BIGRADED COHOMOLOGY OF $\mathbb{Z}/2$-EQUIVARIANT GRASSMANNIANS

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Contents

1. Introduction

Let $\mathbb{R}$ and $\mathbb{R}_-$ denote the two representations of $\mathbb{Z}/2$ on the real line: the first has the trivial action, the second has the sign action. Let $\mathcal{U}$ denote the infinite direct sum

$$\mathcal{U} = \mathbb{R} \oplus \mathbb{R}_- \oplus \mathbb{R} \oplus \mathbb{R}_- \oplus \cdots$$

The subjects of this paper are the infinite Grassmannians $\mathrm{Gr}_k(\mathcal{U})$, regarded as spaces with a $\mathbb{Z}/2$-action. Our goal is to compute the $RO(\mathbb{Z}/2)$-graded cohomology rings $H^*(\mathrm{Gr}_k(\mathcal{U}); (\mathbb{Z}/2)_m)$, where $(\mathbb{Z}/2)_m$ denotes the constant-coefficient Mackey functor. These cohomology rings are a notion of equivariant cohomology that is finer than the classical Borel theory.

Of course our results may be interpreted as giving a calculation of all characteristic classes, with values in the theory $H^*(-; (\mathbb{Z}/2)_m)$, for rank k equivariant bundles. Previous work on related problems has been done by Ferland and Lewis [FL] and by Kronholm [K1, K2], but the present paper provides the first complete computation for any single value of k larger than 1.

The rest of this introduction aims to describe the results of the computation. The context throughout the paper is the category of $\mathbb{Z}/2$-spaces, with equivariant maps. Unless stated otherwise all spaces and maps are in this category.

The theory $H^*(-; (\mathbb{Z}/2)_m)$ is graded by the representation ring $RO(\mathbb{Z}/2)$. That is to say, if V is a virtual representation then the theory yields groups $H^V(-; (\mathbb{Z}/2)_m)$. For the group $\mathbb{Z}/2$ every representation has the form $\mathbb{R}^p \oplus (\mathbb{R}_-)^q$

for some p and q, and this implies that we may regard our cohomology theory as being bigraded. Different authors use different indexing conventions, but we will use the "motivic" indexing described as follows. The representation $V = \mathbb{R}^p \oplus (\mathbb{R}_-)^q$ is denoted $\mathbb{R}^{p+q,q}$, and the corresponding cohomology groups $H^V(-;(\mathbb{Z}/2)_m)$ will be denoted $H^{p+q,q}(-;(\mathbb{Z}/2)_m)$. In this indexing system the first index is called the **topological degree** and the second is called the **weight**. One appeal of this system is that dropping the second index will always give statements that seem familiar from non-equivariant topology.

Before continuing, for ease of reading we will just write $\mathbb{Z}/2$ instead of $(\mathbb{Z}/2)_m$ in coefficients of cohomology groups. In the presence of the bigrading this will never lead to any confusion.

Let $\mathbb{M}_2$ be the bigraded ring $H^{*,*}(pt;\mathbb{Z}/2)$, the cohomology ring of a point. This is the ground ring of our theory; for any $\mathbb{Z}/2$-space X, the ring $H^{*,*}(X;\mathbb{Z}/2)$ is an algebra over $\mathbb{M}_2$. A complete description of $\mathbb{M}_2$ is given in the next section, but for now one only needs to know that there are special elements $\tau \in \mathbb{M}_2^{0,1}$ and $\rho \in \mathbb{M}_2^{1,1}$.

The cohomology ring of the projective space $\mathrm{Gr}_1(\mathcal{U})$ has been known for a while; the motivic analog was computed by Voevodsky, and the same proof works in the $\mathbb{Z}/2$-equivariant setting. A careful proof is written down in [K2, Theorem 4.2]. There is an isomorphism of algebras $H^{*,*}(\mathrm{Gr}_1(\mathcal{U});\mathbb{Z}/2) \cong \mathbb{M}_2[a,b]/(a^2 = \rho a + \tau b)$ where a has bidegree $(1,1)$ and b has bidegree $(2,1)$. In non-equivariant topology one has $\rho = 0$ and $\tau = 1$, so that the above relation becomes $a^2 = b$ and we simply have a polynomial algebra in a variable of degree 1—the familiar answer for the mod 2 cohomology of real projective space.

Note that additively, $H^{*,*}(\mathrm{Gr}_1(\mathcal{U});\mathbb{Z}/2)$ is a free module over $\mathbb{M}_2$ on generators of the following bidegrees:

$$(0,0),(1,1),(2,1),(3,2),(4,2),(5,3),(6,3),(7,4),\dots$$

corresponding to the monomials $1, a, b, ab, b^2, ab^2, b^3, ab^3, \dots$ If one forgets the weights, then one gets the degrees for elements in an additive basis for the singular cohomology $H^*(\mathbb{R}P^\infty;\mathbb{Z}/2)$. So in this case one can obtain the equivariant cohomology groups by taking a basis for the singular cohomology groups, adding appropriate weights, and changing every $\mathbb{Z}/2$ into a copy of $\mathbb{M}_2$. We mention this because it is a theorem of Kronholm [K1] that the same is true in the case of $\mathrm{Gr}_k(\mathcal{U})$ (and for many other spaces as well, though not all spaces). Because we know the singular cohomology groups $H^*(\mathrm{Gr}_k(\mathbb{R}^\infty);\mathbb{Z}/2)$, computing the equivariant version becomes only a question of knowing what weights to attach to the generators. While it might seem that it should be simple to resolve this, the question has been very resistant until now; the present paper provides an answer.

To state our main results, begin by considering the map

$$\eta\colon\ \mathrm{Gr}_1(\mathcal{U}) \times \cdots \times \mathrm{Gr}_1(\mathcal{U}) \longrightarrow \mathrm{Gr}_k(\mathcal{U})$$

that classifies the k-fold direct sum of line bundles. Using the Künneth Theorem, the induced map on cohomology gives

$$\eta^*\colon H^{*,*}(\mathrm{Gr}_k(\mathcal{U});\mathbb{Z}/2) \to H^{*,*}(\mathrm{Gr}_1(\mathcal{U});\mathbb{Z}/2)^{\otimes k}.$$

Since permuting the factors in a k-fold sum yields an isomorphic bundle, the image of η^* lies in the ring of invariants under the action of the symmetric group Σ_k.

That is to say, we may regard η^* as a map

$$\eta^*\colon H^{*,*}(\mathrm{Gr}_k(\mathcal{U});\mathbb{Z}/2) \to \left[H^{*,*}(\mathrm{Gr}_1(\mathcal{U});\mathbb{Z}/2)^{\otimes k}\right]^{\Sigma_k}. \tag{1.1}$$

The first of our results is the following:

Theorem 1.2. *The map in (1.1) is an isomorphism of bigraded rings.*

This is the direct analog of what happens in the nonequivariant case. Let us note, however, that until now neither injectivity nor surjectivity has been known in the present context. It must be admitted up front that in some ways our proof of Theorem 1.2 is not very satisfying: it does not give any reason, based on first principles, why η^* should be an isomorphism. Rather, the proof proceeds by computing the codomain of η^* explicitly and then running a complicated spectral sequence for computing the domain of η^*. By comparing what is happening on the two sides, and appealing to the nonequivariant result at key moments, one can see that there is no choice but for the map to be an isomorphism—even without resolving all the differentials in the spectral sequence (of which there are infinitely many). The argument is somewhat sneaky, but not terribly difficult in the end. However, it depends on a key result proven by Kronholm [K1] that describes the kind of phenomena that take place inside the spectral sequence.

The proof of Theorem 1.2 is the main component of this paper. It is completed in Section 6. Subsequent sections explore some auxilliary issues, that we describe next.

Remark 1.3. In non-equivariant topology there are several familiar techniques for proving Theorem 1.2, perhaps the most familiar being use of the Serre spectral sequence. Since the theorem is really about the identification of characteristic classes, another method that comes to mind is the Grothendieck approach to characteristic classes via the cohomology of projective bundles. The equivariant analogs of both these approaches have been partially explored by Kronholm [K2], but one runs into a fundamental problem: such calculations require the use of local coefficient systems, because the fixed sets of Grassmannians are disconnected. So they involve a level of diffculty that is far beyond what happens in the non-equivariant case, and to date no one has gotten these approaches to work. Cohomology with local coefficients has been little-explored in the equivariant setting, but see [Sh] for work in this direction.

To access the full power of Theorem 1.2 one should compute the ring of invariants $\left[H^{*,*}(\mathrm{Gr}_1(\mathcal{U});\mathbb{Z}/2)^{\otimes k}\right]^{\Sigma_k}$, which is a purely algebraic problem. The proof of Theorem 1.2 only requires understanding an additive basis for this ring. The second part of the paper examines the multiplicative structure.

In regards to the additive basis, we can state one form of our results as follows. Recall that a basis for $H^n(\mathrm{Gr}_k(\mathbb{R}^\infty);\mathbb{Z}/2)$ is provided by the Schubert cells of dimension n, and these are in bijective correspondence with partitions of n into $\leq k$ pieces. For example, a basis for $H^6(\mathrm{Gr}_3(\mathbb{R}^\infty);\mathbb{Z}/2)$ is in bijective correspondence with the set of partitions

$$[6], \quad [51], \quad [42], \quad [411], \quad [33], \quad [321], \quad [222].$$

For any such partition $\sigma = [j_1 j_2 \dots j_k]$, define its *weight* to be

$$w(\sigma) = \sum \lceil \tfrac{j_i}{2} \rceil.$$

So the list of the above seven partitions have corresponding weights $3, 4, 3, 4, 4, 4, 3$. Using this notion, the following result shows how to write down an $\mathbb{M}_2$-basis for $H^{*,*}(\mathrm{Gr}_k(\mathcal{U}); \mathbb{Z}/2)$:

Theorem 1.4. *$H^{*,*}(\mathrm{Gr}_k(\mathcal{U}); \mathbb{Z}/2)$ is a free module over $\mathbb{M}_2$ with a basis S, where the elements of S in topological degree n are in bijective correspondence with partitions of n into at most k pieces. This bijection sends a partition σ to a basis element of bidegree $(n, w(\sigma))$ where $w(\sigma)$ is the weight of σ.*

It is easy to see that for a partition σ of n the weight is also equal to

$$w(\sigma) = \tfrac{1}{2}\big(n + (\#\text{ of odd pieces in }\sigma)\big).$$

Using this description we can reinterpret the theorem as follows:

Corollary 1.5. *The number of free generators for $H^{*,*}(\mathrm{Gr}_k(\mathcal{U}); \mathbb{Z}/2)$ having bidegree (p, q) coincides with the number of partitions of p into at most k pieces where exactly $2q - p$ of the pieces are odd.*

For example, in $H^{7,*}(\mathrm{Gr}_5(\mathcal{U}); \mathbb{Z}/2)$ we have basis elements in weights 4, 5, and 6, corresponding to the partitions

$$\begin{array}{rl} [7], [61], [52], [43], [421], [322], [2221] & \text{(weight 4/one odd piece)} \\ [511], [4111], [331], [3211], [22111] & \text{(weight 5/three odd pieces)} \\ [31111] & \text{(weight 6/five odd pieces).} \end{array}$$

We next describe a little about the ring structure. Unlike what happens in nonequivariant topology, it is not easy to write down a simple description of the ring of invariants in terms of generators and relations—except for small values of k. In essense, the innocuous-looking relation "$a^2 = \rho a + \tau b$" propogates itself viciously into the ring of invariants, leading to some unpleasant bookkeeping. However, we are able to give a minimal set of generators for the algebra, and we investigate the relations in low dimensions.

First, for $1 \leq i \leq k$ there are special classes $w_i \in H^{i,i}(\mathrm{Gr}_k(\mathcal{U}); \mathbb{Z}/2)$ that we call **Stiefel-Whitney classes**; they correspond to the usual Stiefel-Whitney classes in singular cohomology. There are also special classes $c_i \in H^{2i,i}(\mathrm{Gr}_k(\mathcal{U}); \mathbb{Z}/2)$ that we call **Chern classes**; their images in non-equivariant cohomology correspond to the mod 2 reductions of the usual Chern classes of the complexification of a bundle. In some sense these constitute the "obvious" characteristic classes that one might expect. It is not true, however, that these generate $H^{*,*}(\mathrm{Gr}_k(\mathcal{U}); \mathbb{Z}/2)$ as an algebra. This is easy to explain in terms of the ring of invariants. There are two sets of variables $a_1, \dots, a_k$ and $b_1, \dots, b_k$, with Σ_k acting on each as permuation of the indices. The class w_i is the ith elementary symmetric function in the a's, and likewise c_i is the elementary symmetric function in the b's. But there are many other invariants, for example $a_1 b_1 + \cdots + a_k b_k$.

We let $w_j^{(e)}$ be the characteristic class corresponding to the invariant element $\sum a_{i_1} \dots a_{i_j} b_{i_1}^e \dots b_{i_j}^e$. Note that $w_j^{(0)} = w_j$. This particular choice of invariants is not the only natural one, but it seems to be convenient in a number of ways. Among other things, these characteristic classes satisfy a Whitney formula

$$w_j^{(e)}(E \oplus F) = \sum_r w_r^{(e)}(E) \cdot w_{j-r}^{(e)}(F).$$

Using the classes $w_j^{(e)}$ we can write down a minimal set of algebra generators for $H^{*,*}(\mathrm{Gr}_k(\mathcal{U}); \mathbb{Z}/2)$:

Proposition 1.6. *The indecomposables of $H^{*,*}(\mathrm{Gr}_k(\mathcal{U}); \mathbb{Z}/2)$ are represented by $c_1, \ldots, c_k$ together with the classes $w_{2^i}^{(e)}$ for $1 \leq 2^i \leq k$ and $0 \leq e \leq \frac{k}{2^i} - 1$.*

Note that the above result gives a slight surprise when $e = 0$. The equivariant Stiefel-Whitney classes w_i are indecomposable only when i is a power of 2. This phenomenon is familiar in a slightly different (but related) context—see [M, Remark 3.4].

In practice it is unwieldy to write down a complete set of relations for $H^{*,*}(\mathrm{Gr}_k(\mathcal{U}); \mathbb{Z}/2)$. To give a sense of this, however, we do it here for $k = 2$:

Proposition 1.7. *The algebra $H^{*,*}(\mathrm{Gr}_2(\mathcal{U}); \mathbb{Z}/2)$ is the quotient of the ring $\mathbb{M}_2[c_1, c_2, w_1, w_2, w_1^{(1)}]$ by the following relations:*

- $w_1^2 = \rho w_1 + \tau c_1$
- $w_2^2 = \rho^2 w_2 + \rho\tau\big(w_1 c_1 + w_1^{(1)}\big) + \tau^2 c_2$
- $\big[w_1^{(1)}\big]^2 = \rho\big(w_1^{(1)} c_1 + w_1 c_2\big) + \tau(c_1^3 + c_1 c_2)$
- $w_1 w_2 = \rho w_2 + \tau\big(w_1 c_1 + w_1^{(1)}\big)$
- $w_1 w_1^{(1)} = \rho w_1^{(1)} + \tau c_1^2 + w_2 c_1$
- $w_2 w_1^{(1)} = \rho w_2 c_1 + \tau(w_1 c_1^2 + w_1^{(1)} c_1 + w_1 c_2)$.

The classes 1, w_1, w_2, and $w_1^{(1)}$ give a free basis for $H^{*,*}(\mathrm{Gr}_2(\mathcal{U}); \mathbb{Z}/2)$ as a module over the subring $\mathbb{M}_2[c_1, c_2]$.

The forgetful map $H^{*,*}(\mathrm{Gr}_2(\mathcal{U}); \mathbb{Z}/2) \to H^*(\mathrm{Gr}_2(\mathbb{R}^\infty); \mathbb{Z}/2) = \mathbb{Z}/2[w_1, w_2]$ from equivariant to non-equivariant cohomology sends

- $\rho \mapsto 0$, $\tau \mapsto 1$
- $w_1 \mapsto w_1$, $w_2 \mapsto w_2$
- $c_1 \mapsto w_1^2, \quad c_2 \mapsto w_2^2, \quad w_1^{(1)} \mapsto w_1 w_2 + w_1^3$.

(Note that the final line can be read off from the above relations and the first two lines).

The complexity of the above description is discouraging, but the main point is really that (a) it can be done, and (b) it is tedious but mostly mechanical. We discuss both the cases $k = 2$ and $k = 3$ in detail in Section 8.

Remark 1.8. In nonequivariant topology there is the relation $c_i(E \otimes \mathbb{C}) = w_i^2(E)$. The first two relations in Proposition 1.7 should be thought of as deformations of this nonequivariant relation.

One might expect the problem of describing the rings $H^{*,*}(\mathrm{Gr}_k(\mathcal{U}); \mathbb{Z}/2)$ to become more tractable as $k \mapsto \infty$. In some ways it does, but even in this case we have not found a convenient way to write down a complete set of relations. See Proposition 7.14 for more information.

1.9. **Open questions.**

(1) Our computations produce the full set of characteristic classes for equivariant real vector bundles, taking values in $H^{*,*}(-;\mathbb{Z}/2)$. It remains to investigate possible uses for such classes, and in particular their ties to geometry.

(2) In the classical case another way to describe the ring structure on the cohomology of Grassmannians is combinatorially, via Littlewood-Richardson rules. It might be useful to work out equivariant versions of these rules, and to describe the ring structure that way instead of by generators and relations.

(3) There is an interesting duality that appears in our description of the cohomology ring for $H^{*,*}(\mathrm{Gr}_k(\mathcal{U});\mathbb{Z}/2)$. See Corollary 3.4 and the charts preceding it. Is there some geometry underlying this duality?

(4) Although we have computed the bigraded cohomology of the infinite Grassmannians $\mathrm{Gr}_k(\mathcal{U})$, our techniques do not yield the cohomology of the finite Grassmannians (in which $\mathcal{U}$ is replaced by a finite-dimensional subspace). The reason is tied to our inability to resolve all the differentials in the cellular spectral sequence. So computing the cohomology in these cases remains an open problem.

(5) We have not developed any understanding of how to analyze differentials in cellular spectral sequences, since the approach of this paper essentially amounts to a sneaky way of avoiding this. Developing a method for computing such differentials, and connecting them to geometry, is an important area for exploration.

(6) If $\mathbb{C}^\infty$ is given the conjugation action, then the space of complex k-planes $\mathrm{Gr}_k(\mathbb{C}^\infty)$ has simple cohomology, even integrally: $H^{*,*}(\mathrm{Gr}_k(\mathbb{C}^\infty);\mathbb{Z}) = \mathbb{Z}[c_1, c_2, \ldots]$ where the Chern classes c_i have bidegree $(2i, i)$. These are the characteristic classes for Real vector bundles (where 'Real' is in the sense of Atiyah [A2]). One can attempt a similar computation but replacing $\mathbb{C}^\infty$ with $\mathbb{C}\otimes\mathcal{U}$: non-equivariantly this is still $\mathbb{C}^\infty$, but the action is different—it is $\mathbb{C}$-linear rather than conjugate-linear. The computation of $H^{*,*}(\mathrm{Gr}_k(\mathbb{C}\otimes\mathcal{U});\mathbb{Z})$ seems to be an open problem, that could perhaps be tackled by the methods of this paper. See [FL] for some relevant, early computations.

(7) The initial motivation of this work was an interest in motivic characteristic classes for quadratic bundles, generalizing the Stiefel-Whitney classes of Delzant [De] and Milnor [M]; see Section 9 for the connection with the present paper. The original motivic question remains unsolved.

1.10. **Organization of the paper.** Section 2 gives some brief background about the theory $H^{*,*}(-;\mathbb{Z}/2)$. Section 3 gives a first look at the ring of invariants $\left[H^{*,*}(\mathrm{Gr}_1(\mathcal{U});\mathbb{Z}/2)^{\otimes k}\right]^{\Sigma k}$, and we provide an additive basis over the ground ring $\mathbb{M}_2$. We also measure the size of this ring by counting the elements of this free basis that appear in each bidegree.

In Section 4 we describe the equivariant Schubert-cell decomposition of $\mathrm{Gr}_k(\mathcal{U})$. A key point here is counting the number of Schubert cells in each bidegree. We also introduce the associated spectral sequence for computing $H^{*,*}(\mathrm{Gr}_k(\mathcal{U});\mathbb{Z}/2)$, and in Section 5 we discuss Kronholm's theorems about this spectral sequence.

Section 6 contains the main topological part of the paper. Using the results of Sections 3–5 we prove that $H^{*,*}(\mathrm{Gr}_k(\mathcal{U});\mathbb{Z}/2)$ is isomorphic to the expected ring of invariants (Theorem 1.2).

In Section 7 we turn to the multiplicative structure of our ring of invariants. We calculate some relations here, and we identify a minimal set of generators. This section is entirely algebraic. Section 8 then gives a presentation for the ring of invariants in the cases $k = 2$ and $k = 3$.

