DUPLICIAL ALGEBRAS AND LAGRANGE INVERSION

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Dedicated to the memory of Jean-Louis Loday

Abstract. We provide operadic interpretations for two Hopf subalgebras of the algebra of parking functions. The Catalan subalgebra is identified with the free duplicial algebra on one generator, and the Schröder subalgebra is interpreted by means of a new operad, which we call triduplicial.

The noncommutative Lagrange inversion formula is then interpreted in terms of duplicial structures. The generic solution of the noncommutative inversion problem appears as the formal sum of all parking functions. This suggests that combinatorial generating functions derived by functional inversion should be obtainable by evaluating a suitable character on this generic solution. This idea is illustrated by means of the Narayana polynomials, of which we obtain bivariate "super-analogues" by lifting to parking functions a classical character of the algebra of symmetric functions. Other characters, such as evaluation of symmetric functions on a binomial element, are also discussed.

1. Introduction

In its simplest version, the Lagrange inversion formula gives the coefficients of the unique formal power series

$$u(t) = \sum_{n\geq 0} c_n t^{n+1} \tag{1}$$

satisfying the functional equation

$$t = \frac{u}{\varphi(u)} \tag{2}$$

where

$$\varphi(x) = \sum_{n\geq 0} a_n x^n \quad (a_0 \neq 0) \tag{3}$$

is a series with nonzero constant term.

As is well-known, the result is

$$c_n = \frac{1}{n+1}[x^n](\varphi(x))^{n+1}\,. \tag{4}$$

There exist several combinatorial variations of this result, providing descriptions of c_n as a polynomial in the a_n as well as q-analogues or noncommutative versions.

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As any generic identity among formal power series, the formula of Lagrange is equivalent to an identity on symmetric functions [9]. One may identify the a_n with a sequence of algebraically independent generators of the algebra $Sym(X)$ of symmetric functions, for example set $a_n = h_n(X)$ (the complete homogeneous functions as in [17, Ex. 24 p. 35, Ex. 25 p. 132]) so that

$$\varphi(u) = \sum_{n\geq 0} h_n(X)u^n = \prod_{n\geq 1}(1-ux_n)^{-1} =: \sigma_u(X) \tag{5}$$

and the result (4) reads (in λ-ring notation)

$$c_n = \frac{1}{n+1}h_n((n+1)X)\,. \tag{6}$$

What makes this expression interesting is that this symmetric function is the Frobenius characteristic of the permutation representation of the symmetric group $\mathfrak{S}_n$ on the set PF_n of parking functions of length n [5].

It has moreover been shown in [22] that noncommutative versions of the Lagrange formula such as in [4] or [25] could be formulated in terms of noncommutative symmetric functions [3]. Again, the term g_n of degree n in the series g satisfying

$$g = \sum_{n\geq 0} S_n g^n \tag{7}$$

(where the S_n are the noncommutative complete symmetric functions) is the Frobenius characteristic of the natural representation of the 0-Hecke algebra $H_n(0)$ on parking functions.

Now, there is a combinatorial Hopf algebra, **PQSym**, which is based on parking functions. Denoting its natural basis by $\mathbf{F_a}$ as in [20], we shall see that the series g is the image of the sum of all parking functions

$$G = \sum_{\mathbf{a}\in\mathrm{PF}} \mathbf{F_a} \tag{8}$$

by a homomorphism of Hopf algebras. This morphism can be defined at the level of several intermediate Hopf algebras, such as **SQSym** or **CQSym** (both defined in [21]), which have as graded dimensions the sequences of little Schröder numbers and Catalan numbers respectively. We shall see that **CQSym** can be naturally identified with the free duplicial algebra on one generator, and that as an element of **CQSym**, G satisfies a quadratic functional equation allowing to identify it with the formal sum of all binary trees. This forces a bijection between these trees and nondecreasing parking functions, which in turn implies the existence of an involution explaining the symmetry properties of G observed in [22] and established there by means of a different involution.

The operadic interpretation of **CQSym** suggests the existence of a similar one for **SQSym**. We thus introduce a new operad, which we call triduplicial, for which **SQSym** is the free algebra on one generator.

Finally, we illustrate the idea that combinatorial series derived by means of Lagrange inversion should come from some character of the algebra of parking functions. First, we generalize Lassalle's expression [10] of the Narayana polynomials $c_n(t)$ to super-Narayana polynomials $P_n(t,q)$ counting signed parking functions according to certain statistics, and such that $P_n(t,0) = (1+t)c_n(1+t)$. This last polynomial is known to count Schröder paths according to the number of horizontal steps [26, A060693].

These paths can be naturally encoded by a subset of signed parking functions, such that minus signs correspond to horizontal steps. We conclude by investigating the character of symmetric functions defined by evaluation on a binomial element, and obtain a combinatorial interpretation of its lift to parking functions.

This paper is a continuation of [21, 22]. We have only recalled the most basic definitions so as to make it reasonably self-contained.

2. Operads and combinatorial Hopf algebras

Combinatorial Hopf algebras are certain graded bialgebras based on combinatorial objects. This is a heuristic concept, and there is no general agreement on what should be their general definition. For us, they arise as natural generalizations of the algebra of symmetric functions. Surprisingly, most of these algebras also arise in the theory of operads (sometimes with non-obviously equivalent definitions). We shall see that our analysis of the noncommutative Lagrange inversion problem will allow to identify the operads associated with two algebras (**CQSym** and **SQSym**) from [21], for which the operadic interpretation was not known.

2.1. Noncommutative symmetric functions.

The Hopf algebra of noncommutative symmetric functions [3], denoted by **Sym**, or by **Sym**(A) if we consider the realization in terms of an auxiliary alphabet, is the free associative algebra over a sequence $(S_n)_{n\geq 1}$, graded by $\deg(S_n) = n$. For $A = \{a_n | n \geq 1\}$ a totally ordered set of noncommuting indeterminates, we define the generating series

$$\sigma_t(A) := \sum_{n\geq 0} S_n(A)t^n = \prod_{n\geq 1}^{\rightarrow} (1 - ta_i)^{-1} \tag{9}$$

so that $S_n(A)$ is the sum of all nondecreasing words of length n (and $S_0 = 1$).

Bases of $\mathbf{Sym}_n$ are labelled by compositions $I = (i_1, \dots, i_r)$ of n. The natural basis is $S^I = S_{i_1} \cdots S_{i_r}$. The reverse refinement order $I \leq J$ on compositions of n means that the parts of I are sums of consecutive parts of J. The length r of I is denoted by $\ell(I)$. The conjugate composition is denoted by $I^\sim$. The ribbon basis is defined by

$$S^I = \sum_{J\leq I} R_J \quad \Longleftrightarrow R_I = \sum_{J\leq I} (-1)^{\ell(I)-\ell(J)} S^J\,. \tag{10}$$

For two compositions $I = (i_1, \dots, i_r)$ and $J = (j_1, \dots, j_s)$, define

$$I \cdot J = (i_1, \dots, i_r, j_1, \dots, j_s) \text{ and } I \triangleright J = (i_1, \dots, i_r + j_1, \dots, j_s)\,. \tag{11}$$

Then,

$$R_I R_J = R_{I\cdot J} + R_{I\triangleright J}\,. \tag{12}$$

Thus, in the R-basis, the multiplication of **Sym** is the sum of the two operations $\cdot$ and $\triangleright$. For this reason, one may consider **Sym** as a *dialgebra* [14]. Both operations are associative, and furthermore

$$(I *_1 J) *_2 K = I *_1 (J *_2 K) \tag{13}$$

where $*_1$ and $*_2$ are $\cdot$ or $\triangleright$. Such an algebra is called an $As^{(2)}$-algebra [28]. It is easy to see that **Sym** is actually the free $As^{(2)}$-algebra on one generator. This is the simplest example of a combinatorial Hopf algebra associated with an operad.

