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CONJECTURES INVOLVING PRIMES AND QUADRATIC FORMS

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Abstract. We pose many new conjectures involving primes and quadratic forms, which might interest number theorists and stimulate further research. Below are five typical examples:

(i) For any positive integer n, there exists $k \in \{0, \ldots, n\}$ such that $n+k$ and $n+k^2$ are both prime.

(ii) Every integer $n = 12, 13, \ldots$ can be written as $p+q$ with p, $p+6$, $6q-1$ and $6q+1$ all prime.

(iii) For any integer $n \geqslant 6$ there is a prime $p < n$ such that $6n-p$ and $6n+p$ are both prime.

(iv) Any integer $n > 3$ can be written as $p+q$, where p is a prime with $p-1$ and $p+1$ both practical, and q is either prime or practical. Also, each even number $n > 8$ can be written as $p+q+r$, where p is a prime with $p-1$ and $p+1$ both practical, q is a prime with $q-1$ and $q+1$ both practical, and r is a practical number with $r-1$ and $r+1$ both prime.

(v) Each $n = 3, 4, \ldots$ can be written as $p + \sum_{k=1}^{m}(-1)^{m-k}p_k$, where p is a Sophie Germain prime and p_k is the kth prime.

1. Introduction

Primes have been investigated for over two thousand years. Nevertheless, there are many problems on primes remain open. The famous Goldbach conjecture (cf. [CP] and [N]) states that any even integer $n > 2$ can be represented as a sum of two primes. Lemoine's conjecture (see [L]) asserts that any odd integer $n > 6$ can be written as $p + 2q$ with both p and q prime; this is a refinement of the weak Goldbach conjecture (involving sums of three primes) proved by Vinogradov [V] for large odd numbers. Legendre's conjecture states that for any positive integer n there is a prime between n^2 and $(n+1)^2$. Another well known conjecture asserts that for any positive even number d there are infinitely many prime pairs $\{p, q\}$ with $p - q = d$.

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A positive integer n is said to be *practical* if every $m = 1, \ldots, n$ can be written as the sum of some distinct divisors of n. In 1954 B.M. Stewart [St] showed that if $p_1 < \cdots < p_r$ are distinct primes and $a_1, \ldots, a_r$ are positive integers then $m = p_1^{q_1} \cdots p_r^{a_r}$ is practical if and only if $p_1 = 2$ and

$$p_{s+1} - 1 \leqslant \sigma(p_1^{a_1} \cdots p_s^{a_s}) \quad \text{for all} \quad 0 < s < r,$$

where $\sigma(n)$ stands for the sum of all divisors of n. The behavior of practical numbers is quite similar to that of primes. For example, G. Melfi [Me] proved the following Goldbach-type conjecture of M. Margenstern [Ma]: Each positive even integer is a sum of two practical numbers, and there are infinitely many practical numbers m with $m - 2$ and $m + 2$ also practical. Similar to Firoozbakht's conjecture that $(\sqrt[n]{p_n})_{n \geqslant 1}$ is strictly decreasing (where p_n is the nth prime), we conjecture that $(\sqrt[n]{a_n})_{n \geqslant 3}$ is strictly decreasing to the limit 1, where a_n stands for the nth practical number. We also guess that $(\sqrt[n+1]{S(n+1)}/\sqrt[n]{S(n)})_{n \geqslant 7}$ is strictly increasing to the limit 1, where $S(n)$ is the sum of the first n practical numbers. For the author's various conjectures on sequences involving primes, the reader may consult [S13a].

In this paper we formulate many conjectures on primes (and/or practical numbers) as well as binary quadratic forms related to primes. In particular, we find some surprising refinements of Goldbach's conjecture, Lemoine's conjecture, Legendre's conjecture and the twin prime conjecture, and some of our conjectures imply that any positive even integer can be written as difference of two primes infinitely many times. Section 4 contains some conjectures on representations of new types, and Section 5 consists of various conjectures involving alternating sums of consecutive primes. Section 6 is devoted to conjectures involving binary quadratic forms.

Throughout this paper we set $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$. For $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$, by $\{a\}_n$ we mean the least nonnegative residue of a modulo n.

For numbers of representations related to some conjectures in Sections 3-5, the reader may consult [S] for certain sequences in the OEIS.

Many of the conjectures in this paper are certain concrete cases of our following general hypothesis (related to Schinzel's Hypothesis).

General Hypothesis (2012-12-28). *Let $f_1(x, y), \ldots, f_m(x, y)$ be non-constant polynomials with integer coefficients. Suppose that for large $n \in \mathbb{Z}^+$, those $f_1(x, n - x), \ldots, f_m(x, n - x)$ are irreducible, and there is no prime dividing all the products $\prod_{k=1}^m f_k(x, n - x)$ with $x \in \mathbb{Z}$. If $n \in \mathbb{Z}^+$ is large enough, then we can write $n = x + y$ $(x, y \in \mathbb{Z}^+)$ such that $|f_1(x, y)|, \ldots, |f_m(x, y)|$ are all prime.*

2. Conjectures involving one prime variable

Conjecture 2.1 (2012-11-08 and 2012-11-10). *Let n be a positive integer.*

(i) *If* $n \neq 1, 2, 4, 6, 10, 22, 57$, *then there is a prime* $p \leqslant n$ *such that both* $2n - p$ *and* $2n + p - 2$ *are prime. If* $n \neq 1, 2, 3, 5, 8, 87, 108$, *then there is a prime* $p \in (n, 2n)$ *such that* $2n - p$ *and* $2n + p - 2$ *are both prime.*

(ii) *If* $n \neq 1, 2, 9, 21, 191$, *then there is a prime* $p \leqslant n$ *such that both* $2n - p$ *and* $2n+p+2$ *are prime. If* $n \neq 1, 2, 4, 6, 10, 15$, *then there is a prime* $p \in (n, 2n)$ *such that* $2n - p$ *and* $2n + p + 2$ *are both prime.*

(iii) *If* $n > 3$, *then there is a prime* $p < n$ *such that both* $2n - 2p + 1$ *and* $2n + 2p - 1$ *are prime. If* $n > 3$ *and* $n \neq 7, 8, 10, 32$, *then there is a prime* $p < n$ *such that both* $2n - 2p - 1$ *and* $2n + 2p + 1$ *are prime.*

Remark 2.1. Clearly parts (i)-(ii) and (iii) are stronger than Goldbach's conjecture and Lemoine's conjecture respectively. Surprisingly no others have found this conjecture before. We have verified both parts for n up to 7×10^7. Conjecture 2.1 can be further strengthened, for example, we guess that for any integer $n \geqslant 60000$ with $n \neq 65022, 73319, 107733$ there is a prime $p < n$ such that $2n \pm (2p - 1)$ and $2n \pm (2p + 5)$ are all prime.

Conjecture 2.2 (2012-11-09). (i) *For any integer* $n \geqslant 475$ *there is a prime* $p \leqslant n$ *such that both* $2n - p$ *and* $n + (p + 1)/2$ *are prime. For each integer* $n \geqslant 415$ *there is a prime* $p \in (n, 2n)$ *such that* $2n - p$ *and* $n + (p + 1)/2$ *are both prime.*

(ii) *For any integer* $n \geqslant 527$ *there is a prime* $p \leqslant n$ *such that both* $2n + p$ *and* $n - (p + 1)/2$ *are prime. For each integer* $n \geqslant 1133$ *there is a prime* $p \in (n, 2n)$ *such that* $2n + p$ *and* $n - (p + 1)/2$ *are both prime.*

(iii) *For any positive integer* $n \neq 1, 2, 7, 12, 91$, *there is a prime* $p \leqslant n$ *such that* $n + \{n\}_2 - p$ *and* $2n + 2p + 1$ *are both prime. For each* $n = 6, 7, \ldots$ *there is a prime* $p \leqslant n$ *such that* $n + \{n\}_2 + p$ *and* $2n - 2p + 1$ *are both prime.*

(iv) *For any positive integer* $n \neq 1, 2, 7, 8, 91, 92$, *there is a prime* $p \leqslant n$ *such that* $n + \{n\}_2 - p$ *and* $2n + 2p - (-1)^n$ *are both prime. For each* $n = 7, 8, \ldots$ *there is a prime* $p \leqslant n$ *such that* $n + \{n\}_2 + p$ *and* $2n - 2p - (-1)^n$ *are both prime.*

Remark 2.2. We have verified Conjecture 2.2 for n up to 10^7.

Conjecture 2.3 (2012-11-19). (i) *For any integer* $n \geqslant 2720$ *there is a prime* $p < n$ *such that* $2n - p$ *and* $2n + 2p \pm 3$ *are all prime.*

(ii) *For any integer* $n \geqslant 9075$ *there is a prime* $p < n$ *such that* $2n + 1 - 2p$, $2n + p - 2$ *and* $2n + p + 4$ *are all prime.*

Remark 2.3. Conj. 2.3 implies that for any integer $n > 6$ there are primes p and $q < n/2$ such that $p - (1 + \{n\}_2)q = n$; this is similar to Goldbach's conjecture and Lemoine's conjecture. Conj. 2.3 also implies that there are infinitely many sexy prime pairs. (If p and $p + 6$ are both prime, then $\{p, p + 6\}$ is said to be a sexy prime pair.) We have verified Conj. 2.3 for n up to 10^7.

