ON THE SPECTRA OF SIMPLICIAL ROOK GRAPHS

JEREMY L. MARTIN AND JENNIFER D. WAGNER

Abstract. The *simplicial rook graph* $SR(d, n)$ is the graph whose vertices are the lattice points in the nth dilate of the standard simplex in $\mathbb{R}^d$, with two vertices adjacent if they differ in exactly two coordinates. We prove that the adjacency and Laplacian matrices of $SR(3, n)$ have integral spectrum for every n. The proof proceeds by calculating an explicit eigenbasis. We conjecture that $SR(d, n)$ is integral for all d and n, and present evidence in support of this conjecture. For $n < \binom{d}{2}$, the evidence indicates that the smallest eigenvalue of the adjacency matrix is $-n$, and that the corresponding eigenspace has dimension given by the Mahonian numbers, which enumerate permutations by number of inversions.

1. Introduction

Let d and n be nonnegative integers. The *simplicial rook graph* $SR(d, n)$ is the graph with vertices

$$V(d, n) := \left\{ x = (x_1, \dots, x_d) \colon 0 \leq x_i \leq n,\ \sum_{i=1}^{d} x_i = n \right\}$$

with two vertices adjacent if they agree in all but two coordinates. This graph has $N = \binom{n+d-1}{d-1}$ vertices and is regular of degree $\delta = (d-1)n$. Geometrically, let Δ^{d-1} denote the standard simplex in $\mathbb{R}^d$ (i.e., the convex hull of the standard basis vectors $\mathbf{e}_1, \dots, \mathbf{e}_d$) and let $n\Delta^{d-1}$ denote its n^{th} dilate (i.e., the convex hull of $n\mathbf{e}_1, \dots, n\mathbf{e}_d$). Then $V(d, n)$ is the set of lattice points in $n\Delta^{d-1}$, with two points adjacent if their difference is a multiple of $\mathbf{e}_i - \mathbf{e}_j$ for some i, j. Thus the independence number of $SR(d, n)$ is the maximum number of nonattacking rooks that can be placed on a simplicial chessboard with $n + 1$ "squares" on each side. Nivasch and Lev [13] and Blackburn, Paterson and Stinson [2] showed independently that for $d = 3$, this independence number is $\lfloor (2n + 3)/3 \rfloor$.

As far as we can tell, the class of simplicial rook graphs has not been studied before. For some small values of the parameters, $SR(d, n)$ is a well-known graph: $SR(2, n)$ and $SR(d, 1)$ are complete of orders $n + 1$ and d respectively; $SR(3, 2)$ is isomorphic to the octahedron; and $SR(d, 2)$ is isomorphic to the Johnson graph $J(d + 1, 2)$. On the other hand, simplicial rook graphs are not in general vertex-transitive, strongly regular or distance-regular, nor are they line graphs or noncomplete extended p-sums (in the sense of [7, p. 55]). They are also not to be confused with the *simplicial grid graph*, in which two vertices are adjacent only if

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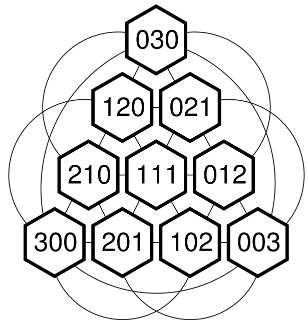

FIGURE 1. The graph $SR(3, 3)$.

their difference vector is exactly $\mathbf{e}_i - \mathbf{e}_j$ (as opposed to some scalar multiple) nor with the *triangular graph* T_n, which is the line graph of K_n [3, p.23], [8, §10.1].

Let G be a simple graph on vertices $[n] = \{1, \dots, n\}$. The *adjacency matrix* $A = A(G)$ is the $n \times n$ symmetric matrix whose (i, j) entry is 1 if ij is an edge, 0 otherwise. The *Laplacian matrix* is $L = L(G) = D - A$, where D is the diagonal matrix whose (i, i) entry is the degree of vertex i. The graph G is said to be *integral* (resp. *Laplacian integral*) if all eigenvalues of A (resp. L) are integers. If G is regular of degree δ, then these conditions are equivalent, since every eigenvector of A with eigenvalue λ is an eigenvector of L with eigenvalue $\delta - \lambda$.

We can now state our main theorem.

Theorem 1.1. *For every $n \geq 1$, the simplicial rook graph $SR(3, n)$ is integral and Laplacian integral, with eigenvalues as follows:*

If $n = 2m + 1$ is odd:

Eigenvalue of A	**Eigenvalue of L**	**Multiplicity**	**Eigenvector**
-3	$4m + 5 = 2n + 3$	$\binom{2m}{2}$	$\mathbf{H}_{a,b,c}$
$-2, -1, \dots, m - 3$	$3m + 5 \dots, 4m + 4$	3	$\mathbf{P}_k$
$m - 1$	$3m + 3$	2	$\mathbf{R}$
$m, \dots, 2m - 1 = n - 2$	$2m + 3 \dots, 3m + 2$	3	$\mathbf{Q}_k$
$4m + 2 = 2n$	0	1	$\mathbf{J}$

If $n = 2m$ is even:

Eigenvalue of A	**Eigenvalue of L**	**Multiplicity**	**Eigenvector**
-3	$4m + 3 = 2n + 3$	$\binom{2m-1}{2}$	$\mathbf{H}_{a,b,c}$
$-2, -1, \dots, m - 4$	$3m + 4, \dots, 4m + 2$	3	$\mathbf{P}_k$
$m - 3$	$3m + 3$	2	$\mathbf{R}$
$m - 1, \dots, 2m - 2 = n - 2$	$2m + 2, \dots, 3m + 1$	3	$\mathbf{Q}_k$
$4m = 2n$	0	1	$\mathbf{J}$

Integrality and Laplacian integrality typically arise from tightly controlled combinatorial structure in special families of graphs, including complete graphs, complete bipartite graphs and hypercubes (classical; see, e.g., [16, §5.6]), Johnson graphs [10], Kneser graphs [11] and threshold graphs [12]. (General references on graph eigenvalues and related topics include [1, 3, 7, 8].) For simplicial rook graphs, lattice geometry provides this combinatorial structure. To prove Theorem 1.1, we construct a basis of $\mathbb{R}^{\binom{n+2}{2}}$ consisting of eigenvectors of $A(SR(3,n))$, as indicated in the tables above. The basis vectors $\mathbf{H}_{a,b,c}$ for the largest eigenspace (Prop. 2.6) are signed characteristic vectors for hexagons centered at lattice points in the interior of $n\Delta^3$ (see Figure 2). The other eigenvectors $\mathbf{P}_k, \mathbf{R}, \mathbf{Q}_k$ (Props. 2.8, 2.9, 2.10) are most easily expressed as certain sums of characteristic vectors of lattice lines.

Theorem 1.1, together with Kirchhoff's matrix-tree theorem [8, Lemma 13.2.4] implies the following formula for the number of spanning trees of $SR(d,n)$.

Corollary 1.2. *The number of spanning trees of $SR(3,n)$ is*

$$\begin{cases} \dfrac{32(2n+3)^{\binom{n-1}{2}} \displaystyle\prod_{a=n+2}^{2n+2} a^3}{3(n+1)^2(n+2)(3n+5)^3} & \text{if } n \text{ is odd,} \\[2em] \dfrac{32(2n+3)^{\binom{n-1}{2}} \displaystyle\prod_{a=n+2}^{2n+2} a^3}{3(n+1)(n+2)^2(3n+4)^3} & \text{if } n \text{ is even.} \end{cases}$$

Based on experimental evidence gathered using Sage [17], we make the following conjecture:

Conjecture 1.3. *The graph $SR(d,n)$ is integral for all d and n.*

We discuss the general case in Section 3. The construction of hexagon vectors generalizes as follows: for each permutohedron whose vertices are lattice points in $n\Delta^{d-1}$, its signed characteristic vector is an eigenvector of eigenvalue $-\binom{d}{2}$ (Proposition 3.1). This is in fact the smallest eigenvalue of $SR(d,n)$ when $n \geq \binom{d}{2}$. Moreover, these eigenvectors are linearly independent and, for fixed d, account for "almost all" of the spectrum as $n \to \infty$, in the sense that

$$\lim_{n\to\infty} \frac{\dim\left(\text{span of permutohedron eigenvectors}\right)}{|V(d,n)|} = 1.$$

When $n < \binom{d}{2}$, the simplex $n\Delta^{d-1}$ is too small to contain any lattice permutohedra. On the other hand, the signed characteristic vectors of *partial permutohedra* (i.e., intersections of lattice permutohedra with $SR(d,n)$) are eigenvectors with eigenvalue $-n$. Experimental evidence indicates that this is in fact the smallest eigenvalue of $A(d,n)$, and that these partial permutohedra form a basis for the corresponding eigenspace. Unexpectedly, its dimension appears to be the *Mahonian number* $M(d,n)$ of permutations in $\mathfrak{S}_d$ with exactly n inversions (sequence #A008302 in Sloane [15]). In Section 3.2, we construct a family of eigenvectors by placing rooks (ordinary rooks, not simplicial rooks!) on Ferrers boards.