A somewhat more interesting example arises if we consider a slightly more general version of the noncommutative Lagrange inversion problem.

2.2. Nondecreasing parking functions.

The versions of [4] and [25] of the noncommutative inversion formula can be interpreted as solving the equation

$$f = S_0 + S_1 f + S_2 f^2 + S_3 f^3 + \cdots\,, \tag{14}$$

where S_0 is now another indeterminate not commuting with the other ones [22].

A *nondecreasing parking function* is a nondecreasing word $\pi = \pi_1 \dots \pi_n$ over $[n]$ such that $\pi_i \le i$. A *parking function* is any rearrangement of such a word.

The solution of (14) can be expressed in the form [22]

$$f_n = \sum_{\pi \in \mathrm{NDPF}_n} S^{\mathrm{ev}(\pi)\cdot 0}\,, \tag{15}$$

where NDPF_n denotes the set of nondecreasing parking functions of length n. The *evaluation* of a word is the sequence $\mathrm{ev}(w) = (|w|_i)$ recording the number of occurences of each letter i. For example,

$$f_0 = S_0\,,\ f_1 = S_1 S_0 = S^{10}\,,\ f_2 = S^{110} + S^{200}\,, \tag{16}$$

$$f_3 = S^{1110} + S^{1200} + S^{2010} + S^{2100} + S^{3000}\,, \tag{17}$$

the nondecreasing parking functions giving f_3 being (in this order) 123, 122, 113, 112, 111.

As shown in [21], there is a combinatorial Hopf algebra based on nondecreasing parking functions. We shall see that it is directly related to Lagrange inversion, and uncover its operadic interpretation.

2.3. Duplicial algebras.

A duplicial algebra [12] is a vector space endowed with two associative operations $\prec$ and $\succ$ such that

$$(x \succ y) \prec z = x \succ (y \prec z)\,. \tag{18}$$

It is known that the free duplicial algebra $\mathcal{D}$ on one generator has a basis labelled by binary trees. The operations $\prec$ and $\succ$ can be respectively identified with the products $\backslash$ (under) and $/$ (over) (see [12]). The dimensions of the homogeneous components are therefore the Catalan numbers $1, 2, 5, 14 \dots$.

Here is another realization of $\mathcal{D}$.

The Hopf algebra **PQSym**, defined in [20], is the linear span of elements $\mathbf{F_a}$ where $\mathbf{a}$ runs over all parking functions. The sums

$$\mathbf{P}^{\pi} := \sum_{\mathbf{a};\mathbf{a}^{\uparrow}=\pi} \mathbf{F_a} \tag{19}$$

where $\mathbf{a}^{\uparrow}$ means the non-decreasing reordering and π runs over the set NDPF of non-decreasing parking functions, span a cocommutative Hopf subalgebra **CQSym** of **PQSym**.

The basis $\mathbf{P}^{\pi}$ is multiplicative:

$$\mathbf{P}^{\alpha}\mathbf{P}^{\beta} = \mathbf{P}^{\alpha\bullet\beta} \tag{20}$$

where $\alpha \bullet \beta$ is the usual shifted concatenation $\alpha \cdot \beta[k]$ if α is of length k. For example, $\mathbf{P}^{12}\mathbf{P}^{113} = \mathbf{P}^{12335}$.

The $S_n = \mathbf{P}^{1^n}$ generate a Hopf subalgebra isomorphic to **Sym**, which is also a quotient of **CQSym** (see below).

Introduce now a second product involving a different kind of shifted concatenation:

$$\mathbf{P}^{\alpha} \prec \mathbf{P}^{\beta} = \mathbf{P}^{\alpha\cdot\beta[\max(\alpha)-1]} =: \mathbf{P}^{\alpha\circ\beta}\,. \tag{21}$$

For example, $\mathbf{P}^{12} \prec \mathbf{P}^{113} = \mathbf{P}^{12224}$.

Proposition 2.1. *Let us denote the usual product of* **CQSym** *by* $\succ$*.*
Then $(\mathbf{CQSym}, \prec, \succ)$ *is the free duplicial algebra on one generator* $x = \mathbf{P}^1$*.*

Proof – It is clear that $\succ$ and $\prec$ are associative. Also,

$$(\mathbf{P}^{\alpha} \succ \mathbf{P}^{\beta}) \prec \mathbf{P}^{\gamma} = \mathbf{P}^{\alpha} \succ (\mathbf{P}^{\beta} \prec \mathbf{P}^{\gamma}) \tag{22}$$

since, denoting by n the length of α and by m the maximum letter of β,

$$(\alpha \bullet \beta) \circ \gamma = \alpha \cdot \beta[n] \cdot \gamma[m+n-1] = \alpha \bullet (\beta \circ \gamma)\,. \tag{23}$$

The other cross-associativity relation is not satisfied, since already $(1 \circ 1) \bullet 1 = 113 \neq 112 = 1 \circ (1 \bullet 1)$.

Thus, $(\mathbf{CQSym}, \prec, \succ)$ is duplicial. It is free on the generator $x = \mathbf{P}^1$, since all non decreasing parking functions can be obtained by iterating $\bullet$ and $\circ$ on 1. Indeed, $\pi \in$ NDPF is either non-connected, that is, of the form $\pi = \pi' \bullet \pi''$ with nontrivial factor, or connected, in which case is can be written $\pi = 1\pi' = 1 \circ \pi'$ for some other π'. Since nondecreasing parking functions of length n are in bijection with binary trees on n nodes, we see that **CQSym** is indeed free. ■

2.4. **The duplicial operad.** The duplicial operad is described in detail in [12]. Here is a brief summary. First, there is a notion of duplicial coproduct, allowing to define primitive elements.

A duplicial bialgebra [12] is a duplicial algebra endowed with a coproduct δ satifsfying

$$\delta(x * y) = x \otimes y + \sum_{(x)} x_{(1)} \otimes (x_{(2)} * y) + \sum_{(y)} (x * y_{(1)}) \otimes y_{(2)} \tag{24}$$

where $\delta x = \sum_{(x)} x_{(1)} \otimes x_{(2)}$ (Sweedler's notation) and $*$ is $\prec$ or $\succ$.

The usual coproduct of **CQSym** is defined in terms of the parkization operation introduced in [21]. Selecting terms in this coproduct, we obtain a duplicial coproduct:

Proposition 2.2. *The coproduct*

$$\delta(\mathbf{P}^{\pi}) = \sum_{k=1}^{n} \mathbf{P}^{\mathrm{park}(\pi_1\cdots\pi_k)} \otimes \mathbf{P}^{\mathrm{park}(\pi_k\cdots\pi_n)} \tag{25}$$

endows **CQSym** *with the structure of a duplicial bialgebra.*

Proof – An immediate verification. ■

The primitive elements are defined by the condition $\delta x = 0$. It is known that the binary operation

$$\{x, y\} = x \prec y - x \succ y \tag{26}$$

is magmatic, and that is preserves the primitive subalgebra of a duplicial algebra. As a consequence, the *Dup*-primitive subalgebra of **CQSym** is generated by $x = \mathbf{P}^1$ for the operation $\{-, -\}$. This implies that (As, Dup, Mag) is a good triple of operads in the sense of Loday [12]. That is, we have $As \circ Mag = Dup$.