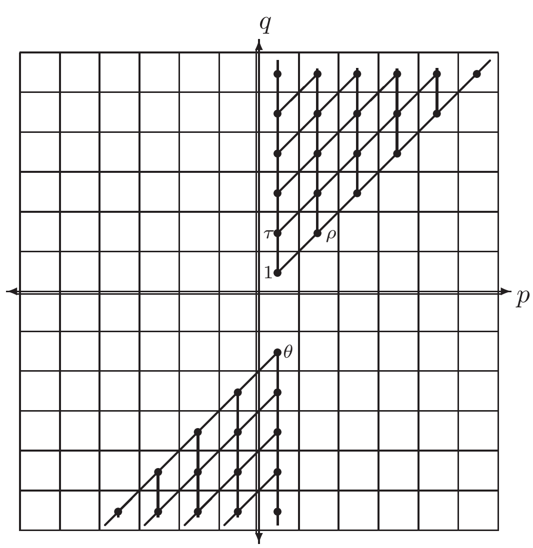
Finally, Section 9 describes the connection between the present work and a certain motivic problem about characteristic classes of quadratic bundles. The results of this section are not needed elsewhere in the paper. An appendix is enclosed which calculates the ring of invariants for Σ_n acting on $\Lambda_{\mathbb{F}_2}(a_1, \ldots, a_n) \otimes_{\mathbb{F}_2} \mathbb{F}_2[b_1, \ldots, b_n]$ by permutation of the indices. This purely algebraic result is needed in the body of the text, and we were unable to find a suitable reference.

Throughout this paper, if X is a $\mathbb{Z}/2$-space then we write $\sigma\colon X \to X$ for the involution. For general background on $RO(G)$-graded equivariant cohomology theories we refer the reader to [Ma].

1.11. **Acknowledgments.** I am grateful to Mike Hopkins for a useful conversation about this subject, and to John Greenlees for expressing some early interest.

2. Background on equivariant cohomology

Recall that $\mathbb{M}_2$ denotes the cohomology ring $H^{*,*}(pt; \mathbb{Z}/2)$. This ring is best depicted via the following diagram:



Each dot represents a $\mathbb{Z}/2$, each vertical line represents a multiplication by τ, and each diagonal line represents multiplication by ρ. In the "positive" range $p, q \geq 0$, the ring is therefore just $\mathbb{Z}/2[\tau, \rho]$. In the negative range there is an element $\theta \in \mathbb{M}_2^{0,-2}$ together with elements that one can formally denote $\frac{\theta}{\tau^k \rho^l} \in \mathbb{M}_2^{-l,-2-k-l}$. After specifying $\theta^2 = 0$ this gives a complete description of the ring $\mathbb{M}_2$. We will refer to the subalgebra $\mathbb{Z}/2[\tau, \rho] \subseteq \mathbb{M}_2$ as the **positive cone**, and the direct sum of all $\mathbb{M}_2^{p,q}$ for $q < 0$ will be called the **negative cone**. See [C] and [D] for more background on this coefficient ring.

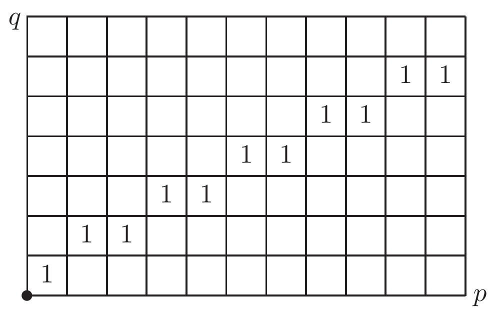
There are natural transformations $H^{p,q}(X;\mathbb{Z}/2) \to H^p_{sing}(X;\mathbb{Z}/2)$ from our bigraded cohomology to ordinary singular cohomology. These are compatible with the ring structure, and when X is a point they send $\tau \mapsto 1$ and $\rho \mapsto 0$. Since everything in the negative cone is a multiple of ρ, it follows that the entire negative cone of $\mathbb{M}_2$ is sent to 0.

2.1. **The graded rank functor.** Let $I \hookrightarrow \mathbb{M}_2$ denote the kernel of the projection map $\mathbb{M}_2 \to \mathbb{Z}/2$. Let M be a bigraded, finitely-generated free module over $\mathbb{M}_2$. Define the **bigraded rank** of M by the formula

$$\operatorname{rank}^{p,q} M = \dim_{\mathbb{Z}/2}(M/IM)^{p,q}.$$

So $\operatorname{rank} M$ should be regarded as a function $\mathbb{Z}^2 \to \mathbb{Z}_{\geq 0}$. Clearly M is determined, up to isomorphism, by its bigraded rank.

It is usually easiest to depict the bigraded rank as a chart. For example, the bigraded rank of $H^{*,*}(\mathrm{Gr}_1(\mathfrak{U});\mathbb{Z}/2)$ is



where the lower left corner is the $(0,0)$ spot and all unmarked boxes are regarded as having a 0 in them.

3. An additive basis for the ring of invariants

Let $R = \mathbb{M}_2[a,b]/(a^2 = \rho a + \tau b)$ where a has degree $(1,1)$ and b has degree $(2,1)$. Fix $k \geq 1$ and let $T_k = R^{\otimes k}$. Let Σ_k act on T_k in the evident way, as permutation of the tensor factors. Define $\mathcal{I}nv_k = [T_k]^{\Sigma_k}$. Our goal in this section is to investigate an additive basis for the algebra $\mathcal{I}nv_k$, regarded as a module over $\mathbb{M}_2$. The multiplicative structure of this ring will be discussed in Section 7.

It will be convenient to rename the variables in the ith copy of R as a_i and b_i. So T_k is the quotient of $\mathbb{M}_2[a_1,\ldots,a_k,b_1,\ldots,b_k]$ by the relations $a_i^2 = \rho a_i + \tau b_i$, for $1 \leq i \leq k$. Let

$$w_i = \sigma_i(a_1,\ldots,a_k)$$

be the ith elementary symmetric function in the a's, and let

$$c_i = \sigma_i(b_1,\ldots,b_k)$$

be the ith elementary symmetric function in the b's. These are the most obvious elements of $\mathcal{I}nv_k$, but there are others as well. For example, the element $a_1b_1 + a_2b_2 + \cdots + a_kb_k$ is invariant under the action of Σ_k. We will need some notation to help us describe these other elements of $\mathcal{I}nv_k$.

If m is a monomial in the a's and b's, write $[m]$ for the smallest homogeneous polynomial in T_k which contains m as one of its terms. By 'smallest' we mean the smallest number of monomial summands. If $H \leq \Sigma_k$ is the stabilizer of m, then $[m]$ is the sum $\sum_{gH\in\Sigma_k/H} gm$. Here are some examples:

(i) $[a_1b_1] = a_1b_1 + a_2b_2 + \ldots + a_kb_k$

(ii) $[a_1b_1b_2] = \sum\limits_{i\neq j} a_ib_ib_j$

(iii) $[a_1b_2b_3] = \sum\limits_{i,\,j,\,k \text{ distinct}} a_ib_jb_k$

(iv) $[a_1a_2] = w_2$.

Notice that

$$[a_1^2b_2] = \sum_{i\neq j} a_i^2b_j = \sum_{i\neq j}(\rho a_i + \tau b_i)b_j = \rho\sum_{i\neq j} a_ib_j + \tau\sum_{i\neq j} b_ib_j = \rho[a_1b_2] + \tau[b_1b_2].$$

A similar computation shows that if m is any monomial with an a_i^2 then $[m]$ is an $\mathbb{M}_2$-linear combination of monomials $[m_j]$ with $\deg m_j < \deg m$.

The following proposition is fairly clear:

Proposition 3.1. *As an $\mathbb{M}_2$-module, $\mathcal{I}nv_k$ is free with basis consisting of all elements $[a_1^{\epsilon_1}\ldots a_k^{\epsilon_k}b_1^{d_1}\ldots b_k^{d_k}]$, where each $d_i \geq 0$ and each $\epsilon_i \in \{0,1\}$.*

Our next task is to count how many of the above basis elements appear in any given bidegree. Write Mon_k for the set of monomials in the variables $a_1,\ldots,a_k,b_1,\ldots,b_k$ having the property that the exponent on each a_i is at most 1. The above proposition implies that $\mathcal{I}nv_k$ has a free basis over $\mathbb{M}_2$ that is in bijective correspondence with the set of orbits Mon_k/Σ_k; this correspondence preserves the bigraded degree. From now on we will refer to this basis as THE free basis for $\mathcal{I}nv_k$. We can easily write down a list of these basis elements in any given bidegree. For instance, here is the list in low dimensions, assuming k is large (with the degrees of the elements given to the left):

$(1,1)$:	$[a_1]$	$(4,3)$:	$[a_1a_2b_1], [a_1a_2b_3]$
$(2,1)$:	$[b_1]$	$(4,4)$:	$[a_1a_2a_3a_4]$
$(2,2)$:	$[a_1a_2]$	$(5,3)$:	$[a_1b_1^2], [a_1b_2^2], [a_1b_1b_2], [a_1b_2b_3]$
$(3,2)$:	$[a_1b_1], [a_1b_2]$	$(5,4)$:	$[a_1a_2a_3b_1], [a_1a_2a_3b_4]$
$(3,3)$:	$[a_1a_2a_3]$	$(5,5)$:	$[a_1a_2a_3a_4a_5]$
$(4,2)$:	$[b_1^2], [b_1b_2]$	$(6,3)$:	$[b_1b_2b_3], [b_1^3], [b_1^2b_2]$

We can count the number of generators in each bidegree in terms of certain kinds of partitions. Given n, k, and j, let $\mathrm{part}_{n,\leq k}[j]$ denote the number of partitions of n into k nonnegative integers such that exactly j of the integers are odd. For example, $\mathrm{part}_{8,\leq 5}[4] = 4$ because it counts the following partitions: 01133, 01115, 11114, and 11123.

Proposition 3.2. *For any p, q, and k one has $\mathrm{rank}^{p,q}(\mathcal{I}nv_k) = \mathrm{part}_{p,\leq k}[2q-p]$.*

Proof. Let w be a monomial in the variables $a_1,\ldots,a_k,b_1,\ldots,b_k$ where each a_i appears at most once. We'll say that w is **pure** if all the symbols in w have the same subscript: e.g., $a_1b_1^3$ is pure, but $a_1a_2b_1^2$ is not. The monomial w can be written in a unique way as $w = w(1)w(2)\cdots w(k)$ where each $w(i)$ is pure and only contains the subscript i.

Regard a_i as having degree 1 and b_i as having degree 2. If v is a pure monomial, let $d(v)$ be its total degree. Finally, if w is any monomial then let $\eta(w)$ be the partition

$$\eta(w) = [d(w(1)),\, d(w(2)),\, \ldots,\, d(w(k))].$$

For example, if $w = a_1a_2a_3b_1^2b_2b_4$ then $\eta(w) = [5312]$.

It is clear that all the Σ_k-cognates of w give rise to the same partition, and so we have a function

$$\mathrm{Mon}_k/\Sigma_k \xrightarrow{\eta} \{\text{partitions with} \leq k \text{ pieces}\}.$$

Moreover, this is a bijection because the partition is enough to recover the invariant element $[w]$: if the ith number in our partition is $2r$ then we write b_i^r, and if it is $2r+1$ we write $a_ib_i^r$, and then we multiply these terms together. For example, given the partition [34678] we would write $[a_1b_1b_2^2b_3^3a_4b_4^3b_5^4]$. This apparently depends on the order in which we listed the numbers in the partition, but this dependence goes away when we take the Σ_k-orbit.

Clearly the topological degree of the monomial w equals the sum of the elements in the partition $\eta(w)$. Also, the number of odd elements of the partition is equal to the number of a_i's in w. But one readily checks that

$$\begin{aligned}\text{weight of } w = \#b_i\text{'s} + \#a_i\text{'s} &= \frac{(\text{topl. degree of } w) - \#a_i\text{'s}}{2} + \#a_i\text{'s} \\ &= \frac{(\text{topl. degree of } w) + \#a_i\text{'s}}{2}.\end{aligned}$$

So the number of odd elements in the partition $\eta(w)$ is $2q-p$, where q is the weight of w and p is the topological degree of w. $\square$

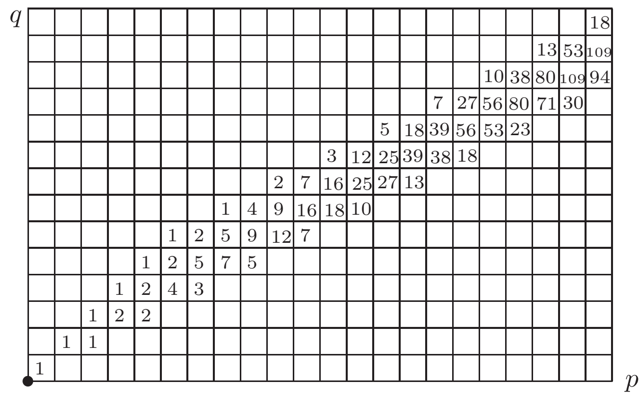
As an example of the above proposition, here is a portion of the bigraded rank function for $\mathcal{I}nv_4$:

q

												5	16	30
										3	11	20	23	11
								2	7	14	16	9		
						1	4	8	11	6				
				1	2	5	7	5						
			1	2	4	3								
		1	2	2										
	1	1												
1														

p

And here is a similar chart for $\mathcal{I}nv_5$:



There are some evident patterns in these charts. For example, if one starts at spot $(2p, p)$ and reads diagonally upwards along a line of slope 1 then the resulting numbers have an evident symmetry. This comes from a symmetry of the $\text{part}_{n,\le k}[j]$ numbers:

Lemma 3.3. *For any n, k, and j, one has* $\text{part}_{n,\le k}[j] = \text{part}_{n+(k-2j),\le k}[k-j]$.

Proof. Suppose $u_1, \ldots, u_k$ is a partition of n in which there are exactly j odd numbers—we can arrange the indices so that these are $u_1, \ldots, u_j$. Subtract 1 from all the odd numbers and add 1 to all the even numbers: this yields the collection of numbers $u_1 - 1, \ldots, u_j - 1, u_{j+1} + 1, \ldots, u_k + 1$. This is a partition of $n + k - 2j$ in which there are exactly $k - j$ odd numbers. One readily checks that this gives a bijection between the two kinds of partitions. $\square$

The diagonal symmetries in our rank charts are as follows:

Corollary 3.4. *For any p, r, and k, one has*

$$\text{rank}^{2p+r,p+r}(\mathcal{I}nv_k) = \text{rank}^{2p+k-r,p+k-r}(\mathcal{I}nv_k).$$

Proof. This is immediate from Proposition 3.2 and Lemma 3.3. $\square$

The numbers in the rank chart for $\mathcal{I}nv_k$ organize themselves naturally into lines of slope $\frac{1}{2}$. To explain this (and because it will be needed later) we introduce the following terminology. A **successor** of a partition α is any partition obtained by adding 2 to exactly one of the numbers in α. For example, 011 has two successors: 013 and 112. If a partition β is obtained from α by a sequence of successors, we say that β is a **descendent** of α. Finally, a partition α will be called **minimal** if it is not a successor of any other partition.

For the set of all partitions consisting of k nonnegative numbers, the following facts are immediate:

(1) There are exactly $k+1$ minimal partitions: $00\ldots0$, $00\ldots01$, $00\ldots011$, $\ldots$, and $11\ldots1$.
(2) Every partition α is a descendent of a unique minimal partition, namely the one obtained by replacing each α_i with either 0 or 1 depending on whether α_i is even or odd.

The partitions consisting of k nonnegative numbers, with exactly j odd numbers, form a tree under the successor operation: and the numbers of such partitions forms the line of slope $\frac{1}{2}$ ascending from spot (j, j) in our rank charts.

The following corollary records the evident bounds on the nonzero numbers in our rank charts. The proof is immediate from the things we have already said, or it could be proven directly from Proposition 3.2.

Corollary 3.5. *The bigraded rank function of $\mathcal{I}nv_k$ is nonzero only in the region bounded by the three lines $y = x$, $y = \frac{1}{2}x$, and $y = \frac{1}{2}x + \frac{k}{2}$. That is to say, the elements of our free basis for $\mathcal{I}nv_k$ appear only in bidegrees (a, b) where $\frac{a}{2} \leq b \leq a$ if $a \leq k$, and $\frac{a}{2} \leq b \leq \frac{1}{2}a + \frac{k}{2}$ if $a \geq k$.*

4. Schubert cells and a spectral sequence

Given a sequence of integers $1 \leq a_1 < a_2 < \cdots < a_k$, define the associated Schubert cell in $\mathrm{Gr}_k(\mathcal{U})$ by

$$\Omega_a = \{V \in \mathrm{Gr}_k(\mathcal{U}) \mid \dim(V \cap \mathcal{U}^{a_i}) \geq i\}.$$

Here $\mathcal{U}^n \subseteq \mathcal{U}$ is simply the subspace of vectors whose rth coordinates all vanish for $r > n$, which we note is closed under the $\mathbb{Z}/2$-action. It will be convenient for us to regard the a-sequence as giving a "$*$-pattern", in which one takes an infinite sequence of empty boxes and places a single $*$ in each box corresponding to an a_i. If the boxes represent the standard basis elements of $\mathcal{U}$, then the $*$'s represent where the jumps in dimension occur for subspaces V lying in the interior of Ω_a. These $*$-patterns will be used several times in our discussion below.

It is somewhat more typical to use a different indexing convention here. Define $\sigma_i = a_i - i$, so that we have $0 \leq \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_k$. Write $\Omega(\sigma)$ for the same Schubert cell as Ω_a, which has dimension equal to $\sum_i \sigma_i$. Define a ***k*-Schubert symbol** to be an increasing sequence $\sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_k$. To get the associated $*$-pattern, skip over σ_1 empty boxes and then place a $*$; then skip over $\sigma_2 - \sigma_1$ empty boxes and place another $*$; then skip over $\sigma_3 - \sigma_2$ empty boxes, and so forth. For example, the Schubert symbol [0235] corresponds to the $*$-pattern $[*\_\_\,*\_\,*\_\_*]$, or the a-sequence $(1, 4, 6, 9)$.

Let $F_r \subseteq \mathrm{Gr}_k(\mathcal{U})$ be the union of all the Schubert cells of dimension less than or equal to r. This filtration gives rise to a spectral sequence on cohomology in the usual way, where the E_1-term is the direct sum $\oplus_\sigma \tilde{H}^{*,*}(S^{a_\sigma, b_\sigma})$ where σ ranges over all k-Schubert symbols and (a_σ, b_σ) is the bidegree of the associated cell. We will next describe an algorithm for producing this bidegree.

Picture the row of symbols $+ - + - + - \cdots$ going on forever, with the initial symbol regarded as the first (rather than the zeroth). These symbols represent the $\mathbb{Z}/2$-action on the standard basis elements of $\mathcal{U}$. For each i in the range $1 \leq i \leq k$, change the a_ith symbol to an asterisk $*$. Then for each i, define u_i to be

$$u_i = \begin{cases} \text{the total number of } + \text{ signs to the left of the } i\text{th asterisk} & \text{if } a_i \text{ is even} \\ \text{the total number of } - \text{ signs to the left of the } i\text{th asterisk} & \text{if } a_i \text{ is odd.} \end{cases}$$

Finally, define the **cell-weight** of the Schubert symbol to be $\sum_i u_i$. We claim that the open Schubert cell corresponding to σ is isomorphic to $\mathbb{R}^{n,k}$ where $n = \sum \sigma_i$ and k is the cell-weight of σ.

Let us say the above in a slightly different way. We think in terms of $*$-patterns, but where the boxes contain alternating $+$ and $-$ signs and the $*$'s eradicate whatever sign was in their box. For the topological dimension of a cell, we count the number of empty boxes to the left of each $*$ and add these up. For the weight we do a fancier kind of counting: if the $*$ replaced a $+$ sign then we count the number of $-$ signs to the left of it, whereas if it replaced a $-$ we count the number of $+$ signs to the left. And again, we add up our answers for each $*$ in the pattern to get the total weight. For example, consider the Schubert symbol $\sigma = [135]$ which has topological dimension 9. The corresponding a-sequence is $(2, 5, 8)$, and this gives the $*$-pattern $+ * + - * - + * + - + - \cdots$ So $u_1 = 1$, $u_2 = 1$, $u_3 = 3$, and therefore the bidegree of $\Omega(\sigma)$ is $(9, 5)$.