2. Proof of the Main Theorem

We begin by reviewing some basic algebraic graph theory; for a general reference, see, e.g., [8]. Let $G = (V,E)$ be a simple undirected graph with N vertices. The *adjacency matrix* $A(G)$ is the $N \times N$ matrix whose (i,j) entry is 1 if vertices i and j are adjacent, 0 otherwise. The *Laplacian matrix* is $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees. These are both real symmetric matrices, so they are diagonalizable, with real eigenvalues, and eigenspaces with different eigenvalues are orthogonal [8, §8.4].

Proposition 2.1. *The graph $SR(d,n)$ has $\binom{n+d-1}{d-1}$ vertices and is regular of degree $(d-1)n$. In particular, its adjacency and Laplacian matrices have the same eigenvectors.*

Proof. Counting vertices is the classic "stars-and-bars" problem (with n stars and $d-1$ bars). For each $x \in V(d,n)$ and each pair of coordinates i,j, there are $x_i + x_j$ other vertices that agree with x in all coordinates but i and j. Therefore, the degree of x is $\sum_{1 \le i < j \le n}(x_i + x_j) = (d-1)\sum_{i=1}^n x_i = (d-1)n$. □

The matrices $A(d,n)$ and $L(d,n)$ act on the vector space $\mathbb{R}^N$ with standard basis $\{\mathbf{e}_{ijk} \colon (i,j,k) \in V(d,n)\}$. We will sometimes consider the standard basis vectors as ordered lexicographically, for the purpose of showing that a collection of vectors is linearly independent.

In the rest of this section, we focus exclusively on the case $d = 3$, and regard n as fixed. We fix $N := \binom{n+2}{2}$, the number of vertices of $SR(3,n)$, and abbreviate $A = A(3,n)$.

2.1. **Basic linear algebra calculations.** Define

$$\mathbf{X}_i := \sum_{j+k=n-i} \mathbf{e}_{ijk}, \qquad \mathbf{J} := \sum_{i+j+k=n} \mathbf{e}_{ijk},$$
$$\mathbf{Y}_j := \sum_{i+k=n-j} \mathbf{e}_{ijk}, \qquad \mathcal{B}_n := \{\mathbf{X}_i, \mathbf{Y}_i, \mathbf{Z}_i \colon 0 \le i \le n\},$$
$$\mathbf{Z}_k := \sum_{i+j=n-k} \mathbf{e}_{ijk}, \qquad \mathcal{B}'_n := \{\mathbf{X}_i, \mathbf{Y}_i, \mathbf{Z}_i \colon 0 \le i \le n-1\}.$$

The vectors $\mathbf{X}_i, \mathbf{Y}_j, \mathbf{Z}_k$ are the characteristic vectors of lattice lines in $n\Delta^2$; see Figure 2. Note that the symmetric group $\mathfrak{S}_3$ acts on $SR(3,n)$ (hence on each of its eigenspaces) by permuting the coordinates of vertices.

Lemma 2.2. *We have*

$$\mathbf{J} = \sum_{i=0}^n \mathbf{X}_i = \sum_{i=0}^n \mathbf{Y}_i = \sum_{i=0}^n \mathbf{Z}_i \qquad \text{and} \qquad n\mathbf{J} = \sum_{i=0}^n i(\mathbf{X}_i + \mathbf{Y}_i + \mathbf{Z}_i).$$

Proof. The first assertion is immediate. For the second, when we expand the sum in terms of the $\mathbf{e}_{ijk}$, the coefficient on each $\mathbf{e}_{ijk}$ is $i + j + k = n$. □

Proposition 2.3. *For every i,j,k, we have*

$$A\mathbf{e}_{ijk} = \mathbf{X}_i + \mathbf{Y}_j + \mathbf{Z}_k - 3\mathbf{e}_{ijk}, \tag{2.1a}$$
$$A\mathbf{J} = 2n\mathbf{J}, \tag{2.1b}$$
$$A\mathbf{X}_i = (n-i-2)\mathbf{X}_i + \sum_{j=0}^{n-i}\Big[\mathbf{Y}_j + \mathbf{Z}_j\Big], \tag{2.1c}$$
$$A\mathbf{Y}_i = (n-i-2)\mathbf{Y}_i + \sum_{j=0}^{n-i}\Big[\mathbf{X}_j + \mathbf{Z}_j\Big], \tag{2.1d}$$
$$A\mathbf{Z}_i = (n-i-2)\mathbf{Z}_i + \sum_{j=0}^{n-i}\Big[\mathbf{X}_j + \mathbf{Y}_j\Big]. \tag{2.1e}$$

Proof. Formula (2.1a) is immediate from the definition of A, and (2.1b) follows because $SR(3,n)$ is $(2n)$-regular. For (2.1c), we have

$$\begin{aligned} A\mathbf{X}_i &= \sum_{j+k=n-i} A\mathbf{e}_{i,j,k} = \sum_{j+k=n-i} [X_i + Y_j + Z_k - 3\mathbf{e}_{i,j,k}] \\ &= (n-i+1)X_i - 3 \sum_{j+k=n-i} \mathbf{e}_{i,j,k} + \sum_{j+k=n-i} [Y_j + Z_k] \\ &= (n-i-2)X_i + \sum_{j=0}^{n-i} [Y_j + Z_j] \end{aligned}$$

and (2.1d) and (2.1e) are proved similarly. □

For future use, we also record (without proof) some elementary summation formulas.

Lemma 2.4. *The following summations hold:*

$$\sum_{i=k+1}^{n-k-1} \big[4i - 2n\big] = 0, \qquad \sum_{i=k+1}^{n-k-1} \big[4i - 2k - 2 - n\big] = (n-2k-1)(n-2k-2),$$

$$\sum_{i=k+1}^{n-j} \big[4i - 2n\big] = 2(n-j-k)(k-j+1), \qquad \sum_{i=k+1}^{n-j} \big[4i - 2k - 2 - n\big] = (n-2j)(n-k-j).$$

Lemma 2.5. *The following summations hold:*

$$\sum_{i=k}^{n-k} \big[4i - 2n\big] = 0, \qquad \sum_{i=k}^{n-k} \big[4i - 3n + 2k - 2\big] = -(n-2k+1)(n-2k+2),$$

$$\sum_{i=k}^{n-j} \big[4i - 2n\big] = 2(j-k)(-n+j+k-1), \qquad \sum_{i=k}^{n-j} \big[4i - 3n + 2k - 2\big] = (2j+2-4k+n)(-n+j+k-1).$$

Having completed these preliminaries, we now construct the eigenvectors of $SR(3,n)$.

2.2. **Hexagon vectors.** Let $(a,b,c) \in V(3,n)$ with $a,b,c > 0$. The corresponding "hexagon vector" is defined as

$$\mathbf{H}_{a,b,c} := \mathbf{e}_{a-1,b,c+1} - \mathbf{e}_{a,b-1,c+1} + \mathbf{e}_{a+1,b-1,c} - \mathbf{e}_{a+1,b,c-1} + \mathbf{e}_{a,b+1,c-1} - \mathbf{e}_{a-1,b+1,c}.$$

Geometrically, this is the characteristic vector, with alternating signs, of a regular lattice hexagon centered at the lattice point (a,b,c) in the interior of $n\Delta^2$ (see Figure 2).

Proposition 2.6. *The vectors* $\{\mathbf{H}_{a,b,c} \colon (a,b,c) \in V(d,n),\ a,b,c > 0\}$ *are linearly independent, and each one is an eigenvector of* A *with eigenvalue* -3.

Proof. The equality $A\mathbf{H}_{a,b,c} = -3\mathbf{H}_{a,b,c}$ is straightforward from (2.1a). The lexicographic leading term of $\mathbf{H}_{a,b,c}$ is $\mathbf{e}_{a-1,b,c+1}$, which is different for each (a,b,c), implying linear independence. □

Proposition 2.7. *Let* $n \geq 1$ *and let* $\mathcal{H}_n = \{\mathbf{H}_{a,b,c} \colon 0 < a,b,c < n\}$. *Then the spaces* $\mathbb{R}\mathcal{H}_n$ *and* $\mathbb{R}\mathcal{B}_n$ *spanned by* $\mathcal{H}_n$ *and* $\mathcal{B}_n$ *are orthogonal complements in* $\mathbb{R}^N$. *In particular,* $\dim \mathbb{R}\mathcal{B}_n = \binom{n+2}{2} - \binom{n-1}{2} = 3n$, *and the set* $\mathcal{B}'_n$ *is a basis for* $\mathbb{R}\mathcal{B}_n$ *(and all linear relations on the* $\mathbf{X}_i, \mathbf{Y}_i, \mathbf{Z}_i$ *are generated by those of Lemma 2.2).*

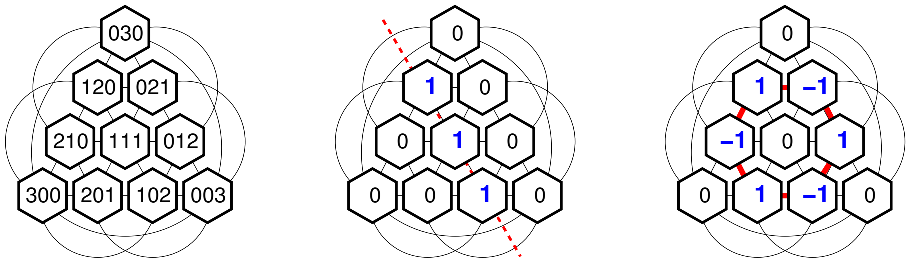

FIGURE 2. (left) The graph $SR(3,3)$. (center) The vector $\mathbf{X}_1$ and the lattice line it supports. (right) $\mathbf{H}_{1,1,1}$.