The first primitive elements are

$$\{x, x\} = \mathbf{P}^{11} - \mathbf{P}^{12}, \tag{27}$$

$$\{\{x, x\}, x\} = \mathbf{P}^{111} - \mathbf{P}^{122} - \mathbf{P}^{113} + \mathbf{P}^{123}, \tag{28}$$

$$\{x, \{x, x\}\} = \mathbf{P}^{111} - \mathbf{P}^{122} - \mathbf{P}^{112} + \mathbf{P}^{123}. \tag{29}$$

As an associative algebra for the product $\succ$, **CQSym** is free over its primitive subspace (which has dimension c_{n-1} in degree n as expected).

The duplicial operad is Koszul, and its dual $Dup^!$ is defined by the same relations, together with two extra ones

$$(x \prec y) \succ z = 0 \quad \text{and} \quad 0 = x \prec (y \succ z) \tag{30}$$

The dimension in degree n is n, and the linear generators are the hook-shaped trees

$$x \succ x \succ \cdots \succ x \prec x \prec \cdots \prec x \prec x. \tag{31}$$

There are known morphisms of operads $Dup \to As^{(2)}$, $Dup \to Dias$ and $2as \to Dup$ [12]. The morphism of operads $Dup \to As^{(2)}$ corresponds to the Hopf algebra morphism $\phi:$ **CQSym** $\to$ **Sym** (defined in [21])

$$\phi(\mathbf{P}^{\pi}) = S^{t(\pi)} \tag{32}$$

where the composition $t(\pi)$ is the packed evaluation of π, i.e., the composition obtained by removing the zeros from $\mathrm{ev}(\pi)$.

2.5. **Free quasi-symmetric functions and dendriform algebras.** A dendriform algebra [13, 14] is an associative algebra A endowed with two bilinear operations $\prec$, $\succ$, satisfying

$$(x \prec y) \prec z = x \prec (y \cdot z)\,, \tag{33}$$

$$(x \succ y) \prec z = x \succ (y \prec z)\,, \tag{34}$$

$$(x \cdot y) \succ z = x \succ (y \succ z)\,, \tag{35}$$

such that the associative multiplication $\cdot$ splits as

$$a \cdot b = a \prec b + a \succ b\,. \tag{36}$$

The algebra of free quasi-symmetric functions **FQSym** [2] (or the Malvenuto-Reutenauer Hopf algebra of permutations [18]) is dendriform.

For a totally ordered alphabet A, $\mathbf{FQSym}(A)$ is the algebra spanned by the non-commutative polynomials

$$\mathbf{G}_\sigma(A) := \sum_{\substack{w\in A^n \\ \mathrm{Std}(w)=\sigma}} w \tag{37}$$

where σ is a permutation in the symmetric group $\mathfrak{S}_n$ and $\mathrm{Std}(w)$ denotes the standardization of the word w. The multiplication rule is

$$\mathbf{G}_\alpha \mathbf{G}_\beta = \sum_{\gamma\in\alpha*\beta} \mathbf{G}_\gamma, \tag{38}$$

where the convolution $\alpha * \beta$ of $\alpha \in \mathfrak{S}_k$ and $\beta \in \mathfrak{S}_l$ is the sum in the group algebra of $\mathfrak{S}_{k+l}$ [18]

$$\alpha * \beta = \sum_{\substack{\gamma=uv \\ \mathrm{Std}(u)=\alpha;\,\mathrm{Std}(v)=\beta}} \gamma\,. \tag{39}$$

The dendriform structure of **FQSym** is given by

$$\mathbf{G}_\alpha \mathbf{G}_\beta = \mathbf{G}_\alpha \prec \mathbf{G}_\beta + \mathbf{G}_\alpha \succ \mathbf{G}_\beta\,, \tag{40}$$

where

$$\mathbf{G}_\alpha \prec \mathbf{G}_\beta = \sum_{\substack{\gamma=uv\in\alpha*\beta \\ |u|=|\alpha|;\,\max(v)<\max(u)}} \mathbf{G}_\gamma\,, \tag{41}$$

$$\mathbf{G}_\alpha \succ \mathbf{G}_\beta = \sum_{\substack{\gamma=u.v\in\alpha*\beta \\ |u|=|\alpha|;\,\max(v)\ge\max(u)}} \mathbf{G}_\gamma\,. \tag{42}$$

There is a scalar product on **FQSym** given by

$$\langle \mathbf{G}_\sigma, \mathbf{G}_\tau \rangle = \delta_{\sigma,\tau^{-1}} \tag{43}$$

so that

$$\mathbf{F}_\sigma := \mathbf{G}_{\sigma^{-1}} \tag{44}$$

is the dual basis of $\mathbf{G}_\sigma$ (as a Hopf algebra, **FQSym** is self-dual).

Now, $x = \mathbf{G}_1$ generates a free dendriform algebra in **FQSym**, **PBT**, the Loday-Ronco algebra of planar binary trees [15]. The natural basis $\mathbf{P}_T$ of this algebra can be interpreted as follows.

In **FQSym**, one can build from the dendriform operations a bilinear map [6]

$$B(F, G) = F \succ \mathbf{F}_1 \prec G \tag{45}$$

so that the terms $B_T(a)$ of the binary tree expansion of the unique solution of the functional equation

$$X = a + B(X, X) \tag{46}$$

are precisely the basis $\mathbf{P}_T$ of the free dendriform algebra on one generator $x = \mathbf{F}_1$ for the choice $a = 1$ ($B_T(a)$ is the result of evaluating the expression encoded by the complete binary tree T with a in the leaves and B in the internal nodes). As for the solution X, it is just the sum of all $\mathbf{G}_\sigma$, that is, the sum of all words.

This interpretation leads to simple derivations of the q-hook length formulas for binary trees [6], as well as to the combinatorial interpretations of various special functions such as the tangent [7] or possibly the Jacobi elliptic functions.

2.6. **Word quasi-symmetric functions and tridendrifom algebras.** A *dendriform trialgebra* [16] (or tridendriform algebra) is an associative algebra whose multiplication $\odot$ splits into three pieces

$$x \odot y = x \prec y + x \circ y + x \succ y\,, \tag{47}$$

where $\circ$ is associative, and

$$(x \prec y) \prec z = x \prec (y \odot z)\,, \tag{48}$$

$$(x \succ y) \prec z = x \succ (y \prec z)\,, \tag{49}$$

$$(x \odot y) \succ z = x \succ (y \succ z)\,, \tag{50}$$

$$(x \succ y) \circ z = x \succ (y \circ z)\,, \tag{51}$$

$$(x \prec y) \circ z = x \circ (y \succ z)\,, \tag{52}$$

$$(x \circ y) \prec z = x \circ (y \prec z)\,. \tag{53}$$

The *packed word* $u = \mathrm{pack}(w)$ associated with a word $w \in A^*$ is obtained by the following process. If $b_1 < b_2 < \ldots < b_r$ are the letters occuring in w, u is the image of w by the homomorphism $b_i \mapsto a_i$. A word u is said to be *packed* if $\mathrm{pack}(u) = u$. We denote by PW the set of packed words. With such a word, we associate the polynomial

$$\mathbf{M}_u := \sum_{\mathrm{pack}(w)=u} w\,. \tag{54}$$

These polynomials span a subalgebra of $\mathbb{K}\langle A\rangle$, called **WQSym** for Word Quasi-Symmetric functions see, *e .g.*, [6]).

