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# A NOTE ON RATIONAL AND ELLIPTIC CURVES ASSOCIATED WITH THE CUBOID FACTOR EQUATIONS.

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Abstract. A rational perfect cuboid is a rectangular parallelepiped whose edges and face diagonals are given by rational numbers and whose space diagonal is equal to unity. It is described by a system of four equations with respect to six variables. The cuboid factor equations were derived from these four equations by symmetrization procedure. They constitute a system of eight polynomial equations, which has been solved parametrically. In the present paper its parametric solution is expressed through intersections or rational and elliptic curves arranged into parametric families.

## 1. Introduction.

Finding a rational perfect cuboid is equivalent to finding a perfect cuboid with all integer edges and diagonals, which is an old unsolved problem (see [1–44]). Here are the equations describing perfect cuboids:

$$\begin{aligned} &p_0 = x_1^2 + x_2^2 + x_3^2 - L^2 = 0, && p_1 = x_2^2 + x_3^2 - d_1^2 = 0, \\ &p_2 = x_3^2 + x_1^2 - d_2^2 = 0, && p_3 = x_1^2 + x_2^2 - d_3^2 = 0. \end{aligned} \tag{1.1}$$

The variables $x_1$, $x_2$, $x_3$ in (1.1) represent three edges of a cuboid, the variables $d_1$, $d_2$, $d_3$ are its face diagonals, and $L$ is its space diagonal. In the case of a rational perfect cuboid we set $L = 1$. Using the polynomials $p_0$, $p_1$, $p_2$, and $p_3$ from (1.1), we write the following eight polynomial equations:

$$\begin{aligned} &p_0 = 0, && \sum_{i=1}^{3} p_i = 0, \\ &\sum_{i=1}^{3} d_i\,p_i = 0, && \sum_{i=1}^{3} x_i\,p_i = 0, \\ &\sum_{i=1}^{3} d_i^2\,p_i = 0, && \sum_{i=1}^{3} x_i^2\,p_i = 0, \\ &\sum_{i=1}^{3} x_i\,d_i\,p_i = 0, && \sum_{i=1}^{3} x_i^2\,d_i^2\,p_i = 0. \end{aligned} \tag{1.2}$$

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The equations (1.2) are called the cuboid factor equations. They were derived from the original cuboid equations (1.1) as a result of a symmetry approach initiated in [45] (see also [46–48]).

It is easy to see that each solution of the original cuboid equations (1.1) is a solution for the factor equations (1.2). Generally speaking, the converse is not true. However, in [47] the following theorem was proved.

**Theorem 1.1.** *Each integer or rational solution of the cuboid factor equations (1.2) such that $x_1 > 0$, $x_2 > 0$, $x_3 > 0$, $d_1 > 0$, $d_2 > 0$, and $d_3 > 0$ is an integer or rational solution for the original cuboid equations (1.1).*

Due to Theorem 1.1 the cuboid factor equations (1.2) are equivalent to the equations (1.1) in studying perfect cuboids. But in this paper, saying a solution of the factor equations, we assume any integer or rational solution, i. e. even such that some of the inequalities $x_1 > 0$, $x_2 > 0$, $x_3 > 0$, $d_1 > 0$, $d_2 > 0$, $d_3 > 0$ or all of them are not fulfilled.

The left hand sides of the cuboid factor equations are multisymmetric polynomials in $x_1$, $x_2$, $x_3$ and $d_1$, $d_2$, $d_3$, i. e. they are invariant with respect to the permutation group $S_3$ acting upon $x_1$, $x_2$, $x_3$, $d_1$, $d_2$, $d_3$, and $L$ as follows:

$$\sigma(x_i) = x_{\sigma i}, \qquad \sigma(d_i) = d_{\sigma i}, \qquad \sigma(L) = L.$$

For the theory of multisymmetric polynomials the reader is referred to [49–69]. According to this theory, each multisymmetric polynomial of $x_1$, $x_2$, $x_3$ and $d_1$, $d_2$, $d_3$ is expressed through the following nine elementary multisymmetric polynomials:

$$\begin{aligned} &x_1 + x_2 + x_3 = E_{10},\\ &x_1\,x_2 + x_2\,x_3 + x_3\,x_1 = E_{20},\\ &x_1\,x_2\,x_3 = E_{30}, \end{aligned} \tag{1.3}$$

$$\begin{aligned} &d_1 + d_2 + d_3 = E_{01},\\ &d_1\,d_2 + d_2\,d_3 + d_3\,d_1 = E_{02},\\ &d_1\,d_2\,d_3 = E_{03}, \end{aligned} \tag{1.4}$$

$$\begin{aligned} &x_1\,x_2\,d_3 + x_2\,x_3\,d_1 + x_3\,x_1\,d_2 = E_{21},\\ &x_1\,d_2 + d_1\,x_2 + x_2\,d_3 + d_2\,x_3 + x_3\,d_1 + d_3\,x_1 = E_{11},\\ &x_1\,d_2\,d_3 + x_2\,d_3\,d_1 + x_3\,d_1\,d_2 = E_{12}. \end{aligned} \tag{1.5}$$

Expressing the left hand sides of the cuboid factor equations (1.2) through the polynomials (1.3), (1.4), and (1.5), one gets polynomial equations with respect to the variables $E_{10}$, $E_{20}$, $E_{30}$, $E_{01}$, $E_{02}$, $E_{03}$, $E_{21}$, $E_{11}$, $E_{12}$, and $L$ (see (3.1) through (3.7) in [48]). These equations were complemented with fourteen identities expressing the mutual algebraic dependence of the elementary multisymmetric polynomials (1.3), (1.4), and (1.5) (see (3.8) in [48]). As a result a system of twenty two equations was obtained. In [48] this huge system of twenty two equations was reduced to the following single polynomial equation for $E_{10}$, $E_{01}$, $E_{11}$, and $L$:

$$(2\,E_{11})^2 + (E_{01}^2 + L^2 - E_{10}^2)^2 - 8\,E_{01}^2\,L^2 = 0. \tag{1.6}$$

The other variables $E_{20}$, $E_{30}$, $E_{02}$, $E_{03}$, $E_{21}$, $E_{12}$ are expressed as rational functions of $E_{10}$, $E_{01}$, $E_{11}$, and $L$ (see formulas (4.1), (4.3), (5.1), (5.2), (4.6), (4.7) in [48]).