Proof. The scalar product $\mathbf{H}_{a,b,c} \cdot \mathbf{X}_i$ is clearly zero if the two vectors have disjoint supports (i.e., $i \notin \{a-1, a, a+1\}$) and is $-1+1=0$ otherwise (geometrically, this corresponds to the statement that any two adjacent vertices in the hexagon occur with opposite signs in $\mathbf{H}_{a,b,c}$; see Figure 2). Therefore $\mathbb{R}\mathcal{H}_n$ and $\mathbb{R}\mathcal{B}_n$ are orthogonal subspaces of $\mathbb{R}^N$, and $\dim \mathbb{R}\mathcal{B}_n \leq 3n$. For the opposite inequality, we induct on n. In the base case $n=1$, the vectors X_0, Y_0, Z_0 form a basis of $\mathbb{R}^3$. For larger n, let M_n be the matrix with columns $X_n, Y_n, Z_n, \ldots, X_0, Y_0, Z_0$ and rows ordered lexicographically, and let $\tilde{M}_n$ be M_n with the columns reordered as

$$X_0,\ Y_n,\ Z_n,\quad X_n,\ Y_{n-1},\ Z_{n-1},\quad \ldots,\quad X_1,\ Y_0,\ Z_0.$$

For example,

$\tilde{M}_3 =$

	X_0	Y_3	Z_3	X_3	Y_2	Z_2	X_2	Y_1	Z_1	X_1	Y_0	Z_0
003	1	0	1	0	0	0	0	0	0	0	1	0
012	1	0	0	0	0	1	0	1	0	0	0	0
021	1	0	0	0	1	0	0	0	1	0	0	0
030	1	1	0	0	0	0	0	0	0	0	0	1
102	0	0	0	0	0	1	0	0	0	1	1	0
111	0	0	0	0	0	0	0	1	1	1	0	0
120	0	0	0	0	1	0	0	0	0	1	0	1
201	0	0	0	0	0	0	1	0	1	0	1	0
210	0	0	0	0	0	0	1	1	0	0	0	1
300	0	0	0	1	0	0	0	0	0	0	1	1

If $a > 0$, then the entries of M_n in row (a,b,c) and columns X_i, Y_i, Z_i equal the entries of M_{n-1} in row $(a-1,b,c)$ and columns X_{i-1}, Y_i, Z_i respectively. Hence $\tilde{M}_n$ has the block form $\left[\begin{array}{c|c} U & * \\ \hline 0 & M_{n-1} \end{array}\right]$, where the entries of $*$ are irrelevant and

$$U = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Since $\operatorname{rank} U = 3$, it follows by induction that $\operatorname{rank} M_n \geq \operatorname{rank} M_{n-1} + 3 = 3n$. Using Lemma 2.2, one can solve for each of $\mathbf{X}_n$, $\mathbf{Y}_n$, and $\mathbf{Z}_n$ as linear combinations of the vectors in $\mathcal{B}'_n$. It follows that $\mathcal{B}'_n$ is a basis, and that the linear relations of Lemma 2.2 generate all linear relations on the vectors $\{\mathbf{X}_i, \mathbf{Y}_i, \mathbf{Z}_i\}$. $\square$

2.3. **Non-Hexagon Eigenvectors.** We now determine the other eigenspaces of A. The vector $\mathbf{J}$ spans an eigenspace of dimension 1; in addition, we will show that there is one eigenspace of dimension 2 (Prop. 2.8) and two families of eigenspaces of dimension 3 (Props. 2.9 and 2.10). Together with the hexagon vectors, these form a complete decomposition of $\mathbb{R}^N$ into eigenspaces of A. Throughout, let σ and ρ denote the permutations (1 2 3) and (1 2) (written in cycle notation), respectively, so that

$$\sigma(\mathbf{X}_i) = \mathbf{Y}_i,\ \ \sigma(\mathbf{Y}_j) = \mathbf{Z}_j,\ \ \sigma(\mathbf{Z}_k) = \mathbf{X}_k,\ \ \rho(\mathbf{X}_i) = \mathbf{Y}_i,\ \ \rho(\mathbf{Y}_j) = \mathbf{X}_j,\ \ \rho(\mathbf{Z}_k) = \mathbf{Z}_k.$$

Proposition 2.8. *Let $n \geq 1$ and $k = \lfloor n/2 \rfloor$. Then*

$$\mathbf{R} := \mathbf{X}_k - \mathbf{Y}_k - \mathbf{X}_{k+1} + \mathbf{Y}_{k+1}$$

is a nonzero eigenvector of A with eigenvalue $n-k-3 = (n-6)/2$ if n is even, or $n-k-2 = (n-3)/2$ if n is odd. Moreover, the $\mathfrak{S}_3$-orbit of $\mathbf{R}$ has dimension 2.

Proof. By (2.1c)...(2.1e),

$$\begin{aligned}
A\mathbf{R} &= (n-k-2)(\mathbf{X}_k - \mathbf{Y}_k) + \sum_{j=0}^{n-k}\Big[\mathbf{Y}_j - \mathbf{X}_j\Big] + (n-k-3)(\mathbf{Y}_{k+1} - \mathbf{X}_{k+1}) + \sum_{j=0}^{n-k-1}\Big[\mathbf{X}_j - \mathbf{Y}_j\Big] \\
&= (n-k-2)(\mathbf{X}_k - \mathbf{Y}_k) + (\mathbf{Y}_{n-k} - \mathbf{X}_{n-k}) + (n-k-3)(\mathbf{Y}_{k+1} - \mathbf{X}_{k+1}) \\
&= \begin{cases} (n-k-2)(\mathbf{X}_k - \mathbf{Y}_k) + (\mathbf{Y}_k - \mathbf{X}_k) + (n-k-3)(\mathbf{Y}_{k+1} - \mathbf{X}_{k+1}) & \text{if } n \text{ is even,} \\ (n-k-2)(\mathbf{X}_k - \mathbf{Y}_k) + (\mathbf{Y}_{k+1} - \mathbf{X}_{k+1}) + (n-k-3)(\mathbf{Y}_{k+1} - \mathbf{X}_{k+1}) & \text{if } n \text{ is odd} \end{cases} \\
&= \begin{cases} (n-k-3)(\mathbf{X}_k - \mathbf{Y}_k) + (n-k-3)(\mathbf{Y}_{k+1} - \mathbf{X}_{k+1}) & \text{if } n \text{ is even,} \\ (n-k-2)(\mathbf{X}_k - \mathbf{Y}_k) + (n-k-2)(\mathbf{Y}_{k+1} - \mathbf{X}_{k+1}) & \text{if } n \text{ is odd} \end{cases} \\
&= \begin{cases} (n-k-3)\mathbf{R} & \text{if } n \text{ is even,} \\ (n-k-2)\mathbf{R} & \text{if } n \text{ is odd} \end{cases}
\end{aligned}$$

as desired. The vectors $\mathbf{R}$ and $\sigma(\mathbf{R}) = \mathbf{Y}_k - \mathbf{Z}_k - \mathbf{Y}_{k+1} + \mathbf{Z}_{k+1}$ are linearly independent; on the other hand, $\rho(\mathbf{R}) = \mathbf{R}$ and $\mathbf{R} + \sigma(\mathbf{R}) + \sigma^2(\mathbf{R}) = 0$, so the $\mathfrak{S}_3$-orbit of $\mathbf{R}$ has dimension 2. $\square$

Proposition 2.9. *For all integers k with $0 \leq k \leq \lfloor \frac{n-3}{2} \rfloor$, the vector*

$$\mathbf{P}_k := -(n-2k-1)(n-2k-2)\mathbf{Z}_{n-k} + \sum_{i=k+1}^{n-k-1}\Big[2(i-k-1)\mathbf{Z}_i + (2i-n)(\mathbf{X}_i + \mathbf{Y}_i)\Big]$$

is a nonzero eigenvector of A with eigenvalue $k-2$. Moreover, the $\mathfrak{S}_3$-orbit of $\mathbf{P}_k$ has dimension 3.