The product on **WQSym** is given by

$$\mathbf{M}_{u'}\mathbf{M}_{u''} = \sum_{u \in u' *_W u''} \mathbf{M}_u\,, \tag{55}$$

where the *convolution* $u' *_W u''$ of two packed words is defined as

$$u' *_W u'' = \sum_{v,w;u=v\cdot w\,\in\,\mathrm{PW},\mathrm{pack}(v)=u',\mathrm{pack}(w)=u''} u\,. \tag{56}$$

It is a dendriform trialgebra. The partial products being given by

$$\mathbf{M}_{w'} \prec \mathbf{M}_{w''} = \sum_{w=u.v\in w' *_W w'',|u|=|w'|;\max(v)<\max(u)} \mathbf{M}_{\mathbf{a}}, \tag{57}$$

$$\mathbf{M}_{w'} \circ \mathbf{M}_{w''} = \sum_{w=u.v\in w' *_W w'',|u|=|w'|;\max(v)=\max(u)} \mathbf{M}_{\mathbf{a}}, \tag{58}$$

$$\mathbf{M}_{w'} \succ \mathbf{M}_{w''} = \sum_{w=u.v\in w' *_W w'',|u|=|w'|;\max(v)>\max(u)} \mathbf{M}_{\mathbf{a}}, \tag{59}$$

It is known [16] that the free dendriform trialgebra on one generator, denoted here by $\mathfrak{T}D$, is a free associative algebra with Hilbert series

$$\sum_{n\ge 0} s_n t^n = \frac{1+t-\sqrt{1-6t+t^2}}{4t} = 1 + t + 3t^2 + 11t^3 + 45t^4 + 197t^5 + \cdots \tag{60}$$

that is, the generating function of the *super-Catalan*, or *little Schröder* numbers, counting *plane trees*.

There is a natural embedding **FQSym** $\hookrightarrow$ **WQSym** given by

$$\mathbf{G}_\sigma \mapsto \sum_{\mathrm{std}(u)=\sigma} \mathbf{M}_u\,. \tag{61}$$

On the polynomial realizations, this is indeed an inclusion.

Both **FQSym** and **WQSym** can be interpreted as operads. The space of n-ary operations of the Zinbiel operad can be naturally identified with $\mathbf{FQSym}_n$ [1], and for **WQSym**, the relevant operad is described in [19].

2.7. **The triduplicial operad.** Thus, the free dendriform and free tridendriform algebras on one generator arise naturally as quotients of **WQSym**, as well as the free tricubical algebra. Applying the same strategy to **PQSym**, we obtain together with **CQSym** (Catalan numbers), the little Schröder numbers (**SQSym**) and segmented compositions (powers of 3) [21]. Having identified **CQSym** with the duplicial operad, we may suspect that **SQSym** should have an operadic interpretation which is to *Dup* what *TriDend* is to *Dend*. This is indeed the case. The new operad will be called *triduplicial*. It is a quotient of the triplicial operad defined in [11] (one more relation).

Definition 2.3. *A triduplicial algebra is a vector space V endowed with three* associative *laws $\prec, \succ, \circ$ such that*
(i) $(x \succ y) \prec z = x \succ (y \prec z)$, *i.e.,* $(V, \prec, \succ)$ *is duplicial,*
(ii) $(x \circ y) \prec z = x \circ (y \prec z)$, *i.e.,* $(V, \prec, \circ)$ *is duplicial,*
(iii) $(x \succ y) \circ z = x \succ (y \circ z)$, *i.e.,* $(V, \circ, \succ)$ *is duplicial,*
(iv) $(x \circ y) \succ z = x \circ (y \succ z)$, *i.e.,* $(V, \succ, \circ)$ *is duplicial.*

Recall from [21] that a *parking quasi-ribbon* is a segmented nondecreasing parking function where the bars only occur at positions $\cdots a|b\cdots$, with $a<b$. These objects encode hypoplactic classes of parking functions. The first ones are

$$\{1\}, \qquad \{11,\ 12,\ 1|2\}, \tag{62}$$

$$\{111,\ 112,\ 11|2,\ 113,\ 11|3,\ 122,\ 1|22,\ 123,\ 1|23,\ 12|3,\ 1|2|3\}. \tag{63}$$

The number of parking quasi-ribbons of length n is the little Schröder number s_n.

With a parking quasi-ribbon $\mathbf{q}$, we associate the elements

$$\mathbf{P}_{\mathbf{q}} := \sum_{\mathsf{P}(\mathbf{a})=\mathbf{q}} \mathbf{F}_{\mathbf{a}}, \tag{64}$$

where $\mathsf{P}(\mathbf{a})$ denotes the hypoplactic class of $\mathbf{a}$. For example,

$$\mathbf{P}_{11|3} = \mathbf{F}_{131} + \mathbf{F}_{311}\,, \qquad \mathbf{P}_{113} = \mathbf{F}_{113}. \tag{65}$$

The $\mathbf{P}_{\mathbf{q}}$ form a basis of a Hopf subalgebra of **PQSym**, denoted by **SQSym** [21]. As an associative algebra, it is isomorphic to $\mathfrak{T}D$, the free tridendriform algebra on one generator, which is itself a Hopf subalgebra of **WQSym** constructed by a similar method. However, this is not an isomorphism of Hopf algebras. Indeed, $\mathfrak{T}D$ is self-dual, but **SQSym** is not. This raised the question of an operadic interpretation of **SQSym**. The following result provides an answer.

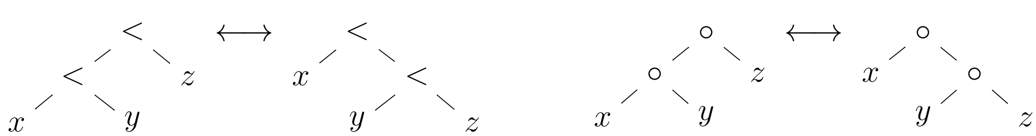
Theorem 2.4. *The free triduplicial algebra on one generator* $\mathbf{T}$ *has the little Schröder numbers as graded dimensions. Its natural basis can be realised by parking quasi-ribbons, if we define* $\prec$ *and* $\succ$ *by shifted concatenation as above, and* $\circ$ *as ordinary shifted concatenation with insertion of a bar:*

$$\mathbf{P}_{\mathbf{q}'} \circ \mathbf{P}_{\mathbf{q}''} = \mathbf{P}_{\mathbf{q}'|\mathbf{q}''[|\mathbf{q}'|]}\,. \tag{66}$$

Proof – It is immediate to check that this defines a triduplicial structure, and that the subspace of **SQSym** generated by $\mathbf{P}_1$ for these three operations contains all the quasi-ribbons $\mathbf{P}_{\mathbf{q}}$. So the free triduplicial algebra has at least the sequence s_n as graded dimensions.

Now, the triduplicial relations can be presented as rewriting rules for evaluation trees. We then have the seven relations, the first three consisting in the associativity of the three operations, the next four ones being the four duplicial relations presented in Definition 2.3.

(67)



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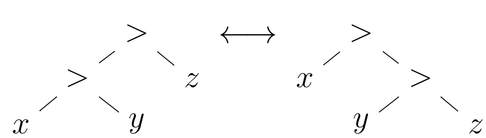


(69)
$$(x \circ y) < z \quad\longleftrightarrow\quad x \circ (y < z) \qquad\qquad (x > y) < z \quad\longleftrightarrow\quad x > (y < z)$$

(70)
$$(x > y) \circ z \quad\longleftrightarrow\quad x > (y \circ z) \qquad\qquad (x \circ y) > z \quad\longleftrightarrow\quad x \circ (y > z)$$

Now consider all these relations as oriented rewriting between trees, each left tree being replaced by its right counterpart. Then the free triduplicial algebra on one generator is spanned (not necessarily freely) by the trees that cannot be rewritten. Analysing the seven relations above, one sees that the trees that cannot be rewritten are of the following form: either the tree consisting in a single node, or a tree with any operation at its root whose left subtree is a leaf, or one of the following two trees:

(71)
$$(\,.\, < A) \circ B \qquad\qquad (\,.\, < A) > B\ ,$$

where A and B are trees that cannot be rewritten. The generating series $S(x)$ of these trees staisfies therefore the functional equation

$$S = 1 + 3xS + 2x^2S^2, \tag{72}$$

whose solution is the generating series of the s_n, so that that the free triduplicial algebra on one generator has at mist the s_n as graded dimensions. Therefore, the parking quasi-ribbons provide a faithful realization of it. ∎

Thus, **T** can be identified with **SQSym**.