Example 4.1. Consider the Grassmannian $\mathrm{Gr}_2(\mathcal{U}^6)$. There are $\binom{6}{2} = 15$ Schubert cells. We list all the $*$-patterns and the bidegrees of the associated cells:

$**+-+-$	$(0,0)$	$+**-+-$	$(2,1)$	$+-*-*-$	$(5,3)$
$*-*-+-$	$(1,1)$	$+*+*+-$	$(3,3)$	$+-*-+*$	$(6,3)$
$*-+*+-$	$(2,1)$	$+*+-*-$	$(4,2)$	$+-+**-$	$(6,3)$
$*-+-*-$	$(3,2)$	$+*+-+*$	$(5,4)$	$+-+*+*$	$(7,5)$
$*-+-+*$	$(4,2)$	$+-**+-$	$(4,2)$	$+-+-**$	$(8,4)$

We need to justify our procedure for determining the weight of a Schubert cell. Given an a-sequence, points in the interior of the associated Schubert cell Ω_a are in bijective correspondence with matrices of a form such as

$$\begin{bmatrix} ? & ? & 1 & 0 & 0 & 0 & 0 & 0 \\ ? & ? & 0 & ? & 1 & 0 & 0 & 0 \\ ? & ? & 0 & ? & 0 & ? & ? & 1 \end{bmatrix}.$$

(The matrix given is for the case of $\mathrm{Gr}_3(\mathcal{U})$ and the a-sequence $(3, 5, 8)$). The matrix in question has 1's in the columns given by the a-sequence, each 1 is followed by only zeros in its row, and each 1 is the only nonzero entry in its column. The set of such matrices is a Euclidean space of dimension equal to the number of "?" symbols. Such a matrix determines a point in $\mathrm{Gr}_k(\mathcal{U})$ by taking the span of its rows, and any k-plane in the interior of Ω_a has a unique basis of the above form. This is all standard from non-equivariant Schubert calculus. In the equivariant case, we have a $\mathbb{Z}/2$-action on the set of such matrices induced by the $\mathbb{Z}/2$-action on $\mathcal{U}$. In our above example, the action is

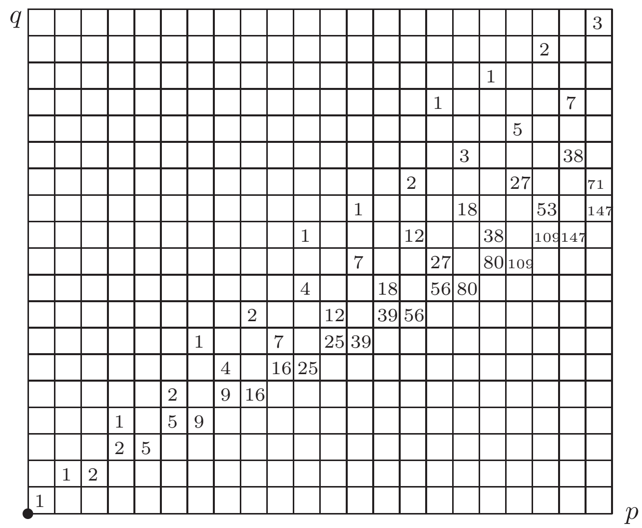
$$\begin{bmatrix} b & c & 1 & 0 & 0 & 0 & 0 & 0 \\ d & e & 0 & f & 1 & 0 & 0 & 0 \\ g & h & 0 & i & 0 & j & k & 1 \end{bmatrix} \mapsto \begin{bmatrix} b & -c & 1 & 0 & 0 & 0 & 0 & 0 \\ d & -e & 0 & -f & 1 & 0 & 0 & 0 \\ g & -h & 0 & -i & 0 & -j & k & -1 \end{bmatrix}.$$

Notice that the matrix on the right is not in our standard form. To convert it to standard form we multiply the third row by -1 to get

$$\begin{bmatrix} b & c & 1 & 0 & 0 & 0 & 0 & 0 \\ d & e & 0 & f & 1 & 0 & 0 & 0 \\ g & h & 0 & i & 0 & j & k & 1 \end{bmatrix} \mapsto \begin{bmatrix} b & -c & 1 & 0 & 0 & 0 & 0 & 0 \\ d & -e & 0 & -f & 1 & 0 & 0 & 0 \\ -g & h & 0 & i & 0 & j & -k & 1 \end{bmatrix}.$$

So as a $\mathbb{Z}/2$-representation we have $\mathbb{R}^{10}$ with five sign changes, and this is $\mathbb{R}^{10,5}$. It is now easy to go from this overall picture to the specific formula for the cell-weight that was given above.

We now know how to compute the bigraded Schubert cell decomposition for any Grassmannian. It is useful to look at a specific example, so here is the Schubert cell picture for $\mathrm{Gr}_5(\mathcal{U})$. Each box gives the number of Schubert cells of the given bidegree.



Note that the numbers appearing along lines of slope $\frac{1}{2}$ are the same as the numbers we saw in the rank chart for $\mathcal{I}nv_5$, except that the lines are arranged differently in the plane. We will need a precise statement:

Proposition 4.2. *Let X be the E_1-term of the cellular spectral sequence for $\mathrm{Gr}_k(\mathcal{U})$ based on the Schubert cell filtration. Then the nonzero entries in the rank chart for X are bordered by the lines $y = x$, $y = \frac{1}{2}x$, and $y = \frac{1}{2}(x + \binom{k+1}{2})$.*

Moreover, for any j, r one has $\mathrm{rank}^{j+2r,j+r} X = 0$ unless $j = \binom{i}{2}$ for some i in the range $1 \leq i \leq k+1$. And finally, if $j = \binom{i}{2}$ then

$$\mathrm{rank}^{j+2r,j+r} X = \mathrm{part}_{2r+\gamma_i,\leq k}[\gamma_i] = \mathrm{rank}^{\gamma_i+2r,\gamma_i+r}(\mathcal{I}nv_k)$$

where γ is the function defined by

$$\gamma_i = \begin{cases} \frac{k+i}{2} & \text{if } k-i \text{ is even} \\ \frac{k+1-i}{2} & \text{if } k-i \text{ is odd.} \end{cases}$$

The mathematical phrasing of the above proposition is somewhat awkward, but it says something very concrete. Namely, the nonzero entries in the rank chart for X are divided into rays of slope $\frac{1}{2}$ emanating from the points $\left(\binom{i}{2}, \binom{i}{2}\right)$ for $1 \leq i \leq k+1$. Starting from $\left(\binom{k+1}{2}, \binom{k+1}{2}\right)$ and working towards the origin along the $y = x$ line, mentally label each vertex with the numbers in the sequence

$$0,\ k,\ 1,\ k-1,\ 2,\ k-2,\ 3,\ k-3,\ \ldots$$

These are the numbers $\gamma_{k+1}, \gamma_k, \ldots$. Then in $\mathrm{rank}(X)$, the rth term from $\left(\binom{i}{2}, \binom{i}{2}\right)$ along the ray of slope $\frac{1}{2}$ is equal to $\mathrm{part}_{2r+\gamma_i,\leq k}[\gamma_i]$.

In order to prove Proposition 4.2 we need to introduce some language for bookkeeping. Define a **successor** of a $*$-pattern to be a pattern made by moving one of the $*$'s two spots to the right (note that one can only do this if the new spot for the $*$ started out empty). In terms of a-sequences, a successor is an a-sequence obtained by adding 2 to one of the a_i's. For example, the a-sequence 123 has exactly two successors, namely 125 and 145. A $*$-pattern (or a-sequence) is said to be **minimal** if it is not the successor of another pattern (or sequence); said differently, a $*$-pattern is minimal if one cannot move any $*$ two places to the left. The sequences 123 and 124 are both minimal, but 125 is not; these correspond to the $*$-patterns $[***]$, $[**+*]$, and $[**+-*]$.

Observe that taking the successors of a $*$-pattern increases the bidegree of the associated Schubert cell by $(2,1)$. This is easy to explain in terms of the following picture, showing an arbitrary $*$-pattern and a successor obtained by moving one of the $*$'s:

$$\cdots \quad \Box \quad * \quad \Box \quad \Box \quad \Box \quad * \quad * \quad \Box \quad \cdots$$

The count of empty boxes to the left of each $*$ is the same for the two patterns, except for the $*$ that got moved: and for that $*$ the count has increased by 2. Likewise, the number of $+/-$ signs in the empty boxes stays the same for each $*$ in the two patterns, except again for the $*$ that got moved: and for that $*$ the number of $+$ and $-$ signs to the left of it each got increased by 1.

The fact that the successor relation increases the bidegree by $(2,1)$ explains why our Schubert cell chart breaks up into rays of slope $\frac{1}{2}$. The number of such rays will be governed by the number of minimal $*$-patterns, so we investigate this next.

It is clear that for a $*$-pattern to be minimal it must be true that any two successive $*$'s have at most one empty space between them. Moreover, as soon as one has an empty space in the $*$-pattern then all successive $*$'s must be separated by one empty space. So for patterns with k asterisks, there are exactly $k+1$ minimal patterns; they are completely described by saying which $*$ has the first blank space after it (the count is $k+1$ because the first blank might appear after the *zeroth* star, which doesn't actually exist). One thing that is easy to verify about these minimal patterns is that the corresponding Schubert cells each have bidegree (p,p), for some values of p; that is, the topological dimension and weight coincide. Recall that computing both the topological dimension and the weight from the $*$-patterns amounts to counting empty boxes to the left of each $*$, with the weight computation involving some restrictions on which boxes get counted. For the minimal $*$-patterns, the placement of the $*$'s results in these restrictions all being vacuous: that is, all empty boxes are counted.

The minimal $*$-patterns correspond to the following a-sequences:

$$(1,2,3,\dots,k),(1,2,\dots,k-1,k+1),(1,2,\dots,k-2,k,k+2),\dots,(1,3,5,\dots,2k-1)$$

and $(2,4,6,\dots,2k)$ (the first k of these follow a common pattern, the final one does not). The associated Schubert symbols are

$$[00\dots0],\ [00\dots01],\ [00\dots12],\ \dots,\ [012\dots(k-1)],\quad \text{and} \quad [123\dots k].$$

The topological dimensions are therefore $\binom{i}{2}$ for $1 \le i \le k+1$, so the minimal $*$-patterns correspond to Schubert cells of bidegree $(\binom{i}{2},\binom{i}{2})$ for i in this range.

Proof of Proposition 4.2. We have determined in the preceding discussion that the successor relation breaks the Schubert-cell chart into $k+1$ rays of slope $\frac{1}{2}$, each ray starting at a point $(\binom{i}{2},\binom{i}{2})$ for $1 \le i \le k+1$. The starting points are the minimal $*$-patterns determined above. What remains to be shown is that the number of cells counted along these rays matches similar rays in the count of partitions we saw in our study of rank($\mathcal{I}nv_k$). This is where the awkward rearrangement of the rays must be accounted for.

We have the classical bijection between Schubert cells and partitions, which associates to any $\ast$-pattern the corresponding Schubert symbol. For the rest of this proof we completely discard this bijection, and instead use a *different* bijection, to be described next. This is the crux of the argument. See Remark 4.4 below for more information about where this new bijection comes from.

Given a partition σ with k nonnegative parts, regard this as two partitions σ^{ev} and σ^{odd} by simply separating the even and odd numbers. For example, if $\sigma = [00123]$ then $\sigma^{ev} = [002]$ and $\sigma^{odd} = [13]$. Note that in both σ^{ev} and σ^{odd} the difference of consecutive pieces (when ordered from least to greatest) will always be even.

Consider a string of empty boxes labelled $1, 2, 3, \ldots$ Take σ^{ev} and convert this to a $\ast$-pattern in what is essentially the usual way, but placing the $\ast$'s only in the *even* boxes of the pattern. If $\sigma^{ev} = [u_1, \ldots, u_r]$ then skip over $\frac{u_1}{2}$ even boxes and place a $\ast$, then skip over $\frac{u_2-u_1}{2}$ even boxes and place a $\ast$, and so on. Likewise, convert σ^{odd} to a $\ast$-pattern in the usual way but placing the $\ast$'s only in the *odd boxes*. If $\sigma^{odd} = [v_1, \ldots, v_r]$ then skip over $\frac{v_1-1}{2}$ odd boxes and place a $\ast$, then skip over $\frac{v_2-v_1}{2}$ odd boxes and place a $\ast$, and so on. This awkward procedure is best demonstrated by an example, so return to $\sigma = [00123]$. Then $\sigma^{ev} = [002]$, which corresponds to the $\ast$-pattern $[\_ \ast \_ \ast \_\_\_ \ast]$, and $\sigma^{odd} = [13]$ which corresponds to the $\ast$-pattern $[\ast \_\_\_ \ast]$. So the combined pattern is $[\ast\ast \_ \ast\ast \_\_ \ast]$.

We have given a function from partitions with k pieces to $\ast$-patterns with k asterisks. It is easy to see that this is a bijection; an example of the inverse should suffice. For the $\ast$-pattern

$$[\ast \_\_ \ast \_ \ast\ast\ast \_\_\_ \ast \_ \ast \_ \ast]$$

the only odd boxes occupied are 1 and 7. The associated σ^{odd} is [15], because $\frac{5-1}{2}$ accounts for the two skipped odd boxes between them. The occupied even boxes are $4, 6, 8, 12, 14, 16$ and so $\sigma^{ev} = [222444]$. The partition associated to the above $\ast$-pattern is therefore $\sigma = [12224445]$.

The point of this strange bijection is the following: it carries the successor relation for $\ast$-patterns to the successor relation for partitions (the latter defined back in Section 3). This is easy to see, and we leave it to the reader—but also see Remark 4.4 below for a strong hint.

Using the above bijection, the minimal $\ast$-patterns of k asterisks correspond to the partitions $[00\ldots0]$, $[00\ldots01]$, $[00\ldots001]$,..., and $[11\ldots1]$ (each with k pieces). For example, if k is even then the $\ast$-pattern with a-sequence $(1, 2, 3, \ldots, k)$ corresponds to the partition $[00\ldots011\ldots1]$ where there are $\frac{k}{2}$ zeros and $\frac{k}{2}$ ones. It is somewhat better to order the partitions as

$$[00\ldots0],\ [11\ldots1],\ [00\ldots01],\ [011\ldots1],\ [00\ldots001],\ [0011\ldots1],\ \ldots \tag{4.3}$$

because in this order the topological degrees of the associated Schubert cells are

$$\binom{k+1}{2}, \binom{k}{2}, \binom{k-1}{2}, \ldots, \binom{2}{2}, \binom{1}{2}.$$

For later use, let $\mu(i)$ be the number of 1's in the ith partition from the list (4.3), with $1 \leq i \leq k+1$. This sequence is $\mu(1) = 0$, $\mu(2) = k$, $\mu(3) = 1$, $\mu(4) = k-1$, and so forth. Note that $\mu(i) = \gamma_{k+2-i}$, for the γ-function defined in Proposition 4.2.

We can now wrap up the argument. We have a bijection between $*$-patterns and partitions, and it preserves the successor relation; it therefore also preserves the trees of descendants. In both settings (of $*$-patterns and partitions) one finds exactly $k+1$ minimal elements—and therefore $k+1$ trees. The minimal partitions are the ones in which each piece is either 0 or 1. For the $*$-patterns we have computed that the minimal elements correspond to cells of bidegree $\left(\binom{i}{2}, \binom{i}{2}\right)$ for $1 \leq i \leq k+1$, and that the partition associated to this $*$-pattern has exactly γ_i pieces equal to 1 (and the rest zeros). We also found that an rth successor of such a $*$-pattern has bidegree $\left(\binom{i}{2} + 2r, \binom{i}{2} + r\right)$.

Let σ_i be the partition associated to the minimal $*$-pattern of bidegree $\left(\binom{i}{2}, \binom{i}{2}\right)$. Then $\operatorname{rank}^{\binom{i}{2}+2r, \binom{i}{2}+r}(X)$ is the number of rth successors of this $*$-pattern, which is equal to the number of rth successors of the partition σ_i. But σ_i contains exactly γ_i odd numbers, so the successors of σ_i are the partitions with exactly γ_i odd numbers. The sum of the numbers in σ_i is equal to γ_i (note that σ_i only contains 0s and 1s), and so the sum of the numbers in an rth successor of σ_i will be $\gamma_i + 2r$. One sees in this way that the number of rth successors of σ_i is equal to $\operatorname{part}_{2r+\gamma_i, \leq k}[\gamma_i]$. This completes the proof. $\square$

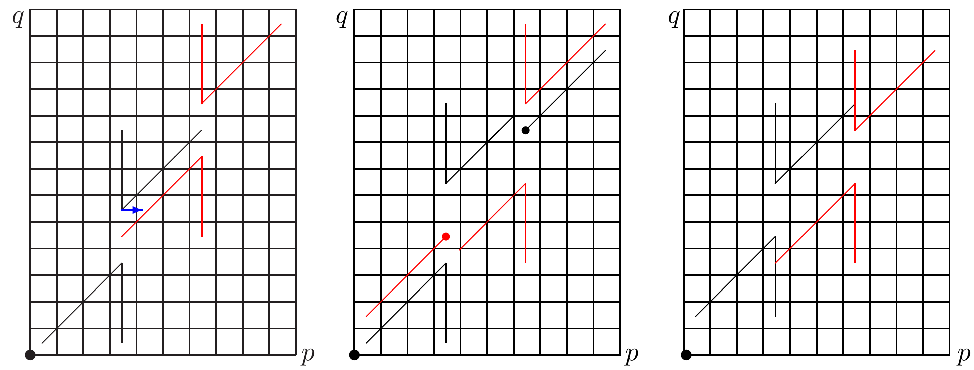
Remark 4.4. Let us return to the classical bijection between $*$-patterns and partitions, via Schubert symbols. We claim that moving an asterisk one spot to the right corresponds to adding 1 to an element of the associated Schubert symbol. Suppose that the a-sequence for the $*$-pattern is $\ldots x, y, z, \ldots$ and that we are promoting y to $y+1$. Clearly this does not effect the beginning or the end of the Schubert symbol. If the original Schubert symbol was $\ldots, u, v, w \ldots$ then $v - u = y - x - 1$ and $w - v = z - y - 1$. The new Schubert symbol will be $\ldots, u, v', w', \ldots$ where $v' - u = (y+1) - x - 1$ and $w' - v' = z - (y+1) - 1$. Clearly this requires $v' = v + 1$ and $w' = w$.

It is not true, however, that moving an asterisk *two* spots to the right corresponds to adding 2 to an element of the associated Schubert symbol. The whole point of the strange bijection from the above proof was to create a situation where this does work, and the previous paragraph suggests why treating the even and odd spots separately accomplishes this.

5. Differentials in the cellular spectral sequence

The main goal of this section is Kronholm's theorem (Theorem 5.1 below), which to date is our best tool for governing what happens inside the cellular spectral sequence.

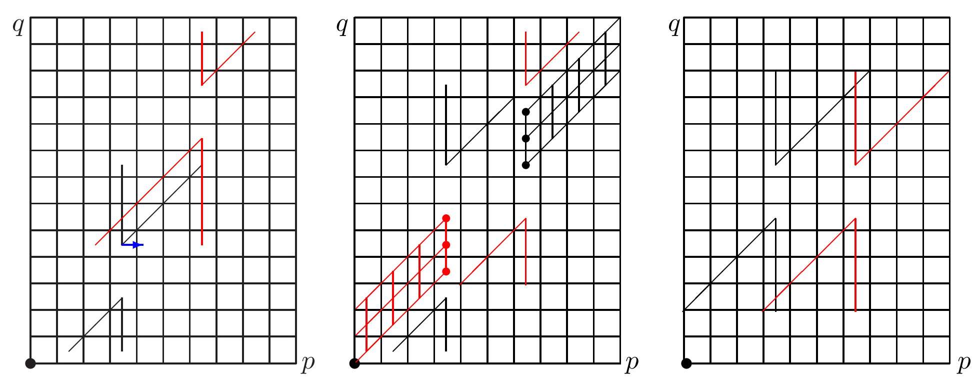
To begin, we give two examples demonstrating the kinds of differentials that can appear in the cellular spectral sequence for $\operatorname{Gr}_k(\mathcal{U})$. The first example consists of the row of three pictures below. In the leftmost picture we have a page of the spectral sequence in which there are two copies of $\mathbb{M}_2$, with generators in bidegrees (a, b) and $(a+3, b+4)$. (Note that one will typically have many more than two copies of $\mathbb{M}_2$, but we focus on this simple situation for pedagogical purposes). There is a differential (shown) that must be a d_3, since it maps a class from filtration degree a into one from filtration degree $a+3$. The differential is only drawn on the *generator* of the first copy of $\mathbb{M}_2$, but the differentials in the cellular spectral sequence are $\mathbb{M}_2$-linear: so the one that is drawn implies several other evident differentials.



In the middle panel we show the E_4-term of the spectral sequence, obtained by taking homology with respect to our differential (warning: not all τ-multiplications are shown here). In our simple example this is the same as E_∞, but note that there are extension problems in deducing the $\mathbb{M}_2$-structure. By a theorem of Kronholm [K1, Theorem 3.2] it turns out that the cohomology we are converging to must be free over $\mathbb{M}_2$, and hence the extensions are resolved as shown in the third panel.

Note the net effect as one passes from the first panel to the third: the two copies of $\mathbb{M}_2$ remain, but their bidegrees have been shifted. The first copy has moved up one weight, and the second copy has moved down one weight.

Our next example shows a very similar phenomenon. Interpreting the pictures requires a little more imagination, though: remember that the pictures only explicitly show the edges of the cones, whereas there are an entire lattice of classes within the cones. The leftmost chart shows a situation where the differential takes the black generator to a class in the interior of the negative cone for the second copy of $\mathbb{M}_2$:



The leftmost chart is again an E_3-term, as the differential maps a class in filtration a to a class in filtration $a+3$. The E_4-page is shown in the second chart. Kronholm's theorem tells us that the cohomology our spectral sequence is converging to is free over $\mathbb{M}_2$, and so the relevant extension problems work out to be as shown in the third chart.

Once again, notice the difference between the first chart and the last chart: the left copy of $\mathbb{M}_2$ has increased its weight by three, whereas the right copy has decreased its weight by three.

Kronholm's theorem generalizes these two examples. It says that the cohomology that the spectral sequence is converging to will be related to the E_1-term by a sequence of "trades" in which two copies of $\mathbb{M}_2$ shift up/down by the same number. The following is a rigorous statement along these lines, which covers all the applications we will need in the present paper:

Theorem 5.1 (Kronholm). *Let X denote the E_1-term of the cellular spectral sequence for $\mathrm{Gr}_k(\mathcal{U})$, and let $Y = H^{*,*}(\mathrm{Gr}_k(\mathcal{U});\mathbb{Z}/2)$. Both X and Y are free as $\mathbb{M}_2$-modules, and for each $p \in \mathbb{Z}$ one has*

$$\sum_q \mathrm{rank}^{p,q}(X) = \sum_q \mathrm{rank}^{p,q}(Y) \quad \textit{and} \quad \sum_c \mathrm{rank}^{c,p+c}(X) = \sum_c \mathrm{rank}^{c,p+c}(Y).$$

Note that the first equality from part (a) says that the number of basis elements in topological dimension p is the same in both E_1 and $H^{*,*}(\mathrm{Gr}_k(\mathcal{U}))$. Relative to our rank charts, the second equality from (a) says that the number of basis elements along any given diagonal is the same in both E_1 and $H^{*,*}(\mathrm{Gr}_k(\mathcal{U});\mathbb{Z}/2)$.