Proof. The upper bound on k is equivalent to $n-2k-2>0$, so the coefficient of $\mathbf{Z}_{n-k}$ in $\mathbf{P}_k$ is nonzero, so $\mathbf{P}_k \neq 0$. By (2.1c)…(2.1e), we have

$$
\begin{aligned}
A\mathbf{P}_k = &-(n-2k-1)(n-2k-2)\left((k-2)\mathbf{Z}_{n-k}+\sum_{i=0}^{k}\Big[\mathbf{X}_i+\mathbf{Y}_i\Big]\right)\\
&+\sum_{i=k+1}^{n-k-1}\left[2(i-k-1)\left((n-i-2)\mathbf{Z}_i+\sum_{j=0}^{n-i}\Big[\mathbf{X}_j+\mathbf{Y}_j\Big]\right)\right.\\
&\left.+(2i-n)\left((n-i-2)(\mathbf{X}_i+\mathbf{Y}_i)+\sum_{j=0}^{n-i}\Big[\mathbf{X}_j+\mathbf{Y}_j+2\mathbf{Z}_j\Big]\right)\right]\\
=&-(n-2k-1)(n-2k-2)(k-2)\mathbf{Z}_{n-k}-(n-2k-1)(n-2k-2)\sum_{i=0}^{k}\Big[\mathbf{X}_i+\mathbf{Y}_i\Big]\\
&+\sum_{i=k+1}^{n-k-1}\Big[(2i-n)(n-i-2)(\mathbf{X}_i+\mathbf{Y}_i)+2(i-k-1)(n-i-2)\mathbf{Z}_i\Big]\\
&+\sum_{i=k+1}^{n-k-1}\sum_{j=0}^{n-i}\Big[(4i-2k-2-n)(\mathbf{X}_j+\mathbf{Y}_j)+(4i-2n)\mathbf{Z}_j\Big].
\end{aligned}
$$

Interchanging the order of summation in the double sum gives

$$
\begin{aligned}
A\mathbf{P}_k = &-(n-2k-1)(n-2k-2)(k-2)\mathbf{Z}_{n-k}\\
&-(n-2k-1)(n-2k-2)\sum_{i=0}^{k}\Big[\mathbf{X}_i+\mathbf{Y}_i\Big]\\
&+\sum_{i=k+1}^{n-k-1}\Big[(2i-n)(n-i-2)(\mathbf{X}_i+\mathbf{Y}_i)+2(i-k-1)(n-i-2)\mathbf{Z}_i\Big]\\
&+\sum_{j=0}^{k}\sum_{i=k+1}^{n-k-1}\Big[(4i-2k-2-n)(\mathbf{X}_j+\mathbf{Y}_j)+(4i-2n)\mathbf{Z}_j\Big]\\
&+\sum_{j=k+1}^{n-k-1}\sum_{i=k+1}^{n-j}\Big[(4i-2k-2-n)(\mathbf{X}_j+\mathbf{Y}_j)+(4i-2n)\mathbf{Z}_j\Big]
\end{aligned}
$$

Applying the summation formulas of Lemma 2.4 gives

$$\begin{aligned}
A\mathbf{P}_k &= -(n-2k-1)(n-2k-2)(k-2)\mathbf{Z}_{n-k} - (n-2k-1)(n-2k-2)\sum_{i=0}^{k}\Big[\mathbf{X}_j + \mathbf{Y}_j\Big] \\
&\quad + \sum_{i=k+1}^{n-k-1}\Big[(2i-n)(n-i-2)(\mathbf{X}_i+\mathbf{Y}_i) + 2(i-k-1)(n-i-2)\mathbf{Z}_i\Big] \\
&\quad + \sum_{j=0}^{k}\Big[(n-2k-1)(n-2k-2)(\mathbf{X}_j+\mathbf{Y}_j)\Big] \\
&\quad + \sum_{j=k+1}^{n-k-1}\Big[(2j-n)(k+j-n)(\mathbf{X}_j+\mathbf{Y}_j) + 2(j-n+k)(j-1-k)\mathbf{Z}_j\Big] \\
&= -(n-2k-1)(n-2k-2)(k-2)\mathbf{Z}_{n-k} \\
&\quad + \sum_{i=k+1}^{n-k-1}\Big[(2i-n)(k-2)(\mathbf{X}_i+\mathbf{Y}_i) + 2(i-k-1)(k-2)\mathbf{Z}_i\Big] \\
&= (k-2)\left(-(n-2k-1)(n-2k-2)\mathbf{Z}_{n-k} + \sum_{i=k+1}^{n-k-1}\Big[(2i-n)(\mathbf{X}_i+\mathbf{Y}_i) + 2(i-k-1)\mathbf{Z}_i\Big]\right) \\
&= (k-2)\mathbf{P}_k
\end{aligned}$$

verifying that $\mathbf{P}_k$ is an eigenvector, as desired.

Consider the vectors $\mathbf{P}_k$, $\sigma(\mathbf{P}_k)$, $\sigma^2(\mathbf{P}_k)$ as elements of the vector space $\mathbb{R}\mathcal{B}_n$, expanded in terms of the basis $\mathcal{B}'_n$ (see Prop. 2.7). In these expansions, the basis vectors $\mathbf{Z}_{n-k}$, $\mathbf{X}_{n-k}$, $\mathbf{Y}_{n-k}$ occur with nonzero coefficients only in $\mathbf{P}_k$, $\sigma(\mathbf{P}_k)$, $\sigma^2(\mathbf{P}_k)$ respectively. This shows that these three vectors are linearly independent. On the other hand, $\rho(\mathbf{P}_k) = \mathbf{P}_k$, so the $\mathfrak{S}_3$-orbit of $\mathbf{P}_k$ has dimension 3. $\square$

Define

$$\begin{aligned}
\mathbf{Q}_k &:= (n-2k+1)(n-2k+2)\mathbf{Z}_k \\
&\qquad + \sum_{j=k}^{n-k}\Big[(2j-n)(\mathbf{X}_j+\mathbf{Y}_j) - 2(n-j-k+1)\mathbf{Z}_j\Big] &\text{(2.2a)} \\
&= (n-2k+1)(n-2k)\mathbf{Z}_k + (2k-n)(\mathbf{X}_k+\mathbf{Y}_k) \\
&\qquad + \sum_{j=k+1}^{n-k}\Big[(2j-n)(\mathbf{X}_j+\mathbf{Y}_j) - 2(n-j-k+1)\mathbf{Z}_j\Big]. &\text{(2.2b)}
\end{aligned}$$

Both of these expressions for $\mathbf{Q}_n$ will be useful in what follows.

Proposition 2.10. *For all integers k with $0 \le k \le \lfloor \frac{n-2}{2} \rfloor$, the vector $\mathbf{Q}_k$ is a nonzero eigenvector of A with eigenvalue $n-k-2$. Moreover, the $\mathfrak{S}_3$-orbit of $\mathbf{Q}_k$ has dimension 3.*

Proof. The statement is vacuously true if $n < 2$. By (2.2a), the coefficient of $\mathbf{Z}_k$ in $\mathbf{Q}_k$ is $(n-2k+1)(n-2k)$. Provided that $k \le \lfloor \frac{n-1}{2} \rfloor$, we have $n > 2k$, so this coefficient is nonzero, as is the vector $\mathbf{Q}_k$. Applying (2.1c)…(2.1e), we have

$$\begin{aligned}
A\mathbf{Q}_k &= (n-2k+1)(n-2k+2)\left((n-k-2)\mathbf{Z}_k + \sum_{j=0}^{n-k}\Big[\mathbf{X}_j+\mathbf{Y}_j\Big]\right)\\
&\quad + \sum_{i=k}^{n-k}\Bigg[(2i-n)\left((n-i-2)(\mathbf{X}_i+\mathbf{Y}_i) + \sum_{j=0}^{n-i}\Big[\mathbf{X}_j+\mathbf{Y}_j+2\mathbf{Z}_j\Big]\right)\\
&\quad -2(n-i-k+1)\left((n-i-2)\mathbf{Z}_i + \sum_{j=0}^{n-i}\Big[\mathbf{X}_j+\mathbf{Y}_j\Big]\right)\Bigg]\\
&= (n-2k+1)(n-2k+2)(n-k-2)\mathbf{Z}_k + (n-2k+1)(n-2k+2)\sum_{j=0}^{n-k}\Big[\mathbf{X}_j+\mathbf{Y}_j\Big]\\
&\quad + \sum_{i=k}^{n-k}\Big[(2i-n)(n-i-2)(\mathbf{X}_i+\mathbf{Y}_i) - 2(n-i-k+1)(n-i-2)\mathbf{Z}_i\Big]\\
&\quad + \sum_{i=k}^{n-k}\sum_{j=0}^{n-i}\Big[(2i-n)(\mathbf{X}_j+\mathbf{Y}_j+2\mathbf{Z}_j) - 2(n-i-k+1)(\mathbf{X}_j+\mathbf{Y}_j)\Big]
\end{aligned}$$

