A FAMILY OF DIGIT FUNCTIONS WITH LARGE PERIODS

VLADIMIR SHEVELEV AND PETER J. C. MOSES

ABSTRACT. For odd $n \ge 3$, we consider a general hypothetical identity for the differences $S_{n,\,0}(x)$ of multiples of n with even and odd digit sums in the base $n-1$ in interval $[0,x)$, which we prove in the cases $n=3$ and $n=5$ and empirically confirm for some other n. We give a verification algorithm for this identity for any odd n. The hypothetical identity allows to give a general recursion for $S_{n,\,0}(x)$ for every integer x depending on the residue of x modulo $p(n)=2n(n-1)^{n-1}$, such that $p(3)=24,\ p(5)=2560,\ p(7)=653184$, *etc.*

1. Introduction

For $x \in \mathbb{N}$ and $n \ge 3$, denote by $S_n(x)$ the sum

$$(1) \qquad S_{n,\,j}(x) = \sum_{0\le r<x:\ r\equiv j \pmod n} (-1)^{s_{n-1}(r)},$$

where $s_{n-1}(r)$ is the digit sum of r in base $n-1$.
Note that, in particular, $S_{3,\,0}(x)$ equals the difference between the numbers of multiples of 3 with even and odd binary digit sums (or multiples of 3 from sequences A001969 and A000069 in [7]) in interval $[0,x)$.

Leo Moser (cf. [3], Introduction) conjectured that always

$$(2) \qquad S_{3,\,0}(x) > 0.$$

Newman [3] proved this conjecture. Moreover, he obtained the inequalities

$$(3) \qquad \frac{1}{20} < S_{3,\,0}(x)x^{-\lambda} < 5,$$

where

$$(4) \qquad \lambda = \frac{\ln 3}{\ln 4} = 0.792481\ldots\ .$$

In connection with this, the qualitative result (2) we call a weak Newman phenomenon (or Moser-Newman phenomenon), while an estimating result of the form (3) we call a strong Newman phenomenon.

1991 *Mathematics Subject Classification.* 11A63.

In 1983, Coquet [1] studied a very complicated continuous and nowhere differentiable fractal function $F(x)$ with period 1 for which

$$S_{3,\,0}(3x) = x^{\lambda}F\left(\frac{\ln x}{\ln 4}\right) + \frac{\eta(x)}{3}, \tag{5}$$

where

$$\eta(x) = \begin{cases} 0, & if\ x\ is\ even, \\ (-1)^{s_2(3x-1)}, & if\ x\ is\ odd. \end{cases} \tag{6}$$

He obtained that

$$\limsup_{x\to\infty,\, x\in\mathbb{N}} S_{3,\,0}(3x)x^{-\lambda} = \frac{55}{3}\left(\frac{3}{65}\right)^{\lambda} = 1.601958421\ldots\ , \tag{7}$$

$$\liminf_{x\to\infty,\, x\in\mathbb{N}} S_{3,\,0}(3x)x^{-\lambda} = \frac{2\sqrt{3}}{3} = 1.154700538\ldots\ . \tag{8}$$

In 2007, Shevelev [4] gave an elementary proof of Coquet's formulas (7)-(8) and his sharp estimates in the form

$$\frac{2\sqrt{3}}{3}x^{\lambda} \le S_{3\,0}(3x,\ 0) \le \frac{55}{3}\left(\frac{3}{65}\right)^{\lambda} x^{\lambda},\ \ x\in\mathbb{N}. \tag{9}$$

In [4] it was found the following simple identity

$$S_{3,\,0}(4x) = 3S_{3,\,0}(x),\ where\ x\ is\ even. \tag{10}$$

Since in the left hand side of (10) the argument $4x \equiv 0 \pmod 8$ then (10) is not a recursion for evaluation of $S_{3,\,0}(x)$. However, in the same work Shevelev found the following recursion for fast calculation of $S_{3,\,0}(x)$:

$$S_{3,0}(x) = 3S_{3,0}\left(\left\lfloor\frac{x}{4}\right\rfloor\right) + \nu(x), \tag{11}$$

where

$$\nu(x) = \begin{cases} 0,\ if\ x \equiv 0,7,8,9,16,17,18,22,23 \pmod{24}; \\ (-1)^{s_2(x)},\ if\ x \equiv 3,4,10,12,20 \pmod{24}; \\ (-1)^{s_2(x)+1},\ if\ x \equiv 1,2,5,6,11,19,21 \pmod{24}; \\ 2(-1)^{s_2(x)},\ if\ x \equiv 15 \pmod{24}; \\ 2(-1)^{s_2(x)+1},\ if\ x \equiv 13,14 \pmod{24}. \end{cases} \tag{12}$$