The equation (1.6) was solved by John Ramsden in [70]. In the case of a rational perfect cuboid, where $L = 1$, omitting some inessential special cases, the general solution of the equation (1.6) is given by the formulas

$$E_{11} = -\frac{b\,(c^2+2-4\,c)}{b^2\,c^2+2\,b^2-3\,b^2\,c+c-b\,c^2\,+2\,b}, \tag{1.7}$$

$$E_{10} = -\frac{b^2\,c^2+2\,b^2-3\,b^2\,c\,-c}{b^2\,c^2+2\,b^2-3\,b^2\,c+c-b\,c^2+2\,b}, \tag{1.8}$$

$$E_{01} = -\frac{b\,(c^2+2-2\,c)}{b^2\,c^2+2\,b^2-3\,b^2\,c+c-b\,c^2+2\,b}. \tag{1.9}$$

Below are the formulas for $E_{12}$, $E_{21}$, $E_{03}$, $E_{30}$, $E_{02}$, $E_{20}$ in (1.3), (1.4), and (1.5):

$$\begin{aligned}
E_{12} = {} & (16\,b^6+32\,b^5-6\,c^5\,b^2+2\,c^5\,b-62\,b^5\,c^6+62\,b^6\,c^6+16\,b^4-\\
&-180\,b^6\,c^5-c^7\,b^3+18\,b^5\,c^7-12\,b^6\,c^7-2\,b^5\,c^8+b^6\,c^8+248\,b^5\,c^2+\\
&+248\,b^6\,c^2-96\,b^6\,c+321\,b^6\,c^4-180\,b^5\,c^3-144\,b^5\,c-360\,b^6\,c^3+\\
&+b^4\,c^8+8\,b^4\,c^6-6\,b^4\,c^7+18\,b^4\,c^5+7\,b^3\,c^6+90\,b^5\,c^5-14\,b^3\,c^5+\\
&+17\,b^2\,c^4+32\,b^4\,c^2+28\,b^3\,c^3-28\,b^3\,c^2-4\,b\,c^3+8\,b^3\,c-57\,b^4\,c^4+\\
&+36\,b^4\,c^3-12\,b^2\,c^3-48\,b^4\,c-c^4)\,(b^2\,c^4-6\,b^2\,c^3+13\,b^2\,c^2-\\
&-12\,b^2\,c+4\,b^2+c^2)^{-1}\,(b\,c-1-b)^{-2}\,(b\,c-c-2\,b)^{-2},
\end{aligned} \tag{1.10}$$

$$\begin{aligned}
E_{21} = {} & \frac{b}{2}\,(5\,c^6\,b-2\,c^6\,b^2+52\,c^5\,b^2-16\,c^5\,b-2\,c^7\,b^2+2\,b^4\,c^8-\\
&-26\,b^4\,c^7-426\,b^4\,c^5-61\,b^3\,c^6+100\,b^3\,c^5+14\,c^7\,b^3-c^8\,b^3-20\,b\,c^2-\\
&-8\,b^2\,c^2-16\,b^2\,c-128\,b^2\,c^4-200\,b^3\,c^3+244\,b^3\,c^2+32\,b\,c^3+\\
&+768\,b^4\,c^4-852\,b^4\,c^3+568\,b^4\,c^2+104\,b^2\,c^3-208\,b^4\,c+8\,c^4+\\
&+16\,b^3-112\,b^3\,c+142\,b^4\,c^6+32\,b^4-2\,c^5)\,(b^2\,c^4-6\,b^2\,c^3+13\,b^2\,c^2-\\
&-12\,b^2\,c-4\,c^3+4\,b^2+c^2)^{-1}\,(b\,c-1-b)^{-2}\,(b\,c-c-2\,b)^{-2},
\end{aligned} \tag{1.11}$$

$$\begin{aligned}
E_{03} = {} & \frac{b}{2}\,(b^2\,c^4-5\,b^2\,c^3+10\,b^2\,c^2-10\,b^2\,c+4\,b^2+2\,b\,c+2\,c^2-\\
&-b\,c^3)\,(2\,b^2\,c^4-12\,b^2\,c^3+26\,b^2\,c^2-24\,b^2\,c+\,8\,b^2-c^4\,b+3\,b\,c^3-\\
&-6\,b\,c+4\,b+c^3-2\,c^2+2\,c)\,(b^2\,c^4-6\,b^2\,c^3+13\,b^2\,c^2-\\
&-12\,b^2\,c+4\,b^2+c^2)^{-1}\,(b\,c-1-b)^{-2}\,(-c+b\,c-2\,b)^{-2},
\end{aligned} \tag{1.12}$$

$$\begin{aligned}
E_{30} = {} & c\,b^2\,(1-c)\,(c-2)\,(b\,c^2-4\,b\,c+2+4\,b)\,(2\,b\,c^2-c^2-4\,b\,c+\\
&+2\,b)\,(b^2\,c^4-6\,b^2\,c^3+13\,b^2\,c^2-12\,b^2\,c+4\,b^2+c^2)^{-1}\times\\
&\times(b\,c-1-b)^{-2}\,(-c+b\,c-2\,b)^{-2},
\end{aligned} \tag{1.13}$$

