DERIVED RAMANUJAN PRIMES: R'_N

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Abstract

In this article, we study the Ramanujan-prime-counting function $\pi_R(x)$ along the lines of Ramanujan's original work on Bertrand's Postulate. We show that the number of Ramanujan primes R_n between x and $2x$ tends to infinity with x. This analysis leads us to define a new sequence of prime numbers, which we call derived Ramanujan primes R'_n. For $n \geq 1$ we define the nth derived Ramanujan prime as the smallest positive integer R'_n with the property that if $x \geq R'_n$ then $\pi_R(x) - \pi_R\left(\frac{x}{2}\right) \geq n$. As an application of the existence of derived Ramanujan primes, we prove analogues for Ramanujan primes of Richert's Theorem and Greenfield's Theorem for primes. We give some new inequalities for both the prime-counting function $\pi(x)$ and for $\pi_R(x)$. Following the recent works of Sondow and Laishram on the bounds of Ramanujan primes, we analyze the bounds of derived Ramanujan primes. Finally, we give another proof of the theorem of Amersi, Beckwith, Miller, Ronan and Sondow, which states that if $c \in (0, 1)$, then the number of primes in the interval (cx, x) tends to infinity with x.

1. Introduction

In 1919 Srinivasa Ramanujan [7] gave an elegant proof of Bertrand's Postulate, which states that there exists a prime number between n and $2n$ for all $n \geq 2$. In the process he showed the existence of a certain sequence of prime numbers, now known as Ramanujan primes. Recall that $\pi(x)$ is the prime counting function, that is, $\pi(x)$ is the number of primes less than or equal to x. In 2009 Jonathan Sondow gave the following definition in [15]:

For $n \geq 1$, the nth Ramanujan prime is the smallest positive integer R_n with the property that if $x \geq R_n$, then $\pi(x) - \pi\left(\frac{x}{2}\right) \geq n$.

As an example, if $n = 1,\ 2,\ 3,\ 4,\ 5,\ 6, \ldots$, then the nth Ramanujan prime $R_n = 2,\ 11,\ 17,\ 29,\ 41,\ 47, \ldots$ (**A104272** in [14]). After that he proved that the nth Ramanujan prime R_n lies between the $2n$th and $4n$th prime for all $n \geq 2$. He also showed that $R_n \sim p_{2n}$ as $n \to \infty$, and that for every $\varepsilon > 0$, there exists $N_0(\varepsilon)$ such that $R_n < (2+\varepsilon)\, n \ln n$ for $n \geq N_0(\varepsilon)$. Shanta Laishram in [5] improved Sondow's result by showing that the nth Ramanujan prime does not exceed the $3n$th prime. In Theorem 1 of [5] Laishram also gave a method to calculate $N_0(\varepsilon)$. Following these theorems, we *denote by* $\pi_R(x)$ *the number of Ramanujan primes which do not exceed* x and we show the existence of derived Ramanujan primes R'_n with the similar definition:

For $n \geq 1$, the nth derived Ramanujan prime is the smallest positive integer R'_n with the property that if $x \geq R'_n$, then $\pi_R(x) - \pi_R\left(\frac{x}{2}\right) \geq n$. In other words, there holds

$$\pi_R(x) - \pi_R\left(\frac{x}{2}\right) \geq 1,\ 2,\ 3,\ 4,\ 5, \ldots, \ if\ x \geq 11,\ 41,\ 59,\ 97,\ 149 \ldots.$$

(*A192820 [14]*)