Remark 5.2. In actuality, Theorem 5.1 as we have stated it is not quite found in [K1]. However, the result is implicit in the proof of [K1, Theorem 3.2].

5.3. **The forgetful map to singular cohomology.** There are natural maps $\Phi\colon H^{p,q}(X;\mathbb{Z}/2) \to H^p(X;\mathbb{Z}/2)$ from equivariant cohomology to singular cohomology. These fit together to give a ring map $\Phi\colon H^{*,*}(X;\mathbb{Z}/2) \to H^*(X;\mathbb{Z}/2)$. When $X = *$ this map is completely determined by the formulas $\Phi(\tau) = 1$, $\Phi(\rho) = 0$. Consequently, Φ induces natural maps

$$H^{*,*}(X;\mathbb{Z}/2)/(\rho) \to H^*(X;\mathbb{Z}/2) \quad \text{and} \quad H^{*,*}(X;\mathbb{Z}/2)[\tau^{-1}] \to H^*(X;\mathbb{Z}/2).$$

If J is a free $\mathbb{M}_2$-module, then $J/\rho J$ is a free $\mathbb{M}_2/\rho = \mathbb{Z}/2[\tau]$-module. Note that τ has topological dimension zero, and so $J/\rho J$ will decompose as a $\mathbb{Z}/2[\tau]$-module into a direct sum over all topological dimensions:

$$J/\rho J = \oplus_p \big[J/\rho J\big]^{p,*}.$$

Note that the submodule $[J/\rho J]^{p,*}$ is only "influenced" by basis elements of J in topological degree p: more precisely, any element of $[J/\rho J]^{p,*}$ is the image under $J \to J/\rho J$ of a $\mathbb{Z}/2[\tau]$-linear combination of basis elements of J in topological degree p. Also, if J has a finite number of free generators in each topological degree then the $\mathbb{Z}/2$-dimension of $[J/\rho J]^{p,q}$ is independent of q for $q \gg 0$ (once q is larger than the weights of all the generators in this topological degree).

Let us apply these ideas when $J = H^{*,*}(\mathrm{Gr}_k(\mathcal{U});\mathbb{Z}/2)$. Then $[J/\rho J]^{p,*}$ is a free $\mathbb{Z}/2[\tau]$-module with a basis corresponding to the equivariant Schubert cells of topological dimension p. While these cells likely have different weights, if we look in $[J/\rho J]^{p,N}$ for N large enough then we will see all of them (more precisely, τ-multiples of all of them). The forgetful map Φ will send these elements to the corresponding non-equivariant Schubert classes in $H^p(\mathrm{Gr}_k(\mathcal{U});\mathbb{Z}/2)$ (recall that $\Phi(\tau) = 1$). This shows that in large enough weights N the map $\Phi\colon [J/\rho J]^{p,N} \to H^p(\mathrm{Gr}_k(\mathcal{U});\mathbb{Z}/2)$ is an isomorphism. This proves part (a) of the following:

Proposition 5.4. *For $p, q \in \mathbb{Z}$ consider the map*

$$\Phi_{p,q} \colon \Big[H^{*,*}(\mathrm{Gr}_k(\mathcal{U}); \mathbb{Z}/2)/(\rho)\Big]^{p,q} \to H^p(\mathrm{Gr}_k(\mathcal{U}); \mathbb{Z}/2).$$

(a) Given p, there exists an $N \in \mathbb{Z}$ such that the map $\Phi_{p,q}$ is an isomorphism for all $q \geq N$.
(b) For any p and q the map $\Phi_{p,q}$ is an injection.

Proof. The proof of part (a) preceded the statement of the proposition. For (b), fix p and q and write $J = H^{*,*}(\mathrm{Gr}_k(\mathcal{U}); \mathbb{Z}/2)$ for simplicity. By (a) we know that for large enough N the map $\Phi_{p,N}$ is an isomorphism. Now just consider the diagram

$$\begin{tikzcd} {[J/\rho J]^{p,q}} \arrow[r, "\cdot\tau^{N-q}"] \arrow[dr, "\Phi"'] & {[J/\rho J]^{p,N}} \arrow[d, "\Phi", "\cong"'] \\ & H^p(\mathrm{Gr}_k(\mathcal{U}); \mathbb{Z}/2), \end{tikzcd}$$

which commutes because $\Phi(\tau) = 1$. Multiplication by τ is an injection on $\mathbb{M}_2/(\rho)$, and hence also on $J/\rho J$. So the diagonal map in the diagram is also injective. $\square$

6. Proof of the main theorem

Throughout this section we let X be the E_1-term of the cellular spectral sequence for $\mathrm{Gr}_k(\mathcal{U})$, we let $Y = H^{*,*}(\mathrm{Gr}_k(\mathcal{U}); \mathbb{Z}/2)$, and we let $Z = \mathcal{I}nv_k$. It will be convenient to keep in mind the diagram

$$X \rightsquigarrow Y \longrightarrow Z,$$

indicating that Y maps to Z and that there is a spectral sequence that starts from X and converges to Y. Our aim is to prove Theorem 1.2, stating that $Y \to Z$ is an isomorphism.

Each of X, Y, and Z is a free module over $\mathbb{M}_2$, and the proof will involve a study of the bigraded rank functions for each. The following lemma collects the key results we will need:

Lemma 6.1.
(a) For every $p \in \mathbb{Z}$,

$$\sum_q \mathrm{rank}^{p,q}(Y) = \sum_q \mathrm{rank}^{p,q}(X) = \sum_q \mathrm{rank}^{p,q}(Z).$$

(b) For every $c \in \mathbb{Z}$,

$$\sum_p \mathrm{rank}^{p,p+c}(Y) = \sum_p \mathrm{rank}^{p,p+c}(X) = \sum_p \mathrm{rank}^{p,p+c}(Z).$$

(c) For every $p, q \in \mathbb{Z}$,

$$\sum_{c \leq q} \mathrm{rank}^{p,c}(Y) \leq \sum_{c \leq q} \mathrm{rank}^{p,c}(Z) \quad \textit{and} \quad \sum_{c \geq q} \mathrm{rank}^{p,c}(Y) \geq \sum_{c \geq q} \mathrm{rank}^{p,c}(Z).$$

We have written the equalities in the first two parts in the order that they will be proven: Y is related to X, and X is related to Z. Phrase in terms of our rank charts, the above results say:

(i) The sum of the numbers in any column is the same for X, Y, and Z.
(ii) The sum of the numbers along any diagonal is the same for X, Y, and Z.

(iii) If one fixes a particular box and adds together the numbers in all boxes directly above it, the sum for Y is always at least the sum for Z. (This is the second inequality in (c)).

We defer the proof of the lemma for just a moment, in order to highlight the structure of the main argument. However, let us point out that the left equalities in (a) and (b) are by Kronholm's Theorem, and the second equalities come from our combinatorial analyses of $\operatorname{rank}^{*,*}(X)$ and $\operatorname{rank}^{*,*}(Z)$. In light of (a), the two inequalities in part (c) are equivalent. The proof of these final inequalities uses some topology, namely the non-equivariant version of Theorem 1.2.

Before proving the next result, we introduce a useful piece of notation. If M is a free $\mathbb{M}_2$-module, then for each $c \in \mathbb{Z}$ let $d_c(M)$ denote the function $\mathbb{Z} \to \mathbb{Z}$ given by $p \mapsto \operatorname{rank}^{p,p-c}(M)$. These are the entries in the rank chart of M along the diagonal line of slope 1 passing through the point $(0, -c)$.

Proposition 6.2. *For all $p, q \in \mathbb{Z}$, $\operatorname{rank}^{p,q}(Y) = \operatorname{rank}^{p,q}(Z)$.*

Proof. We will prove the proposition by establishing that $d_c(Y) = d_c(Z)$ for all $c \in \mathbb{Z}$. First note that this is easy for $c < 0$. In this case we know by direct computation that $\operatorname{rank}^{p,p-c}(Z) = 0$ for all $p \in \mathbb{Z}$ (Corollary 3.5). So $\sum_p \operatorname{rank}^{p,p-c}(Z) = 0$, which implies by Lemma 6.1(b) that $\sum_p \operatorname{rank}^{p,p-c}(Y) = 0$. Since the ranks are all non-negative, this means $\operatorname{rank}^{p,p-c}(Y) = 0$ for all $p \in \mathbb{Z}$.

Next we proceed by induction on c. Assume $c \geq 0$ and that $d_n(Y) = d_n(Z)$ for all $n < c$. Let $p \in \mathbb{Z}$, and consider the inequality

$$\sum_{q \geq p-c} \operatorname{rank}^{p,q}(Y) \geq \sum_{q \geq p-c} \operatorname{rank}^{p,q}(Z)$$

from Lemma 6.1(c). By induction we know that $\operatorname{rank}^{p,q}(Y) = \operatorname{rank}^{p,q}(Z)$ for $q > p - c$, and so we conclude that

$$\operatorname{rank}^{p,p-c}(Y) \geq \operatorname{rank}^{p,p-c}(Z). \tag{6.3}$$

This holds for all $p \in \mathbb{Z}$. But we also know, by Lemma 6.1(b), that

$$\sum_p \operatorname{rank}^{p,p-c}(Y) = \sum_p \operatorname{rank}^{p,p-c}(Z). \tag{6.4}$$

Equations (6.3) and (6.4) can both be true only if $\operatorname{rank}^{p,p-c}(Y) = \operatorname{rank}^{p,p-c}(Z)$ for all $p \in \mathbb{Z}$. That is, $d_c(Y) = d_c(Z)$. □

Remark 6.5. It is worth remarking that Proposition 6.2 has solved one of our main questions. It completely identifies the weights of the free generators for $H^{*,*}(\mathrm{Gr}_k(\mathcal{U}); \mathbb{Z}/2)$ by showing that they agree with the ranks of the generators for the combinatorially-computable ring of invariants $\mathcal{I}nv_k$.

Next we give the

Proof of Lemma 6.1. The first equality in (a) is by Kronholm's Theorem (Theorem 5.1). For the second equality observe that $\sum_q \operatorname{rank}^{p,q}(X)$ is just the number of classical Schubert cells of dimension p inside $\mathrm{Gr}_k(\mathbb{R}^\infty)$. This is the same as the number of partitions of p into at most k pieces, which is the same as $\sum_q \operatorname{part}_{p,\leq k}[q]$. The latter equals $\sum_q \operatorname{rank}^{p,q}(Z)$ by Proposition 3.2.

For (b), the first equality is again by Kronholm's Theorem. The equality $\sum_p \operatorname{rank}^{p,p+c}(X) = \sum_p \operatorname{rank}^{p,p+c}(Z)$ follows from the combinatorial identities in

Proposition 3.2 and Proposition 4.2; to see why, it is best to think pictorially. Proposition 4.2 says that the rank chart for X is concentrated along $k+1$ rays of slope $\frac{1}{2}$, emanating from certain points on the $y=x$ line. Proposition 3.2 says that the rank chart of Z also consists of $k+1$ rays of slope $\frac{1}{2}$—containing the same entries as the ones in X—but which emanate from different points on the $y=x$ line (in other words, the order of the rays in the two charts are both permuted and shifted along the $y=x$ line). From this it follows at once that the diagonals of the two rank charts contain the same entries, only permuted. In particular, the sum of the entries is the same in the two situations.

For (c), it will suffice to prove that $\sum_{c\leq q}\operatorname{rank}^{p,c}(Y) \leq \sum_{c\leq q}\operatorname{rank}^{p,c}(Z)$, since the second inequality follows from this one together with part (a). Consider the diagram

$$\begin{tikzcd}
Y^{p,q} \arrow[rr] \arrow[dr] \arrow[dd, "\Phi"'] & & Z^{p,q} \arrow[dr] \arrow[dd, "\Phi"] & \\
& {[Y/\rho Y]^{p,q}} \arrow[rr] \arrow[dl, dotted] & & {[Z/\rho Z]^{p,q}} \arrow[dl, dotted] \\
H^p(\mathrm{Gr}_k(\mathcal{U});\mathbb{Z}/2) \arrow[rr, "\cong"'] & & \Big[\big[H^*(\mathbb{R}P^\infty;\mathbb{Z}/2)^{\otimes k}\big]^{\Sigma_k}\Big]^p &
\end{tikzcd}$$

where the dotted arrows exist because ρ is sent to zero by Φ. The bottom horizontal map is an isomorphism by the classical theory, and the map $[Y/\rho Y]^{p,q} \to H^p(\mathrm{Gr}_k(\mathcal{U});\mathbb{Z}/2)$ is an injection by Proposition 5.4(b). It follows that $[Y/\rho Y]^{p,q} \to [Z/\rho Z]^{p,q}$ is an injection. However, it is easy to see that if J is a free $\mathbb{M}_2$-module then $\dim_{\mathbb{Z}/2}[J/\rho J]^{p,q} = \sum_{c\leq q}\operatorname{rank}^{p,c}(J)$. Applying this to Y and Z, we have completed the proof. $\square$

At this point we have only proven that Y and Z are free $\mathbb{M}_2$-modules with the same bigraded rank functions. But we have a specific map $Y \to Z$, and our goal is to prove that it is an isomorphism. Since both $Y^{p,q}$ and $Z^{p,q}$ are finite-dimensional over $\mathbb{Z}/2$ for every $p,q\in\mathbb{Z}$, it will be sufficient to prove that $Y\to Z$ is surjective. We begin with the following observation:

Lemma 6.6. *The map $Y/\rho Y \to Z/\rho Z$ is an isomorphism.*

Proof. As in the proof of Lemma 6.1(c), we know that $[Y/\rho Y]^{p,q} \to [Z/\rho Z]^{p,q}$ is an injection. We also know that the $\mathbb{Z}/2$-dimensions of these two spaces are $\sum_{c\leq q}\operatorname{rank}^{p,c}(Y)$ and $\sum_{c\leq q}\operatorname{rank}^{p,c}(Z)$, which are equal by Proposition 6.2. This proves the lemma. $\square$

The desired result will now follow from the purely algebraic lemma below:

Lemma 6.7. *Let M and N be free $\mathbb{M}_2$-modules, and let $f\colon M\to N$ be a map such that $M/\rho M \to N/\rho N$ is an isomorphism. Assume that*

(i) $\operatorname{rank}^{p,q}(M) = \operatorname{rank}^{p,q}(N)$ *for all $p,q\in\mathbb{Z}$.*
(ii) $\dim_{\mathbb{Z}/2} M^{p,q}$ *is finite for all $p,q\in\mathbb{Z}$.*
(iii) There exists an $r\in\mathbb{Z}$ such that $d_c(M)=d_c(N)=0$ for all $c<r$.
(iv) There exists a number u such that $\operatorname{rank}^{p,q}(M)=0$ *for all $p<u$.*

Then f is an isomorphism.

Proof. Pick a free basis $\{e_\alpha\}$ for N consisting of homogeneous elements. For each $s \in \mathbb{Z}$ let $N_s \subseteq N$ be the submodule spanned by all e_α for which the bidegree (p_α, q_α) satisfies $p_\alpha - q_\alpha \leq s$ (these are the basis elements on all diagonals 'higher than' the $p - q = s$ diagonal). Note that $N_s = 0$ for $s < r$, where r is the number specified in condition (iii).

Condition (iv) readily implies the following fact: for every $p, q \in \mathbb{Z}$ there exists an $m \geq 0$ such that $[\rho^m N]^{p,q} \subseteq N_{p-q-1}$. In other words, every element of $N^{p,q}$ that is a multiple of ρ^m is in the $\mathbb{M}_2$-span of basis elements from higher diagonals. (One need only take $m = p - u + 1$ here, where u is from condition (iv)).

We will prove by induction that each N_s is contained in the image of f. We know this for $s < r$ since in that case $N_s = 0$. So assume $s \in \mathbb{Z}$ and $N_{s-1} \subseteq \operatorname{im} f$. Since $M/\rho M \to N/\rho N$ is an isomorphism it follows that $N = (\operatorname{im} f) + \rho N$. Substituting this equation for N into itself, we then find

$$N = (\operatorname{im} f) + \rho N = (\operatorname{im} f) + \rho^2 N = (\operatorname{im} f) + \rho^3 N = \cdots$$

So $N = (\operatorname{im} f) + \rho^n N$ for any $n \geq 1$.

Now let e_α be a basis element lying in N_s, of bidegree (p, q) (so that $p - q \leq s$). We may assume $p - q = s$, for otherwise $e_\alpha \in N_{s-1}$ and so is in the image of f by induction. By the second paragraph of this proof, there exists $m \geq 1$ such that $[\rho^m N]^{p,q} \subseteq N_{s-1}$. But then we have

$$N^{p,q} = (\operatorname{im} f)^{p,q} + [\rho^m N]^{p,q} \subseteq (\operatorname{im} f) + N_{s-1} = \operatorname{im} f$$

where the last equality uses our inductive assumption that $N_{s-1} \subseteq \operatorname{im} f$. We have therefore shown that $e_\alpha \in \operatorname{im} f$, and since this holds for every basis element we have $N_s \subseteq \operatorname{im} f$.

At this point we have shown that f is surjective. The finiteness condition (ii) then implies that f is indeed an isomorphism. □

We now restate Theorem 1.2 from the introduction, and tie up its proof:

Theorem 6.8. *The map* $\eta^* \colon H^{*,*}(\mathrm{Gr}_k(\mathcal{U}); \mathbb{Z}/2) \to \left[H^{*,*}(\mathrm{Gr}_1(\mathcal{U}); \mathbb{Z}/2)^{\otimes k}\right]^{\Sigma_k}$ *is an isomorphism of bigraded rings.*

Proof. This is the map $Y \to Z$ considered throughout this section. Both Y and Z are free $\mathbb{M}_2$-modules that satisfy hypotheses (ii)–(iv) of Lemma 6.7. Proposition 6.2 verifies condition (i) of that lemma. The result therefore follows by that lemma together with Lemma 6.6. □

7. The multiplicative structure of the ring of invariants

At this point in the paper we have proven that our map

$$\eta^* \colon H^{*,*}(\mathrm{Gr}_k(\mathcal{U})) \to [H^{*,*}(\mathrm{Gr}_1(\mathcal{U}))^{\otimes k}]^{\Sigma_k}$$

is an isomorphism of rings. We also have a combinatorial description of the bigraded rank function—that is, we understand the additive structure of these rings, or their structure as $\mathbb{M}_2$-module. In this section we further investigate the ring of invariants, concentrating on the multiplicative structure. Recall that this ring of invariants is denoted $\mathcal{I}nv_k$ for short.

7.1. **First observations.** Recall from Section 3 that we use the notation $w_i = [a_1 \dots a_i]$ and $c_i = [b_1 \dots b_i]$. These are the ith elementary symmetric functions in the a's and b's, respectively. More generally, define the invariant element $wc_{i,j}$ by

$$wc_{i,j} = [a_1 \dots a_i b_{i+1} \cdots b_{i+j}].$$

Note that this only makes sense when $i + j \leq k$. Note also that $wc_{i,0} = w_i$ and $wc_{0,j} = c_j$. Finally, let us observe that the bidegree of $wc_{i,j}$ is $(i, i) + (2j, j) = (i + 2j, i + j)$.

As a warm-up for our investigation let us consider some basic relations. The easiest relation one encounters is

$$w_1^2 = (a_1 + \cdots + a_n)^2 = a_1^2 + \cdots + a_n^2 = (\rho a_1 + \tau b_1) + \cdots + (\rho a_n + \tau b_n) = \rho w_1 + \tau c_1.$$

Analogously,

$$\begin{aligned} w_2^2 = [a_1 a_2]^2 &= \sum_{i<j} (\rho a_i + \tau b_i)(\rho a_j + \tau b_j) \\ &= \rho^2 \sum_{i<j} a_i a_j + \rho\tau \sum_{i<j} (a_i b_j + a_j b_i) + \tau^2 \sum_{i<j} b_i b_j \\ &= \rho^2 w_2 + \rho\tau \sum_{i \neq j} a_i b_j + \tau^2 c_2 \\ &= \rho^2 w_2 + \rho\tau \cdot wc_{1,1} + \tau^2 c_2. \end{aligned}$$

More generally we have the following (the proof is left as an exercise):

Proposition 7.2. *In $\mathcal{I}nv_k$ there is the relation*

$$w_j^2 = \tau^j c_j + \tau^{j-1} \rho wc_{1,j-1} + \tau^{j-2} \rho^2 wc_{2,j-2} + \cdots + \tau \rho^{j-1} wc_{j-1,1} + \rho^j w_j$$

for any $j \leq k$.