Interchanging the order of summation in the double sum gives

$$\begin{aligned}
A\mathbf{Q}_k &= (n-2k+1)(n-2k+2)(n-k-2)\mathbf{Z}_k + (n-2k+1)(n-2k+2)\sum_{j=0}^{n-k}\Big[\mathbf{X}_j+\mathbf{Y}_j\Big]\\
&\quad + \sum_{j=k}^{n-k}\Big[(2j-n)(n-j-2)(\mathbf{X}_j+\mathbf{Y}_j) - 2(n-j-k+1)(n-j-2)\mathbf{Z}_j\Big]\\
&\quad + \sum_{j=0}^{k-1}\sum_{i=k}^{n-k}\Big[(4i-2n)\mathbf{Z}_j + (4i-3n+2k-2)(\mathbf{X}_j+\mathbf{Y}_j)\Big]\\
&\quad + \sum_{j=k}^{n-k}\sum_{i=k}^{n-j}\Big[(4i-2n)\mathbf{Z}_j + (4i-3n+2k-2)(\mathbf{X}_j+\mathbf{Y}_j)\Big]
\end{aligned}$$

Applying the summation formulas of Lemma 2.5 gives

$$\begin{aligned}
A\mathbf{Q}_k &= (n-2k+1)(n-2k+2)(n-k-2)\mathbf{Z}_k + (n-2k+1)(n-2k+2)\sum_{j=0}^{n-k}\Big[\mathbf{X}_j+\mathbf{Y}_j\Big] \\
&\quad + \sum_{j=k}^{n-k}\Big[(2j-n)(n-j-2)(\mathbf{X}_j+\mathbf{Y}_j) - 2(n-j-k+1)(n-j-2)\mathbf{Z}_j\Big] \\
&\quad - \sum_{j=0}^{k-1}\Big[(n-2k+1)(n-2k+2)(\mathbf{X}_j+\mathbf{Y}_j)\Big] \\
&\quad + \sum_{j=k}^{n-k}\Big[2(j-k)(-n+j+k-1)\mathbf{Z}_j + (2j+2-4k+n)(-n+j+k-1)(\mathbf{X}_j+\mathbf{Y}_j)\Big] \\
&= (n-2k+1)(n-2k+2)(n-k-2)\mathbf{Z}_k \\
&\quad + \sum_{j=k}^{n-k}\Big[(n-k-2)(2j-n)(\mathbf{X}_j+\mathbf{Y}_j) - 2(n-j-k+1)(n-k-2)\mathbf{Z}_j\Big] \\
&= (n-k-2)\left((n-2k+1)(n-2k+2)\mathbf{Z}_k + \sum_{j=k}^{n-k}\Big[(2j-n)(\mathbf{X}_j+\mathbf{Y}_j) - 2(n-j-k+1)\mathbf{Z}_j\Big]\right) \\
&= (n-k-2)\mathbf{Q}_k
\end{aligned}$$

as desired.

We now show that the $\mathfrak{S}_3$-orbit of $\mathbf{Q}_k$ has dimension 3. Since $\rho(\mathbf{Q}_k) = \mathbf{Q}_k$, the orbit is spanned by the three vectors $\mathbf{Q}_k$, $\sigma(\mathbf{Q}_k)$, $\sigma^2(\mathbf{Q}_k)$. We consider two cases: $k = 0$ and $k > 0$.

First, if $k = 0$, then the expression (2.2a) for $\mathbf{Q}_0$ becomes (using Lemma 2.2)

$$\begin{aligned}
\mathbf{Q}_0 &= (n+1)(n+2)\mathbf{Z}_0 + \sum_{j=0}^{n}\Big[(2j-n)(\mathbf{X}_j+\mathbf{Y}_j) - 2(n-j+1)\mathbf{Z}_j\Big] \\
&= (n+1)(n+2)\mathbf{Z}_0 - \sum_{j=0}^{n}\Big[n(\mathbf{X}_j+\mathbf{Y}_j) + (2n+2)\mathbf{Z}_j\Big] + 2\sum_{j=0}^{n} j\Big[\mathbf{X}_j+\mathbf{Y}_j+\mathbf{Z}_j\Big] \\
&= (n+1)(n+2)\mathbf{Z}_0 - (4n+2)\mathbf{J} + 2n\mathbf{J} \;=\; (n^2+3n+2)\mathbf{Z}_0 - (2n+2)\mathbf{J} \\
&= \sum_{i+j=n}(n^2+n)\mathbf{e}_{ij0} + \sum_{i,j,k\colon\, k\neq 0}(-2n-2)\mathbf{e}_{ijk}.
\end{aligned}$$

Accordingly we have

$$\begin{aligned}
\sigma(\mathbf{Q}_0) &= \sum_{j+k=n}(n^2+n)\mathbf{e}_{0jk} + \sum_{i,j,k\colon\, i\neq 0}(-2n-2)\mathbf{e}_{ijk}, \\
\sigma^2(\mathbf{Q}_0) &= \sum_{i+k=n}(n^2+n)\mathbf{e}_{i0k} + \sum_{i,j,k\colon\, j\neq 0}(-2n-2)\mathbf{e}_{ijk}.
\end{aligned}$$

Consider the $N \times 3$ matrix with columns $\mathbf{Q}_0, \sigma(\mathbf{Q}_0), \sigma^2(\mathbf{Q}_0)$. By the previous calculation, the 3×3 minor in rows $\mathbf{e}_{n00}, \mathbf{e}_{0n0}, \mathbf{e}_{00n}$ is

$$\begin{vmatrix} n^2+n & -2n-2 & n^2+n \\ n^2+n & n^2+n & -2n-2 \\ -2n-2 & n^2+n & n^2+n \end{vmatrix} = -2(n+1)^3(n+2)^2(n-1)$$

which is nonzero (recall that $n \geq 2$, otherwise the proposition is vacuously true).

On the other hand, if $0 < k \leq \lfloor (n-2)/2 \rfloor$, then (2.2b) expresses $\mathbf{Q}_k, \sigma(\mathbf{Q}_k), \sigma^2(\mathbf{Q}_k)$ as column vectors in the basis $\mathcal{B}'_n$. Let $a = 2k - n$ and $b = (n-2k)(n-2k+1)$; then the 3×3 minor in rows $\mathbf{X}_k, \mathbf{Y}_k, \mathbf{Z}_k$ is

$$\begin{vmatrix} a & a & b \\ a & b & a \\ b & a & a \end{vmatrix} = (2k-n)^3(n-2k-1)(n-2k+2)^2$$

which is nonzero because the assumption $k \leq \lfloor (n-2)/2 \rfloor$ implies $n \geq 2k+2$. $\square$

To sum up the results of Section 2, we have constructed an explicit decomposition of $\mathbb{R}^N$ into eigenspaces of $A(3,n)$ (equivalently, $L(3,n)$). The eigenvectors are the hexagon vectors $\mathbf{H}_{a,b,c}$ and the special vectors $\mathbf{J}$, $\mathbf{R}$, $\mathbf{P}_k$ and $\mathbf{Q}_k$ and their $\mathfrak{S}_3$-orbits.

3. Simplicial rook graphs in arbitrary dimension

We now consider the graph $SR(d,n)$ for arbitrary d and n, with adjacency matrix $A = A(d,n)$. Recall that $SR(d,n)$ has $N := \binom{n+d-1}{d-1}$ vertices and is regular of degree $(d-1)n$. If two vertices $a = (a_1, \dots, a_d)$, $b = (b_1, \dots, b_d) \in V(d,n)$ differ only in their i^{th} and j^{th} positions (and are therefore adjacent), we write $a \underset{i,j}{\sim} b$.

Let $\mathfrak{S}_d$ be the symmetric group of order d, and let $\mathfrak{A}_d \subset \mathfrak{S}_d$ be the alternating subgroup. Let ε be the sign function

$$\varepsilon(\sigma) = \begin{cases} 1 & \text{for } \sigma \in \mathfrak{A}_d, \\ -1 & \text{for } \sigma \notin \mathfrak{A}_d. \end{cases}$$

Let $\tau_{ij} \in \mathfrak{S}_d$ denote the transposition of i and j. Note that $\mathfrak{S}_d = \mathfrak{A}_d \cup \mathfrak{A}_d \tau_{ij}$ for each i, j.

In analogy to the vectors $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ used in the $d = 3$ case, define

$$\mathbf{X}_\alpha^{(i,j)} = \mathbf{e}_\alpha + \sum_{\beta:\ \beta \underset{i,j}{\sim} \alpha} \mathbf{e}_\beta. \tag{3.1}$$

That is, $\mathbf{X}_\alpha^{(i,j)}$ is the characteristic vector of the lattice line through α in direction $\mathbf{e}_i - \mathbf{e}_j$. In particular, if $\alpha \underset{i,j}{\sim} \beta$, then $\mathbf{X}_\alpha^{(i,j)} = \mathbf{X}_\beta^{(i,j)}$. Moreover, the column of A indexed by α is

$$A\mathbf{e}_\alpha = -\binom{d}{2}\mathbf{e}_\alpha + \sum_{1 \leq i < j \leq d} \mathbf{X}_\alpha^{(i,j)}. \tag{3.2}$$

since e_α itself appears in each summand $\mathbf{X}_\alpha^{(i,j)}$.