$$
\begin{aligned}
E_{02}=\frac{1}{2}\,&(28\,b^2\,c^2-16\,b^2\,c-2\,c^2-4\,b^2-b^2\,c^4+4\,b^3\,c^4-12\,b^3\,c^3+\\
&+4\,b\,c^3+24\,b^3\,c-8\,b\,c-2\,b^4\,c^4+12\,b^4\,c^3-26\,b^4\,c^2-8\,b^2\,c^3+\\
&+24\,b^4\,c-16\,b^3-8\,b^4)\,(b\,c-1-b)^{-2}\,(b\,c-c-2\,b)^{-2},
\end{aligned}
\tag{1.14}
$$

$$
\begin{aligned}
E_{20}=\frac{b}{2}\,&(b\,c^2-2\,c-2\,b)\,(2\,b\,c^2-c^2-6\,b\,c+2+4\,b)\times\\
&\times(b\,c-1-b)^{-2}\,(b\,c-c-2\,b)^{-2}.
\end{aligned}
\tag{1.15}
$$

The formulas (1.10), (1.11), (1.12), (1.13), (1.14), (1.15) were derived in [71] by substituting the formulas (1.7), (1.8), and (1.9) along with $L=1$ into the corresponding formulas from [48].

Thus, the right hand sides of the equalities (1.3), (1.4), and (1.5) turned out to be expressed through two arbitrary rational parameters $b$ and $c$. The next step was to resolve these equalities with respect to $x_1$, $x_2$, $x_3$, $d_1$, $d_2$, $d_3$. For this purpose in [71] the following two cubic equations were written:

$$x^3-E_{10}\,x^2+E_{20}\,x-E_{30}=0, \tag{1.16}$$

$$d^3-E_{01}\,d^2+E_{02}\,d-E_{03}=0. \tag{1.17}$$

Note that the left hand sides of the equalities (1.3) are regular symmetric polynomials of the variables $x_1$, $x_2$, $x_3$ (see [72]). Similarly, the left hand sides of the equalities (1.4) are regular symmetric polynomials of the variables $d_1$, $d_2$, $d_3$. For this reason $x_1$, $x_2$, $x_3$ can be found as roots of the cubic equation (1.16). Similarly, $d_1$, $d_2$, $d_3$ are roots of the second cubic equation (1.17). Relying on these facts, in [71] the following two inverse problems were formulated.

**Problem 1.1.** *Find all pairs of rational numbers $b$ and $c$ for which the cubic equations* (1.16) *and* (1.17) *with the coefficients given by the formulas* (1.8), (1.9), (1.12), (1.13), (1.14), (1.15) *possess positive rational roots $x_1$, $x_2$, $x_3$, $d_1$, $d_2$, $d_3$ obeying the auxiliary polynomial equations* (1.5) *whose right hand sides are given by the formulas* (1.7), (1.10), *and* (1.11).

**Problem 1.2.** *Find at least one pair of rational numbers $b$ and $c$ for which the cubic equations* (1.16) *and* (1.17) *with the coefficients given by the formulas* (1.8), (1.9), (1.12), (1.13), (1.14), (1.15) *possess positive rational roots $x_1$, $x_2$, $x_3$, $d_1$, $d_2$, $d_3$ obeying the auxiliary polynomial equations* (1.5) *whose right hand sides are given by the formulas* (1.7), (1.10), *and* (1.11).

Due to Theorem 1.1 the inverse problems 1.1 and 1.2 are equivalent to finding all rational perfect cuboids and to finding at least one rational perfect cuboid respectively. Singularities of the inverse problems 1.1 and 1.2 due to the denominators in the formulas (1.7) through (1.15) were studied in [73]. Some special cases where the equations (1.3), (1.4), (1.5) are solvable with respect to the cuboid variables $x_1$, $x_2$, $x_3$ and $d_1$, $d_2$, $d_3$ were found in [74]. However, none of these special cases have produced a perfect cuboid since the inequalities

$$
\begin{aligned}
&x_1>0, &&x_2>0, &&x_3>0,\\
&d_1>0, &&d_2>0, &&d_3>0
\end{aligned}
\tag{1.18}
$$

required for solving the problems 1.1 and 1.2 are not fulfilled in these special cases.

Again, neglecting the inequalities (1.18), an approach to solving the equations (1.3), (1.4), (1.5) was found in [75]. It exploits the following lemma.

**Lemma 1.1.** *A reduced cubic equation $y^3+y^2+D=0$ has three rational roots if and only if there is a rational number $w$ satisfying the sextic equation*

$$D\,(w^2+3)^3+4\,(w-1)^2\,(1+w)^2=0. \tag{1.19}$$

*In this case the roots of the cubic equation $y^3+y^2+D=0$ are given by the formulas*

$$y_1=-\frac{2\,(w+1)}{w^2+3}, \qquad y_2=\frac{2\,(w-1)}{w^2+3}, \qquad y_3=\frac{1-w^2}{w^2+3}.$$