Next let us consider the products $w_1 w_i$ for various i. For instance, $w_1 w_2 = (a_1 + \ldots + a_n)(a_1 a_2 + \ldots + a_{n-1} a_n)$. When we distribute, we will get terms that look like $a_1^2 a_2$, and also terms that look like $a_1 a_2 a_3$. Note that the former term only appears once, whereas the latter appears $\binom{3}{2} = 3$ times (which is equivalent to once, since we are in characteristic two). So we can write

$$w_1 w_2 = [a_1^2 a_2] + [a_1 a_2 a_3] = [a_1^2 a_2] + w_3.$$

We must be careful when identifying $[a_1^2 a_2]$. We have

$$[a_1^2 a_2] = \sum_{i \neq j} a_i^2 a_j = \sum_{i \neq j} (\rho a_i + \tau b_i) a_j = \rho \sum_{i \neq j} a_i a_j + \tau \sum_{i \neq j} b_i a_j = 0 + \tau wc_{1,1}.$$

Note that $\sum_{i \neq j} a_i a_j = 0$ only because we are in characteristic 2.

As one more example, let's compute $w_1 w_3$. We are looking at the product $(a_1 + \ldots + a_n)(a_1 a_2 a_3 + \ldots)$, and so we have terms that look like $a_1^2 a_2 a_3$ and $a_1 a_2 a_3 a_4$. The former occurs exactly once, the latter $\binom{4}{1} = 4$ times (equivalent to zero times, mod 2). So

$$w_1 w_3 = [a_1^2 a_2 a_3] = \sum_{\substack{j<k \\ i \notin \{j,k\}}} a_i^2 a_j a_k = \sum_{\substack{j<k \\ i \notin \{j,k\}}} (\rho a_i + \tau b_i) a_j a_k = \rho [a_1 a_2 a_3] + \tau [a_1 a_2 b_3].$$

The last equality takes a little thought: we must ask ourselves how many times a typical term $a_1a_2a_3$ appears in the sum $\sum_{\substack{j<k \\ i\notin\{j,k\}}} a_ia_ja_k$, and the answer is that it occurs exactly three times (equivalent to once, mod 2).

The following proposition is easily proven by the above techniques:

Proposition 7.3. *In $\mathcal{I}nv_k$ one has the relations $w_1w_{2i} = \tau \cdot wc_{2i-1,1} + w_{2i+1}$ and $w_1w_{2i+1} = \tau wc_{2i,1} + \rho w_{2i+1}$.*

Note that the first relation from Proposition 7.3 shows that w_{2i+1} is decomposable in $\mathcal{I}nv_k$. Without much trouble this generalizes to the following result. Compare [M, Remark 3.4].

Proposition 7.4. *Let $1 \le j \le k$. Then w_j is indecomposable in $\mathcal{I}nv_k$ if and only if j is a power of 2.*

Proof. If j is not a power of 2, then $\binom{j}{i}$ is odd for some i. Consider the product

$$w_iw_{j-i} = (a_1a_2\dots a_i + \cdots)(a_1a_2\dots a_{j-i} + \cdots).$$

When we distribute, we have some terms which contain one or more squares—these belong to the ideal (ρ,τ) of $\mathcal{I}nv_k$ because of the relation $a_i^2 = \rho a_i + \tau b_i$. A typical term which doesn't involve squares is $a_1a_2\dots a_j$, and this appears exactly $\binom{j}{i}$ times in the big sum. So we can write

$$w_iw_{j-i} \in (\rho,\tau) + w_j.$$

But the elements of (ρ,τ) are by nature decomposable, and so we have that w_j is decomposable.

For the proof that w_{2^r} is indecomposable, we map our ring $\mathcal{I}nv_k$ to a simpler ring where it is easier to prove this. Specifically, consider the map

$$\mathbb{M}_2[a_1,\dots,a_k,b_1,\dots,b_k]/(a_i^2 = \rho a_i + \tau b_i) \to \Lambda_{\mathbb{Z}/2}(a_1,\dots,a_k)$$

that sends ρ, τ, and all the b_i's to zero. Upon taking invariants this gives a map

$$\mathcal{I}nv_k \to \Lambda_{\mathbb{Z}/2}(a_1,\dots,a_k)^{\Sigma_k}$$

that sends each w_i to the ith symmetric function σ_i in the a_j's. But in $\Lambda_{\mathbb{Z}/2}(a_1,\dots,a_k)^{\Sigma_k}$ it is well-known that σ_i is indecomposable when i is a power of 2 (see Proposition A.2 below for a proof). $\square$

7.5. **Generalized Stiefel-Whitney classes.** One of the difficulties in studying the ring $\mathcal{I}nv_k$ is that there does not seem to be a clear choice of which algebra generators to use; every choice seems to have drawbacks. The *wc* classes defined above represent one extreme: they result from making the indices on the a's and b's disjoint. The opposite approach is to make the indices overlap as much as possible, and that leads to the following definition:

$$w_i^{(e)} = [a_1\dots a_ib_1^e\dots b_i^e].$$

Note that this defines an element of $\mathcal{I}nv_k$ for $1 \le i \le k$ and $0 \le e$. It has bidegree $(i,i) + ei(2,1) = (i(2e+1), i(e+1))$, and in terms of our rank charts it lies on the same line of slope $\frac{1}{2}$ as the class w_i. Notice that $w_i^{(0)} = w_i$.

7.6. **Indecomposables.** Let $\epsilon\colon \mathbb{M}_2[a_1,\dots,a_k,b_1,\dots,b_k]/(a_i^2=\rho a_i+\tau b_i)\to\mathbb{M}_2$ be defined by sending each a_i and b_i to zero. We will also write ϵ for the restriction to $\mathcal{I}nv_k$. Let $I_k\subseteq\mathcal{I}nv_k$ be the kernel of $\epsilon\colon\mathcal{I}nv_k\to\mathbb{M}_2$. Then I_k/I_k^2 is a bigraded $\mathbb{M}_2$-module that is readily checked to be free; it is called the **module of indecomposables** for $\mathcal{I}nv_k$ relative to $\mathbb{M}_2$. Our goal is to determine the bigraded rank function for I_k/I_k^2, as well as a basis. In other words, we aim to write down a complete set of representatives for the indecomposables in $\mathcal{I}nv_k$.

Remark 7.7. It is worth stressing that we have set things up so that 'indecomposable' means *relative to* $\mathbb{M}_2$. The elements ρ, τ, and θ are of course indecomposable elements of $\mathcal{I}nv_k$ in the 'absolute' sense, but we do not want to keep track of them. They will not be reflected in the rank function for I_k/I_k^2, which by definition counts the number of basis elements over $\mathbb{M}_2$.

The main result is as follows:

Theorem 7.8.

(a) *The indecomposables of $\mathcal{I}nv_k$ are represented by the classes $c_1,\dots,c_k$ together with the classes $w_{2^i}^{(e)}$ for $1\le 2^i\le k$ and $0\le e\le\frac{k}{2^i}-1$. That is to say, these classes give a free basis for I_k/I_k^2 as an $\mathbb{M}_2$-module.*

(b) *The number of indecomposables for $\mathcal{I}nv_k$ is*

$$3k-(\#\text{ of ones in the binary expansion of }k).$$

(c) *For $1\le 2^i\le k$ and $0\le e\le\frac{k}{2^i}-1$ the classes $wc_{2^i,e2^i}$ and $w_{2^i}^{(e)}$ are equivalent modulo decomposables.*

(d) *For $p,q\in\mathbb{Z}$, $\operatorname{rank}^{p,q}(I_k/I_k^2)=0$ unless $0\le p$ and $0\le q\le k$.*

(e) *When p is odd and $0\le p$,*

$$\operatorname{rank}^{p,q}(I_k/I_k^2)=\begin{cases}1 & \text{if } q=\frac{p+1}{2},\\ 0 & \text{otherwise.}\end{cases}$$

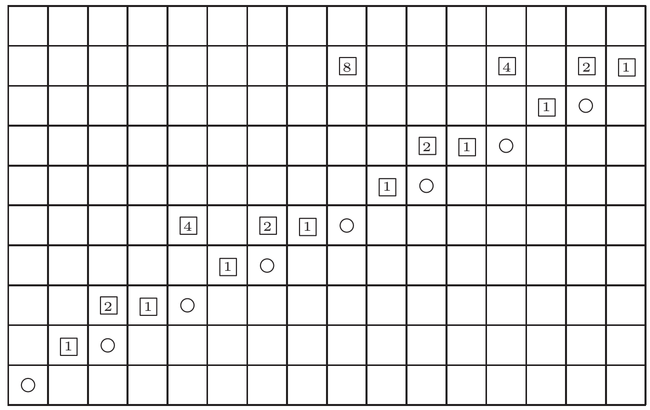
The unique indecomposable in topological dimension p is represented by $w_1^{(\frac{p-1}{2})}$, or equivalently by $wc_{1,\frac{p-1}{2}}$.

(f) *When p is even and positive, write $p=2^i(2e+1)$. Then*

$$\operatorname{rank}^{p,q}(I_k/I_k^2)=\begin{cases}1 & \text{if } q=\frac{p}{2}\text{ or }q=\frac{p}{2}+2^{i-1},\\ 0 & \text{otherwise.}\end{cases}$$

When $q=\frac{p}{2}$, the unique indecomposable in bidegree (p,q) is represented by the Chern class c_q. When $q=\frac{p}{2}+2^{i-1}$ the unique indecomposable is represented by $w_{2^i}^{(e)}$, or equivalently by $wc_{2^i,e\cdot 2^i}$.

To paraphrase the above theorem, in the limiting case $k\to\infty$ there is one indecomposable in every odd topological dimension and two indecomposables in every even topological dimension. The following chart shows the exact bidegrees, with different symbols for different types of indecomposables:



The circles represent the Chern classes, whereas the squares represent the w-classes. The squares with an i inside represent $w_i^{(e)}$ classes, for $0 \leq e$. The pattern here is that the $w_i^{(e)}$ classes start in bidegree (i, i) and then proceed up along the line of slope $\frac{1}{2}$, occuring every i steps along this line, where "step" means a $(2, 1)$ move.

For $\mathcal{I}nv_k$ one cuts the chart off and only takes the classes in weights less than or equal to k. For example, in $\mathcal{I}nv_5$ there will be the following indecomposables (given in order of increasing topological degree):

$$w_1,\ c_1,\ w_2,\ w_1^{(1)},\ c_2,\ w_4,\ w_1^{(2)},\ c_3,\ w_2^{(1)},\ w_1^{(3)},\ c_4,\ w_1^{(4)},\ c_5.$$

Note that Theorem 7.8(b) predicts the number of indecomposables to be $15-2 = 13$, which agrees with the above list.

Our goal is now to prove Theorem 7.8, proceeding by a series of reductions.

Proof of Theorem 7.8. The complexities of $\mathbb{M}_2$ are irrelevant to the considerations at hand. To this end, define $R_k = \mathbb{Z}/2[\tau, \rho, a_1, \ldots, a_k, b_1, \ldots, b_k]/(a_i^2 = \rho a_i + \tau b_i)$. Let $S_k = R_k^{\Sigma_k}$, where the Σ_k-action permutes the a_i's and b_i's but fixes ρ and τ. Let $\epsilon\colon R_k \to \mathbb{Z}/2[\tau, \rho]$ be the map that sends a_i and b_i all to zero, for $1 \leq i \leq k$. Let J_k be the augmentation ideal of S_k, defined as

$$J_k = \ker(S_k \to R_k \xrightarrow{\epsilon} \mathbb{Z}/2[\tau, \rho]).$$

It is easy to see that $\mathcal{I}nv_k \cong S_k \otimes_{\mathbb{Z}/2[\tau,\rho]} \mathbb{M}_2$ and $I_k/I_k^2 \cong (J_k/J_k^2) \otimes_{\mathbb{Z}/2[\tau,\rho]} \mathbb{M}_2$. So the bigraded rank function for J_k/J_k^2 over $\mathbb{Z}/2[\tau, \rho]$ coincides with the bigraded rank function for I_k/I_k^2 over $\mathbb{M}_2$. It will therefore suffice for us to prove the theorem in the former case.

A free basis for J_k/J_k^2 over $\mathbb{Z}/2[\tau, \rho]$ is the same as a vector space basis for $J_k/[J_k^2 + (\rho, \tau)J_k]$ over $\mathbb{Z}/2$. This is the form in which we will study the problem.

Let $\tilde{R}_k = \mathbb{Z}/2[a_1, \ldots, a_k, b_1, \ldots, b_k]/(a_i^2)$ with the evident Σ_k-action, and let $\tilde{S}_k = \tilde{R}_k^{\Sigma_k}$. Consider the diagram

$$\begin{tikzcd} J_k \arrow[r, tail] \arrow[d] & S_k \arrow[r, "\epsilon"] \arrow[d] & \mathbb{Z}/2[\tau, \rho] \arrow[d] \\ \tilde{J}_k \arrow[r, tail] & \tilde{S}_k \arrow[r, "\tilde{\epsilon}"] & \mathbb{Z}/2 \end{tikzcd}$$

where the vertical maps send ρ and τ to zero, and $\tilde{J}_k$ is the kernel of $\tilde{\epsilon}$. It is easy to see that $S_k \to \tilde{S}_k$ is surjective: a $\mathbb{Z}/2$-basis for the target is given by the orbit sums $[m]$ where m is a monomial in the a's and b's, and such an orbit sum lifts into S_k. The same argument shows that $J_k \to \tilde{J}_k$ is surjective. We in fact have a surjection

$$J_k/[J_k^2 + (\rho,\tau)J_k] \twoheadrightarrow \tilde{J}_k/\tilde{J}_k^2,$$

and it is easy to see that this is actually an isomorphism.

We have therefore reduced our problem to understanding the module of indecomposables $\tilde{J}_k/\tilde{J}_k^2$ for the ring $\tilde{S}_k$. This is a fairly routine algebra problem; we give a full treatment in Appendix A for lack of a suitable reference. See Theorem A.1 for the classification of the indecomposables, proving parts (a) and (b). The third statement in Lemma A.6 proves part (c), and parts (d)–(f) are really just restatements of (a) and (c). □

7.9. **Relations.** In general it seems that writing down a complete set of relations for $\mathcal{I}nv_k$ is not practical or useful. See the cases of $k = 2$ and $k = 3$ described in the next section. The relations tend to be numerous and also fairly complicated. One general remark worth making is that there will always be a relation for the square of a $w_i^{(e)}$ class. The square of $[a_1 \dots a_i b_1^e \dots b_i^e]$ will be $[a_1^2 \dots a_i^2 b_1^{2e} \dots b_i^{2e}]$, and each a_j^2 decomposes as $\rho a_j + \tau b_j$. For example,

$$\left[w_1^{(e)}\right]^2 = [a_1^2 b_1^{2e}] = \rho[a_1 b_1^{2e}] + \tau[b_1^{2e+1}] = \rho w_1^{(2e)} + \tau[b_1^{2e+1}]. \tag{7.10}$$

To express this in terms of indecomposables we need to write the power sum $[b_1^{2e+1}]$ as a polynomial in the elementary symmetric functions, via the mod 2 Newton polynomials. This already produces an expression with lots of terms. If $2e > k-1$ then $w_1^{(2e)}$ is not an indecomposable and we also need to rewrite that term. This can be handled via the following result:

Lemma 7.11. *In $\mathcal{I}nv_k$ one has the relation*

$$w_1^{(e)} = w_1^{(e-1)} c_1 + w_1^{(e-2)} c_2 + \cdots + w_1^{(e-k)} c_k$$

for any $e \geq k$.

Proof. This follows from the identities

$$\begin{aligned}
&[a_1 b_1^e] = [a_1 b_1^{e-1}] \cdot [b_1] + [a_1 b_1^{e-1} b_2] \\
&[a_1 b_1^{e-1} b_2] = [a_1 b_1^{e-2}] \cdot [b_1 b_2] + [a_1 b_1^{e-2} b_2 b_3] \\
&\vdots
\end{aligned}$$

We stop when the right-hand term is $[a_1 b_1^{e-(k-1)} b_2 \dots b_k]$, since in this case the monomial $b_1 \cdots b_k$ is a common factor to all the summands in the Σ_k-orbit and can be taken out:

$$[a_1 b_1^{e-(k-1)} b_2 \dots b_k] = [a_1 b_1^{e-k}] \cdot [b_1 \cdots b_k] = w_1^{(e-k)} \cdot c_k.$$

Substituting each identity into the previous one leads to the desired relation. □

Let us work through one example. In $\mathcal{I}nv_3$ there is the indecomposable $w_1^{(2)}$, and according to our above analysis its square is

$$\left[w_1^{(2)}\right]^2 = \rho w_1^{(4)} + \tau[b_1^5] = \rho w_1^{(4)} + \tau[c_1^5 + c_1 c_2^2 + c_1^2 c_3 + c_1^3 c_2 + c_2 c_3]. \tag{7.12}$$

The latter expression comes from working out the appropriate Newton polynomial. For the $w_1^{(4)}$ term we have

$$w_1^{(4)} = w_1^{(3)}c_1 + w_1^{(2)}c_2 + w_1^{(1)}c_3 = \big[w_1^{(2)}c_1 + w_1^{(1)}c_2 + w_1c_3\big]c_1 + w_1^{(2)}c_2 + w_1^{(1)}c_3$$

by two applications of Lemma 7.11. Our final relation is

$$\Big[w_1^{(2)}\Big]^2 = \rho\big[w_1^{(2)}(c_1^2+c_2)+w_1^{(1)}(c_1c_2+c_3)+w_1c_1c_3\big]+\tau[c_1^5+c_1c_2^2+c_1^2c_3+c_1^3c_2+c_2c_3].$$

This gives a fair indication of the level of awkwardness to this approach.

7.13. **The stable case.** The ring of invariants $\mathcal{I}nv_k$ will typically require many relations beyond just those for the squares on the w-classes—see the examples in Section 8. However, things become simpler in the stable case $k \to \infty$. We describe this next.

Recall that $T_k = \mathbb{M}_2[a_1, \ldots, a_k, b_1, \ldots, b_k]/(a_i^2 = \rho a_i + \tau b_i)$. The map $T_{k+1} \to T_k$ that sends a_{k+1} and b_{k+1} to 0 induces a surjection $\mathcal{I}nv_{k+1} \to \mathcal{I}nv_k$ which is an isomorphism in topological degrees less than $k+1$ (the latter is immediate from looking at the standard free bases over $\mathbb{M}_2$). Write $\mathcal{I}nv_\infty$ for the inverse limit of

$$\cdots \twoheadrightarrow \mathcal{I}nv_3 \twoheadrightarrow \mathcal{I}nv_2 \twoheadrightarrow \mathcal{I}nv_1$$

From Theorem 7.8 it follows that the indecomposables of this ring are the classes c_j for $1 \leq j$ and the classes $w_{2^i}^{(e)}$ for $0 \leq i$ and $0 \leq e$.

Proposition 7.14. *There exist a collection of polynomials $R_{i,e}$ such that $\mathcal{I}nv_\infty$ is the quotient of $\mathbb{M}_2[c_j, w_i^{(e)} \,|\, i,j,e \in \mathbb{Z}_{\geq 0}]$ by the relations*

$$\Big[w_i^{(e)}\Big]^2 = R_{i,e}.$$

Remark 7.15. Unfortunately the polynomials $R_{i,e}$ seem cumbersome to work out in general. We saw in (7.12) that $R_{1,e} = \rho w_1^{(2e)} + \tau[N_{2e+1}(c_1, \ldots))]$ where N_{2e+1} is the mod 2 Newton polynomial for writing the $(2e+1)$-power sum as a polynomial in the elementary symmetric functions. The polynomial $R_{2,e}$ is more unpleasant; it has the form

$$R_{2,e} = \rho^2 w_2^{(2e)} + \rho\tau\Big[w_1^{(4e+1)} + w_1^{(2e)}N_{2e+1}(c_1, \ldots)\Big] + \tau^2[b_1^{2e+1}b_2^{2e+1}]$$

where the expression $[b_1^{2e+1}b_2^{2e+1}]$ must be replaced by a certain complicated, Newton-like polynomial in the Chern classes.

Proof of Proposition 7.14. We let $R_{i,e}$ be the polynomials constructed as in Section 7.9—it is clear enough that they exist, it is just not clear how to write down their coefficients in a reasonable way. Consider the surjection

$$\mathbb{M}_2[c_j, w_i^{(e)} \,|\, i,j,e \in \mathbb{Z}_{\geq 0}]/(R_{i,e}) \twoheadrightarrow \mathcal{I}nv_\infty.$$

We claim that the bigraded Poincaré series for these two algebras are identical, and from this it immediately follows that the map is an isomorphism. Both the domain and target are free $\mathbb{M}_2$-modules, so it suffices to instead look at the bigraded rank functions.

The domain has a free $\mathbb{M}_2$-basis consisting of monomials in the variables c_j and $w_i^{(e)}$ that are square-free in the w-classes. So the bigraded rank function is the same as for the algebra

$$\Lambda(w_i^{(e)} \,|\, 0 \leq i, 0 \leq e) \otimes \mathbb{F}_2[c_1, c_2, \ldots].$$

Likewise, the bigraded rank function for $\mathcal{I}nv_\infty$ is the same as the Poincaré series for the algebra L_∞ from Appendix A (L_∞ is just the quotient of $\mathcal{I}nv_\infty$ obtained by killing ρ and τ). But Theorem A.1(c) gives the isomorphism of graded rings $L_\infty \cong \Lambda(w_i^{(e)} \,|\, 0 \leq i, 0 \leq e) \otimes \mathbb{F}_2[c_1, c_2, \ldots]$, so this completes the proof. $\square$

8. Examples

Our purpose in this section is to take a close look at $H^{*,*}(\mathrm{Gr}_2(\mathfrak{U}); \mathbb{Z}/2)$ and $H^{*,*}(\mathrm{Gr}_3(\mathfrak{U}); \mathbb{Z}/2)$, to demonstrate our general results. We also make some remarks about $H^{*,*}(\mathrm{Gr}_4(\mathfrak{U}); \mathbb{Z}/2)$.