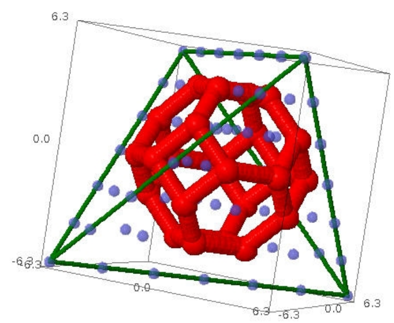

FIGURE 3. A permutohedron vector ($n = 6$, $d = 4$).

3.1. **Permutohedron vectors.** We now generalize the construction of hexagon vectors to arbitrary dimension. The idea is that for each point p in the interior of $n\Delta^{d-1}$ and sufficiently far away from its boundary, there is a lattice permutohedron centered at p, all of whose points are vertices of $SR(d,n)$ (see Figure 3), and the signed characteristic vector of this permutohedron is an eigenvector of $A(d,n)$.

Proposition 3.1. *Let $p, w \in \mathbb{R}^N$ be vectors such that $\{p + \sigma(w) \colon \sigma \in \mathfrak{S}_d\}$ are distinct vertices of $SR(d,n)$. (In particular, the entries of w must all be different.) Define*

$$\mathbf{H}_{p,w} = \sum_{\sigma \in \mathfrak{S}_d} \varepsilon(\sigma) \mathbf{e}_{p+\sigma(w)}.$$

Then $\mathbf{H}_{p,w}$ is an eigenvector of A with eigenvalue $-\binom{d}{2}$. Moreover, for a fixed w, the collection of all such eigenvectors $\mathbf{H}_{p,w}$ is linearly independent.

Proof. By linearity and (3.2), we have

$$\begin{aligned}
A\mathbf{H}_{p,w} &= \sum_{\sigma \in \mathfrak{S}_d} \varepsilon(\sigma) \left(-\binom{d}{2} \mathbf{e}_{p+\sigma(w)} + \sum_{1 \le i < j \le d} \mathbf{X}^{(i,j)}_{p+\sigma(w)} \right) \\
&= -\binom{d}{2} \mathbf{H}_{p,w} + \sum_{\sigma \in \mathfrak{S}_d} \varepsilon(\sigma) \sum_{1 \le i < j \le d} \mathbf{X}^{(i,j)}_{p+\sigma(w)} \\
&= -\binom{d}{2} \mathbf{H}_{p,w} + \sum_{1 \le i < j \le d} \sum_{\sigma \in \mathfrak{S}_d} \varepsilon(\sigma) \mathbf{X}^{(i,j)}_{p+\sigma(w)} \\
&= -\binom{d}{2} \mathbf{H}_{p,w} + \sum_{1 \le i < j \le d} \sum_{\sigma \in \mathfrak{A}_d} \left[\varepsilon(\sigma) \mathbf{X}^{(i,j)}_{p+\sigma(w)} + \varepsilon(\sigma\tau_{ij}) \mathbf{X}^{(i,j)}_{p+\sigma\tau_{ij}(w)} \right] \\
&= -\binom{d}{2} \mathbf{H}_{p,w}.
\end{aligned}$$

(The summand vanishes because $\varepsilon(\sigma) = -\varepsilon(\sigma\tau_{ij})$ and because changing α_i and α_j does not change $\mathbf{X}_\alpha^{(i,j)}$.) For linear independence, it suffices to observe that the lexicographic leading term of $\mathbf{H}_{p,w}$ is $\mathbf{e}_{p+\tilde{w}}$, where $\tilde{w}$ denotes the unique increasing permutation of w, and that these leading terms are different for different p. □

This result says that we can construct a large eigenspace by fitting many congruent permutohedra into the dilated simplex. Depending on the parity of d, the centers of these permutohedra will be points in $\mathbb{Z}^d$ or $(\mathbb{Z}+\frac{1}{2})^d$.

Let d be a positive integer. The *standard offset vector* in $\mathbb{R}^d$ is defined as

$$\mathbf{w} = \mathbf{w}_d = ((1-d)/2, (3-d)/2, \ldots, (d-3)/2, (d-1)/2) \in \mathbb{R}^d. \tag{3.3}$$

Note that $\mathbf{w} \in \mathbb{Z}^d$ if d is odd, and $\mathbf{w} \in (\mathbb{Z}+\frac{1}{2})^d$ if d is even.

Proposition 3.2. *There are*

$$\binom{n - \frac{(d-1)(d-2)}{2}}{d-1}$$

distinct vectors p such that $\mathbf{H}_{p,\mathbf{w}}$ is an eigenvector of $A(d,n)$ (and these eigenvectors are all linearly independent by Prop. 3.1).

Proof. First, suppose that $d = 2c+1$ is odd. In order to satisfy the conditions of Prop. 3.1, it suffices to choose a lattice point $p = (a_1, \ldots, a_d)$ so that $\sum a_i = n$ and $c \leq a_i \leq n-c$ for all i. Subtracting c from each a_i gives a bijection to compositions of $n - cd$ with d nonnegative parts and no part greater than $n-2c$ (that latter condition is extraneous for $d \geq 2$). The number of these compositions is

$$\binom{n-cd+d-1}{d-1} = \binom{n - \frac{(d-1)(d-2)}{2}}{d-1}.$$

Second, suppose that $d = 2c$ is even. Now it suffices to choose a point $p = (a_1 + 1/2, \ldots, a_d + 1/2) \in (\mathbb{Z}+\frac{1}{2})^d$ such that $a_1 + \cdots + a_d = n-c$ and, for each i, $a_i + 1/2 + (1-d)/2 \geq 0$ and $a_i + 1/2 + (d-1)/2 \leq n$, that is, i.e., $c-1 \leq a_1 \leq n-c$. Subtracting $c-1$ from each a_i gives a bijection to compositions of $n-c-d(c-1) = n - d(d-1)/2$ with d nonnegative parts, none of which can be greater than $n-d+1$ (again, the last condition is extraneous). The number of these compositions is

$$\binom{n-d(d-1)/2+d-1}{d-1} = \binom{n - \frac{(d-1)(d-2)}{2}}{d-1}.$$

□

The permutohedron vectors account for "almost all" of the eigenvectors in the following sense: if $\mathcal{H}_{d,n} \subseteq \mathbb{R}^N$ be the linear span of the eigenvectors constructed in Props. 3.1 and 3.2, then for each fixed d, we have

$$\lim_{n\to\infty} \frac{\dim \mathcal{H}_{d,n}}{|V(d,n)|} = \lim_{n\to\infty} \frac{\binom{n-\frac{(d-1)(d-2)}{2}}{d-1}}{\binom{n+d-1}{d-1}} = 1. \tag{3.4}$$

On the other hand, the combinatorial structure of the remaining eigenvectors is not clear.

The next result is a partial generalization of Proposition 2.7.

Proposition 3.3. *Every $\mathbf{H}_{p,\mathbf{w}}$ is orthogonal to every $\mathbf{X}_\alpha^{(i,j)}$.*

Proof. By definition we have

$$\mathbf{X}_\alpha^{(i,j)} \cdot \mathbf{H}_{p,\mathbf{w}} = \left(\mathbf{e}_\alpha + \sum_{\beta:\ \beta \underset{i,j}{\sim} \alpha} \mathbf{e}_\beta \right) \cdot \left(\sum_{\sigma \in \mathfrak{S}_d} \varepsilon(\sigma) \mathbf{e}_{p+\sigma(\mathbf{w})} \right) = \sum_{\sigma:\ p+\sigma(\mathbf{w}) \underset{i,j}{\sim} \alpha} \varepsilon(\sigma).$$

The index set of this summation admits the fixed-point-free involution $(\beta, \sigma) \leftrightarrow (\tau\beta, \tau\sigma)$, and $\varepsilon(\tau\sigma) = -\varepsilon(\sigma)$, so the sum is zero. $\square$

Geometrically, Proposition 3.3 says that if a lattice line meets a lattice permutohedron of the form of Prop. 3.1, then it does so in exactly two points, whose corresponding permutations have opposite signs.

Conjecture 3.4. *The vectors $\mathbf{X}_\alpha^{(i,j)}$ span the orthogonal complement of $\mathcal{H}_{d,n}$.*

This conjecture is equivalent to the statement that every other eigenvector of $A(d,n)$ can be written as a linear combination of the $\mathbf{X}_\alpha^{(i,j)}$. For $n < \binom{d}{2}$, the conjecture is that the $\mathbf{X}_\alpha^{(i,j)}$ span all of $\mathbb{R}^N$. We have verified this statement computationally for $d = 4$ and $n \leq 11$, and for $d = 5$ and $n = 7, 8, 9$. Part of the difficulty is that it is not clear what subset of the $\mathbf{X}_\alpha^{(i,j)}$ ought to form a basis (in contrast to the case $d = 3$, where $\mathcal{B}'_n$ is a natural choice of basis; see Prop. 2.7).