Conjecture 2.4 (2012-11-10)**.** (i) *For each* $n = 6, 7, \ldots$ *there is a prime* $p < n$ *such that both* $6n - p$ *and* $6n + p$ *are prime.*

(ii) *For any given non-constant integer-valued polynomial* $P(x)$ *with positive leading coefficient, if* $n \in \mathbb{Z}^+$ *is sufficiently large then there is a prime* $p < n$ *such that* $6P(n) \pm p$ *are both prime.*

Remark 2.4. We have verified part (i) for n up to 10^8. If we take $P(x)$ in part (ii) to be $x(x+1)/2, x^2, x^3, x^4$, then it suffices to require that n is greater than 1933, 2426, 6772, 24979 respectively.

Conjecture 2.5 (2012-11-10)**.** (i) *For any integer* $n \geqslant 2733$*, there is a prime* $p \leqslant n$ *such that* $n^2 - n + p$ *and* $n^2 + n - p$ *are both prime. For any integer* $n \geqslant 3513$*, there is a prime* $p \in (n, 2n)$ *such that* $n^2 - n + p$ *and* $n^2 + n - p$ *are both prime.*

(ii) *For any integer* $n \geqslant 1829$ *there is a prime* $p \leqslant n$ *such that* $n^2 \pm (n + p)$ *are both prime. For any integer* $n \geqslant 4518$ *there is a prime* $p \in (n, 2n)$ *such that* $n^2 \pm (n + p)$ *are both prime.*

Remark 2.5. Oppermann's conjecture states that for any integer $n > 1$ both of the intervals $(n^2 - n, n^2)$ and $(n^2, n^2 + n)$ contain primes; this is a refinement of Legendre's conjecture. Clearly part (i) of Conj. 2.5 implies the Oppermann conjecture. We have verified both parts of Conj. 2.6 for n up to 10^7.

Conjecture 2.6 (2012-11-08)**.** (i) *For any positive integer* $n \neq 1, 2, 3, 10, 28, 40,$ 218*, there is a prime* $p \leqslant n$ *such that* $(2n)^2 + p$ *is prime. For any positive integer* $n \neq 1, 5, 12, 21, 28$*, there is a prime* $p \leqslant n$ *such that* $(2n - 1)^2 + 2p$ *is prime.*

(ii) *For any integer* $n \neq 1, 2, 3, 6, 7, 57$*, there is a prime* $p \leqslant n$ *such that* $(2n)^2 + p^2$ *is prime. For any integer* $n \neq 1, 2, 4, 17, 19, 57$*, there is a prime* $p \leqslant n$ *such that* $(2n - 1)^2 + (2p)^2$ *is prime.*

(iii) *For any integer* $n \geqslant 142$ *there is a prime* $p \leqslant n$ *such that* $(2n)^4 + p^2$ *is prime. For any integer* $n \neq 1, 24, 39, 47, 89$*, there is a prime* $p \leqslant n$ *such that* $(2n - 1)^4 + (2p)^2$ *is prime.*

(iv) *For any integer* $n \geqslant 1, 2, 3, 5, 6, 11, 22, 35, 40$*, there is a prime* $p \leqslant n$ *such that* $(2n)^4 + p^4$ *is prime. For any integer* $n \neq 1, 33$ *there is a prime* $p \leqslant n$ *such that* $(2n - 1)^4 + (2p)^4$ *is prime.*

Remark 2.6. We have verified all the parts of Conj. 2.6 for n up to 5×10^6. Part (iii) is stronger than the celebrated theorem of J. Friedlander and H. Iwaniec [FI] which asserts that there are infinitely many primes of the form $x^4 + y^2$ with $x, y \in \mathbb{Z}^+$.

Conjecture 2.7 (2012-12-19)**.** *For any odd integer* $n > 1$ *different from* 9 *and* 189*, there is a prime* $p \leqslant n$ *with* $n + (n - p)^4$ *also prime.*

Remark 2.7. We have some other conjectures similar to Conj. 2.7.

Conjecture 2.8 (2012-12-29). *For any positive integer n there is a prime p between n^2 and $(n+1)^2$ with $(\frac{n}{p}) = 1$, where $(-)$ denotes the Legendre symbol. Also, for any integer $n > 1$ we have $(\frac{n(n+1)}{p}) = 1$ for some prime $p \in (n^2, (n+1)^2)$.*

Remark 2.8. We have verified this refinement of Legendre's conjecture for n up to 10^9. We also have some similar conjectures including the following (a) and (b):

(a) For any integer $n > 2$ different from 7 and 17, there is a prime p between n^2 and $(n+1)^2$ with $(\frac{n}{p}) = (\frac{n+1}{p}) = 1$. If $n \in \mathbb{Z}^+ \setminus \{3, 5, 11\}$ is not a square, then $(\frac{n}{p}) = -1$ for some prime $p \in (n^2, (n+1)^2)$.

(b) For any integer $n > 2$ not among 6, 12, 58, there is a prime $p \in (n^2, n^2 + n)$ with $(\frac{n}{p}) = 1$. If $n > 20$ is not a square and different from 37 and 77, there is a prime $p \in (n^2, n^2 + n)$ with $(\frac{n}{p}) = -1$.

Conjecture 2.9 (2012-12-29). (i) *For each integer $n > 8$ with $n \neq 14$, there is a prime p between n and $2n$ with $(\frac{n}{p}) = 1$. If $n \in \mathbb{Z}^+$ is not a square, then there is a prime p between n and $2n$ with $(\frac{n}{p}) = -1$.*

(ii) *For any integer $n > 5$ there is a prime $p \in (n, 2n)$ with $(\frac{2n}{p}) = 1$. For any integer $n > 6$ there is a prime $p \in (n, 2n)$ with $(\frac{-n}{p}) = -1$.*

Remark 2.9. We have verified this refinement of Bertrand's postulate for n up to 5×10^8.

Conjecture 2.10 (2013-01-07). *Let n be a positive integer. Then the interval $[n, 2n]$ contains a Sophie Germain prime. Moreover, whenever $n \geqslant 90$ there is a prime $p \in [n, 2n]$ such that $p+2$ and $2p+1$ are also prime.*

Remark 2.10. Recall that p is called a Sophie Germain prime if p and $2p+1$ are both prime. We have verified Conj. 2.10 for n up to 5×10^8.

Now we introduce two kinds of sandwiches. If p is a prime and $p-1$ and $p+1$ are both practical, then we call $\{p-1, p, p+1\}$ a *sandwich of the first kind*. If $\{p, p+2\}$ is a twin prime pair and $p+1$ is practical, then we call $\{p, p+1, p+2\}$ a *sandwich of the second kind*. For example, $\{88, 89, 90\}$ is a sandwich of the first kind, while $\{59, 60, 61\}$ is a sandwich of the second kind. For the list of those primes in sandwiches of the first kind, see [S, A210479]. We conjecture that $(\sqrt[n]{s_n})_{n \geqslant 9}$ is strictly decreasing to the limit 1, where s_n denotes the central prime in the nth sandwich of the first kind.

Conjecture 2.11 (2013-01-12). (i) *For any integer $n > 8$ the interval $[n, 2n]$ contains a sandwich of the first kind.*

(ii) *For each $n = 7, 8, \ldots$ the interval $[n, 2n]$ contains a sandwich of the second kind.*

(iii) *For any integer $n > 231$ the interval $[n, 2n]$ contains four consecutive integers $p-1, p, p+1, p+2$ with $\{p, p+2\}$ a twin prime pair and $\{p-1, p+1\}$ a twin practical pair.*

(iv) *There are infinitely many quintuples $\{m-2, m-1, m, m+1, m+2\}$ with $\{m-1, m+1\}$ a twin prime pair and $m, m \pm 2$ all practical.*

Remark 2.11. For those middle terms m described in part (iv), the reader may consult [S, A209236].

Conjecture 2.12 (2013-01-20)**.** (i) *For any integer $n > 911$ there is a practical number $k < n$ with $\{kn-1, kn, kn+1\}$ a sandwich of the second kind.*

(ii) *For each integer $n \geqslant 200$, the interval $[1, n]$ contains four consecutive integers $k-1, k, k+1, k+2$ with $k-1$ and $k+1$ both prime, and $k, k+2, kn$ all practical. Moreover, for any integer $n > 26863$ the interval $[1, n]$ contains five consecutive integers $m-2, m-1, m, m+1, m+2$ with $m-1$ and $m+1$ both prime, and $m-2, m, m+2, mn$ all practical.*

Remark 2.12. If p is a prime greater than $\sigma(m)+1$ (where $m \in \mathbb{Z}^+$), then mp is not practical. Thus, part (ii) implies that there are infinitely many quintuples $\{m-2, m-1, m, m+1, m+2\}$ with $\{m-1, m+1\}$ a twin prime pair and $m, m \pm 2$ all practical.

The following conjecture is similar to Conj. 2.4.

Conjecture 2.13 (2013-01-19)**.** *For any integer $n > 2$, there is a practical number $p < n$ such that $n-p$ and $n+p$ are both prime or both practical.*

Remark 2.13. We have verified this conjecture for n up to 10^8.

Now we propose a conjecture of the Collatz type.

Conjecture 2.14 (2013-0-28)**.** (i) *For $n \in \mathbb{Z}^+$ define*

$$f(n) = \begin{cases} (p+1)/2 & \text{if } 4 \mid p+1, \\ p & \text{otherwise}, \end{cases}$$

where p is the least prime greater than n with $2(n+1)-p$ prime. If $a_1 \in \{3, 4, \dots\}$ and $a_{k+1} = f(a_k)$ for $k = 1, 2, 3, \dots$, then $a_N = 4$ for some positive integer N.

(ii) (2013-02-27) *For $n \in \mathbb{Z}^+$ define*

$$g(n) = \begin{cases} q/2 & \text{if } 4 \mid q, \\ q & \text{if } 4 \mid q-2, \end{cases}$$

where q is the least practical number greater than n with $2(n+1)-q$ practical. If $b_1 \in \{4, 5, \dots\}$ and $b_{k+1} = g(b_k)$ for $k = 1, 2, 3, \dots$, then $b_N = 4$ for some positive integer N.