Note that the derived Ramanujan primes denoted in (A192820 [14]) as 2-Ramanujan primes. The existence of R'_n also means that the number of Ramanujan primes between x and $2x$ tends to infinity with x. This proof makes it possible to give some applications to Ramanujan primes of Bertrand's Postulate, Richert's Theorem [8] and Greenfield's Theorem [4] on primes. After that we extend Rosser and Schoenfeld's Theorem $2\pi(x) \geq \pi(2x)$ to the Ramanujan-prime-counting function $\pi_R(x)$ by proving that $2\pi_R(x) \geq \pi_R(2x)$, with the help of Segal's idea [11]. This makes it possible to prove that the nth derived Ramanujan prime lies between the $2n$th Ramanujan prime and the $3n$th Ramanujan prime, and also that $R'_n \sim R_{2n} \sim p_{4n}$. In [16] J. Sondow, J. W. Nicholson and T. D. Noe made an analysis of bounds and runs of Ramanujan primes and showed that if an upper twin prime is Ramanujan, then so is the lower. In [12] V. Shevelev studied some parallel properties of Ramanujan primes and Labos primes and gave generalizations with the construction of two kinds of sieves for them. Recently, N. Amersi, O. Beckwith, S. J. Miller, R. Ronan and J. Sondow [1] gave another generalization of Ramanujan primes which states that for any $c \in (0,1)$, the nth c-Ramanujan prime can be defined as the smallest integer $R_{c,n}$ such that for all $x \geq R_{c,n}$, there are at least n primes in the interval $(cx, x]$. They also showed that $R_{c,n} \sim p_{\frac{n}{1-c}}$ as n tends to infinity. In the last section we give another proof of the existence of c-Ramanujan primes.

2. Derived Ramanujan Primes and Two Applications

We begin this section with a useful corollary of a theorem of Sondow. Then we show the existence of derived Ramanujan primes and analogues of Richert's Theorem and Greenfield's Theorem for Ramanujan primes.

Theorem 1. (Sondow [15]) *For every $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon)$ such that*

$$R_n < (2+\varepsilon)\, n \ln n \qquad (n \geq n_0)\,.$$

Corollary 1. *For all $n \geq N_\varepsilon$ and $x \geq R_n$, there hold the inequalities*

$$\frac{\pi(x)}{2} > \pi_R(x) > \frac{\pi(x)}{2+\varepsilon}. \tag{1}$$

Proof. By Sondow's inequality $R_n > p_{2n}$ for $n > 1$, the left side of (1) must hold because if $\pi_R(x) = n$, then $\pi(x)$ must be greater than $2n$. Now we will prove right side of (1). Let $R_{n+1} > x \geq R_n$, that is, $\pi_R(x) = n$. It is enough to show that $(2+\varepsilon)\, n > \pi(x)$. By Theorem 1 it follows that

$$\pi(x) < \pi(R_{n+1}) \leq \pi((2+\varepsilon)(n+1)\ln(n+1))\,. \tag{2}$$

Now take $(2+\varepsilon)(n+1)\ln(n+1) = k$. For every ε and $n \geq 10$ it is easy to see that the inequality

$$\ln(n+1) < n\left(\ln(k) - \ln(n+1) - 1.2762\right) \tag{3}$$

holds. Hence

$$1.2762n < n \ln k - (n+1)\ln(n+1) \tag{4}$$

and

$$\left(1 + \frac{1.2762n}{n \ln k}\right)\left(1 - \frac{n\ln(k) - (n+1)\ln(n+1)}{n \ln k}\right) < 1. \tag{5}$$

One can check that (5) holds for $n \geq 5$. As we have $1 - \frac{n\ln(k)-(n+1)\ln(n+1)}{n\ln k} = \frac{(n+1)\ln(n+1)}{n \ln k}$ we get

$$\frac{k}{\ln k}\left(1 + \frac{1.2762}{\ln k}\right) < (2+\varepsilon)\, n \tag{6}$$

and by Dusart's inequality [3] for $x > 1$

$$\pi(x) \leq \frac{x}{\ln x}\left(1 + \frac{1.2762}{\ln x}\right) \tag{7}$$

and (2), the inequalities

$$\pi(x) < \pi(R_{n+1}) \leq \pi(k) < (2+\varepsilon)\, n \tag{8}$$

hold for $n \geq 5$, and by computer check also for any $\varepsilon > 0$, with $n \geq N_\varepsilon$ and $x \geq R_n$. □

Theorem 2. *There exists at least one Ramanujan prime between $\frac{x}{2}$ and x, for all $x \geq 11$. Moreover, the number of Ramanujan primes in the interval $\left(\frac{x}{2}, x\right]$, which is $\pi_R(x) - \pi_R\left(\frac{x}{2}\right)$, tends to infinity with x.*