Write $\mathbb{M}_2[\underline{c}] \subseteq H^{*,*}(\mathrm{Gr}_k(\mathfrak{U}); \mathbb{Z}/2)$ for the $\mathbb{M}_2$-subalgebra generated by the c_i's. We have seen that the rank chart for the cohomology ring breaks up naturally into lines of slope $\frac{1}{2}$, and it will be convenient to consider a corresponding decomposition at the level of algebra. To this end, let $F_i \subseteq H^{*,*}(\mathrm{Gr}_k(\mathfrak{U}); \mathbb{Z}/2)$ be the $\mathbb{M}_2$-submodule spanned by the elements of our standard basis having degrees (p, q) for $0 \leq 2q - p \leq i$. Note that $F_0 = \mathbb{M}_2[\underline{c}]$, and in general F_i is an $\mathbb{M}_2[\underline{c}]$-module. Let $Q_i = F_i/F_{i-1}$, and call this module the "i-line". It is a free $\mathbb{M}_2$-module, and the ranks correspond to the ranks of $H^{*,*}(\mathrm{Gr}_k(\mathfrak{U}); \mathbb{Z}/2)$ occuring along the line of slope $\frac{1}{2}$ that passes through (i, i). The 0-line is simply $\mathbb{M}_2[\underline{c}]$. The duality given by Corollary 3.4 says that the ranks along the i-line and the $(k - i)$-line are the same, for every i.

We take the perspective that the 0-line is completely understood, as this is just the polynomial ring over $\mathbb{M}_2$ on the classes $c_1, c_2, \ldots, c_k$. In some sense we then also understand the k-line, by duality. Our next observation is that we can also understand the 1-line (and therefore the $(k - 1)$-line along with it).

Lemma 8.1. *Let $X = H^{*,*}(\mathrm{Gr}_k(\mathfrak{U}); \mathbb{Z}/2)$. Then*

$$\mathrm{rank}^{2p+1,p+1}(X) = \mathrm{rank}^{2p,p}(X) + \mathrm{rank}^{2p-2,p-1}(X) + \cdots + \mathrm{rank}^{2p-2(k-1),p-(k-1)}(X)$$

for any $p \in \mathbb{Z}$.

Proof. We change this into a statement about partitions, using Proposition 3.2. The claim is that

$$\mathrm{part}_{2p+1,\leq k}[1] = \sum_{i=0}^{k-1} \mathrm{part}_{2p-2i,\leq k}[0].$$

We sketch a bijective proof of this. Regard a partition with at most k pieces as a partition having exactly k pieces, but where some pieces are 0. Given a partition of $2p$ into k pieces that are all even, make a partition of $2p + 1$ by adding 1 to the smallest piece. Given a partition of $2p - 1$ into k pieces that are all even, make a partition of $2p + 1$ by adding 3 to the second smallest piece. And so on: given an element of $\mathrm{part}_{2p-2i,\leq k}[0]$, make a partition of $2p + 1$ by adding $2i + 1$ to the ith smallest piece. We leave it to the reader to check that this does indeed give the desired bijection. $\square$

Proposition 8.2. *The 1-line Q_1 is a free $\mathbb{M}_2[\underline{c}]$-module generated by the classes $w_1^{(e)}$ for $0 \leq e \leq k - 1$.*

Proof. We have the evident map

$$\mathbb{M}_2[\underline{c}]\langle w_1, w_1^{(1)}, \ldots, w_1^{(k-1)} \rangle \to Q_1. \tag{8.3}$$

Theorem 7.8 says that X is generated as an $\mathbb{M}_2$-algebra by products of elements c_i and $w_j^{(e)}$. The only such products that can lie on the 1-line are products of c_i's with $w_1^{(e)}$'s. This shows that the map in (8.3) is surjective. But Lemma 8.1 shows that the ranks of the domain and target of (8.3) coincide, hence the map must be an isomorphism. $\square$

In the cohomology of $\mathrm{Gr}_2(\mathcal{U})$ we only have the 0-line, 1-line, and 2-line, and the outer two are dual—so we basically understand everything. In $\mathrm{Gr}_3(\mathcal{U})$ we have the 0-line/3-line and the 1-line/2-line, and again we understand everything. This is why these two cases are fairly easy. When we get to $\mathrm{Gr}_4(\mathcal{U})$ things become more complicated.

Let us now look in detail at $\mathrm{Gr}_2(\mathcal{U})$. The rank calculations can be done by counting partitions using Proposition 3.2, and this is very easy. One finds

$$\mathrm{rank}^{2p,p} = \mathrm{rank}^{2p+2,p+2} = \begin{cases} \frac{p}{2}+1 & \text{if } p \text{ is even,} \\ \frac{p+1}{2} & \text{if } p \text{ is odd,} \end{cases}$$

and

$$\mathrm{rank}^{2p+1,p+1} = p+1.$$

By Theorem 7.8 the indecomposables are the following elements:

$$c_1,\ c_2,\ w_1,\ w_1^{(1)},\ w_2.$$

The 1-line is a free $\mathbb{M}_2[\underline{c}]$-module generated by w_1 and $w_1^{(1)}$, and the rank calculations suggest that the 2-line is the free $\mathbb{M}_2[\underline{c}]$-module generated by w_2. So we guess that the three classes w_1, $w_1^{(1)}$ and w_2 span the cohomology as an $\mathbb{M}_2[\underline{c}]$-module. If this is true, there will be relations specifying the products of any two of the w-classes. A little work shows that

$$\begin{gathered} w_1^2 = \rho w_1 + \tau c_1, \qquad w_2^2 = \rho^2 w_2 + \rho\tau\big(w_1 c_1 + w_1^{(1)}\big) + \tau^2 c_2 \\ \Big[w_1^{(1)}\Big]^2 = \rho\big(w_1^{(1)} c_1 + w_1 c_2\big) + \tau(c_1^3 + c_1 c_2) \end{gathered}$$

and also that

$$\begin{aligned} w_1 w_2 &= \rho w_2 + \tau\big(w_1 c_1 + w_1^{(1)}\big) \\ w_1 w_1^{(1)} &= \rho w_1^{(1)} + \tau c_1^2 + w_2 c_1 \\ w_2 w_1^{(1)} &= \rho w_2 c_1 + \tau(w_1 c_1^2 + w_1^{(1)} c_1 + w_1 c_2). \end{aligned}$$

We have separated the relations into two classes: the relations for the squares of the w-classes will always be present, but the relations amongst square-free monomials in the w-classes depend very much on the value of k.

Once these relations have been verified, we have a surjective algebra map

$$\mathbb{M}_2[c_1, c_2, w_1, w_1^{(1)}, w_2]/(R) \twoheadrightarrow H^{*,*}(\mathrm{Gr}_2(\mathcal{U}); \mathbb{Z}/2)$$

where R is the above list of relations. As an $\mathbb{M}_2[\underline{c}]$-module the domain is free with generators 1, w_1, $w_1^{(1)}$, and w_2, and our rank calculations then show that the Poincaré series for the domain and target agree. So the above map must be an isomorphism.

It remains to verify the relations listed above. The ones for the squares of w_1 and $w_1^{(1)}$ follow readily from (7.10) and Lemma 7.11. For w_2^2 we write

$$\begin{aligned}[a_1a_2]^2 = [a_1^2a_2^2] = [(\rho a_1 + \tau b_1)(\rho a_2 + \tau b_2)] &= \rho^2[a_1a_2] + \rho\tau[a_1b_2] + \tau^2[b_1^2] \\ &= \rho^2 w_2 + \rho\tau\big([a_1][b_1] + [a_1b_1]\big) + \tau^2 c_1^2.\end{aligned}$$

Of the remaining three relations, we leave the first two to the reader and only verify the last:

$$\begin{aligned}[a_1a_2]\cdot[a_1b_1] = [a_1^2a_2b_1] = \rho[a_1a_2b_1] + \tau[a_1b_2^2] &= \rho[a_1a_2][b_1] + \tau\big([a_1][b_1^2] + [a_1b_1^2]\big) \\ &= \rho w_2 c_1 + \tau\big(w_1c_1^2 + w_1^{(2)}\big).\end{aligned}$$

Now use Lemma 7.11 to decompose $w_1^{(2)}$.

Next let us look at the cohomology of $\mathrm{Gr}_3(\mathcal{U})$. The indecomposables are

$$c_1,\ c_2,\ c_3,\ w_1,\ w_1^{(1)},\ w_1^{(2)},\ w_2,$$

and the 1-line is generated over $\mathbb{M}_2[\underline{c}]$ by w_1, $w_1^{(1)}$, and $w_1^{(2)}$. The evident elements of interest on the 2-line are

$$w_2,\ w_1\cdot w_1^{(1)},\ w_1\cdot w_1^{(2)}.$$

Duality between the 1-line and 2-line suggests that we will have three generators as an $\mathbb{M}_2[\underline{c}]$-module, and since these are the only candidates there is not much choice for what can happen. Finally, we expect by duality that the 3-line is the free $\mathbb{M}_2[\underline{c}]$-module generated by w_1w_2. This gives a conjectural description of the cohomology as a module over $\mathbb{M}_2[\underline{c}]$, which we will soon see is correct.

The guess suggests that we should have relations for the products $w_2\cdot w_1^{(1)}$, $w_2\cdot w_1^{(2)}$, and $w_1^{(1)}\cdot w_1^{(2)}$—as well as for the squares of all the w-classes, of course. Some tedious work in the ring of invariants reveals the following relations:

$$\begin{aligned}
&w_1^2 = \rho w_1 + \tau c_1\\
&w_2^2 = \rho^2 w_2 + \rho\tau(w_1c_1 + w_1^{(1)}) + \tau^2 c_2\\
&\Big[w_1^{(1)}\Big]^2 = \rho w_1^{(2)} + \tau\big[c_1^3 + c_1c_2 + c_3\big]\\
&\Big[w_1^{(2)}\Big]^2 = \rho\big[w_1^{(2)}c_1^2 + w_1^{(1)}c_1c_2 + w_1c_1c_3 + w_1^{(2)}c_2 + w_1^{(1)}c_3\big]\\
&\qquad\qquad\qquad + \tau\big[c_1^5 + c_1^3c_2 + c_1^2c_3 + c_1c_2^2 + c_2c_3\big]
\end{aligned}$$

and

$$\begin{aligned}
w_2\cdot w_1^{(1)} &= w_1w_2c_1 + (\rho,\tau)\\
w_2\cdot w_1^{(2)} &= w_1w_2c_1^2 + (\rho,\tau)\\
w_1^{(1)}\cdot w_1^{(2)} &= w_2c_3 + w_2c_1c_2 + w_1w_1^{(1)}c_1^2 + w_1w_1^{(2)}c_1 + (\rho,\tau).
\end{aligned}$$

In the last three cases we are being somewhat lazy and not writing out the entire relations, which are long and complicated. We have instead written "(ρ,τ)" as shorthand for all terms belonging to the ideal (ρ,τ).

Once again, we have now produced a surjective map

$$\mathbb{M}_2[c_1, c_2, w_1, w_1^{(1)}, w_1^{(2)}, w_2]/(R) \twoheadrightarrow H^{*,*}(\mathrm{Gr}_3(\mathcal{U}); \mathbb{Z}/2)$$

where R is the set of relations above. The domain is a free $\mathbb{M}_2[\underline{c}]$-module generated by $1, w_1, w_1^{(1)}, w_1^{(2)}, w_2, w_1 \cdot w_1^{(1)}, w_1 \cdot w_1^{(2)}, w_1w_2$. One can analyze the Poincaré series for the cohomology ring in terms of partitions, and a little work shows that the Poincaré series of the domain and codomain agree. It follows that the above map is an isomorphism of algebras.

Finally, we make some brief remarks about $\mathrm{Gr}_4(\mathcal{U})$. The indecomposables are

$$c_1,\ c_2,\ c_3,\ c_4,\ w_1,\ w_1^{(1)},\ w_1^{(2)},\ w_1^{(3)},\ w_2,\ w_2^{(1)},\ w_4.$$

The 0-line is the polynomial algebra $\mathbb{M}_2[c_1, c_2, c_3, c_4]$, and the 1-line is the free $\mathbb{M}_2[\underline{c}]$-module with basis elements $w_1^{(e)}$ for $0 \leq e \leq 3$. The monomials on the 2-line are

$$w_2,\ w_1w_1^{(1)},\ w_1w_1^{(2)},\ w_1w_1^{(3)},\ w_2^{(1)},\ w_1^{(1)} \cdot w_1^{(2)},\ w_1^{(1)} \cdot w_1^{(3)},\ w_1^{(2)} \cdot w_1^{(3)},$$

with bidegrees

$$(2,2),\ (4,3),\ (6,4),\ (8,5),\ (6,4),\ (8,5),\ (10,6),\ (12,7).$$

The ranks along the 0-line constitute the sequence $S = (1, 1, 2, 3, 5, 6, 9, 11, \ldots)$. If the 2-line were free on the above generators then the ranks along the 2-line would be $P = (1, 2, 5, 9, 15, 23, 34, 47, \ldots)$. This sequence is obtained by adding up eight copies of S with appropriate shifts, according to the topological degrees of the eight monomials listed above: $P = \sum_i (\Sigma^{(p_i-2)/2} S)$ where p_i is the topological degree of the ith element of the list (we subtract two because our 2-line "starts" at w_2). That is,

$$P = S + \Sigma S + \Sigma^2 S + \Sigma^2 S + \Sigma^3 S + \Sigma^3 S + \Sigma^4 S + \Sigma^5 S.$$

Computations with partitions reveals that the actual rank sequence for the 2-line is $(1, 2, 5, 8, 14, 20, 30, 40, 55, \ldots)$. Playing around with the numerology shows that removing a $\Sigma^3 S$ and the $\Sigma^5 S$ from P seems to yield the correct answer; this leads to the guess that there is a dependence relation amongst the two elements $w_1w_1^{(3)}$ and $w_1^{(1)}w_1^{(2)}$, and also that there should be a relation for $w_1^{(2)}w_1^{(3)}$. One can indeed find such relations, although the process is time-consuming. In the first case the relation is

$$w_1w_1^{(3)} + w_1^{(1)}w_1^{(2)} + w_1w_1^{(2)}c_1 + w_2^{(1)}c_1 + w_2c_3 + w_1w_1^{(1)}c_2 + (\rho, \tau) = 0$$

where the last term represents an element in the ideal (ρ, τ) that we have not gone to the trouble of determining.

It again appears that the cohomology of $\mathrm{Gr}_4(\mathcal{U})$ is free as a module over $\mathbb{M}_2[\underline{c}]$, with basis consisting of certain products of w-classes. However, there does not seem to be a canonical choice for the basis: e.g., there is no preferred choice among $w_1w_1^{(3)}$ and $w_1^{(1)}w_1^{(2)}$ for which to include. Also, the relations are getting truly horrendous. We choose to stop here.

9. Connections to motivic phenomena

Let F be a field, not of characteristic 2. For an algebraic variety X over F, a **quadratic bundle** over X is an algebraic vector bundle $E \to X$ together with a pairing $E \otimes_F E \to \mathcal{O}_X$ that is symmetric and restricts to nondegenerate bilinear forms on each fiber. For reasons that we will not explain here, such bundles play the role in motivic homotopy theory that ordinary real vector bundles play in classical algebraic topology (see Remark 9.6 below for a bit more information). It is natural, therefore, to try to understand characteristic classes for quadratic bundles with values in mod 2 motivic cohomology.

One can make a guess at a classifying space for quadratic vector bundles, as follows (this is known to be a true classifying space if one works stably, by a result of [ST]). Equip the affine space $\mathbb{A}^{2n}$ with the quadratic form

$$q_{2n}(x_1, y_1, x_2, y_2, \dots, x_n, y_n) = x_1y_1 + \cdots + x_ny_n$$

and equip $\mathbb{A}^{2n+1}$ with the quadratic form

$$q_{2n+1}(x_1, y_1, x_2, y_2, \dots, x_n, y_n, z) = x_1y_1 + \cdots + x_ny_n + z^2.$$

These are called the **split** quadratic forms. Note that we have $\mathbb{A}^{2n}$ sitting inside $\mathbb{A}^{2n+1}$ as the $z = 0$ subspace, which exhibits q_{2n} as the restriction of q_{2n+1}. We will also regard $\mathbb{A}^{2n+1}$ as sitting inside $\mathbb{A}^{2n+2}$ as the subspace $x_{n+1} = y_{n+1}$, which exhibits q_{2n+1} as the restriction of q_{2n+2}.

From now on we will write $(\mathbb{A}^N, q)$ for either $(\mathbb{A}^{2n}, q_{2n})$ or $(\mathbb{A}^{2n+1}, q_{2n+1})$. Note that we have a series of inclusions

$$(\mathbb{A}^1, q) \hookrightarrow (\mathbb{A}^2, q) \hookrightarrow (\mathbb{A}^3, q) \hookrightarrow \cdots$$

Define the **orthogonal Grassmannian** $\mathrm{OGr}_k(\mathbb{A}^N)$ to be the Zariski open subspace of $\mathrm{Gr}_k(\mathbb{A}^N)$ consisting of the k-planes where q restricts to a nondegenerate form. Taking the colimit over N gives a motivic space $\mathrm{OGr}_k(\mathbb{A}^\infty)$, in the sense of [MV]. It is an interesting (and unsolved) problem to compute the motivic cohomology groups of this space.

Now restrict to the case $F = \mathbb{R}$. From an $\mathbb{R}$-variety X we can consider the set $X(\mathbb{C})$ of $\mathbb{C}$-valued points, regarded as a topological space via the analytic topology. This space has an evident $\mathbb{Z}/2$-action given by complex conjugation, and the assignment $X \mapsto X(\mathbb{C})$ extends to a map of homotopy theories from motivic homotopy theory over $\mathbb{R}$ to $\mathbb{Z}/2$-equivariant homotopy theory. Our goal in this section is only to note the following result:

Theorem 9.1. *There is an equivariant weak homotopy equivalence*

$$[\mathrm{OGr}_k(\mathbb{A}^N)](\mathbb{C}) \simeq \mathrm{Gr}_k(\mathcal{U}^N).$$

(Recall that $\mathcal{U}^N$ denotes the first N summands of the infinite $\mathbb{Z}/2$-representation $\mathcal{U} = \mathbb{R} \oplus \mathbb{R}_- \oplus \mathbb{R} \oplus \mathbb{R}_- \oplus \cdots$).

The above theorem shows that the main problem considered in this paper is indeed the $\mathbb{Z}/2$-equivariant analog of the problem of motivic characteristic classes for quadratic bundles.

We will need a few preliminary results before giving the proof of the theorem. To generalize our previous definition somewhat, if V is any vector space with a quadratic form q then we write $\mathrm{OGr}_k(V)$ for the subspace of $\mathrm{Gr}_k(V)$ consisting of k-planes $W \subseteq V$ such that $q|_W$ is nondegenerate. Sometimes V will be a real vector space and sometimes V will be a complex vector space, and in the latter case our orthogonal Grassmannian will be the space of complex k-planes on which q is nondegenerate. Usually the intent will be clear from context.

Assume V is real and the form q is positive-definite. This form extends to give a complex quadratic form on $V \otimes_{\mathbb{R}} \mathbb{C}$ that we will also call q. The complexification map $c\colon \mathrm{Gr}_k(V) \to \mathrm{Gr}_k(V \otimes_{\mathbb{R}} \mathbb{C})$ has its image contained in $\mathrm{OGr}_k(V \otimes_{\mathbb{R}} \mathbb{C})$. To see this, just observe that if $U \subseteq V$ is any k-plane then there is a basis for U with respect to which q looks like the sum-of-squares form. Extending this basis to

$U \otimes_{\mathbb{R}} \mathbb{C}$ shows that q is nondegenerate here. Similar remarks apply to show that the direct-sum map in part (b) of the following result takes its image in OGr rather than just Gr.

Note that the following result takes place in the non-equivariant setting:

Proposition 9.2. *Let V be a real vector space with a positive-definite quadratic form q.*

(a) The complexification map $\mathrm{Gr}_k(V) \to \mathrm{OGr}_k^{\mathbb{C}}(V \otimes_{\mathbb{R}} \mathbb{C})$ is a weak homotopy equivalence;

(b) Let V' be another real vector space with positive-definite form q'. Then the direct-sum map $\coprod_{a+b=k} \mathrm{Gr}_a(V) \times \mathrm{Gr}_b(V') \to \mathrm{OGr}_k(V \oplus V', q \oplus (-q'))$ is a weak homotopy equivalence.

Proof. Without loss of generality we may assume that $V = \mathbb{R}^n$ and q is the sum-of-squares form. Recall that the symmetry group of this form is the Lie group $O_n = \{A \in M_{n\times n}(\mathbb{R}) \,|\, AA^T = I\}$. The symmetry group for the sum-of-squares form over $\mathbb{C}$ is $O_n(\mathbb{C}) = \{A \in M_{n\times n}(\mathbb{C}) \,|\, AA^T = I\}$. Recall that O_n is a maximal compact subgroup inside of $O_n(\mathbb{C})$; it is therefore known by the Iwasawa decomposition that the inclusion $O_n \hookrightarrow O_n(\mathbb{C})$ is a homotopy equivalence (see [CSM, Theorem 8.1 of Segal's lecture] or [H, Chapter XV, Theorem 3.1]).