Remark 2.14. For example, if in part (i) we start from $a_1 = 45$ then we get the sequence

$$45,\ 61,\ 36,\ 37,\ 24,\ 16,\ 17,\ 10,\ 6,\ 4,\ 5,\ 4,\ \ldots;$$

if in the second part we start from $b_1 = 316$ then we obtain the sequence

$$316, 330, 342, 378, 190, 110, 126, 64, 66, 78,$$
$$40, 42, 54, 28, 30, 16, 18, 10, 8, 6, 4, 6, 4, \ldots$$

3. Conjectures on representations involving two prime variables

Conjecture 3.1 (2012-11-11)**.** (i) *Every $n = 12, 13, \ldots$ can be written as $p + (1 + \{n\}_2)q$ with $q \leqslant n/2$ and $p, q, q + 6$ all prime. Moreover, for any even $n > 8012$ and odd $n > 15727$, there are primes p and $q < p$ with $p-6$ and $q+6$ also prime such that $p + (1 + \{n\}_2)q = n$.*

(ii) *If d_1 and d_2 are integers divisible by 6, then any sufficiently large integer n can be written as $p+(1+\{n\}_2)q$ with $p > q$ and $p, q, p-d_1, q+d_2$ all prime.*

Remark 3.1. Recall that $\{p, p + 6\}$ is called a sexy prime pair if p and $p + 6$ are both prime. We have verified the first assertion and the second assertion in part (i) for n up to 10^9 and 10^8 respectively. If we take (d_1, d_2) in part (ii) to be $(-6, 6), (-6, -6), (6, -6), (\pm 12, \pm 6)$, then it suffices to require that n is greater than 15721, 15733, 15739, 16349 respectively.

Conjecture 3.2. (i) (2012-11-12) *Any integer $n > 62371$ with $n \not\equiv 2 \pmod 6$ can be written as $p + (1 + \{-n\}_6)q$, where p and $q < p$ are primes with $p - 2$ and $q + 2$ also prime. Also, any integer $n > 6896$ with $n \equiv 2 \pmod 6$ can be written as $p - q$ with $q < n/2$ and $p, q, p - 2, q + 2$ all prime.*

(ii) (2012-11-13) *Any integer $n > 66503$ with $n \not\equiv 4 \pmod 6$ can be written as $p + (1 + \{n\}_6)q$, where p and $q < p$ are primes with $p - 4$ and $q + 4$ also prime. Also, any integer $n > 7222$ with $n \equiv 4 \pmod 6$ can be written as $p - q$ with $q < n/2$ and $p, q, p - 4, q + 4$ all prime.*

Remark 3.2. Clearly part (i) of Conj. 3.2 implies the twin prime conjecture. $\{p, p + 4\}$ is said to be a cousin prime pair if p and $p + 4$ are both prime. We have verified Conj. 3.2 for n up to 5×10^7. Maybe it's possible to establish partial results for Conjectures 3.1 and 3.2 similar to Chen's theorem [C] for Goldbach's conjecture.

Conjecture 3.3 (2012-11-13)**.** (i) *For any odd $n > 4676$ and even $n > 30986$, there are primes p and $q < p$ such that $\{3(p - q) \pm 1\}$ is a twin prime pair and $p + (1 + \{n\}_2)q = n$.*

(ii) *For any odd $n > 7658$ and even $n > 41884$, there are primes p and $q < n/2$ such that $\{3(p+q) \pm 1\}$ is a twin prime pair and $n = p - (1+\{n\}_2)q$.*

(iii) *Any even number $n \geqslant 160$ can be written as $p+q$ with $p, q, p-q-1$ all prime. Also, any even number $n \geqslant 280$ can be written as $p+q$ with $p, q, p-q+1$ all prime.*

Remark 3.3. We have verified part (i) for n up to 10^8. Those numbers $n \in [10000, 30986]$ not having the described property in part (i) are 15446, 21494, 23776, 30986. Those numbers $n \in [18000, 41884]$ not having the described property in part (ii) are 21976, 23584, 41884.

Conjecture 3.4 (2012-11-11)**.** (i) *For any integer $n > 14491$ there are primes p and $q < p$ with $pq + 6$ prime such that $p + (1 + \{n\}_2)q = n$. For any integer $n > 22093$ there are primes p and $q < p$ with $pq - 6$ prime such that $p + (1 + \{n\}_2)q = n$.*

(ii) *Let d be any nonzero multiple of 6. Then any sufficiently large integer n can be written as $p + (1 + \{n\}_2)q$ with $p > q$ and $p, q, pq + d$ all prime.*

Remark 3.4. We have verified part (i) for n up to 5×10^7. Here is a weaker version of part (i): Any integer $n > 9$ with $n \neq 13, 14, 41$ can be written as $p + (1 + \{n\}_2)q$, where q is a positive integer not exceeding $n/2$, and p and $pq + 6$ are both prime.

Conjecture 3.5 (2012-12-04)**.** (i) *Any integer $n \geqslant 6782$ can be written as $p+(1+\{n\}_2)q$ with $p > q$ and $p, q, q\pm 6$ all prime. In general, for any $d_1, d_2 \in \mathbb{Z}$ divisible by 6, all sufficiently large integers n can be written as $p + (1 + \{n\}_2)q$ with $p > q$ and $p, q, q + d_1, q + d_2$ all prime.*

(ii) *Any even $n \geqslant 8070$ and odd $n > 18680$ can be written as $p + (1 + \{n\}_2)q$ with $p > q$ and $p, q, 6q \pm 1$ all prime.*

(iii) *Any integer $n \geqslant 4410$ can be written as $p + (1 + \{n\}_2)q$ with $p > q$ and $p, q, 2q \pm 3$ all prime.*

(iv) *Any integer $n \geqslant 16140$ can be written as $p + (1 + \{n\}_2)q$ with $p > q$ and $p, q, 3q \pm 2$ all prime.*

Remark 3.5. We have verified the first assertion in part (i) and the second part for n up to 2×10^7 and 4×10^7 respectively. All the four parts can be further strengthened, for example, we guess that any integer $n \geqslant 186272$ can be written as $p + (1 + \{n\}_2)q$ with $p > q$ and $p, q, q \pm 6, q + 30$ all prime.

Conjecture 3.6 (2012-11-11)**.** (i) *Any integer $n \geqslant 785$ can be written as $p + (1 + \{n\}_2)q$ with $p, q, p^2 + q^2 - 1$ all prime.*

(ii) *Let d be any odd integer with $d \not\equiv 1 \pmod 3$. Then large even numbers can be written as $p+q$ with p, q, p^2+q^2+d all prime. If $d \not\equiv 0 \pmod 5$, then all large odd numbers can be represented as $p + 2q$ with $p, q, p^2 + q^2 + d$ all prime.*

Remark 3.6. We have verified part (i) for n up to 1.4×10^8.

Conjecture 3.7 (2012-11-05)**.** *For any integer $n \geqslant 1188$ there are primes p and q with $p^2 + 3pq + q^2$ prime such that $p + (1 + \{n\}_2)q = n$.*

Remark 3.7. Clearly this conjecture is stronger than Goldbach's conjecture and Lemoine's conjecture. We have verified it for n up to 3×10^8. Those

$1 \leqslant n \leqslant 1187$ not having the required property are listed below:

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 10,\ 32,\ 38,\ 40,\ 51,\ 56,\ 61,\ 66,\ 91,\ 119,\ 131, 148,\ 188,\ 191,$$
$$193,\ 223,\ 226,\ 248,\ 296,\ 356,\ 373,\ 398,\ 428,\ 934,\ 964,\ 1012,\ 1136,\ 1187.$$

In view of Conjecture 3.7, those primes in the form $p^2+3pq+q^2$ are particularly interesting; the reader may consult [S, A218771] for information about such primes.

Conjecture 3.8 (2012-11-07). *For any $a \in \mathbb{N}$ with $a \neq 2$, the set $E(a)$ of positive integers n not in the form $p+(1+\{n\}_2)q$ with $p, q, (2^a+2)(p+q)^2+pq$ all prime, is finite! In particular,*

$$E(0) = \{1 \sim 8,\ 10,\ 13,\ 14,\ 15,\ 22,\ 59,\ 61,\ 62,\ 68,\ 104\},$$
$$E(1) = \{1 \sim 7,\ 9,\ 12,\ 14,\ 15,\ 20,\ 21,\ 27,\ 32,\ 38,\ 61,\ 68,\ 146,\ 188,\ 212,\ 383,\ 746\},$$

and

$$\begin{aligned} E(3) = \{&1 \sim 9,\ 11,\ 12,\ 15,\ 16,\ 18,\ 19,\ 21,\ 22,\ 28,\\ &39,\ 46,\ 52,\ 62,\ 63,\ 121,\ 131,\ 158,\ 226,\ 692\},\\ E(4) = \{&1 \sim 15,\ 17,\ 19,\ 20,\ 22,\ 25,\ 28,\ 35,\ 39,\\ &46,\ 56,\ 58,\ 68,\ 73,\ 122,\ 124,\ 205,\ 227\}. \end{aligned}$$

Moreover,

$$\max E(5) = 2468,\ \max E(6) = 476,\ \max E(7) = 796,\ \max E(8) = 4633,$$
$$\max E(9) = 1642,\ \max E(10) = 2012,\ \max E(11) = 3400,\ \max E(12) = 1996.$$

Remark 3.8. For each $a = 0, 1, 3, 4, \ldots, 12$ we have verified the conjecture for n up to 10^7. Note that the conjecture implies that for each $a = 0, 1, 3, 4, \ldots$ there are infinitely many primes in the form $(2^a+2)(p+q)^2+pq$ with p and q prime.