The idea of Lemma 1.1 belongs to John Ramsden. Its detailed proof is given in [75], where this lemma was applied to (1.16) and (1.17) (see Lemma 2.1 in [75]). As a result two sextic equations of the form (1.19) were derived:

$$D_1\,(w_1^2+3)^3+4\,(w_1-1)^2\,(1+w_1)^2=0, \tag{1.20}$$

$$D_2\,(w_2^2+3)^3+4\,(w_2-1)^2\,(1+w_2)^2=0. \tag{1.21}$$

The $D$-parameters $D_1$ and $D_2$ of the sextic equations (1.20) and (1.21) depend on the same two rational numbers $b$ and $c$ as $E_{11}$, $E_{10}$, $E_{01}$, $E_{12}$, $E_{21}$, $E_{03}$, $E_{30}$, $E_{02}$, $E_{20}$ in the formulas (1.7) through (1.15). Therefore the equations (1.20) and (1.21) define two algebraic surfaces or, which is equivalent, two algebraic functions $w_1(b,c)$ and $w_2(b,c)$. Here are the explicit formulas for the $D$-parameters $D_1$ and $D_2$ in the sextic equations (1.20) and (1.21):

$$\begin{gathered}
D_1=-\frac{2}{27}\,(7812\,b^4\,c^4-216\,b^2\,c^4-52\,b^2\,c^3+1764\,b^3\,c^4-1200\,b^4\,c^3-\\
-1848\,b^4\,c^2+720\,b^4\,c-36\,c^4\,b-1512\,b^3\,c^3-36\,c^8\,b^3+288\,b^3\,c^2-\\
-108\,c^6\,b^2+380\,c^5\,b^2+378\,c^7\,b^3-231\,c^8\,b^4-300\,c^7\,b^4+3906\,c^6\,b^4-\\
-13\,c^7\,b^2-8904\,c^5\,b^4-882\,c^6\,b^3+18\,c^6\,b-1319\,b^6\,c^8+20952\,b^5\,c^3-\\
-11952\,b^5\,c^2+2592\,b^5\,c-48372\,b^6\,c^4+31620\,b^6\,c^3-10552\,b^6\,c^2+\\
+816\,b^6\,c+1494\,b^5\,c^8-5238\,b^5\,c^7-4\,c^5+7905\,b^6\,c^7-24186\,b^6\,c^6+\\
+288\,b^6+43740\,b^6\,c^5+7686\,b^5\,c^6+576\,b^7+128\,b^8-15372\,b^5\,c^4-\\
-1080\,b^7\,c^8-3546\,b^7\,c^6+51\,c^9\,b^6+400\,b^8\,c^8-162\,c^9\,b^5+8640\,b^7\,c^2-\\
-3456\,b^7\,c+2808\,b^7\,c^7-1560\,b^8\,c^7+3940\,b^8\,c^6+216\,c^9\,b^7-960\,b^8\,c-\\
-6240\,b^8\,c^3+9\,c^{10}\,b^6+7880\,b^8\,c^4+4\,c^{10}\,b^8-6732\,b^8\,c^5+45\,c^9\,b^4+\\
+3200\,b^8\,c^2-11232\,b^7\,c^3+7092\,b^7\,c^4-18\,c^{10}\,b^7-60\,c^9\,b^8)^2\,(2\,c^2+\\
+2\,b^4\,c^4-12\,b^4\,c^3+26\,b^4\,c^2-24\,b^4\,c+8\,b^4-6\,b^3\,c^4+18\,b^3\,c^3-\\
-36\,b^3\,c+24\,b^3+3\,b^2\,c^4+8\,b^2\,c^3-36\,b^2\,c^2+16\,b^2\,c+12\,b^2-6\,b\,c^3+\\
+12\,b\,c)^{-3}\,(b^2\,c^4-6\,b^2\,c^{-3}+13\,b^2\,c^2-12\,b^2\,c+4\,b^2+c^2)^{-2},
\end{gathered} \tag{1.22}$$

$$
\begin{gathered}
D_2=-\frac{2\,b^2}{27}\,(832\,b^2\,c^2-1440\,b^2\,c^4-840\,b^2\,c^3+4788\,b^3\,c^4+396\,b\,c^3+\\
+720\,b^3\,c+808\,b^4\,c^4+3032\,b^4\,c^3-2576\,b^4\,c^2-96\,b^4\,c+448\,b^4-\\
-504\,c^4\,b-4176\,b^3\,c^3-9\,c^8\,b^3+72\,b^3\,c^2-720\,c^6\,b^2+2288\,c^5\,b^2+\\
+1044\,c^7\,b^3-322\,c^8\,b^4+758\,c^7\,b^4+404\,c^6\,b^4-210\,c^7\,b^2-2464\,c^5\,b^4-\\
-2394\,c^6\,b^3+72\,c^4+252\,c^6\,b+3168\,b^6\,c^8+441\,c^9\,b^5-7056\,b^5\,c+\\
+57960\,b^6\,c^4-47232\,b^6\,c^3+25344\,b^6\,c^2-8064\,b^6\,c-1809\,b^5\,c^8+\\
+14472\,b^5\,c^2+3951\,b^5\,c^7-72\,c^5+36\,c^6-11808\,b^6\,c^7+1440\,b^5+\\
+28980\,b^6\,c^6-49032\,b^6\,c^5-4410\,b^5\,c^6+8820\,b^5\,c^4-15804\,b^5\,c^3+\\
+1152\,b^6-504\,c^9\,b^6-45\,c^9\,b^3-6\,c^9\,b^4+104\,c^8\,b^2+36\,c^{10}\,b^6+\\
+14\,c^{10}\,b^4-45\,c^{10}\,b^5-99\,c^7\,b)^2\,(6\,b^4\,c^4-36\,b^4\,c^3+78\,b^4\,c^2-72\,b^4\,c+\\
+24\,b^4-12\,b^3\,c^4+36\,b^3\,c^3-72\,b^3\,c+48\,b^3+5\,b^2\,c^4+16\,b^2\,c^3-\\
-68\,b^2\,c^2+32\,b^2\,c+20\,b^2-12\,b\,c^3+24\,b\,c+6\,c^2)^{-3}\,(b^2\,c^4-6\,b^2\,c^3+\\
+13\,b^2\,c^2-12\,b^2\,c+4\,b^2+c^2)^{-2}.
\end{gathered}
\tag{1.23}
$$

The main result of [75] is expressed by the following two theorems.