Proof. By P. Dusart's [3] inequalities

$$\frac{x}{\ln x}\left(1+\frac{1.2762}{\ln x}\right) \underset{x>1}{>} \pi(x) \underset{x\geq 599}{\geq} \frac{x}{\ln x}\left(1+\frac{1}{\ln x}\right) \tag{9}$$

and Corollary 1 we obtain for all $x \geq 599$

$$\frac{x}{2\ln x}\left(1+\frac{1.2762}{\ln x}\right) > \frac{\pi(x)}{2} > \pi_R(x) \geq \frac{\pi(x)}{3} > \frac{x}{3\ln(x)}\left(1+\frac{1}{\ln(x)}\right). \tag{10}$$

Therefore the inequalities

$$\pi_R(x) - \pi_R\left(\frac{x}{2}\right) > \frac{x}{3\ln x}\left(1+\frac{1}{\ln x}\right) - \frac{x}{4\ln\frac{x}{2}}\left(1+\frac{1.2762}{\ln\frac{x}{2}}\right) \geq \frac{x}{\ln x}\left(\frac{1}{12}-\frac{0.3}{\ln x}\right) \tag{11}$$

hold for all $x \geq 599$, where the right side of the last inequality tends to infinity with x. To verify that there exists at least one Ramanujan prime between $\frac{x}{2}$ and x for $11 \leq x \leq 599$, it is enough to see that there exists one of the Ramanujan primes 11, 17, 29, 47, 71, 127, 241 and 461 between $\frac{x}{2}$ and x. □

Since $\pi_R(x) - \pi_R\left(\frac{x}{2}\right)$ is greater than the monotone increasing function in (11), the number of Ramanujan primes between $\frac{x}{2}$ and x tends to infinity with x. As a result, derived Ramanujan primes exist.

n	R'_n	n	R'_n	n	R'_n	n	R'_n	n	R'_n
1	11	11	263	21	599	31	1009	41	1373
2	41	12	307	22	641	32	1019	42	1423
3	59	13	367	23	643	33	1021	43	1427
4	97	14	373	24	647	34	1031	44	1439
5	149	15	401	25	653	35	1049	45	1481
6	151	16	409	26	719	36	1051	46	1487
7	227	17	569	27	751	37	1061	47	1549
8	229	18	571	28	821	38	1063	48	1553
9	233	19	587	29	937	39	1217	49	1559
10	239	20	593	30	941	40	1367	50	1567

Table 1. The First 50 Derived Ramanujan Primes

In 1948 Hans-Egon Richert [8] proved that each natural number $n \geq 7$ can be expressed as a sum of distinct primes. His method has been generalized by Sierpinski, who showed the following theorem.

Theorem 3. (Sierpinski [13]) *Let $m_1, m_2, \ldots$ be an infinite increasing sequence of natural numbers such that for a certain natural number k the inequality*

$$m_{i+1} \leq 2m_i \quad \textit{for} \quad i > k \tag{12}$$

holds. If there exists an integer $a \geq 0$ and natural numbers r and $s_{r-1} \geq m_{k+r}$ such that each of the numbers

$$a+1, \quad a+2, \ \ldots \ , \quad a+s_{r-1}$$

is the sum of different numbers of the sequence $m_1, m_2, \ldots, m_{k+r-1}$, then for $s_r = s_{r-1} + m_{k+r}$ each of the numbers

$$a+1, \quad a+2, \ \ldots \ , \quad a+s_r$$

is the sum of different numbers of the sequence $m_1, m_2, \ldots, m_{k+r}$, and moreover $s_r \geq m_{k+r+1}$.

Corollary 2. *Each natural number $n \geq 123$ can be expressed as a sum of distinct Ramanujan primes.*