Conjecture 3.9 (2012-11-07 and 2012-11-08).

(i) *Each odd integer $n > 15$ can be written as $p+2q$ with p, q, p^2+60q^2 all prime. Any odd integer greater than 1424 can be represented as $p+2q$ with $p, q, p^4+(2q)^4$ all prime.*

(ii) *For any give $a, b \in \mathbb{Z}^+$ sufficiently large odd numbers can be represented as $p+2q$ with p, q and $p^{2^b}+(2^a-1)(2q)^{2^b}$ all prime. Let $E^*(a)$ denote the set of positive odd integers not of the form $p+2q$ with $p, q, p^2+4(2^a-1)q^2$ all prime. Then*

$$E^*(3) = \{1,\ 3,\ 5,\ 7,\ 31,\ 73\},$$

and

$$\max E^*(1)=3449,\ \max E^*(2)=1711,\ \max E^*(5)=6227,\ \max E^*(6)=1051,$$
$$\max E^*(7)=2239,\ \max E^*(8)=2599,\ \max E^*(9)=7723,$$
$$\max E^*(10)=781,\ \max E^*(11)=1163,\ \max E^*(12)=587.$$

Remark 3.9. We have verified part (i) for n up to 10^8. It is still open whether there are infinitely many primes in the form x^4+y^4 with $x,y\in\mathbb{Z}^+$.

Conjecture 3.10 (2012-11-11). *For any integer $n\geqslant 7830$ there are primes p and $q<p$ with p^4+q^4-1 prime such that $p+(1+\{n\}_2)q=n$.*

Remark 3.10. This is similar to Conjecture 3.5(i).

Conjecture 3.11 (2012-11-08). *For any integer $n\geqslant 9608$ there are primes p and q with $(p+q)^4+(pq)^2$ prime such that $p+q=2n$.*

Remark 3.11. We have verified the conjecture for n up to 3×10^7. As $pq\leqslant(\frac{p+q}{2})^2$, Conjecture 3.10 implies that for any integer $n\geqslant 9608$ there is a positive integer $m\leqslant n^2$ such that $(2n)^4+m^2$ is prime (compare this with Conj. 2.6(iii)). This also holds for all $n\leqslant 9607$.

Conjecture 3.12 (joint with Olivier Gerard). *For any integer $n\geqslant 400$ with $n\neq 757,\ 1069,\ 1238$, there are odd primes p and q with $(\frac{p}{q})=(\frac{q}{p})=1$ such that $p+(1+\{n\}_2)q=n$.*

Remark 3.12. We have verified Conj. 3.12 for n up to 10^8. See [GS] for the announcement of this conjecture.

Conjecture 3.13. *For any integer m define $s(m)$ as the smallest positive integer s such that for any $n=s,s+1,\ldots$ there are primes $p>q>2$ with $(\frac{p-(1+\{n\}_2)m}{q})=(\frac{q+m}{p})=1$ and $p+(1+\{n\}_2)q=n$; if such a positive integer s does not exist then $s(m)$ is defined as 0.*

(i) (2012-11-22) *We have* $s(m) > 0$ *for all* $m \in \mathbb{Z}$. *In particular,*

$$s(0) = 1239,\ s(1) = 1470,\ s(-1) = 2192,\ s(2) = 1034,\ s(-2) = 1292,$$
$$s(3) = 1698,\ s(-3) = 1788,\ s(4) = 848,\ s(-4) = 1458,$$
$$s(5) = 1490,\ s(-5) = 2558,\ s(6) = 1115,\ s(-6) = 1572,$$
$$s(7) = 1550,\ s(-7) = 932,\ s(8) = 825,\ s(-8) = 2132,$$
$$s(9) = 1154,\ s(-9) = 1968,\ s(10) = 1880,\ s(-10) = 1305,$$
$$s(11) = 1052,\ s(-11) = 1230,\ s(12) = 2340,\ s(-12) = 1428,$$
$$s(13) = 2492,\ s(-13) = 2673,\ s(14) = 1412,\ s(-14) = 1638,$$
$$s(15) = 1185,\ s(-15) = 1230,\ s(16) = 978,\ s(-16) = 1605,$$
$$s(17) = 1154,\ s(-17) = 1692,\ s(18) = 1757,\ s(-18) = 2292,$$
$$s(19) = 1230,\ s(-19) = 2187,\ s(20) = 2048,\ s(-20) = 1372,$$
$$s(21) = 1934,\ s(-21) = 1890,\ s(22) = 1440,\ s(-22) = 1034,$$
$$s(23) = 1964,\ s(-23) = 1322,\ s(24) = 1428,\ s(-24) = 2042,$$
$$s(25) = 1734,\ s(-25) = 1214,\ s(26) = 1260,\ s(-26) = 1230,$$
$$s(27) = 1680,\ s(-27) = 1154,\ s(28) = 1652,\ s(-28) = 1808,$$
$$s(29) = 1112,\ s(-29) = 1670,\ s(30) = 1820,\ s(-30) = 1284,$$

$$s(31) = 1614,\ s(-31) = 1404,\ s(32) = 1552,\ s(-32) = 1808,$$
$$s(33) = 1230,\ s(-33) = 1914,\ s(34) = 1200,\ s(-34) = 1832,$$
$$s(35) = 1480,\ s(-35) = 1094,\ s(36) = 1572,\ s(-36) = 1397,$$
$$s(37) = 1622,\ s(-37) = 1220,\ s(38) = 1452,\ s(-38) = 2064,$$
$$s(39) = 1848,\ s(-39) = 1440,\ s(40) = 1262,\ s(-40) = 1397,$$
$$s(41) = 2384,\ s(-41) = 1262,\ s(42) = 1536,\ s(-42) = 2838,$$
$$s(43) = 1542,\ s(-43) = 1550,\ s(44) = 2012,\ s(-44) = 1683,$$
$$s(45) = 1274,\ s(-45) = 2544,\ s(46) = 1432,\ s(-46) = 1368,$$
$$s(47) = 1710,\ s(-47) = 2132,\ s(48) = 1392,\ s(-48) = 1734,$$
$$s(49) = 1790,\ s(-49) = 1334,\ s(50) = 2138,\ s(-50) = 1364.$$

(ii) (2012-11-24) *For any integer* $m \geqslant 3720$ *with* $8 \mid m$, *we have* $s(-m) = m + 1$. *For any integer* $a \geqslant 10$ *we have*

$$s(2^a) = \begin{cases} 2^{a+1} - 2 & \text{if } 2 \mid a, \\ 2^a + 1 & \text{if } 2 \nmid a. \end{cases}$$

(iii) (2012-11-25) *For any integer* $m \geqslant 1573$ *with* $m \equiv 1 \pmod{12}$, *we have* $s(m) = 2m - 2$.

Remark 3.13. (i) That $s(0)=1239$ is a slight variant of Conjecture 3.11.

(ii) For any positive even integer m, we have $s(\pm m)\geqslant m+1$, for, if p and $q<p$ are odd primes with $p+q=m$ then

$$\left(\frac{p-m}{q}\right)=\left(\frac{-q}{q}\right)=0 \ \text{ and } \left(\frac{q+(-m)}{p}\right)=\left(\frac{-p}{p}\right)=0.$$

(iii) Let $m\in\mathbb{Z}^+$ with $m\equiv 1 \pmod 3$. Then $s(m)\geqslant 2m-2$. In fact, if there are odd primes p and $q<p$ such that $p+2q=2m-3$ and $(\frac{p-2m}{q})=1$, then $(\frac{-3}{q})=(\frac{p+2q-2m}{q})=(\frac{p-2m}{q})=1$, hence $q\equiv 1 \pmod 3$ and $p=2(m-q)-3\equiv 2(m-1)\equiv 0 \pmod 3$ which is impossible since $p>q>2$.

Conjecture 3.14 (2012-12-30)**.** *Any integer $n>5$ can be written as $p+(1+\{n\}_2)q$, where p is an odd prime and q is a prime not exceeding $n/2$ such that $(\frac{q}{n})=1$ if $2\nmid n$, and $(\frac{(q+1)/2}{n+1})=1$ if $2\mid n$, where $(-)$ denotes the Jacobi symbol.*

Remark 3.14. We have verified this refinement of Goldbach's conjecture and Lemoine's conjecture for n up to 10^9.

Conjecture 3.15 (2013-01-19)**.** (i) *Any even integer $2n>4$ can be written as $p+q$, where p and q are primes with $p+1$ and $q-1$ both practical. We may require additionally that $q-p-1$ is prime if $n>29663$.*

(ii) *For each integer $n>8$, we can write $2n-1=p+q=2p+(q-p)$, where p and $q-p$ are both prime, and q is practical.*

Remark 3.15. We have verified the first assertion in part (i) and the second part for n up to 10^8 and 10^7 respectively.

4. Conjectures involving representations of new types

Recall that those $T_n=n(n+1)/2$ ($n\in\mathbb{N}$) are called triangular numbers. A conjecture of the author [S09] states that any positive integer can be written as $p+T_x$ with $x\in\mathbb{N}$, where p is either zero or prime.

Conjecture 4.1 (2013-01-05)**.** *Any integer $n>48624$ with $n\neq 76106$ can be written as $x+y$ ($x,y\in\mathbb{N}$) with $\{6x-1,6x+1\}$ a twin prime pair and y a triangular number.*

Remark 4.1. We have verified this conjecture for n up to 10^9 (for numbers of related representations, see [S, A187785]), and guess that 723662 is the unique value of $n>76106$ which really needs $y=0$ in the described representation.