**Theorem 1.2.** *Each rational point of the algebraic surface* (1.20)*, except for points belonging to some algebraic subvariety of this surface, determine six rational numbers* $x_1$*,* $x_2$*,* $x_3$ *and* $d_1$*,* $d_2$*,* $d_3$ *obeying the cuboid factor equations* (1.2) *as well as the equations* (1.3)*,* (1.4)*,* (1.5) *whose right hand sides are given by the formulas* (1.7) *through* (1.15)*.*

**Theorem 1.3.** *Each rational point of the algebraic surface* (1.21)*, except for points belonging to some algebraic subvariety of this surface, determine six rational numbers* $x_1$*,* $x_2$*,* $x_3$ *and* $d_1$*,* $d_2$*,* $d_3$ *obeying the cuboid factor equations* (1.2) *as well as the equations* (1.3)*,* (1.4)*,* (1.5) *whose right hand sides are given by the formulas* (1.7) *through* (1.15)*.*

The rational numbers $x_1$, $x_2$, $x_3$ and $d_1$, $d_2$, $d_3$ in Theorem 1.2 are given by explicit formulas expressing them as rational functions of $b$, $c$ and $w_1$:

$$
x_i=x_i(b,c,w_1), \qquad\qquad d_i=d_i(b,c,w_1). \tag{1.24}
$$

But the formulas for $x_i(b,c,w_1)$ and $d_i(b,c,w_1)$ in (1.24) are very huge. For this reason they are nor presented here. They are given in a machine readable form in the ancillary file **Solution_1.txt** to [76]. As for the exceptional subvariety mentioned in Theorem 1.2, it is determined by the denominator of (1.22) and by the denominators in those huge formulas for $x_1$, $x_2$, $x_3$ and $d_1$, $d_2$, $d_3$.

The rational numbers $x_1$, $x_2$, $x_3$ and $d_1$, $d_2$, $d_3$ in Theorem 1.3 are also given by explicit formulas expressing them as rational functions of $b$, $c$ and $w_2$:

$$
x_i=x_i(b,c,w_2), \qquad\qquad d_i=d_i(b,c,w_2). \tag{1.25}
$$

These formulas for $x_i(b,c,w_2)$ and $d_i(b,c,w_2)$ in (1.25) are also very huge. For this reason we do not present them here. They are given in a machine readable form in the ancillary file **Solution_2.txt** to [76]. The exceptional subvariety in Theorem 1.3 is determined by the denominator of (1.23) and by the denominators in those huge formulas for $x_1$, $x_2$, $x_3$ and $d_1$, $d_2$, $d_3$.

The algebraic surfaces (1.20) and (1.21) are not independent. They were studied in [76]. As it was shown in [76], these two algebraic surfaces are birationally equivalent. This birational equivalence is established by two rational functions

$$w_2 = w_2(b, c, w_1), \qquad\qquad w_1 = w_1(b, c, w_2). \tag{1.26}$$

The formulas (1.26) should be treated modulo the equations (1.20) and (1.21) respectively. Then they produce two mutually inverse transformations.

The functions (1.24) and (1.25) are also not independent. They are related to each other through the birational transformations (1.26):

$$\begin{aligned} x_i(b, c, w_1) &= x_i(b, c, w_2(b, c, w_1)), \\ d_i(b, c, w_1) &= d_i(b, c, w_2(b, c, w_1)), \end{aligned} \tag{1.27}$$

$$\begin{aligned} x_i(b, c, w_2) &= x_i(b, c, w_1(b, c, w_2)), \\ d_i(b, c, w_2) &= d_i(b, c, w_1(b, c, w_2)). \end{aligned} \tag{1.28}$$

The explicit formulas for the rational functions $w_2(b, c, w_1)$ and $w_1(b, c, w_2)$ in (1.26) are very huge. They are not presented here, but they are given in a machine readable form in the ancillary file **Conversion_formulas.txt** to [76].

Due to (1.26), (1.27), and (1.28) the algebraic surfaces given by the sextic equations (1.20) and (1.21) are closely related to each other. Below we study both of them. Since $D_1 = D_1(b, c)$ and $D_2 = D_2(b, c)$ in (1.20) and (1.21), they are surfaces in $\mathbb{R}^3$. The main goal of the present paper is to embed these surfaces into $\mathbb{R}^4$ and show that each point of any one of them lies in the intersection of some genus zero curve and some genus one curve specific to this point.

Note that some genus one curves associated with perfect cuboids were considered in [77]. However, they correspond to some very special solutions of the cubic equations (1.16) and (1.17). Genus one curves considered in this paper cover the general case in the theory of the cuboid factor equations.

## 2. The structure of $D_1$ and $D_2$ and the $\alpha$-parameters.

Let's consider the formulas (1.22) and (1.23) for $D_1$ and $D_2$. Looking at these formulas, one can detect the following structure of the expressions for $D_1$ and $D_2$:

$$D_1 = -\frac{(P_1)^2}{(Q_1)^3}, \qquad\qquad D_2 = -\frac{(P_2)^2}{(Q_2)^3}. \tag{2.1}$$