In 2008, Drmota and Stoll [2] proved a generalized weak Newman phenomenon, showing that (2) is valid for $S_{n,\,0}(x)$ for every $n \ge 3$, at least beginning with $x \ge x_0(n)$. A year before, Shevelev [5] proved a strong form of this generalization, but yet only in "full" intervals of the form $[0,(n-1)^{2p})$. Recently Shevelev and Moses [6] in the case of odd $n \geq 3$ and $p \geq \frac{n-1}{2}$ found the relation

$$\sum_{k=0}^{\frac{n-1}{2}}(-1)^k\binom{n}{2k}S_{n,\,0}((n-1)^{2p-2k}) = \begin{cases} 0, & if\ p \geq \frac{n+1}{2}, \\ (-1)^n, & if\ \ p = \frac{n-1}{2}. \end{cases} \tag{13}$$

In the case of $p = \frac{n-1}{2}$, (13) could be rewrite in the form

$$\sum_{j=0}^{\frac{n-1}{2}}(-1)^j\binom{n}{2j+1}S_{n,\,0}((n-1)^{2j}) = 1. \tag{14}$$

Numerous experiments show that, most likely, the following more general relation takes place:

$$\sum_{j=0}^{\frac{n-1}{2}}(-1)^j\binom{n}{2j+1}S_{n,\,0}((n-1)^{2j}x) =$$

$$\sum_{j=0}^{n-1}S_{n,\,j}(x),\ x \geq 1,\ n \equiv 1 \pmod 2. \tag{15}$$

In particular, we verified (15) for $n = 3, 5, 7, ..., 35$ and $1 \leq x \leq 1000$. It is clear that (14) is a special case of (15) for $x = 1$, since

$$S_{n,\,j}(1) = \begin{cases} 1, & if\ j = 0, \\ 0, & if\ \ 1 \leq j \leq n-1. \end{cases} \tag{16}$$

Below we show that (15) allows with the uniform positions to find a recursion for $S_{n,\,0}(x)$ for every odd $n \geq 3$. In the two first sections we prove identity (15) in cases $n = 3$ and $n = 5$. In Section 4 we give a general verification algorithm for the identity (15) which allows to prove the identity (15) for $n = 7, 9, ..., etc.$ In Section 5 we give a simplification of the conjectural equality (15). In Section 6 we prove the recursion in case $n = 3$ and in Section 7 we give the recursion in case $n = 5$. After these sections, in supposition that (15) is true, it will be clear how to find the further recursions for odd $n \geq 7$.

2. The identity in case $n = 3$

Note that, by (1),

$$S_{3,\,j}(x) = \sum_{0 \leq r < x:\ r \equiv j \pmod 3}(-1)^{s_2(r)}$$

which yields that

$$\sum_{0 \leq r < 2x:\ r \equiv 2j \pmod 6}(-1)^{s_2(r)},\ j = 0, 1, 2. \tag{17}$$

On the other hand,

$$S_{3,\,j}(2x)=\sum_{0\le r<2x:\ r\equiv j \pmod 6}(-1)^{s_2(r)}+$$

$$\sum_{0\le r<2x:\ r\equiv j+3 \pmod 6}(-1)^{s_2(r)},\ j=0,1,2. \tag{18}$$

Using (18), for $j=0,1,2$, we consecutively find

$$S_{3,\,0}(2x)=\sum_{0\le r<2x:\ r\equiv 0 \pmod 6}(-1)^{s_2(r)}-$$

$$\sum_{0\le r<2x:\ r\equiv 2 \pmod 6}(-1)^{s_2(r)}, \tag{19}$$

$$S_{3,\,1}(2x)=-\sum_{0\le r<2x:\ r\equiv 0 \pmod 6}(-1)^{s_2(r)}+$$

$$\sum_{0\le r<2x:\ r\equiv 4 \pmod 6}(-1)^{s_2(r)}, \tag{20}$$

$$S_{3,\,2}(2x)=\sum_{0\le r<2x:\ r\equiv 2 \pmod 6}(-1)^{s_2(r)}-$$

$$\sum_{0\le r<2x:\ r\equiv 4 \pmod 6}(-1)^{s_2(r)}. \tag{21}$$

Now the application of (17) to (19)-(21) yields the relations

$$S_{3,\,0}(2x)=S_{3,\,0}(x)-S_{3,\,1}(x), \tag{22}$$

$$S_{3,\,1}(2x)=-S_{3,\,0}(x)+S_{3,\,2}(x), \tag{23}$$

$$S_{3,\,2}(2x)=S_{3,\,1}(x)-S_{3,\,2}(x). \tag{24}$$

For $n=3$, the left hand side of (15) is $3S_{3,\,0}(x)-S_{3,\,0}(4x)$ and, using (22)-(24), we have

$$3S_{3,\,0}(x)-S_{3,\,0}(4x)=3S_{3,\,0}(x)-S_{3,\,0}(2x)+S_{3,\,1}(2x)=$$

$$3S_{3,\,0}(x)-S_{3,\,0}(x)+S_{3,\,1}(x)-S_{3,\,0}(x)+S_{3,\,2}(x)=$$

$$S_{3,\,0}(x)+S_{3,\,1}(x)+S_{3,\,2}(x)$$

which proves (15) in the case $n=3$.