Recall that for two subsets X and Y of $\mathbb{Z}$, their sumset is given by

$$X+Y=\{x+y:\ x\in X \text{ and } y\in Y\}.$$

Conjecture 4.2 (2013-01-03). *Let*

$$\begin{aligned} A =& \{x \in \mathbb{Z}^+ :\ 6x-1 \text{ and } 6x+1 \text{ are both prime}\},\\ B =& \{x \in \mathbb{Z}^+ :\ 6x+1 \text{ and } 6x+5 \text{ are both prime}\},\\ C =& \{x \in \mathbb{Z}^+ :\ 2x-3 \text{ and } 2x+3 \text{ are both prime}\}. \end{aligned}$$

Then

$$A+B = \{2,3,\ldots\},\ B+C = \{5,6,\ldots\}, \quad \text{and } A+C = \{5,6,\ldots\} \setminus \{161\}.$$

Remark 4.2. We have verified Conj. 4.2 for n up to 3×10^8. In view of Conjectures 3.1 and 3.2, we should have $2A = A + A \supseteq \{702, 703, \ldots\}$, $2B \supseteq \{492, 493, \ldots\}$ and $2C \supseteq \{4006, 4007, \ldots\}$.

Conjecture 4.3 (2012-12-22). (i) *Any integer $n \geqslant 12$ can be written as $p+q$ with p, $p+6$, $6q-1$ and $6q+1$ all prime.*

(ii) *Each integer $n > 6$ with $n \neq 319$ can be written as $p+q$ with p, $p+6$, $3q-2+\{n\}_2$ and $3q+2-\{n\}_2$ all prime.*

(iii) *Any integer $n > 3$ not among* 11, 64, 86, 629 *can be written as $x+y$ $(x, y \in \mathbb{Z}^+)$ with $3x \pm 2$ and $6y \pm 1$ all prime.*

Remark 4.3. We have verified part (i) for n up to 10^9. For numbers of representations related to part (i), see [S, A199920]. Note that part (i) of Conj. 4.3 implies that there are infinitely many twin primes and also infinitely many sexy primes, because for any $m = 2, 3, \ldots$ the interval $[m! + 2, m! + m]$ of length $m-2$ contains no prime.

Conjecture 4.4. (i) (2012-11-30 and 2012-12-01) *Any integer $n > 7$ can be written as $p+q$, where q is a positive integer, and p and $2pq+1$ are both prime. In general, for each $m \in \mathbb{N}$ any sufficiently large integer n can be written as $x+y$ $(x, y \in \mathbb{Z}^+)$ with $x-m$, $x+m$ and $2xy+1$ all prime.*

(ii) (2012-11-29) *Any integer $n > 357$ can be written as $x+y$ $(x, y \in \mathbb{Z}^+)$ with $2xy \pm 1$ twin primes. In general, for each positive odd integer m, any sufficiently large integer n can be written as $x+y$ with $x, y \in \mathbb{Z}^+$ such that $2xy-m$ and $2xy+m$ are both prime.*

Remark 4.4. (a) We have verified the first assertion in part (i) of Conj. 4.4 for n up to 10^9 and the first assertion in part (ii) for n up ro 2×10^8. Concerning the general statement in part (i), for $m = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ it suffices to require that n is greater than

623, 28, 151, 357, 199, 307, 357, 278, 697, 263

respectively. We also guess that for every odd integer $m \not\equiv 5 \pmod 6$, any sufficiently large integer n can be written as $p+q$ $(q \in \mathbb{Z}^+)$ with p and $2pq+m$

both primes, e.g., when $m = 3$ it suffices to require $n > 1$. Concerning the general statement in part (ii), for $m = 3, 5, 7, 9, 11$ it suffices to require that n is greater than 5090, 222, 1785, 548, 603 respectively. Note that if $x, y \in \mathbb{Z}^+$ and $x + y = n$ then $2n - 2 \leqslant 2xy \leqslant n^2/2$ since

$$n - 1 = x + y - 1 \leqslant xy \leqslant \left(\frac{x+y}{2}\right)^2 = \frac{n^2}{4}.$$

(b) Given finitely many positive integers $x_1, \ldots, x_k$ and distinct odd primes $q_1, \ldots, q_k$ greater than $\max\{x_1, \ldots, x_k\}$, by the Chinese Remainder Theorem there are infinitely many $n \in \mathbb{Z}^+$ such that $q_i | 2x_i(n - x_i) + 1$ for all $i = 1, \ldots, k$. Thus, part (i) of Conj. 4.4 implies that for any $m \in \mathbb{N}$ there are infinitely many positive integers x with $x - m$ and $x + m$ both prime. That any positive even integer can be expressed as difference of two primes infinitely many times is a well known unsolved problem.

Conjecture 4.5. (i) (2012-12-01) *For each $m \in \mathbb{N}$, any sufficiently large integer n with m or n odd can be written as $x + y$, where x and y are positive integers with $x - m, x + m$ and $xy - 1$ all prime. In particular, in the case $m = 0$ it suffices to require that $n \neq 1, 3, 85$; when $m = 1$ it suffices to require that n is not among*

$$1,\ 2,\ 3,\ 4,\ 40,\ 125,\ 155,\ 180,\ 470,\ 1275,\ 2185,\ 3875;$$

in the case $m = 2$ it suffices to require that $n > 7$ and $n \neq 13$.

(ii) (2012-11-27) *Any integer $n > 3120$ can be written as $x + y$ with $x, y \in \mathbb{Z}^+$ and $\{xy - 1, xy + 1\}$ a twin prime pair. In general, for each positive integer m, any sufficiently large integer n with $(m - 1)n$ even can be written as $x + y$ with $x, y \in \mathbb{Z}^+$ such that $xy - m$ and $xy + m$ are both prime.*

Remark 4.5. We have verified the first assertion in part (ii) for n up to 2×10^8. Amarnath Murthy [Mu] conjectured that any integer $n > 3$ can be written as $x + y$ ($x, y \in \mathbb{Z}^+$) with $xy - 1$ prime. Concerning part (ii) for $m = 2, 3, 4, 5$, it suffices to require $n \geqslant 696$, $n \geqslant 1037$, $n \geqslant 4682$ and $n \geqslant 2779$ respectively.

Conjecture 4.6 (2012-11-27)**.** *For any positive integer m, each sufficiently large integer n with $(m - 1)n$ even can be written as $x + y$ with $x, y \in \mathbb{Z}^+$ such that $xy + mn$ and $xy - mn$ are both prime. In particular, for any integer $n > 6$ with $n \neq 24$ there are $x, y \in \mathbb{Z}^+$ with $x + y = n$ such that $xy + n = (x+1)(y+1) - 1$ and $xy - n = (x - 1)(y - 1) - 1$ are both prime; for any even integer $n > 10$ there are $x, y \in \mathbb{Z}^+$ with $x + y = n$ such that $xy + 2n = (x + 2)(y + 2) - 4$ and $xy - 2n = (x - 2)(y - 2) - 4$ are both prime.*

Remark 4.6. We also guess that any integer $n \geqslant 507$ can be written as $x + y$ ($x, y \in \mathbb{Z}^+$) with $xy \pm 3n$ both prime.

Conjecture 4.7. (i) (2012-11-03 and 2012-11-04) *Any integer $n>1$ different from 8 can be written as $x+y$, where x and y are positive integers with x^2+xy+y^2 prime. Also, any integer $n>1$ can be written as $x+y$, where x and y are positive integers with $x^2+3xy+y^2$ prime.*

(ii) (2012-11-27) *Any $n\in\mathbb{Z}^+$ not among $1,8,10,18,20,41,46,58,78,116,440$ can be written as $x+y$ with $x,y\in\mathbb{Z}^+$ such that $n^2-xy=x^2+xy+y^2$ and $n^2+xy=x^2+3xy+y^2$ are both prime.*

(iii) (2012-11-27) *For any $a=4,5,6,\ldots$ and positive odd integer m, each sufficiently large integer n can be written as $x+y$ with $x,y\in\mathbb{Z}^+$ such that mn^a-xy and mn^a+xy are both prime. In particular, for any integer $n\geqslant 4687$ there are $x,y\in\mathbb{Z}^+$ such that n^4+xy and n^4-xy are both prime.*

Remark 4.7. It is known (cf. [IR] and [Cox]) that any prime $p\equiv 1 \pmod 3$ can be written uniquely in the form x^2+xy+y^2 with $x,y\in\mathbb{Z}^+$, and any prime $p\equiv\pm1 \pmod 5$ can be written uniquely in the form $x^2+3xy+y^2$ with $x,y\in\mathbb{Z}^+$. Ming-Zhi Zhang (cf. [G, p. 161]) asked whether any odd integer greater than one can be written as $x+y$ with $x,y\in\mathbb{Z}^+$ and x^2+y^2 prime.

Conjecture 4.8. (i) (2012-11-28) *Any positive integer $n\neq 1,6,16,24$ can be written as $x+y$ $(x,y\in\mathbb{Z}^+)$ with $(xy)^2+1$ prime. In general, for any $a\in\mathbb{N}$, each sufficiently large integer n can be written as $x+y$, where x and y are positive integers with $(xy)^{2^a}+1$ prime.*

(ii) (2012-11-29) *Any integer $n>1$ can be written as $x+y$ $(x,y\in\mathbb{Z}^+)$ with $(xy)^2+xy+1$ prime. In general, for any prime p, each sufficiently large integer n can be written as $x+y$, where x and y are positive integers with $((xy)^p-1)/(xy-1)$ prime.*

Remark 4.8. We also guess that any integer $n>1157$ can be written as $x+y$ $(x,y\in\mathbb{Z}^+)$ with $(xy)^2+xy+1$ and $(xy)^2+xy-1$ both prime. Concerning the general assertion in part (i), for $a=2,3,4$ it suffices to require that n is greater than 22, 386, 748 respectively. Concerning the general assertion in part (ii), for $p=5,7,11,13$ it suffices to require that n is greater than 28, 46, 178, 108 respectively.