Proof. Let $m_i = R_i$, $k = 0$, $r = 10$, $a = 122$ and $s_9 = 97$. There exists at least one Ramanujan prime between x and $2x$ for $x \geq 11$ by Theorem 2. So we get $R_i < R_{i+1} < 2R_i$ for all natural numbers $i \geq 2$ and this implies the condition (12). From Table 2 it can be seen that each number from 123 to 224 is the sum of different Ramanujan primes $R_1, R_2, \ldots, R_9$. So each natural number greater than 123 can be expressed as a sum of distinct Ramanujan primes. $\square$

a+j	Expression	a+j	Expression	a+j	Expression
123	$71+41+11$	157	$71+67+17+2$	191	$101+71+17+2$
124	$67+29+17+11$	158	$71+59+17+11$	192	$71+67+41+11+2$
125	$71+41+11+2$	159	$71+59+29$	193	$59+47+41+29+17$
126	$67+59$	160	$71+59+17+11+2$	194	$71+59+47+17$
127	$67+47+11+2$	161	$71+59+29+2$	195	$71+67+29+17+11$
128	$71+29+17+11$	162	$67+47+29+17+2$	196	$71+67+47+11$
129	$71+47+11$	163	$59+47+29+17+11$	197	$71+67+59$
130	$71+59$	164	$71+47+29+17$	198	$71+67+47+11+2$
131	$67+47+17$	165	$59+47+29+17+11+2$	199	$71+67+59+2$
132	$71+59+2$	166	$71+67+17+11$	200	$71+59+41+29$
133	$67+47+17+2$	167	$71+67+29$	201	$71+59+41+17+11+2$
134	$59+47+17+11$	168	$71+67+17+11+2$	202	$71+67+47+17$
135	$71+47+17$	169	$71+67+29+2$	203	$67+59+47+17+11+2$
136	$59+47+17+11+2$	170	$71+59+29+11$	204	$71+67+47+17+2$
137	$71+47+17+2$	171	$71+59+41$	205	$71+59+47+17+11$
138	$71+67$	172	$71+59+29+11+2$	206	$71+59+47+29$
139	$67+59+11+2$	173	$71+59+41+2$	207	$71+67+41+17+11$
140	$71+67+2$	174	$67+59+29+17+2$	208	$71+67+59+11$
141	$71+59+11$	175	$67+59+47+2$	209	$71+67+41+17+11+2$
142	$71+41+17+11+2$	176	$59+47+41+29$	210	$71+67+41+29+2$
143	$71+59+11+2$	177	$59+47+41+17+11+2$	211	$71+59+41+29+11$
144	$67+47+17+11+2$	178	$71+67+29+11$	212	$67+47+41+29+17+11$
145	$67+59+17+2$	179	$71+67+41$	213	$71+67+47+17+11$
146	$71+47+17+11$	180	$71+67+29+11+2$	214	$71+67+59+17$
147	$71+59+17$	181	$71+67+41+2$	215	$71+67+47+17+11+2$
148	$71+47+17+11+2$	182	$71+59+41+11$	216	$71+67+59+17+2$
149	$71+67+11$	183	$67+59+29+17+11$	217	$71+59+47+29+11$
150	$67+41+29+11+2$	184	$71+67+29+17$	218	$71+59+47+41$
151	$71+67+11+2$	185	$71+67+47$	219	$71+67+41+29+11$
152	$71+41+29+11$	186	$71+67+29+17+2$	220	$71+59+47+41+2$
153	$101+41+11$	187	$71+67+47+2$	221	$71+67+41+29+11+2$
154	$71+41+29+11+2$	188	$71+59+47+11$	222	$97+71+41+11+2$
155	$71+67+17$	189	$71+59+29+17+11+2$	223	$71+59+47+29+17$
156	$67+59+17+11+2$	190	$71+67+41+11$	224	$67+59+41+29+17+11$

Table 2. Expressions of Natural Numbers between 123 and 224 as Sums of Different Ramanujan Primes

In [4] L. Greenfield and S. Greenfield showed that the integers $\{1,\ 2,\ \ldots,\ 2k\}$ can be arranged in k disjoint pairs such that the sum of the elements in each pair is prime. Similar result can be shown for Ramanujan primes with their method.