The denominators $Q_1$ and $Q_2$ in (2.1) are given by the formulas

$$\begin{gathered} Q_1 = \frac{3}{2}\,(2\,c^2 + 2\,b^4\,c^4 - 12\,b^4\,c^3 + 26\,b^4\,c^2 - 24\,b^4\,c + 8\,b^4 - \\ -\,6\,b^3\,c^4 + 18\,b^3\,c^3 - 36\,b^3\,c + 24\,b^3 + 3\,b^2\,c^4 + 8\,b^2\,c^3 - \\ -36\,b^2\,c^2 + 16\,b^2\,c + 12\,b^2 - 6\,b\,c^3 + 12\,b\,c), \end{gathered} \tag{2.2}$$

$$
\begin{aligned}
Q_2=\frac{3}{2}\,(6\,b^4\,c^4-36\,b^4\,c^3+78\,b^4\,c^2-72\,b^4\,c+24\,b^4-12\,b^3\,c^4+\\
+36\,b^3\,c^3-72\,b^3\,c+48\,b^3+5\,b^2\,c^4+16\,b^2\,c^3-68\,b^2\,c^2+\\
+32\,b^2\,c+20\,b^2-12\,b\,c^3+24\,b\,c+6\,c^2).
\end{aligned}
\tag{2.3}
$$

The numerators $P_1$ and $P_2$ in (2.1) are given by similar formulas:

$$
\begin{aligned}
P_1=\frac{1}{2}\,(7812\,b^4\,c^4-216\,b^2\,c^4-52\,b^2\,c^3+1764\,b^3\,c^4-1200\,b^4\,c^3-\\
-1848\,b^4\,c^2+720\,b^4\,c-36\,c^4\,b-1512\,b^3\,c^3-36\,c^8\,b^3+288\,b^3\,c^2-\\
-108\,c^6\,b^2+380\,c^5\,b^2+378\,c^7\,b^3-231\,c^8\,b^4-300\,c^7\,b^4+3906\,c^6\,b^4-\\
-13\,c^7\,b^2-8904\,c^5\,b^4-882\,c^6\,b^3+18\,c^6\,b-1319\,b^6\,c^8+20952\,b^5\,c^3-\\
-11952\,b^5\,c^2+2592\,b^5\,c-48372\,b^6\,c^4+31620\,b^6\,c^3-10552\,b^6\,c^2+\\
+816\,b^6\,c+1494\,b^5\,c^8-5238\,b^5\,c^7-4\,c^5+7905\,b^6\,c^7-24186\,b^6\,c^6+\\
+288\,b^6+43740\,b^6\,c^5+7686\,b^5\,c^6+576\,b^7+128\,b^8-15372\,b^5\,c^4-\\
-1080\,b^7\,c^8-3546\,b^7\,c^6+51\,c^9\,b^6+400\,b^8\,c^8-162\,c^9\,b^5+8640\,b^7\,c^2-\\
-3456\,b^7\,c+2808\,b^7\,c^7-1560\,b^8\,c^7+3940\,b^8\,c^6+216\,c^9\,b^7-960\,b^8\,c-\\
-6240\,b^8\,c^3+9\,c^{10}\,b^6+7880\,b^8\,c^4+4\,c^{10}\,b^8-6732\,b^8\,c^5+45\,c^9\,b^4+\\
+3200\,b^8\,c^2-11232\,b^7\,c^3+7092\,b^7\,c^4-18\,c^{10}\,b^7-60\,c^9\,b^8)\times\\
\times(b^2\,c^4-6\,b^2\,c^{-3}+13\,b^2\,c^2-12\,b^2\,c+4\,b^2+c^2)^{-1},
\end{aligned}
\tag{2.4}
$$

$$
\begin{aligned}
P_2=\frac{b}{2}\,(832\,b^2\,c^2-1440\,b^2\,c^4-840\,b^2\,c^3+4788\,b^3\,c^4+396\,b\,c^3+\\
+720\,b^3\,c+808\,b^4\,c^4+3032\,b^4\,c^3-2576\,b^4\,c^2-96\,b^4\,c+448\,b^4-\\
-504\,c^4\,b-4176\,b^3\,c^3-9\,c^8\,b^3+72\,b^3\,c^2-720\,c^6\,b^2+2288\,c^5\,b^2+\\
+1044\,c^7\,b^3-322\,c^8\,b^4+758\,c^7\,b^4+404\,c^6\,b^4-210\,c^7\,b^2-2464\,c^5\,b^4-\\
-2394\,c^6\,b^3+72\,c^4+252\,c^6\,b+3168\,b^6\,c^8+441\,c^9\,b^5-7056\,b^5\,c+\\
+57960\,b^6\,c^4-47232\,b^6\,c^3+25344\,b^6\,c^2-8064\,b^6\,c-1809\,b^5\,c^8+\\
+14472\,b^5\,c^2+3951\,b^5\,c^7-72\,c^5+36\,c^6-11808\,b^6\,c^7+1440\,b^5+\\
+28980\,b^6\,c^6-49032\,b^6\,c^5-4410\,b^5\,c^6+8820\,b^5\,c^4-15804\,b^5\,c^3+\\
+1152\,b^6-504\,c^9\,b^6-45\,c^9\,b^3-6\,c^9\,b^4+104\,c^8\,b^2+36\,c^{10}\,b^6+\\
+14\,c^{10}\,b^4-45\,c^{10}\,b^5-99\,c^7\,b)\,(b^2\,c^4-6\,b^2\,c^3+\\
+13\,b^2\,c^2-12\,b^2\,c+4\,b^2+c^2)^{-2}.
\end{aligned}
\tag{2.5}
$$

Due to (2.1) and the formulas (2.2), (2.3), (2.4), (2.5) we can write the sextic equations (1.20) and (1.21) in the following way:

$$
\left(\frac{w_1^2+3}{Q_1}\right)^3=\left(\frac{2\,(w_1^2-1)}{P_1}\right)^2,\qquad\qquad \left(\frac{w_2^2+3}{Q_2}\right)^3=\left(\frac{2\,(w_2^2-1)}{P_2}\right)^2.
\tag{2.6}
$$