3. The identity in case $n = 5$

In the same way, instead of (22)-(24), we find the following relations

$$S_{5,0}(4x) = S_{5,0}(x) - S_{5,1}(x) + S_{5,2}(x) - S_{5,3}(x), \tag{25}$$

$$S_{5,1}(4x) = -S_{5,0}(x) + S_{5,1}(x) - S_{5,2}(x) + S_{5,4}(x), \tag{26}$$

$$S_{5,2}(4x) = S_{5,0}(x) - S_{5,1}(x) + S_{5,3}(x) - S_{5,4}(x), \tag{27}$$

$$S_{5,3}(4x) = -S_{5,0}(x) + S_{5,2}(x) - S_{5,3}(x) + S_{5,4}(x), \tag{28}$$

$$S_{5,4}(4x) = S_{5,1}(x) - S_{5,2}(x) + S_{5,3}(x) - S_{5,4}(x). \tag{29}$$

For $n = 5$, the left hand side of (15) is

$$5S_{5,0}(x) - 10S_{5,0}(16x) + S_{5,0}(256x). \tag{30}$$

Using (25)-(29), we easily find

$$S_{5,0}(16x) = 4S_{5,0}(x) - 3S_{5,1}(x) + S_{5,2}(x) + S_{5,3}(x) - 3S_{5,4}(x), \tag{31}$$

$$S_{5,1}(16x) = -3S_{5,0}(x) + 4S_{5,1}(x) - 3S_{5,2}(x) + S_{5,3}(x) + S_{5,4}(x), \tag{32}$$

$$S_{5,2}(16x) = S_{5,0}(x) - 3S_{5,1}(x) + 4S_{5,2}(x) - 3S_{5,3}(x) + S_{5,4}(x), \tag{33}$$

$$S_{5,3}(16x) = S_{5,0}(x) + S_{5,1}(x) - 3S_{5,2}(x) + 4S_{5,3}(x) - 3S_{5,4}(x), \tag{34}$$

$$S_{5,4}(16x) = -3S_{5,0}(x) + S_{5,1}(x) + S_{5,2}(x) - 3S_{5,3}(x) + 4S_{5,4}(x). \tag{35}$$

Now using (31)-(35), we find

$$\begin{gathered} S_{5,0}(256x) = 36S_{5,0}(x) - 29S_{5,1}(x) + \\ 11S_{5,2}(x) + 11S_{5,3}(x) - 29S_{5,4}(x). \end{gathered} \tag{36}$$

Finally, for the expression (30), using (31) and (36), we have

$$\begin{gathered} 5S_{5,0}(x) - 10S_{5,0}(16x) + S_{5,0}(256x) = \\ S_{5,0}(x) + S_{5,1}(x) + S_{5,2}(x) + S_{5,3}(x) + S_{5,4}(x). \end{gathered} \tag{37}$$

It is the identity (15) in the case $n = 5$.

4. General problem

Quite analogously to systems (22)-(24), (25)-(29) we can write the system for any $n \geq 3$. For odd n, we have

$$S_{n,\,0}((n-1)x) = S_{n,\,0}(x) - S_{n,\,1}(x) + ... + S_{n,\,n-3}(x) - S_{n,\,n-2}(x),$$

$$S_{n,\,1}((n-1)x) = -S_{n,\,0}(x) + S_{n,\,1}(x) - ... - S_{n,\,n-3}(x) + S_{n,\,n-1}(x),$$

$$S_{n,\,2}((n-1)x) = S_{n,\,0}(x) - S_{n,\,1}(x) + ... - S_{n,\,n-4}(x) + S_{n,\,n-2}(x) - S_{n,\,n-1}(x),$$

....

$$S_{n,\,n-2}((n-1)x) = -S_{n,\,0}(x) + S_{n,\,2}(x) - ... - S_{n,\,n-2}(x) + S_{n,\,n-1}(x),$$

$$(38)\qquad S_{n,\,n-1}((n-1)x) = S_{n,\,1}(x) - S_{n,\,2}(x) + ... + S_{n,\,n-1}(x).$$

It is easy to see that the right hand side of the i-th equality for $S_{n,\,i}((n-1)x)$, $i = 0, 1, ..., n-1$, of the system (38) satisfies the rules: 1) the signs alternate, beginning with $(-)^i$; 2) there is no summand $S_{n,\,n-1-i}(x)$. Using, as usual, the convention $\sum_a^b = 0$, if $b < a$, one can write the system (38) in the form

$$(39)\qquad (-1)^i S_{n,\,i}((n-1)x)) = \sum_{j=0}^{n-i-2} (-1)^j S_{n\,j}(x) - \sum_{j=n-i}^{n-1} (-1)^j S_{n\,j}(x).$$

Thus the general problem is to prove that (39) yields (15).