For the sake of completeness, let us mention that as in the case of Dup, the dual operad $TriDup^!$ is defined by the same relations, together with four extra ones

$$(x \prec y) \succ z = 0 \qquad \text{and} \qquad 0 = x \prec (y \succ z), \tag{73}$$
$$(x \prec y) \circ z = 0 \qquad \text{and} \qquad 0 = x \prec (y \circ z). \tag{74}$$

The sequence of dimensions is therefore $2^n - 1$, as for the dual of TriDend [28].

2.8. **Duplicial operations on parking functions.** The free dendriform and tridendriform algebras on one generator arise naturally as subalgebras of **FQSym** and **WQSym**, which are themselves dendriform and tridendriform in a natural way. Similarly, the duplicial structure of the Catalan algebra **CQSym** is actually inherited from a duplicial structure on **PQSym**. The $\succ$ operation is the usual product (given by the ordinary shifted shuffle). The $\prec$ operation is also a kind of shifted shuffle, with a normalization factor.

Proposition 2.5. *If* $\max(\mathbf{a}) = m$, *let*

$$\mathbf{F_a} \prec \mathbf{F_b} = \frac{|\mathbf{a}|_m!|\mathbf{b}|_1!}{(|\mathbf{a}|_m + |\mathbf{b}|_1)!}\mathbf{F}_{\mathbf{a} \sqcup\!\sqcup \mathbf{b}[m-1]} \,. \tag{75}$$

Then, **PQSym** *is duplicial for* $\prec$ *and* $\succ$.

Proof – Direct verification. ∎

3. Lagrange inversion

3.1. **A bilinear duplicial equation.** Let $G \in$ **PQSym** be the formal sum of all parking functions

$$G = \sum_{\mathbf{a} \in \mathrm{PF}} \mathbf{F_a} \,. \tag{76}$$

Actually, G belongs to **CQSym**, and

$$G = \sum_{\pi \in \mathrm{NDPF}} \mathbf{P}^\pi \,. \tag{77}$$

Proposition 3.1. *Define a bilinear map* B *on* **CQSym** *by formula (45), interpreting now* $\prec$ *and* $\succ$ *as the duplicial operations. Then,* G *satisfies the functional equation*

$$G = 1 + B(G, G) \tag{78}$$

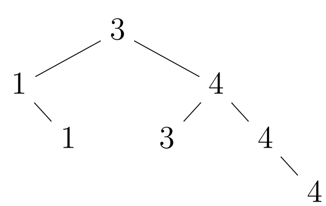
and each term $B_T(1)$ *of the tree expansion of the solution is a single* $\mathbf{P}^\pi$, *thus forcing a bijection beween binary trees and nondecreasing parking functions.*

Proof – Proposition 2.1 shows that **CQSym** is the free duplicial algebra on one generator. So we have already seen that each nondecreasing parking function has a unique expression of the form $\mathbf{P}^\alpha \succ \mathbf{P}^1 \prec \mathbf{P}^\beta$, where α and β are (possibly empty) nondecreasing parking functions.

The same is true of binary trees, if one interprets $\prec$ and $\succ$ as the over-under operations [15, 12], whence the correspondence. ∎

The bijection between binary trees and nondecreasing parking functions can be described as follows. Starting with a tree T, its vertices are recursively labeled by integers, and the tree is flattened so as to read a word. The label of the root is the number m of vertices of its left subtree, plus one. The labels of the right subtree are its original ones shifted by $m - 1$.

For example,



$$\longrightarrow 1133444 = 11 \succ 1 \prec 1222 \,.$$

3.2. **Some Tamari intervals.** This correspondence has an interesting compatibility with the Tamari order:

Proposition 3.2. *Nondecreasing parking functions with the same packed evaluation I form an interval, whose cardinality is the coefficient of S^I in g.*

Proof – It is enough to show that the nondecreasing parking functions with packed evaluation $J \leq I$ for the reverse refinement order form an interval. Now, $1^{i_1} \succ \ldots \succ 1^{i_r}$ is maximal among those for the Tamari order (this is the maximal element of the product in **PBT** of the trees encoded by the 1^{i_k}), and $1^n = 1^{i_1} \prec \ldots \prec 1^{i_r}$ is its minimal element. This corresponds to S^I through the embedding of **Sym** given by $S_n \mapsto 1^n$. Thus this is an interval, and the intervals composed of nondecreasing parking functions with the same packed evaluation I correspond to the expansion of the ribbons $R_{\bar{I}^\sim}$ (conjugate mirror of I) on the basis of trees. ∎

In other words, our bijection between binary trees and nondecreasing parking functions has the property that the trees having the same canopy [27, 15] correspond to nondecreasing parking functions with the same packed evaluation.

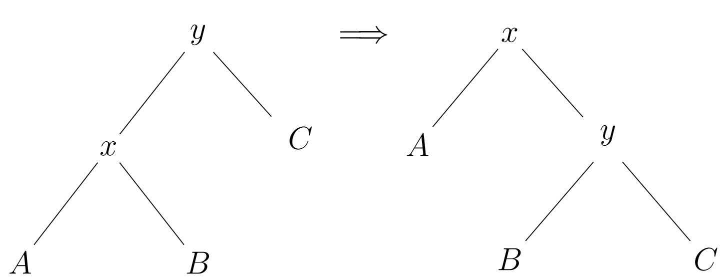
For example, the coefficients of

$$g_4 = S^4 + 3S^{31} + 2S^{22} + S^{13} + 3S^{211} + 2S^{121} + S^{112} + S^{1111} \tag{79}$$

can be read on Figure 1, where one can also easily check the interval property.

Each interval consists in the nonequivalent bracketings of a duplicial product involving the operations $\prec$ and $\succ$ in the same order. For example, the interval $1123 - 1124 - 1134$ consists in the bracketings of the word $1 \prec 1 \succ 1 \succ 1$, which are $1 \prec ((1 \succ 1) \succ 1) = 1123$, $(1 \prec (1 \succ 1)) \succ 1 = 1124$, and $((1 \prec 1) \succ 1) \succ 1 = 1124$. The cover relation is the application of one associativity relation. The other edges of the Hasse diagram are obtained by changing one $\succ$ into a $\prec$. Under the bijection with trees, both operations correspond to a rotation:

(80)



where A, B, and C are subtrees, the difference between both cases being whether the subtree B is or not empty. For example, $1113 = 1 \prec ((1 \prec 1) \succ 1)$ and $1123 = 1 \prec ((1 \succ 1) \succ 1)$.

3.3. **An involution on CQSym.** The natural involution (mirror symmetry) on binary trees induces an involution ι on nondecreasing parking functions, which is similar to (but different from) the involution ν of [22].