Corollary 3. *For all integers $k \geq 17$ the numbers $\{1,\ 2,\ \ldots,\ 2k\}$ can be arranged in k disjoint pairs such that the sum of the elements in each pair is a Ramanujan prime.*

Proof. From Table 3 it can be seen for $k = 17$ that our assumption is true. There exists at least one Ramanujan prime between $2k$ and $4k$ for $k \geq 3$ by Theorem 2. Now let $j \geq 17$ and $2k + j$ be a Ramanujan prime. Therefore $\{j,\ j+1,\ \ldots,\ 2k-1,\ 2k\}$ can be paired as sum of each pair will be equal to $2k+j$, namely $\{j, 2k\}$, $\{j+1, 2k-1\}$, $\{j+2, 2k-2\}$, ..., $\left\{\left\lfloor \frac{j+2k}{2} \right\rfloor, \left\lfloor \frac{j+2k}{2} \right\rfloor + 1\right\}$. Also, by induction $\{1,\ 2,\ \ldots,\ j-1\}$ can be arranged in disjoint pairs if $j-1 \geq 34$. So it is enough to show that we can always find such an odd natural number j or equivalently that there exist a Ramanujan prime in the interval $(2k+34, 4k)$. One can easily check that $\{1,\ 2,\ \ldots,\ 2k\}$ can be arranged in k disjoint pairs as $k \leq 17$ only for $k \in \{5, 6, 8, 9, 11, 12, 14, 15, 17\}$. Some certain arrangements given in Table 3. From Table 3 it can be seen if $j - 1 \in M = \{10,\ 12,\ 16,\ 18,\ 22,\ 24,\ 28,\ 30\}$ or by induction hypothesis if ≥ 34 that there is a way to pair the set. So there is no solution if and only if $j - 1 \in N = \{2,\ 4,\ 6,\ 8,\ 14,\ 20,\ 26,\ 32\}$. But as $R_9\,(2) = 233$ there must be least 9 choices for j if $k \geq 117$. So all solutions can not be from N. By Table 3 our statement is also verified for $17 \leq k < 117$. □

k	5	6	8	9	11	12	14	15	17
	$1,10$	$1,10$	$1,16$	$1,10$	$1,10$	$1,10$	$1,10$	$1,10$	$1,10$
	$2,9$	$2,9$	$2,15$	$2,9$	$2,9$	$2,9$	$2,9$	$2,9$	$2,9$
	$3,8$	$3,8$	$3,14$	$3,8$	$3,8$	$3,8$	$3,8$	$3,8$	$3,8$
	$4,7$	$4,7$	$4,13$	$4,7$	$4,7$	$4,7$	$4,7$	$4,7$	$4,7$
	$5,6$	$5,12$	$5,12$	$5,6$	$5,6$	$5,6$	$5,6$	$5,6$	$5,6$
		$6,11$	$6,11$	$11,18$	$11,18$	$11,18$	$11,18$	$11,18$	$11,18$
			$7,10$	$12,17$	$12,17$	$12,17$	$12,17$	$12,17$	$12,17$
			$8,9$	$13,16$	$13,16$	$13,16$	$13,16$	$13,16$	$13,16$
				$14,15$	$14,15$	$14,15$	$14,15$	$14,15$	$14,15$
					$19,22$	$19,22$	$19,28$	$19,28$	$19,22$
					$20,21$	$20,21$	$20,27$	$20,27$	$20,21$
						$23,24$	$21,26$	$21,26$	$23,24$
							$22,25$	$22,25$	$25,34$
							$23,24$	$23,24$	$26,33$
								$29,30$	$27,32$
									$28,31$
									$29,30$

Table 3. Partitions of Sets $\{1, 2, \ldots, 2k\}$ for Certain Numbers k up to 17

3. Some Inequalities for $\pi(x)$

In this section we prove some inequalities for the prime-counting function by using Dusart's inequalities to show $2\pi_R(x) > \pi_R(2x)$ and get better bounds for derived Ramanujan primes.

Lemma 1. *For $x \geq 569$ the inequality*

$$\pi(2x) - \pi(x) \leq 2\left(\pi(x) - \pi\left(\frac{x}{2}\right)\right)$$

holds.