5. A simplification of the conjecture

Note that in the sum $\sum_{j=0}^{n-1} S_{n,\,j}(x)$ the index of summing j runs all residues modulo n. Therefore, we have

$$\sum_{j=0}^{n-1} S_{n,\,j}(x) = S_{1,\,0}(x) = \sum_{0 \leq i < x} (-1)^{s_{n-1}(i)} =$$

$$(40)\qquad \begin{cases} 0, & if\ x\ is\ even, \\ (-1)^{s_{n-1}(x-1)}, & if\ x\ is\ odd. \end{cases}$$

Thus the conjectural relation (15) is equivalent to the equality

$$\sum_{j=0}^{\frac{n-1}{2}} (-1)^j \binom{n}{2j+1} S_{n,\,0}((n-1)^{2j}x) =$$

$$(41)\qquad \begin{cases} 0, & if\ x\ is\ even, \\ (-1)^{s_{n-1}(x-1)}, & if\ x\ is\ odd. \end{cases}$$

In particular, for $x = 1$, we again have (14). Note that (41) means that its left hand side taken with sign $(-1)^{s_{n-1}(x-1)}$ is periodic with period 2:

$$(-1)^{s_{n-1}(x-1)}\sum_{j=0}^{\frac{n-1}{2}}(-1)^j\binom{n}{2j+1}S_{n,\,0}((n-1)^{2j}x)=$$

$$\begin{cases}0, & if\ x\ is\ even,\\ 1, & if\ \ x\ is\ odd.\end{cases} \tag{42}$$

6. Recursion for $S_{3,0}(x)$

Here we prove (11)-(12). Let us write (42) for $n=3$ and $x := \lfloor\frac{x}{4}\rfloor$. We have

$$(-1)^{s_2(\lfloor\frac{x}{4}\rfloor-1)}(3S_{3,\,0}(\lfloor\frac{x}{4}\rfloor)-S_{3,\,0}(4\lfloor\frac{x}{4}\rfloor)=$$

$$\begin{cases}0, & if\ \lfloor\frac{x}{4}\rfloor\ is\ even,\\ 1, & if\ \ \lfloor\frac{x}{4}\rfloor\ is\ odd.\end{cases} \tag{43}$$

Note that $\lfloor\frac{x}{4}\rfloor$ is even, if $x=0,1,2,3,8,9,10,11,...$ and odd for other integers. Thus we obtain

Lemma 1. *The sequence $\{A_3(x)\}$, where*

$$A_3(x)=(-1)^{s_2(\lfloor\frac{x}{4}\rfloor-1)}(3S_{3,\,0}(\lfloor\frac{x}{4}\rfloor)-S_{3,\,0}(4\lfloor\frac{x}{4}\rfloor), \tag{44}$$

is periodic with the period 8, *such that*

$$A_3(x)=\begin{cases}0, & if\ x\equiv 0,1,2,3, \pmod 8,\\ 1, & if\ \ x\equiv 4,5,6,7 \pmod 8.\end{cases} \tag{45}$$

Consider the difference

$$\Delta_3(x)=S_{3,\,0}(x)-S_{3,\,0}(4\lfloor\frac{x}{4}\rfloor). \tag{46}$$

Lemma 2. *We have*

$$\Delta_3(x)=\begin{cases}(-1)^{s_2(x-1)}, & if\ x\equiv 1,7\ or\ 10 \pmod{12}\\ (-1)^{s_2(x-2)}, & if\ x\equiv 2\ or\ 11 \pmod{12}\\ (-1)^{s_2(x-3)}, & if\ x\equiv 3 \pmod{12}\\ 0, otherwise.\end{cases} \tag{47}$$

Proof. Let $x=12t+j,\ j=0,1,...,11$. Consider 3 cases.

$a)\ j=0,1,2\ or\ 3.$

Then

$$\Delta_3(x)=S_{3,0}(12t+j)-S_{3,0}(12t)=$$

$$\begin{cases}0, & if\ j=0,\\ (-1)^{s_2(x-j)}, & if\ \ j=1,2,3.\end{cases}$$

b) $j=4,5,6$ or 7.

Then

$$\Delta_3(x)=S_{3,0}(12t+j)-S_{3,0}(12t+4)=$$

$$\begin{cases} 0, \quad if\ j=4,5,6, \\ (-1)^{s_2(x-1)}, \quad if \quad j=7. \end{cases}$$

c) $j=8,9,10$ or 11.

Then

$$\Delta_3(x)=S_{3,0}(12t+j)-S_{3,0}(12t+8)=$$

$$\begin{cases} 0, \quad if\ j=8,9, \\ (-1)^{s_2(x-1)}, \quad if \quad j=10, \\ (-1)^{s_2(x-2)}, \quad if \quad j=11 \end{cases}$$

and (47) follows. ■

Now from (44)-(47) we easily deduce the following result.