The space $\mathrm{Gr}_k(\mathbb{R}^n)$ is homeomorphic to $O_n/[O_k \times O_{n-k}]$. Likewise, $\mathrm{OGr}_k(\mathbb{C}^n)$ is homeomorphic to $O_n(\mathbb{C})/[O_k(\mathbb{C}) \times O_{n-k}(\mathbb{C})]$. The map in part (a) is the evident comparison map between these homogeneous spaces. Consider the two fiber bundles

$$\begin{tikzcd}
O_k \times O_{n-k} \arrow[r] \arrow[d] & O_n \arrow[r] \arrow[d] & O_n/[O_k \times O_{n-k}] \arrow[d] \\
O_k(\mathbb{C}) \times O_{n-k}(\mathbb{C}) \arrow[r] & O_n(\mathbb{C}) \arrow[r] & O_n(\mathbb{C})/[O_k(\mathbb{C}) \times O_{n-k}(\mathbb{C})]
\end{tikzcd}$$

(written horizontally). The left and middle vertical maps are weak equivalences, therefore the right map is as well. This proves (a).

For (b) recall that a nondegenerate quadratic form on an n-dimensional real vector space is classified by its signature: the pair of integers (a, b) such that $a+b = n$, representing the number of positive and negative entries in any diagonalization of the form. Let $O(a, b)$ be the symmetry group for the quadratic form of signature (a, b). This Lie group contains $O(a) \times O(b)$ in the evident way, and it is known that this is a maximal compact subgroup. Consequently, the inclusion $O(a) \times O(b) \hookrightarrow O(a, b)$ is a weak homotopy equivalence by the Iwasawa decomposition.

We can assume $V = \mathbb{R}^n$ and $V' = \mathbb{R}^{n'}$, with both q and q' being the sum-of-squares form. The group $O(n, n')$ acts on $\mathrm{OGr}_k(V \oplus V')$ in the evident way. It is easy to see that the action decomposes the orthogonal Grassmannian into a disjoint union of orbits, one for every possible signature (a, b) with $a + b = k$. The path component corresponding to such a signature is the homogeneous space

$$O(n, n')/[O(a, b) \times O(n - a, n' - b)].$$

The map in part (b) coincides with the disjoint union of the evident maps

$$\begin{array}{c} \Big[O(n)/[O(a)\times O(n-a)]\Big]\times\Big[O(n')/[O(b)\times O(n'-b)]\Big] \\ \Big\downarrow\cong \\ [O(n)\times O(n')]/\Big[[O(a)\times O(n-a)]\times[O(b)\times O(n'-b)]\Big] \\ \Big\downarrow \\ O(n,n')/[O(a,b)\times O(n-a,n'-b)]. \end{array}$$

At this point one proceeds exactly in the proof of part (a): write down a map between two fiber bundles, where two of the three maps are already known to be weak homotopy equivalences. □

We next move into the equivariant setting. By an **orthogonal representation** of $\mathbb{Z}/2$ we mean a pair (V,q) where V is a real vector space and $q\colon V\to\mathbb{R}$ is a positive-definite quadratic form on V such that $q(\sigma x)=q(x)$ for all $x\in V$. The main examples for us will be where $V=\mathbb{R}^n$, q is the standard sum-of-squares form, and $\mathbb{Z}/2$ acts on V by changing signs on some subset of the standard basis elements.

Let $V_{\mathbb{C}}=V\otimes_{\mathbb{R}}\mathbb{C}$, with the $\mathbb{Z}/2$ action induced by that on V. The complexification map $\mathrm{Gr}_k(V)\to\mathrm{OGr}_k(V_{\mathbb{C}})$ sending $U\subseteq V$ to $U_{\mathbb{C}}\subseteq V_{\mathbb{C}}$ is clearly equivariant, where the $\mathbb{Z}/2$-actions on domain and codomain are induced by those on V and $V_{\mathbb{C}}$.

Corollary 9.3. *For any orthogonal representation V of $\mathbb{Z}/2$, the map of $\mathbb{Z}/2$-spaces $\mathrm{Gr}_k(V)\to\mathrm{OGr}_k(V_{\mathbb{C}})$ is an equivariant weak equivalence.*

Proof. Taking Proposition 9.2(a) under consideration, it suffices to prove that the induced map of fixed sets is a weak equivalence. Let $V^{\mathbb{Z}/2}$ and $V^{-\mathbb{Z}/2}$ denote the $+1$ and -1 eigenspaces for the involution on V. These are orthogonal with respect to the inner product on V. A subspace $U\subseteq V$ is fixed under the $\mathbb{Z}/2$ action if and only if U equals the direct sum $(U\cap V^{\mathbb{Z}/2})\oplus(U\cap V^{-\mathbb{Z}/2})$. From this we get a homeomorphism

$$\mathrm{Gr}_k(V)^{\mathbb{Z}/2}\cong\coprod_i \mathrm{Gr}_i(V^{\mathbb{Z}/2})\times\mathrm{Gr}_{k-i}(V^{-\mathbb{Z}/2}),$$

which sends $U\subseteq V$ to the pair $(U\cap V^{\mathbb{Z}/2},U\cap V^{-\mathbb{Z}/2})$. In the same way, one obtains a homeomorphism

$$\mathrm{OGr}_k(V_{\mathbb{C}})^{\mathbb{Z}/2}\cong\coprod_i \mathrm{OGr}_i(V_{\mathbb{C}}^{\mathbb{Z}/2})\times\mathrm{OGr}_{k-i}(V_{\mathbb{C}}^{-\mathbb{Z}/2}).$$

Since the inclusions $\mathrm{Gr}_i(V^{\mathbb{Z}/2})\hookrightarrow\mathrm{OGr}_i(V_{\mathbb{C}}^{\mathbb{Z}/2})$ and $\mathrm{Gr}_j(V^{-\mathbb{Z}/2})\hookrightarrow\mathrm{OGr}_j(V_{\mathbb{C}}^{-\mathbb{Z}/2})$ are (non-equivariant) weak equivalences by Proposition 9.2(a), this completes the proof. □

The above corollary has been included for completeness, but it actually does not give us what we need. The $\mathbb{Z}/2$ action on $V\otimes_{\mathbb{R}}\mathbb{C}$ is complex linear, whereas we will find that we actually need to consider conjugate linear actions. We do this next.

Let W be a complex vector space with a nondegenerate quadratic form q. Let $\sigma\colon W\to W$ be a conjugate-linear map such that $\sigma^2=1$. That is, $\sigma(zx)=\bar{z}\sigma(x)$ for every $z\in\mathbb{C}$ and $x\in W$. Also assume that $q(\sigma x)=\overline{q(x)}$ for every $x\in W$.

The space $\mathrm{OGr}_k(W)$ then has a $\mathbb{Z}/2$-action induced by σ: if $J \subseteq W$ is a complex subspace such that $q|_J$ is nondegenerate, then $\sigma(J)$ is another complex subspace on which q restricts to be nondegenerate. Our next task is to analyze the fixed space $\mathrm{OGr}_k(W)^{\mathbb{Z}/2}$.

Remark 9.4. Let (V, q) be an orthogonal representation for $\mathbb{Z}/2$, and let W be the vector space $V \otimes_{\mathbb{R}} \mathbb{C}$ with the action given by $\sigma(v \otimes z) = \sigma(v) \otimes \bar{z}$. Then (W, q) satisfies the conditions of the above paragraph. In this case we will use the notation $W = V \otimes_{\mathbb{R}} \overline{\mathbb{C}}$. The bar over the $\mathbb{C}$ just reminds us that $\mathbb{Z}/2$ acts on that factor by conjugation.

Returning to the case of a general W, note that as a real vector space W decomposes as $W^{\mathbb{Z}/2} \oplus W^{-\mathbb{Z}/2}$, where the summands are the subspaces on which σ acts as the identity and as multiplication by -1. Moreover, multiplication by i maps $W^{\mathbb{Z}/2}$ isomorphically onto $W^{-\mathbb{Z}/2}$. Finally, one easily checks that q is real-valued on both $W^{\mathbb{Z}/2}$ and $W^{-\mathbb{Z}/2}$.

If $J \subseteq W$ is any complex subspace that is fixed by σ then we have the decomposition $J = (J \cap W^{\mathbb{Z}/2}) \oplus (J \cap W^{-\mathbb{Z}/2})$, and multiplication by i interchanges the two summands. In this way we get a map

$$\mathrm{OGr}_k(W, q)^{\mathbb{Z}/2} \longrightarrow \mathrm{Gr}_k(W^{\mathbb{Z}/2}), \qquad J \mapsto J \cap W^{\mathbb{Z}/2}$$

and the image is readily checked to land in $\mathrm{OGr}_k(W^{\mathbb{Z}/2}, q)$. Conversely, if $M \subseteq W^{\mathbb{Z}/2}$ is any k-dimensional real subspace such that $q|_M$ is nondegenerate then $M \oplus iM \subseteq W$ is a k-dimensional complex subspace with the same property. So we also get a map $\mathrm{OGr}_k(W^{\mathbb{Z}/2}) \to \mathrm{OGr}_k(W, q)^{\mathbb{Z}/2}$. It is routine to check that these maps are inverse isomorphisms. Thus, we have proven the following:

Proposition 9.5. *In the above setting, there is a homeomorphism* $\mathrm{OGr}_k(W, q)^{\mathbb{Z}/2} \cong \mathrm{OGr}_k(W^{\mathbb{Z}/2}, q)$.

We are now ready to prove the main theorem of this section:

Proof of Theorem 9.1. Write q_{sp} for the split quadratic form on $\mathbb{C}^N$, and q_{ss} for the sum-of-squares quadratic form on $\mathbb{C}^N$. The theorem concerns the space $\mathrm{OGr}_k(\mathbb{C}^N, q_{sp})$ where the $\mathbb{Z}/2$-action is induced by complex conjugation. Let $x_1, y_1, x_2, y_2, \ldots$ denote our standard coordinates on $\mathbb{C}^N$, with the convention that when N is odd then the last of the y_j's is just zero. By changing coordinates we can change q_{sp} into q_{ss}. Precisely, define a map $\phi\colon \mathbb{C}^N \to \mathbb{C}^N$ by

$$\phi(x_1, y_1, x_2, y_2, \ldots) = (x_1 + iy_1, x_1 - iy_1, x_2 + iy_2, x_2 - iy_2, \ldots).$$

Then we have $q_{sp}(\phi(v)) = q_{ss}(v)$ for any $v \in \mathbb{C}^N$. This gives us an identification of non-equivariant spaces $\mathrm{OGr}_k(\mathbb{C}^N, q_{sp}) \cong \mathrm{OGr}_k(\mathbb{C}^N, q_{ss})$. To extend this to an equivariant identification, note that if the target of ϕ is given the conjugation action then the domain of ϕ gets the action that both conjugates all coordinates AND changes the signs of the y-coordinates. In terms of previously-established notation, this is the equivariant homeomorphism

$$\mathrm{OGr}_k(\mathbb{C}^N, q_{sp}) \cong \mathrm{OGr}_k(\mathcal{U}^N \otimes \overline{\mathbb{C}}, q_{ss}).$$

Consider the complexification map

$$c\colon \mathrm{Gr}_k(\mathcal{U}^N) \to \mathrm{OGr}_k(\mathcal{U}^N \otimes \overline{\mathbb{C}}, q_{ss}).$$

We have seen in Proposition 9.2(a) that this is a non-equivariant weak equivalence. To analyze what is happening on fixed sets, let $W = \mathcal{U}^N \otimes \overline{\mathbb{C}}$. Note that $W^{\mathbb{Z}/2} = \{(r_1, ir_2, r_3, ir_4, \ldots, (i)r_N) \,|\, r_1, \ldots, r_N \in \mathbb{R}\}$, where the last coordinate has the i in front when N is even. Note as well that we can decompose $W^{\mathbb{Z}/2} = W_+^{\mathbb{Z}/2} \oplus W_-^{\mathbb{Z}/2}$ where

$$W_+^{\mathbb{Z}/2} = \{(r_1, 0, r_3, 0, \ldots) \,|\, r_i \in \mathbb{R}\}, \qquad W_-^{\mathbb{Z}/2} = \{(0, ir_2, 0, ir_4, \ldots) \,|\, r_i \in \mathbb{R}\}.$$

The form q_{ss} is positive definite on the first summand and negative definite on the second.

Let $\mathcal{U}_+^N$ and $\mathcal{U}_-^N$ be the subspaces spanned by the odd- and even-numbered basis elements, respectively. So $\mathcal{U}_+^N = (\mathcal{U}^N)^{\mathbb{Z}/2}$ and $\mathcal{U}_-^N = (\mathcal{U}^N)^{-\mathbb{Z}/2}$. Note the following maps:

$$\begin{tikzcd}
\mathrm{Gr}_k(\mathcal{U}^N)^{\mathbb{Z}/2} \arrow[r, "c"] \arrow[d, equal] & \mathrm{OGr}_k(W, q_{ss})^{\mathbb{Z}/2} & \mathrm{OGr}_k(W^{\mathbb{Z}/2}, q_{ss}) \arrow[l, "\cong"'] \\
\coprod_{a+b=k} \mathrm{Gr}_a(\mathcal{U}_+^N) \times \mathrm{Gr}_b(\mathcal{U}_-^N) \arrow[rr, dotted] & & \coprod_{a+b=k} \mathrm{Gr}_a(W_+^{\mathbb{Z}/2}) \times \mathrm{Gr}_b(W_-^{\mathbb{Z}/2}) \arrow[u, "\sim"']
\end{tikzcd}$$

The map on the right is the evident one, and is a weak homotopy equivalence by Proposition 9.2(b). The dotted map is the obvious homeomorphism, obtained by identifying $\mathcal{U}_+^N = W_+^{\mathbb{Z}/2}$, $i \cdot \mathcal{U}_-^N = W_-^{\mathbb{Z}/2}$. One readily checks that the diagram commutes, and this verifies that c induces a weak homotopy equivalence of fixed sets. Thus, c is an equivariant weak equivalence. $\square$

Remark 9.6. The non-equivariant part of Theorem 9.1 (equivalently, Proposition 9.2(a)) gives the homotopy equivalence of spaces $\mathrm{OGr}_k(\mathbb{C}^N) \simeq \mathrm{Gr}_k(\mathbb{R}^N)$. This is a classical result: for example, see [A1, remarks in Section 1.5] and [S, discussion of real Grassmannians throughout Chapter 5]. Notice that this gives some corroboration to the idea that quadratic bundles are the motivic analogs of real vector bundles.

Appendix A. The deRham ring of invariants in characteristic two

Let $K_n = \Lambda(a_1, \ldots, a_n) \otimes \mathbb{F}_2[b_1, \ldots, b_n]$, and let Σ_n act on K_n by simultaneous permutation of the a_i's and b_j's. Let $L_n = K_n^{\Sigma_n}$. We call L_n the "deRham ring of invariants". Note that there is an augmentation $\epsilon\colon K_n \to \mathbb{F}_2$ sending all the a_i's and b_j's to zero, and this restricts to an augmentation of L_n. Let $I \subseteq L_n$ be the augmentation ideal. Our first aim in this section is to give a vector space basis for the module of indecomposables I/I^2. Said differently, we give a minimal set of generators for the ring L_n.

Note that K_{n+1} maps to K_n by sending a_{n+1} and b_{n+1} to zero, and this homomorphism induces an algebra map $L_{n+1} \to L_n$. That is, if $f(a, b)$ is a polynomial expression in the a's and b's that is invariant under the Σ_{n+1}-action, then eliminating all monomials with an a_{n+1} or b_{n+1} produces a polynomial that is invariant under Σ_n. From this description it is also clear that $L_{n+1} \to L_n$ is surjective: if $f(a, b)$ is a Σ_n-invariant then one can make a Σ_{n+1}-invariant by adding on appropriate monomial terms that all have a_{n+1} or b_{n+1}.

Let L_∞ be the inverse limit of the system

$$\cdots \longrightarrow L_3 \longrightarrow L_2 \longrightarrow L_1.$$

The second goal of this section is to give a complete description of the ring L_∞.

These results are presumably well-known amongst algebraists. See Section 7 of [R] for the case of $\mathbb{F}_2[a_1, \ldots, a_k, b_1, \ldots, b_k]$, which can be used to deduce some of our results. See also [GSS, Section 2] for some related work. Rather than use the machinery of [R], however, we have chosen to give a 'low-tech' treatment which is perhaps more illuminating for our present purposes.

If $m \in K_n$ is a monomial in the a_i's and b_j's, write $[m]$ for the smallest polynomial that contains m as one of its terms and is invariant under the Σ_n-action. Here 'smallest' is measured in terms of the number of monomial summands. We can also describe $[m]$ as

$$[m] = \sum_{\sigma \in \Sigma_n/H} \sigma.m$$

where H is the stabilizer of m in Σ_n.

Using the above noation, write $\alpha_{i,e} = [a_1 \ldots a_{2^i} b_1^e \ldots b_{2^i}^e]$ for $1 \le 2^i \le n$ and $0 \le e$. Also, write $\sigma_i(a)$ and $\sigma_i(b)$ for the elementary symmetric functions in the a's and b's, respectively. So $\sigma_i(a) = [a_1 \ldots a_i]$, for example.

We can now state the main result:

Theorem A.1.

(a) L_n is minimally generated by the classes $\sigma_i(b)$ for $1 \le i \le n$ together with the classes $\alpha_{i,e}$ for $1 \le 2^i \le n$ and $0 \le e \le \frac{n}{2^i} - 1$. That is to say, these classes give a vector space basis for I/I^2.

(b) The number of indecomposables for L_n is

$$3n - (\#\text{ of ones in the binary expansion for } n).$$

(c) $L_\infty \cong \Lambda\big(\alpha_{i,e} \,|\, 0 \le i, 0 \le e\big) \otimes \mathbb{F}_2[\sigma_1(b), \sigma_2(b), \ldots]$.

The Online Encyclopedia of Integer Sequences [OE] was useful in discovering the formula in part (b).

The proof of this theorem will be given after establishing several lemmas. The first result we give is not directly needed for the proof, but is included for two reasons: it provides some context that helps explain the more complicated theorem above, and we actually need the result in the proof of Proposition 7.4. The result is probably well-known, but we are not aware of a reference.

Proposition A.2. *Let Σ_n act on $\Lambda_{\mathbb{F}_2}(a_1, \ldots, a_n)$ by permutation of indices. Then*

$$\Lambda(a_1, \ldots, a_n)^{\Sigma_n} = \Lambda(\sigma_1, \sigma_2, \sigma_4, \ldots, \sigma_{2^k})/R$$

where k is the largest integer such that $2^k \le n$ and R is the ideal generated by all products $\sigma_{2^{i_1}}\sigma_{2^{i_2}} \cdots \sigma_{2^{i_s}}$ where $2^{i_1} + 2^{i_2} + \cdots + 2^{i_s} > n$.

Proof. It is easy to see that the classes $1, \sigma_1, \ldots, \sigma_n$ form a vector space basis for the ring of invariants over $\mathbb{F}_2$. Put a grading on $\Lambda(a_1, \ldots, a_n)$ by having the degree of each a_i be 1. Then the ring of invariants is also graded; the dimension of each homogeneous piece equals 1 in degrees from 0 through n, and zero in degrees larger than n.

It is also easy to see that $\sigma_i^2 = 0$ for each i, and so we get a map of rings

$$\Lambda(\sigma_1, \dots, \sigma_n)/R \twoheadrightarrow \Lambda(a_1, \dots, a_n)^{\Sigma_n}.$$

The next thing to note is that $\sigma_r \cdot \sigma_s = \binom{r+s}{r}\sigma_{r+s}$. This is an easy computation: distributing the product in $[a_1 \dots a_r]\cdot[a_1 \dots a_s]$ one finds that the products of monomials are all zero if the monomials have any variables in common. The products that are not zero have the form $a_{i_1} \dots a_{i_{r+s}}$, and such a monomial appears exactly $\binom{r+s}{r}$ times.

If r is not a power of 2 then there exists an i such that $\binom{r}{i}$ is odd, which implies that $\sigma_r = \sigma_i \cdot \sigma_{r-i}$. So such classes are decomposable. We therefore have a map

$$\Lambda(\sigma_1, \sigma_2, \sigma_4, \dots, \sigma_{2^k})/R \twoheadrightarrow \Lambda(a_1, \dots, a_n)^{\Sigma_n}.$$

This is a map of graded algebras, and the Poincaré Series for the domain and target are readily checked to coincide. Since the map is a surjection, it must be an isomorphism. □

We next establish a series of lemmas directly dealing with the situation of Theorem A.1. We begin by introducing some notation and terminology. If $I = \{i_1, \dots, i_k\}$ then write a_I for $a_{i_1}a_{i_2}\cdots a_{i_k}$. Likewise, if d_I is a function $I \to \mathbb{Z}_{\geq 0}$ then write $b_I^{d_I}$ for the monomial $b_{i_1}^{d_{i_1}} b_{i_2}^{d_{i_2}} \cdots b_{i_k}^{d_{i_k}}$. If m is a monomial in the a's and b's, then the variables a_i and b_i are said to be **bound** in m if $a_i b_i$ divides m. If a_i divides m but b_i does not, we will say that a_i is **free** in m (and in the opposite situation we'll say that b_i is free). Any monomial may be written uniquely in the form

$$m = a_I b_I^{d_I} a_J b_K^{e_K}$$

where the indices in I represent all the bound variables: so $I \cap J = I \cap K = J \cap K = \emptyset$. Finally, recall that $[m]$ denotes the smallest invariant polynomial containing m as one of its terms.