Proof. By P. Dusart's [3] inequalities

$$\frac{x}{\ln x - 1} \underset{x \geq 5393}{\leq} \pi(x) \underset{x \geq 60184}{\leq} \frac{x}{\ln x - 1.1} \tag{13}$$

it is enough to show that

$$\frac{x}{\ln \frac{x}{2} - 1.1} + \frac{2x}{\ln 2x - 1.1} \leq \frac{3x}{\ln x - 1}. \tag{14}$$

Therefore we deduce that

$$\frac{x}{\ln \frac{x}{2} - 1.1} + \frac{2x}{\ln 2x - 1.1} \leq \frac{x}{\ln x - 1.8} + \frac{2x}{\ln x - 0.41} = \frac{3x\ln x - 4.01x}{\ln^2 x - 2.21\ln x + 0.738} \tag{15}$$

and for $x \geq \exp 4.72631 \geq 112.877$

$$\frac{3x\ln x - 4.01x}{\ln^2 x - 2.21\ln x + 0.738} \leq \frac{3x}{\ln x - 1}. \tag{16}$$

By computer check we also verify our statement for $569 \leq x \leq 60184$. □

In [10] Rosser and Schoenfeld showed that for $x \geq 20.5$ the inequality $\pi(2x) - \pi(x) > \frac{3}{5}\frac{x}{\ln x}$ holds. In [3] Dusart improved this result and showed that the inequality $\pi(2x) - \pi(x) > \frac{x}{\ln x} - \frac{0.7x}{\ln^2 x}$ holds for $x \geq 1328.5$. In [5] Laishram showed that $\pi(x) - \pi\left(\frac{x}{2}\right) > \frac{x}{2\ln x} - \frac{0.010182x}{\ln^2 x}$ for $x \geq 21088222$ by using Dusart's inequality $|\vartheta(x) - x| \leq \frac{0.006788x}{\ln x}$, where $\vartheta(x)$ denotes Chebyshev function, equal to $\sum\limits_{p \leq x} \ln p$. In [3] Dusart gave better inequalities for $\vartheta(x)$. Following Laishram's proof we will improve the bound for $\pi(x) - \pi\left(\frac{x}{2}\right)$ to get a better bound in Lemma 4.

Lemma 2. *For any $x \geq 75374781$ the inequality*

$$\pi(x) - \pi\left(\frac{x}{2}\right) > \frac{x}{2\ln x}\left(1 - \frac{31.24}{\ln^3 x}\right)$$

holds.

Proof. By P. Dusart's [3] inequality

$$|\vartheta(x) - x| \leq \frac{10x}{\ln^3 x} \tag{17}$$

for any $x \geq 32321$ we get

$$\pi(x) - \pi\left(\frac{x}{2}\right) \geq \frac{\vartheta(x) - \vartheta\left(\frac{x}{2}\right)}{\ln x} \geq \frac{x}{\ln x}\left(1 - \frac{10}{\ln^3 x} - \frac{1}{2}\left(1 + \frac{10}{\ln^3 \frac{x}{2}}\right)\right) \tag{18}$$

and for $x \geq 75374781$

$$\geq \frac{x}{2\ln x}\left(1 - \frac{31.24}{\ln^3 x}\right) \tag{19}$$

holds. □

4. Bounds for Derived Ramanujan Primes

To prove a similar result to J. B. Rosser and L. Schoenfeld's inequality [9] $2\pi(x) > \pi(2x)$ for Ramanujan primes, namely, $2\pi_R(x) > \pi_R(2x)$, we will use the idea of S. L. Segal [11].

Lemma 3. *Let k and l be positive integers. The following two conditions are equivalent:*
(i) $R_k + R_l \leq R_{k+l-1}$.
(ii) If $R_{k-1} \leq x < R_k$ and $R_{l-1} \leq y < R_l$, then the inequality

$$\pi_R(x+y) \leq \pi_R(x) + \pi_R(y)$$

holds.

Proof. **(i)⇒(ii):** By the conditions on x and y it is easy to see that $x + y < R_k + R_l$ and $\pi_R(x+y) \leq \pi_R(R_k + R_l - 1)$. Likewise, one can check that

$$\pi_R(R_{k+l-2}) = k + l - 2 = \pi_R(R_{k-1}) + \pi_R(R_{l-1}) \leq \pi_R(x) + \pi_R(y). \tag{20}$$

By (i) we get $R_k + R_l - 1 \leq R_{k+l-1} - 1$ and easily

$$\pi_R(x+y) \leq \pi_R(R_k + R_l - 1) \leq \pi_R(R_{k+l-1} - 1) \tag{21}$$

$$= \pi_R(R_{k+l-2}) \leq \pi_R(x) + \pi_R(y). \tag{22}$$

(ii)⇒(i): Set $x = R_k - \frac{1}{2}$ and $y = R_l - \frac{1}{2}$. Therefore we get $\pi_R(x) + \pi_R(y) = k + l - 2$ and $\pi_R(x+y) = \pi_R(R_k + R_l - 1)$. By (ii) we deduce that $k + l - 2 \geq \pi_R(R_k + R_l - 1)$ and $R_{k+l-1} - 1 \geq R_k + R_l - 1$. □

Theorem 4. *For $x \geq 11$ the inequality*

$$2\pi_R(x) > \pi_R(2x)$$

holds.