Theorem 3.

$$S_{3,\,0}(x)=3S_{3,\,0}(\lfloor\frac{x}{4}\rfloor)+\Delta_3(x)-(-1)^{s_2(\lfloor\frac{x}{4}\rfloor-1)}A_3(x), \tag{48}$$

where $A_3(x)$ and $\Delta_3(x)$ are defined by (45) and (47) respectively.

Formula (48) gives a recursion for $S_{3,\,0}(x)$. Let us show that it coincides with the recursion (11)-(12), i.e.,

$$\Delta_3(x)-(-1)^{s_2(\lfloor\frac{x}{4}\rfloor-1)}A_3(x)=\nu(x), \tag{49}$$

where $\nu(x)$ is defined by (12). This follows from the following two lemmas.

Lemma 4. *The sequence*

$$\{(-1)^{s_2(x)+s_2(\lfloor\frac{x}{4}\rfloor-1)}A_3(x)\} \tag{50}$$

is periodic with period 8.

Proof. In cases $x\equiv i \pmod 8$, $i=0,1,2,3$ the terms of the sequence are zeros. If $x\equiv i \pmod 8$, $i=4,5,6,7$, put $x=8t+i$. Then $A_3(x)=1$ and we have

$$(-1)^{s_2(x)+s_2(\lfloor\frac{x}{4}\rfloor-1)}=(-1)^{s_2(8t+i)+s_2(2t)}=$$

$$(-1)^{s_2(8t+i)+s_2(8t)}=(-1)^{s_2(i)}$$

and the lemma follows. ■

Note that period of sequence (50) is

$$\{0,0,0,0,-1,1,1,-1\}. \tag{51}$$

Lemma 5. *The sequence*

$$\{(-1)^{s_2(x)}\Delta_3(x)\} \tag{52}$$

is periodic with period 12.

Proof. According to (47), we have

$$(-1)^{s_2(x)}\Delta_3(x) =$$

$$\begin{cases} (-1)^{s_2(x)+s_2(x-1)}, & if\ x \equiv 1, 7\ or\ 10 \pmod{12} \\ (-1)^{s_2(x)+s_2(x-2)}, & if\ x \equiv 2\ or\ 11 \pmod{12} \\ (-1)^{s_2(x)+s_2(x-3)}, & if\ x \equiv 3 \pmod{12} \\ 0, otherwise. \end{cases} \tag{53}$$

Let $x = 12t + i,\ 0 \le i \le 11$. Let, firstly, $i = 1, 7, 10$. In cases $i = 1$ and $i = 7$, we, evidently, have $(-1)^{s_2(x)+s_2(x-1)} = -1$, while in case $i = 10$,

$$(-1)^{s_2(12t+10)+s_2(12t+9)} = (-1)^{s_2(12t+1010_2)+s_2(12t+1001_2)} = 1.$$

Let now $i = 2, 11$. In case $i = 2$, we, evidently, have $(-1)^{s_2(x)+s_2(x-2)} = -1$ and also in case $i = 11$, we find

$$(-1)^{s_2(12t+11)+s_2(12t+9)} = (-1)^{s_2(12t+1011_2)+s_2(12t+1001_2)} = -1;$$

finally, if $i = 3$, then, evidently, we have $(-1)^{s_2(x)+s_2(x-3)} = 1$. In other cases, the terms of the sequence are zeros. ■

Thus period of sequence (52) is

$$\{0,-1,-1,1,0,0,0,-1,0,0,1,-1\}. \tag{54}$$

Subtracting the tripled period (51) from the doubled period (54), we obtain the period of length 24 of the left hand side of (49) multiplied by $(-1)^{s_2(x)}$. It is

$$\{0,-1,-1,1,1,-1,-1,0,0,0,1,-1,$$

$$1,-2,-2,2,0,0,0,-1,1,-1,0,0\}. \tag{55}$$

It is left to note that, according to (12), $(-1)^{s_2(x)}\nu(x)$ is periodic with the same period. ■

7. On recursion for $S_{n,0}(x)$

Let (42) be true. Let us write (42) for $x := \lfloor\frac{x}{(n-1)^{n-1}}\rfloor$. We have

$$(-1)^{s_{n-1}(\lfloor\frac{x}{(n-1)^{n-1}}\rfloor-1)}((-1)^{\frac{n-1}{2}}S_{n,\,0}((n-1)^{n-1}\lfloor\frac{x}{(n-1)^{n-1}}\rfloor)+$$

$$\sum_{j=0}^{\frac{n-3}{2}}(-1)^j\binom{n}{2j+1}S_{n,\,0}((n-1)^{2j}\lfloor\frac{x}{(n-1)^{n-1}}\rfloor)) =$$

(56)
$$\begin{cases}0, & if\ \lfloor\frac{x}{(n-1)^{n-1}}\rfloor\ is\ \ even,\\ 1, & if\ \ \lfloor\frac{x}{(n-1)^{n-1}}\rfloor\ \ is\ \ odd.\end{cases}$$