Conjecture 4.9 (2012-12-06)**.** (i) *Any integer $n\geqslant 15000$ with $n\neq 33142$, 37723, 55762 can be written as $p+q$, where q is a positive integer, and $p,p\pm6,2pq+1$ are all prime. In general, for any $d_1,d_2\in\mathbb{Z}$ divisible by 6 all sufficiently large integers n can be written as $p+q$, where q is a positive integer, and $p,p+d_1,p+d_2,2pq+1$ are all prime.*

(ii) *Any integer $n\geqslant 73179$ can be written as $p+q$, where q is a positive integer, and $p,6p\pm1,2pq+1$ are all prime.*

(iii) *Any integer $n\geqslant 90983$ can be written as $p+q$, where q is a positive integer, and $p,2p\pm3,2pq+1$ are all prime.*

(iv) *Any integer $n\geqslant 92618$ can be written as $p+q$, where q is a positive integer, and $p,3p\pm2,2pq+1$ are all prime.*

Remark 4.9. Note that if p and $p \pm 6$ are all prime then $p-6, p, p+6$ form a three-term AP (arithmetic progression) of primes. In 1939 van der Corput [Co] proved that there are infinitely many three-term AP of primes. It is interesting to compare Conj. 4.9 with Conj. 3.5.

Conjecture 4.10 (2012-12-01). (i) *Any integer $n > 10$ can be written as $p+q$ ($q \in \mathbb{Z}^+$) with p, $p+6$ and $p^2+3pq+q^2 = n^2+pq$ all prime. Also, any integer $n > 2$ with $n \neq 8, 37$ can be written as $p+q$ ($q \in \mathbb{Z}^+$) with p and $p^2+pq+q^2 = n^2-pq$ both prime. Moreover, any integer $n > 600$ different from* 772, 1177, 1621, 2162 *can be written as $p+q$ ($q \in \mathbb{Z}^+$) with p and $n^2 \pm pq$ all prime.*

(ii) *Any integer $n > 2572$ with $n \neq 6892$ can be written as $p+q$ ($q \in \mathbb{Z}^+$ with p and $pq \pm n$ all prime.*

(iii) *Any integer $n \neq 1$, 2, 13, 16, 46, 95, 157 can be written as $p+q$ ($q \in \mathbb{Z}^+$ with p and $(pq)^2+pq+1$ all prime. Also, if $n > 2$ is an integer with $n \neq 64$ and $5 \nmid n$, then we may write n as $p+q$ ($q \in \mathbb{Z}^+$) with p and $(2pq)^2+1$ both prime.*

(iv) *Any integer n not among* 1, 2, 5, 10, 34, 68 *can be written as $p+q$ ($q \in \mathbb{Z}^+$ with p and $(2pq)^4+1$ both prime.*

Remark 4.10. There are some other variants of some statements in Conj. 4.10.

Conjecture 4.11 (2012-12-09). (i) *Any integer $n > 2$ can be written as $x^2 + y$ ($x, y \in \mathbb{Z}^+$) with $2xy-1$ prime. In other words, for each $n = 3, 4, \ldots$ there is a prime in the form $2k(n-k^2)-1$ with $k \in \mathbb{Z}^+$.*

(ii) *Let $m \in \mathbb{Z}^+$ and $r \in \{\pm 1\}$. Then any sufficiently large integer n can be written as x^2+y ($x, y \in \mathbb{Z}^+$) with $mxy+r$ prime.*

Remark 4.11. We have verified part (i) for n up to 3×10^9. When $n = 1691955723$, the number 411 is the only positive integer k with $2k(n-k^2)-1$ prime, and $411/\log^2 n \approx 0.910246$. The author ever thought that one may require $x < \log^2 n$ in part (i), but Jack Brennen found that $n = 4630581798$ is a counterexample, and the least $k \in \mathbb{Z}^+$ with $2k(n-k^2)-1$ prime is $500 \approx 1.00943 \log^2 n$.

Conjecture 4.12 (2012-12-15). (i) *Each integer $n > 3$ can be written as $x + y$ ($x, y \in \mathbb{Z}^+$) with $3x \pm 1$ and $xy-1$ all prime. Also, any integer $n > 2$ can be written as $x+y$ ($x, y \in \mathbb{Z}^+$) with $3x \pm 1$ and $3xy-1$ all prime, and any integer $n > 2$ not equal to 63 can be written as $x+y$ ($x, y \in \mathbb{Z}^+$) with $2x \pm 1$ and $2xy+1$ all prime.*

(ii) *Each integer $n > 7$ can be written as $x+y$ ($x, y \in \mathbb{Z}^+$) with $3x \pm 2$ and $2xy+1$ all prime. Also, any integer $n > 7$ with $n \neq 17$ can be written as $x+y$ ($x, y \in \mathbb{Z}^+$) with $2x \pm 3$ and $2xy+1$ all prime.*

(iii) *Each integer $n > 2$ with $n \neq 28$ can be written as $x+y$ ($x, y \in \mathbb{Z}^+$) with $2x+1$, $2y-1$ and $2xy+1$ all prime. Also, any integer $n > 2$ with $n \neq 9$, 96 can be written as $x+y$ ($x, y \in \mathbb{Z}^+$) with $2x+1$, $2y-1$ and $2xy-1$ all prime.*

Remark 4.12. We have verified the first assertion in part (i) for n up to 10^9, and all the remaining statements in Conj. 4.12 for n up to 10^8. Note that part (iii) implies the Goldbach conjecture since $2(x+y)=(2x+1)+(2y-1)$. By the argument in Remark 4.4(b), parts (i)-(ii) of Conj. 4.12 imply that there are infinitely many twin primes, cousin primes and sexy primes.

Conjecture 4.13 (2012-12-16)**.** (i) *Each odd integer $n>1$ can be written as $x+y$ ($x,y\in\mathbb{Z}^+$) with $3x\pm1$ and x^2+y^2 all prime. Also, any odd integer $n>3$ can be written as $x+y$ ($x,y\in\mathbb{Z}^+$) with $3x\pm2$ and x^2+y^2 all prime.*

(ii) *Each odd integer $n>10$ can be written as $x+y$ ($x,y\in\mathbb{Z}^+$) with $x\pm3$ and x^2+y^2 all prime. Also, any odd integer $n>3$ can be written as $x+y$ ($x,y\in\mathbb{Z}^+$) with $2x\pm3$ and x^2+y^2 all prime, and any odd integer $n>13$ not among* 47, 209, 239, 253 *can be written as $x+y$ ($x,y\in\mathbb{Z}^+$) with x, $x\pm6$ and x^2+y^2 all prime.*

(iii) *Any even integer $n>2$ can be written as $p+q$ ($q\in\mathbb{Z}^+$) with p, $2p+1$ and $(p-1)^2+q^2$ prime.*

(iv) *Each integer $n>2$ can be written as $x+y$ ($x,y\in\mathbb{Z}^+$) with $2xy+1$ and $x^2+y^2-3\{n-1\}_2$ both prime. Also, any integer $n>2$ can be written as $x+y$ ($x,y\in\mathbb{Z}^+$ and $x<n/2$) with $x^2+y^2-3\{n-1\}_2$ prime and*

$$\left(\frac{x}{n+3\{n-1\}_2}\right)=1.$$

Remark 4.13. We have verified all the assertions in Conj. 4.13 for n up to 10^8. Concerning part (iv) we remark that $(x+y)^2+1=x^2+y^2+(2xy+1)=x^2+y^2+2(xy-1)+3$.

Conjecture 4.14 (2012-12-14)**.** (i) *Every $n\in\mathbb{Z}^+$ can be written as $x+y$ ($x,y\in\mathbb{N}$) with x^3+2y^3 prime. In general, for each positive odd integer m, any sufficiently large integer can be written as $x+y$ ($x,y\in\mathbb{N}$) with x^m+2y^m prime.*

(ii) *Any even integer $n>1194$ can be written as $x+y$ ($x,y\in\mathbb{Z}^+$) with x^3+2y^3 and $2x^3+y^3$ both all prime. Also, any integer $n>25537$ can be written as $p+q$ ($q\in\mathbb{Z}^+$) with p, $p\pm6$ and p^3+2q^3 all prime.*

(iii) *Each integer $n>527$ can be written as $x+y$ ($x,y\in\mathbb{Z}^+$) with $2x+1$, $2y-1$ and x^3+2y^3 all prime. Also, any integer $n>1544$ can be written as $x+y$ ($x,y\in\mathbb{Z}^+$) with $2x-1$, $2y+1$ and x^3+2y^3 all prime.*

(iv) *Any integer $n>392$ can be written as $x+y$ ($x,y\in\mathbb{Z}^+$) with $3x+2$, $3x+4$ and x^3+2y^3 all prime, and any integer $n>1737$ can be written as $x+y$ ($x,y\in\mathbb{Z}^+$) with $6x+1$, $6x+5$ and x^3+2y^3 all prime.*

(v) *Any odd integer $n>2060$ can be written as $2p+q$ with p, q and $p^3+2((q-1)/2)^3$ all prime. Also, any positive integer not among* 1, 49, 53, 567 *can be written as $x+y$ ($x,y\in\mathbb{Z}^+$) with $2xy+1$ and x^3+2y^3 both prime.*

Remark 4.14. In 2001 D. R. Heath-Brown [HB] proved that there are infinitely many primes in the form x^3+2y^3 with $x,y\in\mathbb{Z}^+$. We have verified the first the assertion in part (i) of Conj. 4.14 for n up to 10^9.