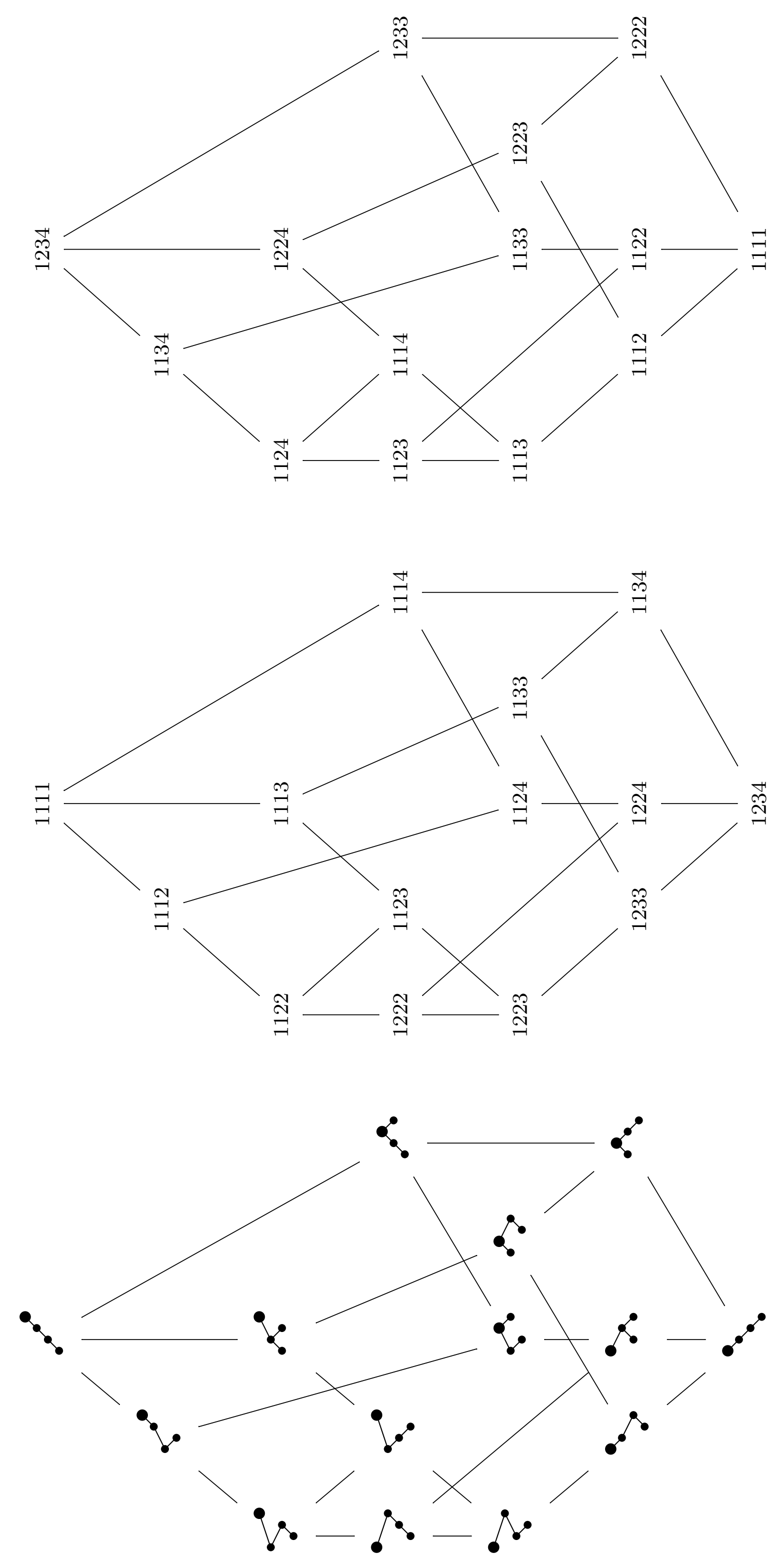


FIGURE 1. The Tamari order on trees, on nondecreasing parking functions regarded as noncrossing partitions, and via the new bijection.

(81)

π	$\iota(\pi)$
1	1
12	11
123	111
113	112
122	122

π	$\iota(\pi)$
1234	1111
1134	1112
1224	1122
1124	1113
1114	1123
1233	1222
1133	1223

Defining a basis $\mathbf{Q}^\pi$ by

$$\mathbf{Q}^\pi = \iota(\mathbf{P}^\pi) \tag{82}$$

we have

$$\mathbf{Q}^{\pi'}\mathbf{Q}^{\pi''} = \mathbf{Q}^{\pi' \prec \pi''}\,. \tag{83}$$

Now, there is a Hopf algebra morphism ϕ : **CQSym** $\to$ **Sym** mapping $\mathbf{P}^\pi$ to $S^{t(\pi)}$, where the composition $t(\pi)$ is the packed evaluation of π. This maps sends G to the noncommutative Lagrange series g [22],

$$\phi(G) = g\,,\ \text{the unique solution of } g = \sum_{n\geq 0} S_n g^n\,. \tag{84}$$

The involution ι is mapped to $S^I \mapsto S^{I^\sim}$. This provides a simple proof of the symmetry of g observed in [22, Sec. 7].

4. Characters

4.1. The formula of Lassalle for the Narayana polynomials.

It has been observed by M. Lassalle [10] that the Narayana polynomials $c_n(q)$ could be expressed in the form

$$qc_n(q) = \frac{1}{n+1} h_n((n+1)q) \tag{85}$$

in λ-ring notation, with the assumption that $x = 1-q$ is of rank 1.

Otherwise said,

$$(1-x)c_n(1-x) = g_n(1-x), \tag{86}$$

the image of the symmetric function $g_n(X)$ by the character of Sym sending the power sum p_n to $1-x^n$. This character is also well-defined on **Sym** [8], and we shall see that it can be lifted to **PQSym**, as well as the two-parameter extension

$$p_n \mapsto p_n\left(\frac{1-x}{1-t}\right) = \frac{1-x^n}{1-t^n}\,. \tag{87}$$

This result can then be given a combinatorial interpretation by following a chain of morphisms whose composition builds up a character of **PQSym**.

4.2. **From parking functions to permutations.** The polynomial realization of the dual $\mathbf{PQSym}^*$ given in [20] implies the existence of an embedding of Hopf algebras

$$i:\ \mathbf{G}_\sigma \longmapsto \sum_{\mathrm{std}(\mathbf{a})=\sigma} \mathbf{G}_{\mathbf{a}} \tag{88}$$

which is actually an inclusion of the polynomial realizations:

$$\mathbf{G}_\sigma(A) = \sum_{\mathrm{std}(\mathbf{a})=\sigma} \mathbf{G}_{\mathbf{a}}(A)\,. \tag{89}$$

Thus, the dual map

$$i^*:\ \mathbf{F}_{\mathbf{a}} \longmapsto \mathbf{F}_{\mathrm{std}(\mathbf{a})} \tag{90}$$

is a surjective morphism of Hopf algebras.

4.3. **From permutations to compositions.** The restriction of i^* to **SQSym** takes its values in **Sym**, and precisely,

$$i^*(\mathbf{P}_{\mathbf{q}}) = R_I \tag{91}$$

where I is the shape of the quasi-ribbon $\mathbf{q}$.

Further restricting to **CQSym** $\subset$ **SQSym** yields

$$i^*(\mathbf{P}^\pi) = S^I \tag{92}$$

where $I = t(\pi)$ is the packed evaluation of π.

4.4. **From compositions to scalars or polynomials.** At the level of **Sym**, we have many characters at our disposal. Since characters take their values in a commutative algebra (our ground field $\mathbb{K}$), they come actually from characters of *Sym*. An important and classical example is evaluation on the virtual alphabet

$$\mathsf{A} = \frac{1-x}{1-q} \text{ given by } p_n(\mathsf{A}) = \frac{1-x^n}{1-q^n}\,. \tag{93}$$

This character can be lifted in several ways to **FQSym**. One of them, denoted by

$$\mathsf{A} = \frac{|1-x}{1-q} \tag{94}$$

is described in [23] (see (96) below).