Denote the left hand side of (56) by $A_n(x)$. Then, similar to (45), we have

$$A_n(x) =$$

(57)
$$\begin{cases}0, & if\ x \equiv 0, ..., (n-1)^{n-1}-1, \quad (\mathrm{mod}\ 2(n-1)^{n-1}),\\ 1, & if\ \ x \equiv (n-1)^{n-1}, ..., 2(n-1)^{n-1}-1, \quad (\mathrm{mod}\ 2(n-1)^{n-1}).\end{cases}$$

Furthermore, we consider the difference

(58)
$$\Delta_n(x) = S_{n,\,0}(x) - S_{n,\,0}((n-1)^{n-1}\lfloor\frac{x}{(n-1)^{n-1}}\rfloor).$$

Lemma 6. *$(-1)^{s_{n-1}(x)}\Delta_n(x)$ is periodic with period $n(n-1)^{n-1}$.*

Proof. Indeed, let

$$x = n(n-1)^{n-1}t + j,\ j = 0, 1, ..., n(n-1)^{n-1}-1.$$

Let j such that

$$\lfloor\frac{j}{(n-1)^{n-1}}\rfloor = m,\ 0 \le m \le n-1.$$

Then

$$j = (n-1)^{n-1}m + k,\ 0 \le k \le (n-1)^{n-1}-1.$$

We have

$$\Delta_n(x) = S_{n,\,0}(n(n-1)^{n-1}t + j) - S_{n,\,0}(n(n-1)^{n-1}t + (n-1)^{n-1}m) =$$

$$S_{n,\,0}(n(n-1)^{n-1}t + (n-1)^{n-1}m + k) - S_{n,\,0}(n(n-1)^{n-1}t + (n-1)^{n-1}m) =$$

(59)
$$\sum_{i:(n-1)^{n-1}m+1\le 5i\le (n-1)^{n-1}m+k-1}(-1)^{s_4(n(n-1)^{n-1}t+5i)}.$$

Note that

$$5i = (n-1)^{n-1}m + l,\ 1 \le l \le k-1 \le (n-1)^{n-1}-2.$$

Therefore, the summands in (59) multiplied by $(-1)^{s_{n-1}(x)}$ have the form

$$(-1)^{s_{n-1}(n(n-1)^{n-1}t+(n-1)^{n-1}m+k)+s_{n-1}(n(n-1)^{n-1}t+(n-1)^{n-1}m+l)}$$

and, since $l < k \leq (n-1)^{n-1} - 1$, this equal

$$(-1)^{s_{n-1}(n(n-1)^{n-1}t+(n-1)^{n-1}m)+s_{n-1}(k)+s_{n-1}(n(n-1)^{n-1}t+(n-1)^{n-1}m)+s_{n-1}(l)} =$$

$$(-1)^{s_{n-1}(k)+s_{n-1}(l)}.$$

Therefore, the summands of (59) not depend on t and thus the sum (59), i.e., $\Delta_n(x)$ not depends on t. ■

Lemma 7. *The sequence*

$$\{(-1)^{s_{n-1}(x)+s_{n-1}(\lfloor\frac{x}{(n-1)^{n-1}}\rfloor-1)}A_n(x)\} \tag{60}$$

is periodic with period $2(n-1)^{n-1}$.

Proof. In cases $x \equiv i \pmod{2(n-1)^{n-1}}$, $i = 0, 1, ..., (n-1)^{n-1} - 1$ the terms of the sequence are zeros. If $x \equiv i \pmod{2(n-1)^{n-1}}$, $i = (n-1)^{n-1}, ..., 2(n-1)^{n-1} - 1$, put $x = 2(n-1)^{n-1}t + i$. Then $A_n(x) = 1$ and we have

$$(-1)^{s_{n-1}(x)+s_{n-1}(\lfloor\frac{x}{(n-1)^{n-1}}\rfloor-1)} = (-1)^{s_{n-1}(2(n-1)^{n-1}t+i)+s_{n-1}(2t)} =$$

$$(-1)^{s_{n-1}(2(n-1)^{n-1}t+i)+s_{n-1}(2(n-1)^{n-1}t)} = (-1)^{s_{n-1}(i)}$$

and the lemma follows. ■

Now we obtain the following result.

Theorem 8. *If the conjectural relation (15) is true, then we have*

$$S_{n,\,0}(x) = \sum_{j=0}^{\frac{n-3}{2}}(-1)^{\frac{n-3}{2}-j}\binom{n}{2j+1}S_{n,\,0}((n-1)^{2j}\lfloor\frac{x}{(n-1)^{n-1}}\rfloor)+\nu_n(x), \tag{61}$$

where $\nu_n(x)$ *multiplied by* $(-1)^{s_{n-1}(x)}$ *is periodic with period* $2n(n-1)^{n-1}$.