Conjecture 4.15 (2012-12-14)**.** (i) *Each odd integer $n > 1$ can be written as $x + y$ $(x, y \in \mathbb{Z}^+)$ with $x^4 + y^2$ prime. Moreover, any odd integer $n > 1621$ can be written as $x + y$ $(x, y \in \mathbb{Z}^+)$ with $x^4 + y^2$ and $x^2 + y^4$ both prime.*

(ii) *Each odd integer $n > 15050$ can be written as $p + 2q$ with p, q and $p^4 + (2q)^2$ all prime. Also, any odd integer $n > 16260$ can be written as $p + 2q$ with p, q and $p^2 + (2q)^4$ all prime.*

(iii) *Any integer $n > 3662$ can be written as $x+y$ $(x, y \in \mathbb{Z}^+)$ with $3(xy)^3 \pm 1$ both prime, and each integer $n > 7425$ can be written as $x + y$ $(x, y \in \mathbb{Z}^+)$ with $2(xy)^4 \pm 1$ both prime. Also, any integer $n > 22$ can be written as $x + y$ $(x, y \in \mathbb{Z}^+)$ with $(xy)^4 + 1$ prime.*

Remark 4.15. Recall that there are infinitely many primes in the form $x^4 + y^2$ by [FI].

Conjecture 4.16 (2012-12-16)**.** (i) *Any odd integer $n > 5$ can be written as $p + q$ $(q \in \mathbb{Z}^+)$ with p, $p + 6$ and $p^2 + 3q^2$ all prime, and any odd integer $n > 35$ can be written as $p + q$ $(q \in \mathbb{Z}^+)$ with p, $p + 2$ and $p^2 + 3q^2$ all prime. Also, any integer $n > 1$ not among 8, 22, 78 can be written as $x + y$ $(x, y \in \mathbb{Z}^+)$ with $3xy - 1$ and $x^2 + 3y^2 + \{n - 1\}_2$ both prime.*

(ii) *Any odd integer $n > 1$ with $n \neq 47$ can be written as $x + y$ $(x, y \in \mathbb{Z}^+)$ with $6x \pm 1$ and $x^4 + 3y^4$ all prime.*

(iii) *Any odd integer $n > 1$ other than 13 and 21 can be written as $p+q$ $(p, q \in \mathbb{Z}^+)$ with p and $p^6 + 3q^6$ both prime.*

Remark 4.16. We omit some other less elegant conjectures of the same nature.

Conjecture 4.17 (2012-12-16)**.** *Let m be a positive integer. Then any sufficiently large odd integer n can be written as $x + y$ $(x, y \in \mathbb{Z}^+)$ with $x^m + 3y^m$ prime (and hence there are infinitely many primes of the form $x^m + 3y^m$), and any sufficiently large even integer n can be written as $x + y$ $(x, y \in \mathbb{Z}^+)$ with $x^m + 3y^m + 1$ prime (and hence there are infinitely many primes of the form $x^m + 3y^m + 1$). In particular, when $m \in \{1, 2, 3, 6\}$ every positive integer can be written as $x + y$ $(x, y \in \mathbb{N})$ with $x^m + 3y^m + \{n - 1\}_2$ prime; for $m = 4, 5, 18$ each positive odd integer can be written as $x + y$ $(x, y \in \mathbb{N})$ with $x^m + 3y^m$ prime.*

Remark 4.17. We have verified the conjecture in the case $m = 18$ for all positive odd integers not exceeding 2×10^6, and the reader may consult [S, A220572] for the behavior of the number of ways to write $2n - 1 = x + y$ $(x, y \in \mathbb{N})$ with $x^{18} + 3y^{18}$ prime. Conj. 4.17 can be strengthened in various ways, for example, any sufficiently large odd integer n can be written as $p + q$ $(p, q \in \mathbb{Z}^+)$ with p, $p \pm 6$ and $p^m + 3q^m$ all prime, and $n > 9$ suffices in the case $m = 1$. Also, any sufficiently large integer n can be written as $x + y$ $(x, y \in \mathbb{Z}^+)$ with $3xy - 1$ and $x^m + 2y^m + \{n - 1\}_2$ both prime; in particular, for $m = 1, 2, 3, 4$ any odd integer $n > 1$ can be written as $x + y$ $(x, y \in \mathbb{Z}^+)$ with $3xy - 1$ and $x^m + 3y^m$ both prime.

For each $m = 1, 2, 3, \ldots$, let $f(m)$ be the smallest positive odd integer s such that any odd integer $n \geqslant s$ can be written as $x + y$ $(x, y \in \mathbb{N})$ with $x^m + 3y^m$ prime, and define $f(m) = 0$ if such an s does not exist. Our computation lead us to guess the following 20 initial values of $f(m)$:

$$f(1) = f(2) = f(3) = f(4) = f(5) = f(6) = 1,\ f(7) = 33,\ f(8) = 11,$$
$$f(9) = 25,\ f(10) = 31,\ f(11) = 49,\ f(12) = 37,\ f(13) = 73,\ f(14) = 147,$$
$$f(15) = f(16) = 49,\ f(17) = 153,\ f(18) = 1,\ f(19) = 239,\ f(20) = 85.$$

Conjecture 4.18 (2012-12-18)**.** (i) *Any integer $n > 210$ can be written as $x + y$ $(x, y \in \mathbb{Z}^+)$ with $p = 3xy - 1$ and $2p - 1 = 6xy - 1$ both prime.*

(ii) *Any odd integer $n > 1$ with $n \neq 43$ can be written as $x + y$ $(x, y \in \mathbb{Z}^+)$ with $2xy + 1$ and $x^2 + y$ both prime.*

(iii) *For each positive integer m, any sufficiently large integer n can be written as $x+y$ $(x, y \in \mathbb{Z}^+)$ with $(xy)^m+3$ prime. In particular, for $m = 1, 2, 3, 4, 5, 6$ it suffices to require that n is greater than* 2, 176, 466, 788, 1058, 440 *respectively.*

Remark 4.18. We have verified part (i) for n up to 2.7×10^7.

Conjecture 4.19 (2012-12-18)**.** (i) *For each positive integer n, there is an integer $k \in \{0, \ldots, n\}$ such that $n + k$ and $n + k^2$ are both prime. Moreover, for any integer $n > 971$, there is a positive integer $k < \sqrt{n} \log n$ such that $n + k$ and $n + k^2$ are both prime. Also, for any integer $n > 43181$ there is a positive integer $k \leqslant \sqrt{n}$ such that $n + k^2$ is prime.*

(ii) *If a positive integer n is not among*

1, 16, 76, 166, 316, 341, 361, 411, 481, 556, 656, 766, 866, 1456,

then $n \pm k$ and $n + k^2$ are all prime for some $k = 0, \ldots, n - 1$, i.e., there is a prime $p \leqslant n$ such that $2n - p$ and $n + (n - p)^2$ are both prime.

(iii) *For any odd integer $n > 1$, there is an integer $0 \leqslant k < n$ such that $n+k$ and $k^2+(n-k)^2$ are both prime. For any odd integer $n > 5$, there is an integer $0 \leqslant k < n$ such that $n + k^2$ and $k^2 + (n - k)^2$ are both prime. Also, for any integer $n > 146$ there is an integer $0 \leqslant k < n$ such that $n + k^2$ and $k + n^2$ are both prime.*

(iv) *For any integer $n > 1$ there is an integer $0 \leqslant k < n$ such that $n + k$ and $2k(n - k) + 1$ are both prime. Also, for any integer $n > 182$ there is an integer $0 \leqslant k < n$ such that $n + k^2$ and $2k(n - k) + 1$ are both prime.*

Remark 4.19. We have verified this conjecture for n up to 10^9. Note that Bertrand's postulate proved by Chebyshev in 1850 states that for any $n \in \mathbb{Z}^+$ there is a prime in the interval $[n, 2n]$. For $a(n) = |\{0 \leqslant k < n :\ n+k \text{ and } n+k^2 \text{ are both prime}\}|$, the reader may consult [S, A185636]. Clearly Part (ii) of Conj. 4.19 is stronger than the Goldbach conjecture. We also conjecture that for each $n \in \mathbb{Z}^+$ there is an integer $k \in \{0, \ldots, n - 1\}$ with $n + k^3$ practical, moreover we may require $k \leqslant \sqrt{n} \log n$ whenever $n \neq 74,\ 138,\ 166,\ 542$.

Conjecture 4.20 (2012-12-18). *For each $m = 3, 4, 5, \ldots$, if $n \in \mathbb{Z}^+$ is sufficiently large, then $n + p_m(k)$ is prime for some $k = 0, \ldots, n-1$, where $p_m(k) = (m-2)k(k-1)/2 + k$ is an m-gonal number. In particular, for $m = 3, \ldots, 20$ it suffices to require that n is greater than*

$$1,\ 1,\ 1,\ 1,\ 14,\ 1,\ 1,\ 38,\ 1,\ 1,\ 1,\ 1,\ 9,\ 20,\ 1,\ 33,\ 14,\ 1$$

respectively.

Remark 4.20. Note that squares are just 4-gonal numbers.