4.5. **Super-Narayana polynomials.** Recall from [24] that a signed parking function is a pair $(\mathbf{a},\varepsilon)$ where $\mathbf{a}$ is an ordinary parking function and ε a word of the same length over the alphabet $\{\pm 1\}$. There are therefore $2^n(n+1)^{n-1}$ signed parking functions of length n (cf. [26, A097629]). There is a Hopf algebra, denoted by $\mathbf{PQSym}^{(2)}$ based on signed parking functions. The product in its $\mathbf{F}_{(\mathbf{a},\varepsilon)}$ basis is given by the shifted shuffle of signed words.

Definition 4.1. *A signed inversion of a signed parking function* $(\mathbf{a},\varepsilon)$ *is a pair* (i,j) *with* $i<j$ *such that either* $\varepsilon_i a_i > \varepsilon_j a_j$ *(an ordinary inversion) or* $\varepsilon_i a_i = \varepsilon_j a_j$ *and the common sign* ε_i *is* -1*.*

Similarly, a signed descent is an index i *such that either* $\varepsilon_i a_i > \varepsilon_{i+1}a_{i+1}$ *or* $\varepsilon_i a_i = \varepsilon_{i+1}a_{i+1}$ *and the common sign* ε_i *is* -1*.*

The signed major index $\operatorname{smaj}(\mathbf{a},\varepsilon)$ *is the sum of the signed descents.*

Theorem 4.2. *Let*

$$P_n(t,q) = g_n\left(\frac{1-x}{1-q}\right)_{t=-x} = \sum_{i,j} a_n(i,j)t^iq^j \tag{95}$$

(where g_n *is interpreted as a commutative symmetric function). Then,* $a_n(i,j)$ *is equal to the number of signed parking functions* $(\mathbf{a},\varepsilon)$ *of length* n *with* i *minus signs and* j *signed inversions, or with signed major index* j*. In particular, the coefficient of* t^k *in* $P_n(t,0)$ *is equal to the number of nondecreasing signed parking functions* $(\mathbf{a},\varepsilon)$ *with* k *minus signs.*

Proof – The signed inversions (resp. descents) of $(\mathbf{a},\varepsilon)$ are the ordinary inversions (resp. descents) of $(\operatorname{std}(\mathbf{a}),\varepsilon)$. Hence, sinv and smaj have the same distribution over signed parking functions of length n.

Now, on the one hand, it has been shown in [23] that

$$(q)_n\mathbf{F}_\sigma\left(\frac{|1-x}{1-q}\right) = \sum_{\varepsilon\in\{\pm1\}^n}(-x)^{m(\varepsilon)}q^{\operatorname{maj}(\sigma,\varepsilon)}\,, \tag{96}$$

where $m(\varepsilon)$ is the number of minus signs, and maj is computed w.r.t. the usual order of $\mathbb{Z}$.

On the other hand, the map $s:\ \mathbf{PQSym}\to\mathbf{PQSym}^{(2)}$ defined by

$$\mathbf{F}_{\mathbf{a}} \longmapsto \sum_{\varepsilon\in\{\pm1\}^n}\mathbf{F}_{(\mathbf{a},\varepsilon)} \tag{97}$$

is a morphism of algebras, and the map $\phi_{q,x}:\ \mathbf{PQSym}^{(2)}\to\mathbb{K}(q)[x]$ defined by

$$\mathbf{F}_{(\mathbf{a},\varepsilon)} \longmapsto (-x)^{m(\varepsilon)}\frac{q^{\operatorname{smaj}(\mathbf{a},\varepsilon)}}{(q_n)} \tag{98}$$

is a character of $\mathbf{PQSym}^{(2)}$ (also if one replaces smaj by sinv).

Thanks to the characterization of signed inversions (or descents) as ordinary inversions (or descents) of the standardization, we see that

$$\phi_{q,x}\circ s(\mathbf{F}_{\mathbf{a}}) = \sum_{\varepsilon}(-x)^{m(\varepsilon)}\frac{q^{\operatorname{maj}(\operatorname{std}(\mathbf{a}),\varepsilon)}}{(q_n)} = \mathbf{F}_{\operatorname{std}(\mathbf{a})}\left(\frac{|1-x}{1-q}\right)\,. \tag{99}$$

Thus,

$$\phi_{q,x}\circ s(G) = g\left(\frac{1-x}{1-q}\right) \tag{100}$$

where g can now be interpreted as an ordinary (commutative) symmetric function.
∎

For example, in

$$P_2(t,q) = (1+2q)t^2 + (3+3q)t + (2+q) \tag{101}$$

the coefficient of t^0 comes from $11, 12, 21$, the coefficient of t^1 from $\bar{1}1, \bar{1}2, \bar{2}1, 1\bar{1}, 1\bar{2}, 2\bar{1}$ and the coefficient of t^2 from $\bar{1}\bar{1}, \bar{1}\bar{2}, \bar{2}\bar{1}$. Similarly, in

$$\begin{aligned} P_3(t,q) = {} & \left(5\,q^2 + 5\,q + 5\,q^3 + 1\right) t^3 + \left(10\,q^3 + 16\,q^2 + 16\,q + 6\right) t^2 \\ & + \left(6\,q^3 + 16\,q^2 + 16\,q + 10\right) t + q^3 + 5\,q^2 + 5\,q + 5, \end{aligned} \tag{102}$$

the constant term in q can be explained as follows. There are 5 parking functions of standardized 123, 3 for 132 and 312, 2 for 213 and 231 and 1 for 321. To get an increasing word, 123 can be signed as 123 and $\bar{1}23$, contributing 5 times $1+t$. Similarly, 132 does not contribute and 312, yields $3\times(t+t^2)$ corresponding to $\bar{3}12$ and $\bar{3}\bar{1}2$. 231 does not contribute and 213 gives $2\times(t+t^2)$. Finally, 321 contributes a term t^2+t^3 and we end up with

$$\begin{aligned} 5(1+t) + 5(t+t^2) + (t^2+t^3) = 5 + 10t + 6t^2 + t^3 &= (t+1)((t+1)^2 + 3(t+1) + 1) \\ &= (t+1)c_3(t+1)\,. \end{aligned} \tag{103}$$

The corresponding signed parking functions are given by an encoding of Schröder paths extending the classical encoding of Dyck paths by nondecreasing parking functions. A Dyck path is a sequence of steps $u=(1,1)$ (up) and $d=(1,-1)$ (down), going from $(0,0)$ to $(2n,0)$ without crossing the horizontal axis. Encoding each up step by the number of its diagonal, we obtain a nondecreasing parking function. For example, *uuududdudd* yields 11124.

Now, Schröder paths are obtained from Dyck paths by replacing some peaks (factors *ud*) by a horizontal step $h=(2,0)$. If we replace k by $\bar{k}$ when a peak with up step in diagonal k has been replaced by a horizontal step, we obtain an encoding of Schröder paths by (some) signed nondecreasing parking functions. Sorting these words according to the order of the signed alphabet, we obtain precisely those signed parking functions which have no signed inversion. For example, replacing the first and last peaks in the previous example by horizontal steps, we obtain the path *uuhuddhd*, encoded by $11\bar{1}2\bar{4}$, whose sorting is $\bar{4}\bar{1}112$ (see Figure 2).