Proof. Indeed, by (56)-(58), we obtain (61) with

$$\nu_n(x) = \Delta_n(x) + (-1)^{\frac{n-1}{2}+s_{n-1}(\lfloor\frac{x}{(n-1)^{n-1}}\rfloor-1)}A_n(x).$$

Then, by Lemmas 6-7, $(-1)^{s_{n-1}(x)}\nu_n(x)$ is periodic with period equal the least common multiple of numbers $2(n-1)^{n-1}$ and $n(n-1)^{n-1}$. ■

As a corollary, in the case $n = 3$ we again obtain Theorem 3 for $\nu(x) = \nu_3(x)$ but without detailed representation of $\Delta_3(x)$ and $\nu(x)$.

Remark 9. It follows from the proof that, if for some

$$j = j_i,\ i = 1, ..., k,\ 1 \leq j_1 < j_2 < ... < j_k \leq \frac{n-3}{2},$$

to replace in (61) $S_{n,\,0}((n-1)^{2j}\lfloor\frac{x}{(n-1)^{n-1}}\rfloor)$ by $S_{n,\,0}(\lfloor\frac{x}{(n-1)^{n-1-2j}}\rfloor)$ and to denote the new sum by $\Sigma(j_1, ..., j_k)$, then also the following form of Theorem 8 is valid

Theorem 10. *If the conjectural relation (15) is true, then we have*

$$S_{n,\,0}(x) = \Sigma(j_1, ..., j_k) + \nu_n^{(j_1,...,j_k)}(x), \tag{62}$$

where $\nu_n^{(j_1,...,j_k)}(x)$ *multiplied by* $(-1)^{s_{n-1}(x)}$ *is periodic with period* $2n(n-1)^{n-1}$.

Thus we have $2^{\frac{n-3}{2}}$ different formulas of type (62). In particular, in case $n = 3$ we have only formula, in case $n = 5$ we have two different formulas, etc.

8. Application of Theorem 8 in case $n = 5$

Since the conjectural identity (15) was proved in case $n = 5$, then, by Theorem 8, we conclude that

$$(-1)^{s_4(x)}\nu_5(x) = (-1)^{s_4(x)}(S_{5,\,0}(x) - 10S_{5,\,0}(16\lfloor\frac{x}{256}\rfloor) + 5S_{5,\,0}(\lfloor\frac{x}{256}\rfloor)) \tag{63}$$

is periodic with period 2560. If to write the period, then (63) gives a recursion for $S_{5,\,0}(x)$. The computer calculations show that the period with positions $\{0, ..., 2559\}$ contains all numbers from interval $[-35, 35]$. Here we give several sequences of positions in $[0, 2559]$ with these numbers $g \in [-35, 35]$.

$$g = -35 : \{251, 252, 254\},$$
$$g = -34 : \{246, 249, 1531, 1532, 1534\},$$
$$g = -33 : \{241, 243, 244, 1526, 1529\},$$
$$g = -32 : \{237, 239, 1521, 1523, 1524\},$$
$$g = -31 : \{231, 232, 234, 1517, 1519\},$$
$$g = -30 : \{197, 199, 200, 217, 219, 220, 226, 229, 511, 1511, 1512, 1514,$$
$$2497, 2499, 2500, 2557, 2559\},$$
$$...$$
$$g = 30 : \{196, 198, 216, 218, 227, 228, 230, 1513,$$
$$1515, 2496, 2498, 2556, 2558\},$$
$$g = 31 : \{233, 235, 1516, 1518, 1520\},$$
$$g = 32 : \{236, 238, 240, 1522, 1525\},$$
$$g = 33 : \{242, 245, 1527, 1528, 1530\},$$
$$g = 34 : \{247, 248, 250, 1533, 1535\},$$
$$g = 35 : \{253, 255\}.$$

Besides, by Theorem 10, also

$$(64)\quad (-1)^{s_4(x)}\nu_5^{(1)}(x)=(-1)^{s_4(x)}(S_{5,\,0}(x)-10S_{5,\,0}(\lfloor\frac{x}{16}\rfloor)+5S_{5,\,0}(\lfloor\frac{x}{256}\rfloor))$$

is periodic with period 2560. Again, if to write the period, then (64) gives another recursion for $S_{5,\,0}(x)$. The computer calculations show that the period with positions $\{0,...,2559\}$ contains all numbers from interval $[-9,9]$. Several sequences of positions in $[0,2559]$ with these numbers $h\in[-9,9]$ are the following:

$$h=-9:\{2411,2412,2414,2491,2492,2494\},$$

$$h=-8:\{1131,1132,1134,1211,1212,1214,2406,2409,2486,2489\},$$

$$...$$

$$h=8:\{1133,1135,1213,1215,2407,2408,2410,2487,2488,2490\},$$

$$h=9:\{2413,2415,2493,2495\}.$$

Finally, note that the sequence of the numbers of different values of $\nu_3(x),\ \nu_5^{(1)}(x),\ \nu_5(x), etc.$ begins with $\{5,19,71,...\}$.