**Lemma 2.1.** *If $x$ and $y$ are two rational numbers obeying the equality $x^3=y^2$, then there is a third rational number $\alpha$ such that $x=\alpha^2$ and $y=\alpha^3$.*

*Proof.* Each rational number is presented as an irreducible ratio of two integer numbers. Expanding these numbers into prime factors we find that each rational number is a product of distinct prime numbers to some integer powers which can be either positive or negative. For $x$ and $y$ this fact yields

$$x = \pm\, p_1^{\beta_1} \cdot \ldots \cdot p_m^{\beta_m}, \qquad\qquad y = \pm\, q_1^{\gamma_1} \cdot \ldots \cdot q_n^{\gamma_n}. \tag{2.7}$$

Substituting (2.7) into the equality $x^3 = y^2$, we find that $x > 0$, $m = n$ and the prime factors $p_1, \ldots, p_m$ should coincide with the prime factors $q_1, \ldots, q_n$ up to some permutation. Performing this permutation upon the prime factors $q_1, \ldots, q_n$, for the exponents $\beta_1, \ldots, \beta_m$ and $\gamma_1, \ldots, \gamma_n$ in (2.7) we derive

$$3\,\beta_i = 2\,\gamma_i, \ \text{ where } \ i = 1, \ldots, n. \tag{2.8}$$

Due to (2.8) there are some unique integer numbers $\alpha_1, \ldots, \alpha_n$ such that

$$\beta_i = 2\,\alpha_i, \qquad\qquad \gamma_i = 3\,\alpha_i. \tag{2.9}$$

Using the numbers $\alpha_1, \ldots, \alpha_n$, we define the number $\alpha$ by setting

$$\alpha = \pm\, p_1^{\alpha_1} \cdot \ldots \cdot p_n^{\alpha_n}. \tag{2.10}$$

The sign of $\alpha$ in (2.10) is chosen to be coinciding with the sign of $y$ in (2.7). Then from (2.9), using $x > 0$ and (2.7), we derive the required equalities $x = \alpha^2$ and $y = \alpha^3$. Lemma 2.1 is proved. □

Now we can apply Lemma 2.1 to the equations (2.6). As a result we obtain the following two theorems whose proofs are obvious.

**Theorem 2.1.** *For each rational point $(b, c, w_1)$ of the algebraic surface (1.20) in $\mathbb{R}^3$, where $w_1 \neq \pm 1$, there is a rational number $\alpha_1$ such that*

$$w_1^2 + 3 = Q_1\,\alpha_1^2, \qquad\qquad 2\,(w_1^2 - 1) = P_1\,\alpha_1^3. \tag{2.11}$$

*The numbers $P_1$ and $Q_1$ in (2.11) are given by the formulas (2.2) and (2.4).*

**Theorem 2.2.** *For each rational point $(b, c, w_2)$ of the algebraic surface (1.21) in $\mathbb{R}^3$, where $w_2 \neq \pm 1$, there is a rational number $\alpha_2$ such that*

$$w_2^2 + 3 = Q_2\,\alpha_2^2, \qquad\qquad 2\,(w_2^2 - 1) = P_2\,\alpha_2^3. \tag{2.12}$$

*The numbers $P_2$ and $Q_2$ in (2.12) are given by the formulas (2.3) and (2.5).*

## 3. Genus zero curves.

Let's consider the equations (2.11) separately. Since $Q_1 = Q_1(b, c)$, the first equation (2.11) defines a hypersurface in $\mathbb{R}^4$. Similarly, $P_1 = P_1(b, c)$. Therefore the second equation (2.11) defines another hypersurface in $\mathbb{R}^4$. Thus, we get an embedding of the algebraic surface (1.20) into $\mathbb{R}^4$ where this surface is presented as the intersection of two algebraic hypersurfaces. In a similar way, the equations (2.12) define an embedding of the algebraic surface (1.21) into $\mathbb{R}^4$ where this surface is presented as the intersection of two algebraic hypersurfaces.

Now assume that the rational numbers $b$ and $c$ are chosen to be constants. Then $P_1$ and $Q_1$ in (2.11) are also two rational constants. Under this assumption the first equation (2.11) defines a genus zero curve in $\mathbb{R}^2$, while the second equation defines a genus one curve. The same is true for the equations (2.12), provided $b$ and $c$ are considered as constants.

**Definition 3.1.** A genus zero algebraic curve is called a rational curve if it is birationally equivalent to a line (see [78]).

**Theorem 3.1.** *An algebraic curve in $\mathbb{R}^2$ given by the quadratic equation*

$$w^2 + 3 = Q\,\alpha^2, \tag{3.1}$$

*where $Q$ is a rational number, is birationally equivalent to a line over $\mathbb{Q}$ if and only if it has at least one rational point $(w_0, \alpha_0)$ in $\mathbb{R}^2$.*

*Proof.* Necessity. Assume that the curve (3.1) is birationally equivalent to a line over $\mathbb{Q}$. Then there are two rational functions with rational coefficients

$$w = w(t), \qquad\qquad \alpha = \alpha(t) \tag{3.2}$$

whose domain is some Zariski open subset $D \subseteq \mathbb{Q} \subset \mathbb{R}$ and such that they satisfy the equality (3.1) identically. Choosing some point $t_0 \in D$ and substituting $t = t_0$ into (3.2), we get two rational numbers $w_0 = w(t_0)$ and $\alpha_0 = \alpha(t_0)$ satisfying the equality (3.1). They constitute a rational point $(w_0, \alpha_0)$ of our curve (3.1).