Proof. By Lemma 3 it is enough to show $2R_n \leq R_{2n-1}$. But that is equivalent to $\pi_R(2R_n - 1) \leq \pi_R(R_{2n} - 1)$, i.e., $2R_n \leq R_{2n}$. There we will use the idea of the proof of Theorem 2 in [15] and we will show that the inequality

$$\pi(2R_n) - \pi(R_n) \leq 2n \tag{23}$$

holds. By Lemma 1 we easily deduce that

$$\pi(2R_n) - \pi(R_n) \leq 2\left(\pi(R_n) - \pi\left(\frac{R_n}{2}\right)\right) = 2n. \qquad \square \tag{24}$$

Lemma 4. *The nth Ramanujan prime satisfies the inequality*

$$R_n < \frac{8}{3} n \ln n$$

for any $n \geq 5315$.

Proof. It is enough to show that $\pi(x) - \pi\left(\frac{x}{2}\right) > n$ if $x \geq \frac{8}{3} n \ln n$. We have

$$\frac{x}{\ln x} \geq \frac{8n \ln n}{3 \ln\left(\frac{8}{3} n \ln n\right)} > 2.011n \tag{25}$$

for all $n \geq 2193650$. By Lemma 2 we deduce that

$$\pi(x) - \pi\left(\frac{x}{2}\right) \geq \frac{x}{2\ln x}\left(1 - \frac{31.24}{\ln^3 x}\right) \geq 1.0055n\left(1 - \frac{31.24}{\ln^3 x}\right) \tag{26}$$

where $1 - \frac{31.24}{\ln^3 x} > \frac{1}{1.0055}$ for $x \geq 75374781$. As $\pi(x) - \pi\left(\frac{x}{2}\right) > n$ for $x \geq 75374781$ and $R_{2113924} = 75374791$, we may take $R_{m+1} > x \geq R_m$ for $m \geq 2113924$. So our statement is true for $n \geq 2113924$. By computer check we see that our statement is also true for $5315 \leq n < 2113924$. □

Theorem 5. *The nth derived Ramanujan prime satisfies the inequalities*

$$R_{2n} \leq R'_n < R_{3n} \tag{27}$$

for any $n \geq 1$.

Proof. For $n = 1$, the inequalities hold. If $n > 1$, to prove the left side of (27), it is enough to show that $\pi_R(R_{2n}) - \pi_R\left(\frac{R_{2n}}{2}\right) \leq n$. By Theorem 4 we can see that

$$2\pi_R\left(\frac{R_{2n}}{2}\right) \geq \pi_R(R_{2n}) = 2n \tag{28}$$

holds. As $R_{2\cdot 1} = 11$, by (28) the inequality

$$\pi_R(R_{2n}) - \pi_R\left(\frac{R_{2n}}{2}\right) \leq 2n - n = n \tag{29}$$

holds for any $n \geq 1$. Now by Sondow's Theorem and Rosser's Theorem we deduce that

$$4n \ln 4n \leq p_{4n} < R_{2n} \leq R'_n. \tag{30}$$

Let us now show the right side of (27), namely $R'_n < R_{3n}$. Similarly, it is enough to show that $\pi_R(R_{3n}) - \pi_R\left(\frac{R_{3n}}{2}\right) > n$, that is, $2n > \pi_R\left(\frac{R_{3n}}{2}\right)$. This inequality holds if and only if $\pi_R(R_{2n}) > \pi_R\left(\frac{R_{3n}}{2}\right)$, that is, $2R_{2n} > R_{3n}$. By Sondow's Theorem and Rosser's Theorem we get

$$2R_{2n} > 2p_{4n} > 8n \ln 4n.$$

By Lemma 4 we have the inequality $8n \ln 3n > R_{3n}$ for any $n \geq 5315$. As $8n \ln 4n \geq 8n \ln 3n$ for all $n \geq 1$, the inequality $2R_{2n} > R_{3n}$ holds for all $n \geq 5315$. By computer check we can see that the right side of the inequality holds also for $5315 > n \geq 1$. □