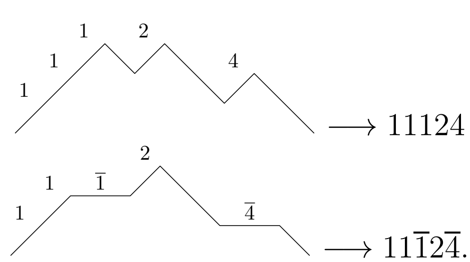


FIGURE 2. Encoding of Dyck and Schröder paths.

The coefficients of the polynomials $P_n(t) := P_n(t,0)$ form the sequence (triangle $T(n,k)$) [26, A060693]. Since $P_n(-t)$ is the coefficient of z^n in $g((1-t)z)$, unique solution of the functional equation

$$g = \sum_{n\geq 0} S_n(1-t)z^n g^n = 1 + (1-t)\frac{zg}{1-zg} \tag{104}$$

we have the generating series

$$\sum_{n\geq 0} z^n P_n(t) = \frac{1-tz-\sqrt{(1-tz)^2-4z}}{2z} \tag{105}$$

which coincides with that of [26, A060693] (up to the constant term, which is 1 for us).

Corollary 4.3. *The coefficient of t^k in $P_n(t)$ is the number of Schröder paths of semilength n with k horizontal steps. In particular, we recover the formula of Lassalle, since it is known that the generating polynomial for horizontal steps in Schröder paths is $(t+1)c_n(t+1)$. Hence*

$$tc_n(t) = P_n(t-1)\,. \tag{106}$$

∎

4.6. **A character of SQSym.** The character yielding the Narayana polynomials has a simple description at the level of **SQSym**. Let $b(\mathbf{q})$ be the number of bars of a parking quasi-ribbon $\mathbf{q}$. Then, $\chi(\mathbf{P_q}) = (1+t)t^{b(\mathbf{q})}$ is a character of **SQSym**.

Let again $G(z) = \sum_{\mathbf{q}} z^{|\mathbf{q}|}\mathbf{P_q} = \sum_{\mathbf{a}} z^{|\mathbf{a}|}\mathbf{F_a}$ be the sum of all parking functions (with a homogeneity variable z). Then,

$$\chi(G(z)) = g(z(1-x))|_{x=-t} \tag{107}$$

so that

$$\chi(G_n) = (1+t)c_n(1+t)\,. \tag{108}$$

Thus, $c_n(1+t)$ counts parking quasi-ribbons according to the number of bars. As we have already seen, the classical interpretation of $c_n(1+t)$ is in terms of Schröder paths (the coefficient of t^k is the number of paths of semilength n containing exactly k peaks but no peaks at level one [26, A126216]). Thus, parking quasi-ribbons with k bars are in bijection with these paths.

A simpler interpretation still in terms of Schröder paths is: the number of paths with k horizontal steps, and no peak after the last horizontal step.

Replacing the last horizontal step by a peak gives the missing paths, wich explains the factor $(1+t)$.

4.7. **Other characters.** Any character of **Sym** (hence also of *Sym*) can be lifted to **PQSym**, thanks to the following property:

Lemma 4.4. *The map ψ : **PQSym** $\to$ **Sym** defined by*

$$\psi(\mathbf{F_a}) = \frac{1}{n!}S^{t(\mathbf{a})} \quad \textit{for } \mathbf{a}\in \mathrm{PF}_n \tag{109}$$

where $t(\mathbf{a})$ is the packed evaluation of $\mathbf{a}$, is a morphism of algebras.

Proof – Let $\mathbf{a} \in \mathrm{PF}_n$ andb $\mathbf{b} \in \mathrm{PF}_m$, $I = t(\mathbf{a})$ and $J = t(\mathbf{b})$. Then, the product $\mathbf{F_a F_b}$ contains $\frac{(n+m)!}{n!m!}$ terms $\mathbf{F_c}$, which all have packed evaluation $t(\mathbf{c}) = I \cdot J$. ∎

Of course, this is not a morphism of coalgebras, as this would imply by duality an embedding of $QSym$ into a free associative algebra.

As an illustration of this principle, let us consider the classical character of Sym, given by evaluation on a binomial element. Recall that the symmetric functions of a binomial element α are given by

$$e_n(\alpha) = \binom{\alpha}{n} \text{ or, equivalently, } h_n(\alpha) = \binom{\alpha+n-1}{n},\ p_n(\alpha) = \alpha\,. \tag{110}$$

Thus,

$$\frac{1}{n!}S^I(\alpha) = \prod_k \frac{1}{i_k!} \cdot \frac{\prod_k \prod_{j_k=1}^{i_k}(\alpha + j_k - 1)}{n!} = \chi(S^I) \cdot \frac{1}{n!} Z_I(\alpha) \tag{111}$$

where χ is the character of **Sym** given by $S_n \mapsto \frac{1}{n!}$, and

$$Z_I(\alpha) = \sum_{\sigma \in \mathrm{Stab}(w)} \alpha^{\#\mathrm{cycles}(\sigma)} \tag{112}$$

is the cycle enumerator of the stabilizer of any word w of (packed) evaluation I. Dividing by $\chi(S^I)$, we have:

Proposition 4.5. *The map $\psi_\alpha :$ **PQSym** $\to \mathbb{K}[\alpha]$ defined by*

$$\psi_\alpha(\mathbf{F_a}) = \frac{1}{n!} Z_{t(\mathbf{a})}(\alpha) \tag{113}$$

is a character of **PQSym**. ∎

The image of the sum of all parking functions of length n is thus

$$\psi_\alpha(G_n) = g_n(\alpha) = \frac{1}{n+1} h_n((n+1)\alpha) = \frac{1}{n+1}\binom{(n+1)\alpha + n - 1}{n} = \frac{P_n(\alpha)}{n!} \tag{114}$$

where

$$P_n(\alpha) = \alpha \prod_{k=1}^{n-1} ((n+1)\alpha + k)\,. \tag{115}$$

From the previous considerations, we obtain a combinatorial interpretation of the coefficients of $P_n(\alpha)$:

Corollary 4.6. *The coefficient of α^k in $P_n(\alpha)$ is equal to the number of pairs $(\mathbf{a}, \sigma)$ where $\mathbf{a} \in \mathrm{PF}_n$ and σ is a permutation in $\mathfrak{S}_n$ with k cycles, such that $\mathbf{a}\sigma = \mathbf{a}$.* ∎

The first polynomials P_n are

$$P_1(\alpha) = \alpha, \tag{116}$$
$$P_2(\alpha) = 3\alpha^2 + \alpha, \tag{117}$$
$$P_3(\alpha) = 16\alpha^3 + 12\alpha^2 + 2\alpha, \tag{118}$$
$$P_4(\alpha) = 125\alpha^4 + 150\alpha^3 + 55\alpha^2 + 6\alpha. \tag{119}$$

We can observe a curious property of the reciprocal polynomials evaluated at $q-1$: if we set

$$Q_n(q) = (q-1)^{n-1} P_n\left(\frac{1}{q-1}\right) = \prod_{k=2}^{n}((n+1-k)q+k) \tag{120}$$

we obtain a q-analogue of $(n+1)^{n-1}$ whose coefficients form the triangle

$$\begin{array}{cccc} 1 & & & \\ 2 & 1 & & \\ 6 & 8 & 2 & \\ 24 & 58 & 37 & 6 \end{array} \tag{121}$$

The first column is clearly $n!$. The second column is sequence A002538 in the OEIS [26], which counts the number of edges in the Hasse diagram of the Bruhat order on $\mathfrak{S}_n$. Since the first column is the number of vertices, one may wonder whether the other numbers also have an interpretation in this context. This will be left as an open problem.

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