathbf{70}},\mathbf{2}) \oplus (\mathbf{20},\mathbf{2}) \oplus (\mathbf{189},\mathbf{1}) \oplus (\mathbf{35},\mathbf{1}) \oplus (\mathbf{35},\mathbf{3})$, where some non-trivial $SL(6)$ representations are

$\mathbf{20}:$  $\mathbf{21}:$ $\mathbf{70}:$  $\mathbf{84}:$  $\mathbf{105}:$  $\mathbf{189}:$  (4.14)

A representative of the solution of the constraint on Λ can be taken as $u_a{}^b = w_{[abc]}{}^\alpha = 0$, $\phi^{12} = \phi^{22} = 0$, $\phi^{11} \neq 0$.

Now we consider this particular solution in the section equation in the representation $\bar{R}_4 = \mathbf{351'}$, which decomposes as $\mathbf{351'} \to (\overline{\mathbf{21}},\mathbf{1}) \oplus (\overline{\mathbf{15}},\mathbf{3}) \oplus (\overline{\mathbf{84}},\mathbf{2}) \oplus (\mathbf{6},\mathbf{2}) \oplus (\mathbf{105},\mathbf{1})$. The relevant $SL(6) \times SL(2)$ representations of the linear section condition (considering $u = w = 0$) are only $(\overline{\mathbf{15}},\mathbf{3}) \oplus (\mathbf{6},\mathbf{2})$. This implies $\partial^{ab} = 0$ and $\partial_a^2 = 0$. Six directions remain.

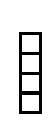
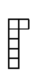
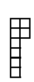
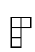
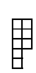
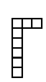
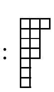
We can check that under dimensional reduction this reduces to the pure spinor constraint and associated linear section condition. This entails doing a branching into $SO(5,5)$ and essentially keeping only the $\mathbf{10_{-2}} \subset \mathbf{27}$ and the $\mathbf{16_{-3}} \subset \mathbf{78}$. From the purity constraint, $(\mathbf{78} \otimes_S \mathbf{78})|_{\mathbf{650}} = 0$, one recovers $(\mathbf{16_{-3}} \otimes \mathbf{16_{-3}})|_{\mathbf{10_{-6}}}$ and then from the section condition $(\mathbf{27} \otimes \mathbf{78})|_{\mathbf{351}} = 0$ one indeed recovers $(\mathbf{10_{-2}} \otimes \mathbf{16_{-3}})|_{\overline{\mathbf{16}}_{\mathbf{-5}}} = 0$.

4.5. $n = 7$

For E_7, the "pure spinor" should be in $\bar{R}_3 = \mathbf{912}$. Start by decomposing various modules in $SL(8)$ modules:

$$
\begin{aligned}
\mathbf{56} &\longrightarrow \mathbf{28} \oplus \overline{\mathbf{28}} \\
\mathbf{133} &\longrightarrow \mathbf{63} \oplus \mathbf{70} \\
\mathbf{912} &\longrightarrow \mathbf{36} \oplus \overline{\mathbf{36}} \oplus \mathbf{420} \oplus \overline{\mathbf{420}} \\
\mathbf{1463} &\longrightarrow \mathbf{1} \oplus \mathbf{336} \oplus \overline{\mathbf{336}} \oplus \mathbf{720} \oplus \mathbf{70} \\
\mathbf{8645} &\longrightarrow \mathbf{63} \oplus \mathbf{378} \oplus \overline{\mathbf{378}} \oplus \mathbf{2352} \oplus \mathbf{945} \oplus \overline{\mathbf{945}} \oplus \mathbf{3584}
\end{aligned}
\tag{4.15}
$$

The Young tableaux for some non-obvious representations are:

$\mathbf{70}:$  $\mathbf{420}:$  $\mathbf{336}:$  $\mathbf{720}:$  $\mathbf{378}:$  $\mathbf{2352}:$  $\mathbf{945}:$  $\mathbf{3584}:$  (4.16)

The section condition as a bilinear condition on the derivatives reads $\partial_{ac}\partial^{bc} - \frac{1}{8}\delta_a^b \partial_{cd}\partial^{cd} = 0$, $\partial_{[ab}\partial_{cd]} + \frac{1}{24}\varepsilon_{abcdefgh}\partial^{ef}\partial^{gh} = 0$. A representative of the solution can be taken as the linear subspace spanned by ∂_{a8}, which breaks to $SL(7)$. Consider an object Λ in $\mathbf{912}$, constrained by $\Lambda^2|_{\mathbf{1463}} = 0$. One solution is that Λ only sits as an $SL(7)$ singlet λ_{88} in the $\mathbf{36}$. Consider now a linear condition $\Lambda\partial|_{\mathbf{8645}\oplus\mathbf{133}} = 0$. With Λ in **36** as above, the constraints are $\Lambda_{ab}\partial^{cd} = 0$ ($\mathbf{945}\oplus\mathbf{63}$), $\Lambda_{a[b}\partial_{cd]} = 0$ (**378**). This is solved by ∂_{ab} as above.

5. Conclusions

In this paper, we have studied a couple of different, but connected, aspects of generalised diffeomorphisms in the U-duality (exceptional) framework. We have examined their algebraic structure and reducibility in a U-duality covariant formalism, and demonstrated how to understand and formulate the section condition in a linear way.

One of the most striking observations here is the appearance of the representations forming tensor hierarchies or Borcherds algebras, connected to form fields of different degrees in the dimensionally reduced theory, as representations describing the infinite reducibility of the generalised diffeomorphisms. The representation contents of these structures have varied slightly between different authors, but we believe that our predictions, that are algebraically unique, will provide the generic structure. This question certainly merits further attention.

Although we have not been able to give a consistent algebra of generalised diffeomorphisms based on E_8, it is striking that the algebra is as close to working as it is. It is also remarkable that the natural extrapolation of the reducibility of gauge parameters produces the correct counting, including the dual diffeomorphisms. Even if E_8 in itself is a complicated algebra, it is still finite-dimensional, and may provide a relatively simple means of studying dual gravity without the introduction of infinite-dimensional algebras — the caveat of course being that the structure must be modified in some way to make sense algebraically.

We think that the covariant treatment in the present paper opens a route to a classification of generalised geometries. Another urgent question is the extension to supergeometries. In order to take that step, a tensor calculus with a spin connection has to be invented. Such a formulation is at present unknown, but will be needed since fermions transform under the compact subgroup $K(E_{n(n)})$ [66].

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