9. Recursions for $S_{3,\,1}(x)$ and $S_{3,\,2}(x)$

Using (22)-(24), it is easy to show that the form $3y(x)-y(4x)$ is invariant with respect to $S_{3,\,i}(x),\ i=0,1,2$. This means that together with

$$(65)\qquad 3S_{3,\,0}(x)-S_{3,\,0}(4x)=S_{3,\,0}(x)+S_{3,\,1}(x)+S_{3,\,2}(x),$$

we have also

$$(66)\qquad 3S_{3,\,1}(x)-S_{3,\,1}(4x)=S_{3,\,0}(x)+S_{3,\,1}(x)+S_{3,\,2}(x),$$

$$(67)\qquad 3S_{3,\,2}(x)-S_{3,\,2}(4x)=S_{3,\,0}(x)+S_{3,\,1}(x)+S_{3,\,2}(x).$$

Using (66)-(67), as in Section 6, we can prove that the expressions

$$(68)\qquad (-1)^{s_2(x)}(S_{3,\,1}(x)-3S_{3,\,1}(\lfloor\frac{x}{4}\rfloor)),$$

and

$$(69)\qquad (-1)^{s_2(x)}(S_{3,\,2}(x)-3S_{3,\,2}(\lfloor\frac{x}{4}\rfloor)),$$

are eventually priodic with the same period as $(-1)^{s_2(x)}\nu(x)$ (12), i.e., the period (55), such that for $S_{3,\,2}(x)$ the period starts at $x=8$, while for $S_{3,\,1}(x)$ the period starts at $x=16$. This means that, for $S_{3,\,i}(x),\ i=1,2,$ the same recursions hold as the recursion for $S_{3,\,0}(x)$ (11) with the same function $\nu(x)$ (12):

$$(70)\qquad S_{3,1}(x)=3S_{3,1}\left(\left\lfloor\frac{x}{4}\right\rfloor\right)+\nu(x),\ x\ge16,$$

with the initials

$$S_{3,1}(x)=\begin{cases}0, & if\ x=0,1,\\ -1, & if\ x=2,3,4,\\ -2, & if\ x=5,6,7,11,12,13,\\ -3, & if\ x=8,9,10,14,15.\end{cases} \tag{71}$$

$$S_{3,2}(x)=3S_{3,2}\left(\left\lfloor\frac{x}{4}\right\rfloor\right)+\nu(x),\ x\geq 8, \tag{72}$$

with the initials

$$S_{3,2}(x)=\begin{cases}0, & if\ x=0,1,2,6,7,\\ -1, & if\ x=3,4,5.\end{cases} \tag{73}$$

For example, by (70), (71) and (12), we have

$$S_{3,1}(20)=3S_{3,1}(5)+\nu(20)=3\cdot(-2)+(-1)^{s_2(20)}=-5;$$

analogously, by (72), (73) and (12), we find

$$S_{3,2}(20)=3S_{3,2}(5)+\nu(20)=3\cdot(-1)+(-1)^{s_2(20)}=-2.$$

10. A generalization

A generalization of the conjectural equality (15) is the following

$$\sum_{j=0}^{\frac{n-1}{2}}(-1)^j\binom{n}{2j+1}S_{n,\,i}((n-1)^{2j}x)=$$

$$\sum_{j=0}^{n-1}S_{n,\,j}(x),\ i=0,...,n-1,\ x\geq 1,\ n\equiv 1 \pmod 2. \tag{74}$$

If this conjecture is valid, then, as in the previous sections, we can obtain the same recursions for every digit function $S_{n,\,i}(x),\ i=1,...,n-1$, as for $S_{n,\,0}(x)$ (cf. Theorems 8, 10). The question on initials in cases $i\geq 1$ we here remain open.

References

[1] J. Coquet, A summation formula related to the binary digits, *Invent. Math.* **73** (1983),107-115.

[2] M. Drmota, and T. Stoll, Newman's phenomenon for generalized Thue-Morse sequence, *Discrete Math.* 308 (2008) no.7, 1191-1208.

[3] D. J. Newman, On the number of binary digits in a multiple of three, *Proc. Amer. Math.* Soc. **21** (1969),719-721.

[4] V. Shevelev, Two algorithms for exact evalution of the Newman digit sum, and a new proof of Coquet's theorem, *arXiv* 0709.0885 [math.NT].

[5] V. Shevelev, On monotonic strengthening of Newman-like phenomenon on (2m+1)-multiples in base 2m, *arXiv* 0709.0885 [math.NT].

[6] V. Shevelev, and P. Moses, Tangent power sums and their applications, *arXiv* 1207.0404 [math.NT]

.

[7] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences* http://oeis.org.

Department of Mathematics, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel. e-mail: shevelev@bgu.ac.il

United Kingdom. e-mail: mows@mopar.freeserve.co.uk