3.2. **The smallest eigenvalue.** For a matrix M with real spectrum, let $\tau(M)$ denote its smallest eigenvalue, and for a graph H, let $\tau(G) = \tau(A(G))$. The invariant $\tau(G)$ of a graph is important in spectral graph theory; for instance, it is related to the independence number [8, Lemma 9.6.2].

Proposition 3.5. *Suppose that $d \geq 1$ and $n \geq \binom{d}{2}$. Then $\tau(SR(d,n)) = -\binom{d}{2}$.*

Proof. By the construction of Proposition 3.2, there is at least one eigenvector with eigenvalue $-\binom{d}{2}$ when $n \geq \binom{d}{2}$. The following argument that $-\binom{d}{2}$ is in fact the smallest eigenvalue was suggested to the authors by Noam Elkies. The edges of $SR(d,n)$ in direction (i,j) form a spanning subgraph $SR(d,n)_{i,j}$ isomorphic to $K_{n+1} + K_n + K_{n-1} + \cdots + K_1$, where $+$ means disjoint union. The eigenvalues of K_n are $n-1$ and -1, and the spectrum of $G + H$ is the union of the spectra of G and H, so $\tau(SR(d,n)_{i,j}) = -1$. Since the edge set of $SR(d,n)$ is the disjoint union of the edge sets of the $SR(d,n)_{i,j}$, we have $A(d,n) = \sum_{(i,j)} A(SR(d,n)_{i,j})$, and in general $\tau(M+N) \geq \tau(M) + \tau(N)$, so $\tau(SR(d,n)) \geq -\binom{d}{2}$ as desired. $\square$

The case $n < \binom{d}{2}$ is more complicated. Experimental evidence indicates that the smallest eigenvalue of $SR(d,n)$ is $-n$, and moreover that the multiplicity of this eigenvalue equals the number $M(d,n)$ of permutations in $\mathfrak{S}_d$ with exactly n inversions. The numbers $M(d,n)$ are well known in combinatorics as the *Mahonian numbers*, or as the coefficients of the q-factorial polynomials; see [15, sequence #A008302]. In the rest of this section, we construct $M(d,n)$ linearly independent eigenvectors of eigenvalue $-n$; however, we do not know how to rule out the possibility of additional eigenvectors of equal or smaller eigenvalue

We review some basics of rook theory; for a general reference, see, e.g., [5]. For a sequence of positive integers $c = (c_1, \ldots, c_d)$, the *skyline board* $\mathrm{Sky}(c)$ consists of a sequence of d columns, with the i^{th} column containing c_i squares. A *rook placement* on $\mathrm{Sky}(c)$ consists of a choice of one square in each column. A rook placement is *proper* if all d squares belong to different rows.

An *inversion* of a permutation $\pi = (\pi_1, \dots, \pi_d) \in \mathfrak{S}_d$ is a pair i, j such that $i < j$ and $\pi_i > \pi_j$. Let $\mathfrak{S}_{d,n}$ denote the set of permutations of $[d]$ with exactly n inversions.

Definition 3.6. Let $\pi \in \mathfrak{S}_{d,n}$. The *inversion word* of π is $a = a(\pi) = (a_1, \dots, a_d)$, where

$$a_i = \#\{j \in [d] \colon i < j \text{ and } \pi_i > \pi_j\}.$$

Note that a is a weak composition of n with d parts, hence a vertex of $SR(d, n)$. A permutation $\sigma \in \mathfrak{S}_{d,n}$ is π-*admissible* if σ is a proper skyline rook placement on $\mathrm{Sky}(a_1 + 1, \dots, a_d + d)$; that is, if

$$x(\sigma) = a(\pi) + \mathbf{w} - \sigma(\mathbf{w}) = a(\pi) + \mathrm{id} - \sigma$$

is a lattice point in $n\Delta^{d-1}$. Note that the coordinates of $x(\sigma)$ sum to n, so admissibility means that its coordinates are all nonnegative. The set of all π-admissible permutations is denoted $\mathrm{Adm}(\pi)$; that is,

$$\mathrm{Adm}(\pi) = \{\sigma \in \mathfrak{S}_d \colon a_i - \sigma_i + i \geq 0 \quad \forall i = 1, \dots, d\}.$$

The corresponding *partial permutohedron* is

$$\mathrm{Parp}(\pi) = \{x(\sigma) \colon \sigma \in \mathrm{Adm}(\pi)\}.$$

That is, $\mathrm{Parp}(\pi)$ is the set of permutations corresponding to lattice points in the intersection of $n\Delta^{d-1}$ with the standard permutohedron centered at $a(\pi) + \mathbf{w}$. The *partial permutohedron vector* is the signed characteristic vector of $\mathrm{Parp}(\pi)$, that is,

$$\mathbf{F}_\pi = \sum_{\sigma \in \mathrm{Parp}(\pi)} \varepsilon(\sigma) \mathbf{e}_{x(\sigma)}.$$

Example 3.7. Let $d = 4$ and $\pi = 3142 \in \mathfrak{S}_d$. Then π has $n = 3$ inversions, namely 12, 14, 34. Its inversion word is accordingly $a = (2, 0, 1, 0)$. The π-admissible permutations are the proper skyline rook placements on $\mathrm{Sky}(2{+}1, 0{+}2, 1{+}3, 0{+}4) = \mathrm{Sky}(3, 2, 4, 4)$, namely 1234, 1243, 2134, 2143, 3124, 3142, 3214, 3241 (see Figure 4). The corresponding lattice points $x(\sigma)$ can be read off from the rook placements by counting the number of empty squares above each rook, obtaining respectively 2010, 2001, 1110, 1101, 0120, 0102, 0030, 0003; these are the neighbors of a in $\mathrm{Parp}(\pi)$. Thus $\mathbf{F}_\pi = \mathbf{e}_{2010} - \mathbf{e}_{2001} - \mathbf{e}_{1110} + \mathbf{e}_{1101} + \mathbf{e}_{0120} - \mathbf{e}_{0102} - \mathbf{e}_{0030} + \mathbf{e}_{0003}$; see Figure 5.

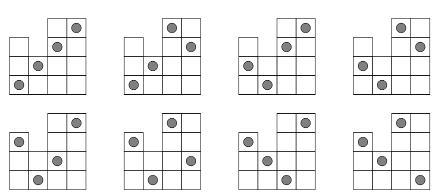

FIGURE 4. Rook placements on the skyline board $\mathrm{Sky}(3, 2, 4, 4)$.

Theorem 3.8. *Let $\pi \in \mathfrak{S}_{d,n}$ and $A = A(d, n)$. Then $\mathbf{F}_\pi$ is an eigenvector of A with eigenvalue $-n$. Moreover, for every pair d, n with $n < \binom{d}{2}$, the set $\{\mathbf{F}_\pi \colon \pi \in \mathfrak{S}_{d,n}\}$ is linearly independent. In particular, the dimension of the $(-n)$-eigenspace of A is at least the Mahonian number $M(d, n)$.*

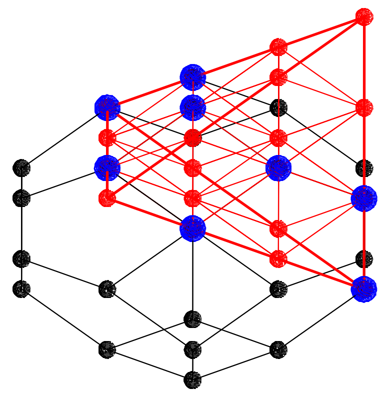

FIGURE 5. The partial permutohedron Parp(3142) in $SR(4,3)$.

Proof. **First**, we show that the $\mathbf{F}_\pi$ are linearly independent. This follows from the observation that the lexicographically leading term of $\mathbf{F}_\pi$ is $\mathbf{e}_{a(\pi)}$, and these terms are different for all $\pi \in \mathfrak{S}_{d,n}$.

Second, let $\sigma \in \mathrm{Adm}(\pi)$. Then the coefficient of $\mathbf{e}_{x(\sigma)}$ in $\mathbf{F}_\pi$ is $\varepsilon(\sigma) \in \{1,-1\}$. We will show that the coefficient of $\mathbf{e}_{x(\sigma)}$ in $A\mathbf{F}_\pi$ is $-n\varepsilon(\sigma)$, i.e., that

$$\varepsilon(\sigma)\sum_{\rho}\varepsilon(\rho) = -n, \tag{3.5}$$

the sum over all ρ such that $\rho \sim \sigma$ and $\rho \in \mathrm{Parp}(\pi)$. (Here and subsequently, $\sim$ denotes adjacency in $SR(d,n)$.) Each such rook placement ρ is obtained by multiplying σ by the transposition $(i\ j)$, that is, by choosing a rook at (i,σ_i), choosing a second rook at (j,σ_j) with $\sigma_j > \sigma_i$, and replacing these two rooks with rooks in positions (i,σ_j) and (j,σ_i). For each choice of i, there are $(a_i+i)-\sigma_i$ possible j's, and $\sum_i(a_i+i-\sigma_i) = n$. Moreover, the sign of each such ρ is opposite to that of σ, proving (3.5).