Corollary 4. *For* $n > 0$*, the nth derived Ramanujan prime satisfies*

$$p_{4n} < R'_n < p_{9n}. \tag{31}$$

Proof. Use Theorem 5 together with Sondow's and Laishram's bounds

$$p_{2n} < R_n < p_{3n}. \tag{32}$$

□

Note that the right side of (31) can be replaced by $R'_n < p_{8n}$ for $n \geq 5315$ if we combine Lemma 4, Theorem 5 and Rosser's Theorem.

In [12] Shevelev showed that

$$\pi_R(x) \sim \frac{\pi(x)}{2} \sim \frac{x}{2\ln x} \tag{33}$$

holds following Sondow's $R_n \sim p_{2n}$ result. Combining (33) with Sondow's method in [15] it is easy to see the following corollary. Denote by $\pi_{R'}(x)$ the derived-Ramanujan-prime-counting function.

Corollary 5. *As $n \to \infty$ the asymptotic formula $R'_n \sim R_{2n} \sim p_{4n}$ holds, and given $\varepsilon > 0$ there exists N_ε such that $R'_n < (4+\varepsilon)\, n \ln n$ for $n \geq N_\varepsilon$. Moreover*

$$\pi_{R'}(x) \sim \frac{\pi_R(x)}{2} \sim \frac{\pi(x)}{4} \sim \frac{x}{4\ln x}.$$

5. The Number of Primes between $(1-\varepsilon)\,x$ and x

In [1, Theorem 2.2] N. Amersi, O. Beckwith, S. J. Miller, R. Ronan and J. Sondow proved that for $c \in (0,1)$ the number of primes in the interval (cx, x) tends to infinity as $x \to \infty$. We will give another proof of this theorem.

Theorem 6. *For any fixed $\epsilon > 0$, the number of primes between $(1-\varepsilon)\,x$ and x tends to infinity as $x \to \infty$.*

Proof. Let $R_{n+1} > x \geq R_n$ and therefore $\pi(x) - \pi\left(\frac{x}{2}\right) \geq n$. The number of primes between $(1-\varepsilon)\,x$ and x tends to infinity as $x \to \infty$ if and only if $\pi((1-\varepsilon)\,x) - \pi\left(\frac{x}{2}\right) < n - f(n)$ where $f(n)$ is a steadily increasing function. But as

$$\pi((1-\varepsilon)\,x) - \pi\left(\frac{x}{2}\right) < \pi((1-\varepsilon)\,R_{n+1}) - \pi\left(\frac{R_n}{2}\right) \tag{34}$$

holds, it is enough to show that

$$\pi((1-\varepsilon)\,R_{n+1}) - \pi\left(\frac{R_n}{2}\right) < n - f(n), \tag{35}$$

or by the equality $n = \pi(R_n) - \pi\left(\frac{R_n}{2}\right)$ to prove that $f(x)$ is not greater than $\pi(R_n) - \pi((1-\varepsilon)\,R_{n+1})$. By Sondow's Theorem [15] we know that for all $\varepsilon > 0$ there exists $N(\varepsilon)$ such that the inequalities

$$(2+\varepsilon)\, n \ln n > R_n > p_{2n} \tag{36}$$

hold for $n > N(\varepsilon)$. Hence by Corollary 1 and (13)

$$\pi(R_n) - \pi((1-\varepsilon)\,R_{n+1}) > 2n - \frac{\left(2-\varepsilon-\varepsilon^2\right)(n+1)\ln(n+1)}{\ln\left(\left(2-\varepsilon-\varepsilon^2\right)(n+1)\ln(n+1)\right) - 1} \tag{37}$$

holds. We can set $f(n)$ equal to the right side of the inequality because it tends to infinity as $n \to \infty$. □

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