Sufficiency. Assume that $(w_0, \alpha_0)$ is some rational point of the curve (3.1). Then the following equality holds for $w_0$ and $\alpha_0$:

$$w_0^2 + 3 = Q\,\alpha_0^2. \tag{3.3}$$

Subtracting (3.3) from (3.1), we derive

$$(w - w_0)\,(w + w_0) = Q\,(\alpha - \alpha_0)\,(\alpha + \alpha_0). \tag{3.4}$$

The equality (3.4) can be transformed to the following one:

$$\frac{w - w_0}{\alpha + \alpha_0} = Q\,\frac{\alpha - \alpha_0}{w + w_0}. \tag{3.5}$$

Using the equality (3.5), we introduce the parameter $t$ by setting

$$\frac{\alpha - \alpha_0}{w + w_0} = t, \qquad\qquad \frac{w - w_0}{\alpha + \alpha_0} = Q\,t. \tag{3.6}$$

The equations (3.6) are easily resolved with respect to $w$ and $\alpha$:

$$w = \frac{w_0 + 2\,Q\,\alpha_0\,t + Q\,w_0\,t^2}{1 - Q\,t^2}, \qquad\qquad \alpha = \frac{\alpha_0 + 2\,w_0\,t + Q\,\alpha_0\,t^2}{1 - Q\,t^2}. \tag{3.7}$$

The functions (3.7) are that very rational functions $w(t)$ and $\alpha(t)$ which implement the birational equivalence of the curve (3.1) and a line. Theorem is proved. $\square$

Note that $Q$ in (3.1) is a rational number. Therefore $Q = (M\,m^2)/(N\,n^2)$, where $M$ and $N$ are relatively prime square free integer numbers. Let $(w_0, \alpha_0)$ be a rational point of the algebraic curve (3.1). Then $w_0$ and $(\alpha_0\,m)/(N\,n)$ are two rational numbers. We can express these numbers as

$$w_0 = \frac{X}{Z}, \qquad\qquad \frac{\alpha_0\,m}{N\,n} = \frac{Y}{Z}, \tag{3.8}$$

where $X$, $Y$, and $Z$ are three integers. Applying $Q = (M\,m^2)/(N\,n^2)$ and (3.8) to (3.1), we derive the following Diophantine equation:

$$X^2 - M\,N\,Y^2 + 3\,Z^2 = 0. \tag{3.9}$$

The Diophantine equation (3.9) is an instance of the Legendre equation (see [79], [80], and [81]). There is a criterion for the equation (3.9) to have a solution. This criterion can be derived from the following Legendre theorem.

**Theorem 3.2** (**Legendre**). *A quadratic Diophantine equation with square free and pairwise relatively prime coefficients*

$$A\,X^2 + B\,Y^2 + C\,Z^2 = 0 \tag{3.10}$$

*has a nonzero solution if and only if its coefficients are not all of the same sign and if $-B\,C$, $-C\,A$, and $-A\,B$ are squares modulo $A$, $B$, and $C$ respectively.*

Due to (3.8) the numbers $M$ and $N$ in (3.9) are relatively prime and square free. Therefore their product $M\,N$ is also square free. The equation (3.9) is a special instance of the equation (3.10). There are special solvability criteria for it (see [80]).

**Theorem 3.3.** *If $M\,N$ is nonzero modulo 3, then the Diophantine equation (3.9) has a nonzero solution if and only if the following three conditions are fulfilled:*

*1) $M\,N > 0$, 2) $-3$ is a square modulo $M\,N$, 3) $M\,N$ is a square modulo 3.*

**Theorem 3.4.** *If $M\,N$ is zero modulo 3, then the Diophantine equation (3.9) has a nonzero solution if and only if the following three conditions are fulfilled:*

*1) $M\,N > 0$, 2) $-3$ is a square modulo $M\,N$, 3) $\dfrac{M\,N}{3}$ is a square modulo 3.*

Applying Theorems 3.3 and 3.4, we can study the rationality of genus zero curves from (2.11) and (2.12) for particular numeric values of $b$ and $c$:

$$w_1^2 + 3 = Q_1\,\alpha_1^2, \qquad\qquad w_2^2 + 3 = Q_2\,\alpha_2^2. \tag{3.11}$$

The results of such numeric studies for the curves (3.11) are presented in the ancillary file **Quadratic_curves.txt** attached to this arXiv submission.

Looking through the numeric output file **Quadratic_curves.txt**, we see that some genus zero curves are rational over $\mathbb{Q}$. Others are not rational. So, both types of curves arise in the theory of cuboid factor equations.

## 4. Genus one curves.

Genus one algebraic curves in (2.11) and (2.12) are more regular than genus zero curves (3.11). They are given by the following cubic equations:

$$2\,(w_1^2-1)=P_1\,\alpha_1^3, \qquad\qquad 2\,(w_2^2-1)=P_2\,\alpha_2^3. \tag{4.1}$$

**Definition 4.1.** A genus one algebraic curve is called an elliptic curve if it has at least one rational point either finite or at infinity (see [82]).

The parameters $P_1$ and $P_2$ in (4.1) are given by the formulas (2.4) and (2.5). But despite the values of these parameters, both curves (4.1) have rational points

$$(w_1,\alpha_1)=(\pm\,1,0), \qquad\qquad (w_2,\alpha_2)=(\pm\,1,0). \tag{4.2}$$

The points (4.2) are exceptional in the sense of Theorems 2.1 and 2.2. Nevertheless, they are sufficient to say that both curves (4.1) are elliptic curves.

## 5. Concluding remarks.

Summarizing the results of Section 3 and Section 4 above, we conclude that each rational point of the algebraic surfaces (1.20) and (1.21) is produced through the intersection of some elliptic curve and some rational curve. This fact could be used for to apply powerful tools of the modern theory of elliptic curves in studying the equations (1.20) and (1.21) and thus in studying perfect cuboids.

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