Third, let $y = (y_1,\dots,y_d) \in V(d,n) \setminus \mathrm{Parp}(\pi)$. Then the coefficient of $e_{x(\sigma)}$ in $\mathbf{F}_\pi$ is 0. We will show that the coefficient of $\mathbf{e}_{x(\sigma)}$ in $A\mathbf{F}_\pi$ is also 0, i.e., that

$$\sum_{\sigma\in N}\varepsilon(\sigma) = 0. \tag{3.6}$$

where $N = \{\rho : x(\rho) \sim y\} \cap \mathrm{Parp}(\pi)$. In order to prove this, we will construct a sign-reversing involution on N.

Let $a = a(\pi)$ and let $b = (b_1,\dots,b_d) = (a_1+1-y_1, a_2+2-y_2,\dots,a_d+d-y_d)$. Note that $b_i \le a_i+i$ for every i; therefore, we can regard b as a rook placement on $\mathrm{Sky}(a_1+1,\dots,a_d+d)$. (It is possible that $b_i \le 0$ for one or more i; we will consider that case shortly.) To say that $y \notin \mathbf{F}_\pi$ is to say that b is not a proper π-skyline rook placement; on the other hand, we have $\sum b_i = \binom{d+1}{2}$ (as would be the case if b were proper). Hence the elements of N are the proper π-skyline rook skyline placements obtained from b by moving one rook up and one other rook down, necessarily by the same number of squares. Let $b(i\uparrow q, j\downarrow r)$ denote the rook placement obtained by moving the i^{th} rook up to row q and the j^{th} rook down to row r.

We now consider the various possible ways in which b can fail to be proper.

Case 1: $b_i \leq 0$ for two or more i. In this case $N = \emptyset$, because moving only one rook up cannot produce a proper π-skyline rook placement.

Case 2: $b_i \leq 0$ for exactly one i. The other rooks in b cannot all be at different heights, because that would imply that $\sum b_i \leq 0 + (2 + \cdots + d) < \binom{d+1}{2}$. Therefore, either $N = \emptyset$, or else $b_j = b_k$ for some j, k and there are rooks at all heights except q and r for some $q, r < b_j = b_k$.

Then $b(i \uparrow q, j \downarrow r)$ is proper if and only if $b(i \uparrow q, k \downarrow r)$ is proper, and likewise $b(i \uparrow r, j \downarrow q)$ is proper if and only if $b(i \uparrow r, k \downarrow q)$ is proper. Each of these pairs is related by the transposition $(j\ k)$, so we have the desired sign-reversing involution on N.

Case 3: $b_i \geq 1$ for all i. Then the reason that b is not proper must be that some row has no rooks and some row has more than one rook. There are several subcases:

Case 3a: For some $q \neq r$, there are two rooks at height q, no rooks at height r, and one rook at every other height. But this is impossible because then $\sum b_i = \binom{d+1}{2} + q - r \neq \binom{d+1}{2}$.

Case 3b: There are four or more rooks at height q, or three at height q and two or more at height r. In both cases $N = \emptyset$.

Case 3c: We have $b_i = b_j = b_k$; no rooks at heights q or r for some $q < r$; and one rook at every other height. Then

$$N \subseteq \left\{ \begin{array}{lll} b(i \uparrow r, j \downarrow q), & b(j \uparrow r, i \downarrow q), & b(k \uparrow r, i \downarrow q), \\ b(i \uparrow r, k \downarrow q), & b(j \uparrow r, k \downarrow q), & b(k \uparrow r, j \downarrow q). \end{array} \right\}$$

For each column of the table above, its two rook placements are related by a transposition (e.g., $(j\ k)$ for the first column) and either both or neither of those rook placements are proper (e.g., for the first column, depending on whether or not $b_i \leq r$). Therefore, we have the desired sign-reversing involution on N.

Case 3d: We have $b_i = b_j = q$; $b_k = b_\ell = r$, and one rook at every other height except heights s and t. Now the desired sign-reversing involution on N is toggling the rook that gets moved down; for instance, $b(j \uparrow s, k \downarrow t)$ is proper if and only if $b(j \uparrow s, \ell \downarrow t)$ is proper.

This completes the proof of (3.6), which together with (3.5) completes the proof that $\mathbf{F}_\pi$ is an eigenvector of $A(d,n)$ with eigenvalue $-n$. $\square$

Conjecture 3.9. *If $n \leq \binom{d}{2}$, then in fact $\tau(SR(d,n)) = -n$, and the dimension of the corresponding eigenspace is the Mahonian number $M(d,n)$.*

We have verified this conjecture, using Sage, for all $d \leq 6$. It is not clear in general how to rule out the possibility of a smaller eigenvalue, or of additional $(-n)$-eigenvectors linearly independent of the $\mathbf{F}_\pi$.

The proof of Theorem 3.8 implies that every partial permutohedron $\mathrm{Parp}(\pi)$ induces an n-regular subgraph of $SR(d,n)$. Another experimental observation is the following:

Conjecture 3.10. *For every $\pi \in \mathfrak{S}_{d,n}$, the induced subgraph $SR(d,n)|_{\mathrm{Parp}(\pi)}$ is Laplacian integral.*

We have verified this conjecture, using Sage, for all permutations of length $d \leq 6$. We do not know what the eigenvalues are, but these graphs are not in general strongly regular (as evidenced by the observation that they have more than 3 distinct eigenvalues).

4. Corollaries, alternate methods, and further directions

4.1. **The independence number.** The independence number of $SR(d,n)$ can be interpreted as the maximum number of nonattacking "rooks" that can be placed on a simplicial chessboard of side length $n+1$. By [8, Lemma 9.6.2], the independence number $\alpha(G)$ of a δ-regular graph G on N vertices is at most $-\tau N/(\delta-\tau)$, where τ is the smallest eigenvalue of $A(G)$. For $d=3$ and $n\geq 3$, we have $\tau=-3$, which implies that the independence number $\alpha(SR(d,n))$ is at most $3(n+2)(n+1)/(4n+6)$. This is of course a weaker result (except for a few small values of n) than the exact value $\lfloor (3n+3)/2\rfloor$ obtained in [13] and [2].

Question 4.1. What is the independence number of $SR(d,n)$? That is, how many nonattacking rooks can be placed on a simplicial chessboard?

Proposition 3.5 implies the upper bound

$$\alpha(SR(d,n)) \leq \frac{d(d+1)}{(2n+d)(d-1)}\binom{n+d-1}{d-1}$$

for $n\geq\binom{d}{2}$, but this bound is not sharp (for example, the bound for $SR(4,6)$ is $\alpha\leq 21$, but computation indicates that $\alpha=16$).

4.2. **Equitable partitions.** One approach to determining the spectrum of a graph uses the theory of *interlacing* and *equitable partitions* [9], [8, chapter 9]. Let $X=\{O_1,\dots,O_k\}$ be the set of orbits of vertices of G under the group of automorphisms of G. For each two orbits O_i,O_j, define $f(i,j)=|N(x)\cap O_j|$ for any $x\in O_i$. The choice of x does not matter, so that the function f is well-defined (albeit not necessarily symmetric); that is to say, the orbits form an *equitable partition* of $V(G)$. Let $P(G)$ be the $k\times k$ square matrix with entries $f(i,j)$. Then every eigenvalue of P is also an eigenvalue of $A(G)$ [8, Thm. 9.3.3].

When $G=SR(n,d)$, the spectrum of $P(G)$ is typically a proper subset of that of $A(G)$. For example, when $n=3$ and $d=3$, the matrix $A(G)$ has spectrum $6,1,1,1,0,0,-2,-2,-2,-3$ by Theorem 1.1, but the automorphism group has only three orbits, so $P(G)$ is a 3×3 matrix and must have a strictly smaller set of eigenvalues. In fact its spectrum is $6,1,-2$, which is not a tight interlacing of that of $A(G)$ in the sense of Haemers.

Therefore, these methods may not be sufficient to describe the spectrum of $SR(n,d)$ in general. On the other hand, in all cases we have checked computationally ($d=4, n\leq 30$; $d=5, n\leq 25$), the matrices $P(SR(n,d))$ have integral spectra, which is consistent with Conjecture 1.3.

Question 4.2. Is $SR(d,n)$ determined up to isomorphism by its spectrum?

For $SR(3,3)$, the answer to the question is "yes," for the following reason. A regular graph is integral if and only if its complement is integral, by [8, Lemma 8.5.1]. Thus the complement $\overline{SR(3,3)}$ is 3-regular and integral. There are exactly thirteen such graphs, as classified by Bussemaker, Cvetković, and Schwenk [4, 6, 14]; see also [1, pp. 50–51]. Only two of these have ten vertices, namely $\overline{SR(3,3)}$ and the Petersen graph, which are not cospectral. For more on the general problem of which graphs are determined by their spectra, see [18, 19].

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Department of Mathematics, University of Kansas, Lawrence, KS 66045
E-mail address: `jmartin@math.ku.edu`

Department of Mathematics and Statistics, Washburn University, Topeka, KS 66621
E-mail address: `jennifer.wagner1@washburn.edu`