Lemma A.3. *Let $m = a_I b_I^{d_I} a_J b_K^{e_K}$. Then $[m]$ is decomposable in L_n if any of the following conditions are satisfied:*

(1) $I \neq \emptyset$ and $J \neq \emptyset$ (i.e., some of the a's are bound and some are free).
(2) $J = \emptyset$ and $d_{i_1} \neq d_{i_2}$ for some $i_1, i_2 \in I$.
(3) $J = K = \emptyset$ and $\#I$ is not a power of 2.
(4) $I = K = \emptyset$ and $\#J$ is not a power of 2.

Proof. For (1) first assume that $K = \emptyset$, and consider the product $[a_I b_I^{d_I}] \cdot [a_J]$. Distributing this into sums of products of monomials, such products vanish if I and J intersect. A typical term that remains is $a_I a_J b_I^{d_I}$, and it is clear that this term occurs exactly once. In other words,

$$[a_I b_I^{d_I}] \cdot [a_J] = [a_I a_J b_I^{d^I}].$$

To finish the proof of (1) we do an induction on the size of $\#K$. If $m = a_I b_I^{d_I} a_J b_K^{e_K}$ then consider the product $[a_I b_I^{d_I}] \cdot [a_J b_K^{e_K}]$. Distributing this into sums of products of monomials, we find that

$$[a_I b_I^{d_I}] \cdot [a_J b_K^{e_K}] = [m] + \Big(\text{terms of the form } [a_I b_I^{d'_I} a_J b_{K'}^{e_{K'}}] \text{ where } \#K' < \#K\Big).$$

The latter terms come from products where the indices in I match some of those in K. By induction these latter terms are all decomposable in L_n, so $[m]$ is also decomposable.

For (2), we again first assume that $K = \emptyset$ so that we are looking at $[a_1 \dots a_s b_1^{d_1} \cdots b_s^{d_s}]$. By rearranging the labels we may assume $d_1 \geq d_2 \geq \cdots \geq d_s$. Let r be the smallest index for which $d_r = d_s$, and consider the product

$$[a_1 \dots a_{r-1} b_1^{d_1} \dots b_{r-1}^{d_{r-1}}] \cdot [a_r \dots a_s b_r^f \cdots b_s^f]$$

where $f = d_s$. Once again considering the pairwise product of monomials, all such terms vanish except for ones of the form $a_{i_1} \dots a_{i_s} b_{i_1}^{d_1} \dots b_{i_{r-1}}^{d_{r-1}} b_{i_r}^f \dots b_{i_s}^f$. The fact that f is the smallest degree on the b_i's guarantees that this term appears exactly once in the sum, and hence

$$[a_1 \dots a_{r-1} b_1^{d_1} \dots b_{r-1}^{d_{r-1}}] \cdot [a_r \dots a_s b_r^f \cdots b_s^f] = [a_1 \dots a_s b_1^{d_1} \dots b_s^{d_s}].$$

To complete the proof of (2) we perform an induction on $\#K$. Consider a monomial

$$m = a_I b_I^{d_I} b_K^{e_K} = a_1 \dots a_s b_1^{d_1} \dots b_s^{d_s} b_{s+1}^{e_1} \dots b_{s+k}^{e_k}.$$

Again arrange things so that $d_1 \geq d_2 \geq \cdots \geq d_s$ and let r be the smallest index for which $d_r = d_s$. If we again write $f = d_s$, then one readily checks that

$$[a_1 \dots a_{r-1} b_1^{d_1} \dots b_{r-1}^{d_{r-1}}] \cdot [a_r \dots a_s b_r^f \cdots b_s^f b_{s+1}^{e_1} \dots b_{s+k}^{e_k}] = [m] + \sum [a_I b_I^{d'_I} b_{K'}^{e_{K'}}]$$

where for each term in the sum K' is a proper subset of K. These terms inside the sum correspond to pairs of monomials in the product for which a b_i for $1 \leq i \leq r-1$ matches a b_{s+j} for $1 \leq j \leq k$. However, by induction on $\#K$ each $[a_I b_I^{d'_I} b_{K'}^{e_{K'}}]$ is decomposable, hence $[m]$ is also decomposable.

To prove (3) it suffices (in light of (2)) to show that $[a_1 \dots a_k b_1^e \dots b_k^e]$ is decomposable whenever k is not a power of 2. This assumption guarantees that $\binom{k}{i}$ is odd for some i in the range $1 \leq i \leq k-1$. We claim that

$$[a_1 \dots a_i b_1^e \dots b_i^e] \cdot [a_{i+1} \dots a_k b_{i+1}^e \dots b_k^e] = [a_1 \dots a_k b_1^e \dots b_k^e].$$

To see this, note that all terms in the product vanish except for ones of the form $a_{i_1} \dots a_{i_k} b_{i_1}^e \dots b_{i_k}^e$, and such a term appears exactly $\binom{k}{i}$ times. Use that $\binom{k}{i}$ is odd.

The proof of (4) is the same as for (3), it is really the special case $e = 0$. $\square$

Lemma A.4. *L_n is generated as an algebra by the elements $\sigma_i(b)$ for $1 \leq i \leq n$ together with the classes $[m]$ where $m = a_I b_I^{d_I} a_J$ (that is, where m has no free b's).*

Proof. Let $Q \subseteq L_n$ denote the subalgebra generated by the elements from the statement of the lemma. We will prove that if $m = a_I b_I^{d_I} a_J b_K^{e_K}$ is an arbitrary monomial then $[m]$ is equivalent modulo decomposables to an element of Q. This readily yields the result by an induction on degree.

First consider the case where $I = J = \emptyset$, so that $m = b_K^{e_K}$. Note that $\mathbb{Z}/2[b_1, \dots, b_n] \subseteq L_n$, and we know $\mathbb{Z}/2[b_1, \dots, b_n]^{\Sigma_n}$ is a polynomial algebra on the $\sigma_i(b)$ for $1 \leq i \leq n$. It follows at once that $[m]$ is equivalent modulo decomposables to a multiple of a $\sigma_i(b)$.

The next stage of the proof is done by induction on $\#K$. The base case $K = \emptyset$ is trivial, as such monomials lie in Q by definition. So assume $K \neq \emptyset$ and consider the product $[a_I b_I^{d_I} a_J] \cdot [b_K^{e_K}]$. This product decomposes into a sum $[m] + [m_1] + [m_2] + \cdots$ where each m_i has fewer free b's than m. Therefore $[m]$ is equivalent to $\sum_i [m_i]$ modulo decomposables, and each $[m_i]$ is equivalent to an element of Q by induction. $\square$

Corollary A.5. *L_n is generated as an algebra by the following elements:*

(1) $\sigma_i(b)$ *for* $1 \le i \le n$*;*
(2) $[a_1 \dots a_{2^i} b_1^e \dots b_{2^i}^e]$ *for* $1 \le 2^i \le n$ *and* $e \ge 0$*.*

Proof. Lemma A.4 gives the generators $\sigma_i(b)$ and $[a_I b_I^{d_I} a_J]$. Using Lemma A.3(1) we reduce the second class to all elements $[a_I b_I^{d_I}]$ and $[a_J]$. Finally, Lemma A.3(2,3,4) further reduces the class to the set of elements in the statement of the corollary. □

We need one more lemma before completing the proof of Theorem A.1. For $x, y \in L_n$ let us write $x \equiv y$ to mean x and y are equivalent modulo decomposables (that is, $x - y \in I^2$).

Lemma A.6. *If* $r \ge k$ *and* $n \ge r + k$ *then*

$$[a_1 \dots a_k b_1^e \dots b_k^e b_{k+1} \dots b_{k+r}] \equiv [a_1 \dots a_k b_1^{e+1} \dots b_k^{e+1} b_{k+1} \dots b_r].$$

Consequently, provided $ke \le n$ *one has that*

$$[a_1 \dots a_k b_1^e \dots b_k^e] \equiv [a_1 \dots a_k b_1 \dots b_{ke}].$$

If $k + ke \le n$ *we also have*

$$[a_1 \dots a_k b_1^e \dots b_k^e] \equiv [a_1 \dots a_k b_{k+1} \dots b_{k+ke}].$$

Proof. For the first statement consider the product

$$[a_1 \dots a_k b_1^e \dots b_k^e] \cdot [b_1 \dots b_r].$$

The product contains $[a_1 \dots a_k b_1^e \dots b_k^e b_{k+1} \dots b_{k+r}]$ and $[a_1 \dots a_k b_1^{e+1} \dots b_k^{e+1} b_{k+1} \dots b_r]$, as well as other terms that look like $[a_1 \dots a_k b_1^{d_1} \dots b_k^{d_k} b_{k+1} \dots b_{k+i}]$ in which the d_i's are not all equal. But such terms are all decomposable by Lemma A.3(2).

The second statement follows from the first using an induction:

$$\begin{aligned}
[a_1 \dots a_k b_1 \dots b_{ke}] &\equiv [a_1 \dots a_k b_1^2 \dots b_k^2 b_{k+1} \dots b_{ke-k}] \\
&\equiv [a_1 \dots a_k b_1^3 \dots b_k^3 b_{k+1} \dots b_{ke-2k}] \\
&\equiv \dots \\
&\equiv [a_1 \dots a_k b_1^e \dots b_k^e].
\end{aligned}$$

Finally, for the third statement we consider the product

$$[a_1 \dots a_k] \cdot [b_1 \cdots b_{ke}].$$

This is a sum of terms $[m_i]$ where $[a_1 \dots a_k b_1 \dots b_{ke}]$ appears exactly once, $[a_1 \dots a_k b_{k+1} \dots b_{k+ke}]$ appears exactly once, and all other m_i's have at least one free a and one bound a. But Lemma A.3(1) then tells us that these other m_i's are all decomposable. □

Corollary A.7. *If* $e > \frac{n}{k} - 1$ *then* $[a_1 \dots a_k b_1^e \dots b_k^e]$ *is decomposable in* L_n*.*

Proof. Let $N = ke + k$, which is larger than n by assumption. We begin by considering the element $[a_1 \dots a_k b_1^e \dots b_k^e]$ in L_N. Lemma A.6 gives that

$$[a_1 \dots a_k b_1^e \dots b_k^e] \equiv [a_1 \dots a_k b_{k+1} \dots b_{k+ke}].$$

Now apply the homomorphism $L_N \to L_n$, and note that since $ke+k > n$ the element on the right maps to zero (every monomial term has at least one index that is larger than n). This proves that $[a_1 \dots a_k b_1^e \dots b_k^e]$ is decomposable in L_n. □

At this point we have verified that L_n is generated, as an algebra, by the classes $\sigma_i(b)$ for $1 \leq i \leq n$ together with the classes $[a_1 \dots a_{2^i} b_1^e \dots b_{2^i}^e]$ for $1 \leq 2^i \leq n$ and $0 \leq e \leq \frac{n}{2^i} - 1$. It remains to verify that these classes are a minimal set of algebra generators—or equivalently, that they give a $\mathbb{Z}/2$-basis for I/I^2. The approach will be to first grade the algebras in a convenient way. Then we identify the indecomposables in L_∞, which can be done by a counting argument. Finally, we observe that $L_\infty \to L_n$ is an isomorphism in degrees less than or equal to n, and use this to deduce the desired facts about the indecomposables in L_n.

Grade the algebra $K_n = \Lambda(a_1, \dots, a_n) \otimes \mathbb{F}_2[b_1, \dots, b_n]$ by having the degree of each a_i be 1 and the degree of each b_i be 2. Then L_n inherits a corresponding grading. The invariant element $\sigma_i(b)$ has degree $2i$, whereas the element $\alpha_{i,e} = [a_1 \dots a_{2^i} b_1^e \dots b_{2^i}^e]$ has degree $2^i + 2e \cdot 2^i = 2^i(2e+1)$. Notice that for every positive integer r the set $\{\alpha_{i,e} \,|\, 0 \leq i, 0 \leq e\}$ has exactly one element of degree r.

Proposition A.8. *The map* $\Lambda(\alpha_{i,e} \,|\, 0 \leq i, 0 \leq e) \otimes \mathbb{F}_2[\sigma_i \,|\, i \geq 0] \to L_\infty$ *is an isomorphism.*

Proof. We have already proven in Corollary A.5 that the map is a surjection. The injectivity will be deduced from a counting argument. For convenience, let D denote the domain of the map from the statement of the proposition. Let $S = \mathbb{F}_2[v_1, v_2, \dots]$ where v_i has degree i. We will prove that the Poincaré series for D and L_∞ both coincide with the Poincaré series for S. Since D and L_∞ will therefore have identical Poincaré series, the surjection $D \twoheadrightarrow L_\infty$ must in fact be an isomorphism.

Note that S has a basis over $\mathbb{F}_2$ consisting of monomials

$$v_{i_1} v_{i_2} \cdots v_{i_r} v_1^{2e_1} v_2^{2e_2} \cdots v_k^{2e_k}$$

with each $e_j \geq 0$, where the i_u's are distinct. There is an evident bijection between the elements of this basis and the basis for D consisting of monomials in the $\alpha_{i,e}$'s and σ_i's: we replace each v_{i_r} with the unique $\alpha_{i,e}$ having degree i_r, and we replace each v_i^{2e} with σ_i^e This identifies the Poincaré series for S and D.

Recall that L_∞ has a $\mathbb{Z}/2$-basis consisting of the invariants $[a_{i_1} \dots a_{i_r} b_{j_1}^{e_1} \dots b_{j_s}^{e_s}]$ where there is allowed to be overlap between the i- and j-indices. Say that a monomial is **pure** if it only contains a's and b's of a single index. So b_i^e and $a_i b_i^e$ are pure, but $a_1 a_2 b_1^2$ is not. An arbitrary monomial m can be written uniquely (up to permutation of the factors) as

$$m = m_1 \cdot m_2 \cdots m_t$$

where each m_i is pure and the indices appearing in m_i and m_j are different for every $i \neq j$. For example,

$$a_1 a_2 a_3 a_4 b_1^4 b_2 b_4 b_5^2 = (a_1 b_1^4) \cdot (a_2 b_2) \cdot (a_3) \cdot (a_4 b_4) \cdot (b_5^2). \tag{A.9}$$

For a pure monomial m, let $d(m)$ be its degree and let $\eta(m) = v_{d(m)}$. Finally, for an arbitrary monomial m as above define $\eta(m) = \eta(m_1) \cdots \eta(m_t) = v_{d(1)} \cdot v_{d(2)} \cdots v_{d(t)}$. For example, for the monomial in (A.9) we have $\eta(m) = v_1 v_3^2 v_9 v_{10}$.

Note that if σ is a permutation of the indices then $\eta(\sigma m) = \eta(m)$. One readily checks that the function η gives a bijection between our basis for L_∞ and the standard monomial basis for S; it should be enough to see the inverse in one example, e.g.

$$v_1^3 v_2^2 v_3^2 v_6 v_{10} = \eta([a_1 a_2 a_3 \cdot b_4 b_5 \cdot a_6 b_6 a_7 b_7 \cdot b_8^3 \cdot b_9^5]).$$

Clearly η preserves the homogeneous degrees of the elements, so the Poincaré series for L_∞ and S coincide. This completes our proof. $\square$

Lemma A.10. *The surjections $L_{n+1} \twoheadrightarrow L_n$ and $L_\infty \twoheadrightarrow L_n$ are isomorphisms in degrees less than or equal to n.*

Proof. This is clear from our description of the additive basis for L_n. $\square$

Proof of Theorem A.1. We have already proven (c) in Proposition A.8, so it only remains to prove (a) and (b). For (a) we have proven in Corollaries A.5 and A.7 that L_n is generated by the given classes, so we need only show that those classes are independent modulo I^2. However, all of the classes in question are in degrees less than n. If there were a relation among them in L_n, this relation would lift to L_∞ by Lemma A.10. Yet in L_∞ the classes are obviously independent modulo I^2.

Finally, we prove (b). In our list of indecomposables there are n of the form $\sigma_i(b)$ $(1 \leq i \leq n)$. The ones of the form $[a_1 \dots a_{2^i}]$ number $\lfloor \log_2(n) \rfloor$ since we must have $2^i \leq n$. The ones of the form $[a_1 b_1^e]$ number $\lfloor n-1 \rfloor$, the ones of the form $[a_1 a_2 b_1^e b_2^e]$ number $\lfloor \frac{n}{2} - 1 \rfloor$, etc. So we have the formula

$$\#(\text{indecomposables in } L_n) = n + \lfloor \log_2(n) \rfloor + (n-1) + \lfloor \tfrac{n}{2} - 1 \rfloor + \lfloor \tfrac{n}{4} - 1 \rfloor + \cdots$$

where the series stops when $\frac{n}{2^i}$ becomes smaller than 1. Thus, excluding the first two terms we have $\lfloor \log_2(n) \rfloor$ terms, all of which have a "-1" in them. These negative ones together cancel the $\lfloor \log_2(n) \rfloor$ term, leaving

$$\#(\text{indecomposables in } L_n) = 2n + \lfloor \tfrac{n}{2} \rfloor + \lfloor \tfrac{n}{4} \rfloor + \cdots$$

Let $\alpha(n) = \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{4} \rfloor + \cdots$. We complete the proof of (b) by showing that

$$\alpha(n) = n - (\text{number of ones in the binary expansion of } n).$$

We do this by induction on n, the case $n = 1$ being trivial. For the general case write $n = 2^k + n'$ where $n' < 2^k$. Then

$$\begin{aligned} \alpha(n) = (2^{k-1} + 2^{k-2} + \cdots + 1) + \alpha(n') = 2^k - 1 + \alpha(n') &= n - n' - 1 + \alpha(n') \\ &= n - (n' - \alpha(n') + 1). \end{aligned}$$

By induction, $n' - \alpha(n')$ is the number of ones in the binary expansion of n'—which is also one less than the number in the binary expansion of n. This completes the proof. $\square$

References

[A1] M. F. Atiyah, *K-theory*, W. A. Benjamin, Inc., 1967.

[A2] M. F. Atiayh, *K-theory and reality*, Quart. J. Math. Oxford Ser. (2) **17** (1966), 367–386.

[CSM] R. Carter, G. Segal, and I. MacDonald, *Lectures on Lie groups and Lie algebras*, Cambridge University Press, 1995.

[C] J. Caruso, *Operations in $\mathbb{Z}/p$-cohomology*, Math. Proc. Cambridge Philos. Soc. **126** (1999), no. 3, 521–541.

[De] A. Delzant, *Définition des classes Stiefel-Whitney d'un module quadratique sur un corps de caractéristique différente de* 2, C. R. Acad. Sci. Paris **255** (1962), 1366–1368.

[D] D. Dugger, *An Atiyah-Hirzebruch spectral sequence for KR-theory*, *K*-theory **35** (2005), no. 3–4, 213–256.

[FL] K. K. Ferland and L.G. Lewis, *The RO(G)-graded equivariant ordinary homology of G-cell complexes with even-dimensional cells for $G = \mathbb{Z}/p$*, Mem. Amer. Math. Soc. **167** (2004), no. 794.

[GSS] C. Giusti, P. Salvatore, D. Sinha, *The mod-two cohomology rings of symmetric groups*, J. Topol. **5** (2012), no. 1, 169–198.

[H] G. Hochschild, *The structure of Lie groups*, Holden-Day Inc., San Francisco, 1965.

[K1] W. Kronholm, *A freeness theorem for $RO(\mathbb{Z}/2)$-graded cohomology*, Topology Appl. **157** (2010), no. 5, 902–915.

[K2] W. Kronholm, *The $RO(G)$-graded Serre spectral sequence*, Homology Homotopy Appl. **12** (2010), no. 1, 75–92.

[Ma] J.P. May, *Equivariant homotopy and cohomology theory* (with contributions by Cole, Comezana, Costenoble, Elmendorff, Greenlees, Lewis, Piacenza, Triantafillou, and Waner), CBMS Regional Conference Series in Mathematics **91**, American Mathematical Society, Providence, RI, 1996.

[M] J. Milnor, *Algebraic K-theory and quadratic forms*, Invent. Math. **9** 1969/1970, 318–344.

[MV] F. Morel and V. Voevodsky, *$\mathbb{A}^1$-homotopy theory of schemes*, Inst. Hautes Études Sci. Publ. Math., No. 90 (1999), 45–143.

[OE] Online Encyclopedia of Integer Sequences, published electronically at http://oeis.org, 2012.

[R] D. Rydh, *A minimal set of generators for the ring of multisymmetric functions*, Ann. Inst. Fourier (Grenoble) **57** (2007), no. 6, 1741–1769.

[Sh] M. Shulman, *Equivariant local coefficients and the $RO(G)$-graded cohomology of classifying spaces*, PhD thesis, University of Chicago, 2010.

[ST] M. Schlichting and G. S. Tripathi, *Geometric representation of Hermitian K-theory in $\mathbb{A}^1$-homotopy theory*, preprint, 2012.

[S] D. Sullivan, *Geometric topology: localization, periodicity, and Galois symmetry* (the 1970 MIT notes), K-monographs in Mathematics **8**, Springer, Dordrecht, 2005.

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