Conjecture 4.21 (2012-12-20). (i) *For every positive integer n there exists $k \in \{1, \ldots, n\}$ such that $n + k$ and $kn + 1$ are both prime. For any integer $n > 101$ there is an integer $0 < k < n$ such that $kn - 1$ is a Sophie Germain prime.*

(ii) *For any integer $n > 3$, there exists $k \in \{1, \ldots, n\}$ such that $p = kn + 1$ is a prime with $(\frac{n}{p}) = 1$, also $k(n-k) - 1$ and $kn + 1$ are both prime for some $k = 1, \ldots, n$.*

(iii) *For any integer $n > 1$ there exists $k \in \{1, \ldots, n\}$ such that $3k \pm 1$ and $kn + 1$ are both prime.*

(iv) *For any odd integer $n > 1$ there exists $k \in \{1, \ldots, n\}$ such that $kn + 1$ and $k^2 + (n-k)^2$ are both prime.*

(v) *For a given positive odd integer m and sufficiently large integer n, there is an integer $k \in \{1, \ldots, n\}$ such that $k \pm m$ and $kn + 1$ are all prime. In particular, for any integer $n > 8$ with $n \neq 34$, there exists $k \in \{1, \ldots, n\}$ such that $k \pm 3$ and $kn + 1$ are all prime; also, for any integer $n > 3$ not among 5, 8, 14, 53, 82, there exists $k \in \{1, \ldots, n\}$ such that $k \pm 1$ and $kn + 1$ are all prime.*

Remark 4.21. In 2001 A. Murthy [Mu] conjectured that for any integer $n > 1$ there is $0 \leqslant k < n$ with $kn + 1$ prime.

Conjecture 4.22 (2013-01-07). (i) *For integer $n > 17261$, there is an integer $0 < k < \sqrt{n} \log n$ such that both $kn - 1$ and $kn + 1$ are prime. Consequently, for each $n = 128, 129, \ldots$ there is a positive integer $k < n$ with $kn - 1$ and $kn + 1$ both prime.*

(ii) *For any integer $n > 1$ there is an integer $k \in \{0, \ldots, n-1\}$ such that $2k + 3$, $n(n-k) - 1$ and $n(n+k) - 1$ are all prime.*

Remark 4.22. We have verified Conj. 4.22 for n up to 3×10^7.

Conjecture 4.23 (2013-01-14). (i) *Every $n \in \mathbb{Z}^+$ can be represented as the sum of a practical number and a triangular number. Also, for each $n \in \mathbb{Z}^+$ there is a practical number $m \in [n, 2n)$ with $m - n$ a triangular number.*

(ii) *Each odd number $n > 1$ can be written as the sum of a Sophie Germain prime and a triangular number.*

(iii) *Any odd number* $n > 1$ *can be written as* $p + q$*, where* p *is prime,* q *is practical, and* $p^4 + q^4$ *is prime. We may also replace* $p^4 + q^4$ *by* $p^2 + q^2$*.*

Remark 4.23. We have verified the first assertion and the second assertion in part (i) for n up to 10^8 and 4.2×10^6 respectively. Parts (ii) and (iii) have been verified for n below 10^8.

Conjecture 4.24 (2013-01-28)**.** (i) *Any even number greater than* 4 *can be written as* $x + y$ $(x, y > 0)$*, where* $\{x-1, x, x+1\}$ *is a sandwich of the second kind, and* $x^3 + y^3$ *is practical. In general, for each* $m = 2, 3, \ldots,$ *all sufficiently large even numbers can be written in the form* $x + y$ $(x, y > 0)$*, where* $\{x-1, x, x+1\}$ *is a sandwich of the second kind, and* $x^m + y^m$ *is practical.*

(ii) *Any positive even integer can be written as* $p + q$*, where* p *and* q *are practical numbers with* $p^6 + q^6$ *also practical.*

Remark 4.24. It seems that for any $m, n \in \mathbb{Z}^+$ with $m > 1$ or $n > 337$, we may write $2n = p + q$ with $p, q, p^{3m} + q^{3m}$ all practical.

Conjecture 4.25 (2013-01-12)**.** (i) *Any integer* $n > 8$ *can be written as* $p+q$*, where* p *is prime or practical, and* $q, q \pm 4$ *are all practical. Also, each integer* $n > 5$ *can be written as* $p+q$*, where* p *and* $p+6$ *are both prime or both practical, and* q *is practical.*

(ii) *Any integer* $n > 10$ *can be written as* $x + y$ *with* $6x \pm 1$ *both prime, and* y *and* $y + 6$ *both practical.*

(iii) *Let* $n > 1$ *be an integer. Then* n *can be written as* $x^2 + y$ $(x, y \in \mathbb{Z}^+)$ *with* $2x$ *and* $2xy$ *both practical. Also, we may write* $n = x^3 + y$ $(x, y \in \mathbb{Z}^+)$ *with* $2x$ *and* $4xy$ *both practical.*

Remark 4.25. Note that if x is practical and $y \in \{1, \ldots, x\}$ then xy is also practical.

Conjecture 4.26 (2013-01-23)**.** (i) *Each* $n = 4, 5, \ldots$ *can be written as* $p+q$*, where* p *is a prime with* $p-1$ *and* $p+1$ *both practical, and* q *is either prime or practical.*

(ii) *Any odd number* $n > 7$ *not among* 223, 875, 899, 923 *can be written as* $2p + q$ *with* p *and* q *both prime, and* $p - 1$ *and* $p + 1$ *both practical.*

(iii) *Any odd number* $n \geqslant 5$ *not equal to* 55 *can be written as* $p + q$*, where* p *and* $p + 2$ *are twin primes, and* $p + 1$ *and* q *are both practical.*

Remark 4.26. (a) We have verified part (i) for n up to 10^8. For numbers of the described representations in part (i), see [S, A210480]. It follows from part (i) that any integer $n \geqslant 3$ can be written as $p + q$ with p prime or practical, and q and $q + 2$ both practical. We also conjecture that any integer $n > 4$ can be represented by $p+q/2$, where p and q are practical numbers smaller than n (cf. [S, A214841]).

(b) As there is an interval of any given length containing no primes (or practical numbers), part (i) or (ii) implies that there are infinitely many sandwiches of the first kind. Similarly, part (iii) implies that there are infinitely many sandwiches of the second kind.

Conjecture 4.27 (2013-01-29)**.** *Let*

$$S = \{\text{prime } p: \ p-1 \text{ and } p+1 \text{ are both practical}\}$$

and

$$T = \{\text{practical number } q: \ q-1 \text{ and } q+1 \text{ are both prime}\}.$$

(i) *Any integer $n > 11$ can be written as $(1 + \{n\}_2)p + q + r$ with $p, q \in S$ and $r \in T$.*

(ii) *Each integer $n > 6$ can be written as $p+q+r$ with $p, q \in S$ and $6r \in T$. Also, every $n = 3, 4, \ldots$ can be represented as $x + y + z$ with $6x, 6y, 6z \in T$, and any even number greater than 10 is a sum of four elements of S.*

(iii) *Any integer $n > 7$ is the sum of an element of S, an element of T and a square. Also, each $n = 3, 4, \ldots$ can be written as the sum of an element of S and two triangular numbers.*

Remark 4.27. Clearly part (i) is much stronger than Goldbach's weak conjecture for odd numbers. We have verified part (i) for n up to 10^7. For numbers of representations related to part (i), see [S, A210681]. Our calculation suggests that $\sum_{p\in S} 1/p \approx 0.994$ and that the number of elements of S not exceeding x is asymptotically equivalent to $cx/\log^3 x$, where c is a constant in the interval (5.86, 5.87).

Conjecture 4.28 (2013-01-30)**.** (i) *Each integer $n > 5$ can be written as the sum of a prime p with $p-1$ and $p+1$ both practical, a prime q with $q+2$ also prime, and a Fibonacci number.*

(ii) *Any integer $n \geqslant 10$ can be written as $(1 + \{n\}_2)p + q + 2^k$, where p is a prime with $p-1$ and $p+1$ both practical, $\{q, q+2\}$ is a twin prime pair, and k is a positive integer.*

Remark 4.28. We have verified part (i) for n up to 2×10^6.

Conjecture 4.29 (2013-01-30)**.** *Let $a \leqslant b \leqslant c$ be positive integers and let S be the set of those primes p with $p-1$ and $p+1$ both practical. Then all integers $n \geqslant 3(a+b+c)$ with $n \equiv a+b+c \pmod 2$ can be written as $ap+bq+cr$ with $p, q, r \in S$, if and only if (a, b, c) is among the following six triples:*

$$(1, 2, 3),\ (1, 2, 4),\ (1, 2, 8),\ (1, 2, 9),\ (1, 3, 5),\ (1, 3, 8).$$

In particular, any even number greater than 16 can be written as $p + 2q + 3r$ with $p, q, r \in S$.

Remark 4.29. Besides the six triples listed in Conj. 4.29, there are also several triples (a, b, c) such that all sufficiently large integers $n \equiv a+b+c \pmod 2$ can be expressed as $ap + bq + cr$ with $p, q, r \in S$. For example, any even number greater than 48 can be written as $p + 4q + 9r$ with $p, q, r \in S$.

Conjecture 4.30 (2013-01-30). *Any odd number $n > 8$ with $n \not\equiv \pm 1 \pmod{12}$ and $n \neq 201,\ 447$ can be written as the sum of three elements of the set S defined in Conj.* 4.27 *or* 4.29.

Remark 4.30. No element of S can be congruent to 1 or -1 modulo 12. In fact, if $p > 3$ and $p \equiv 1 \pmod{12}$, then neither 3 nor 4 divides $p+1$, hence $p+1$ is not practical since 4 cannot be a sum of some distinct divisors of $p+1$. Similarly, if $p \equiv -1 \pmod{12}$ then $p-1$ is not practical.

5. Conjectures involving alternating sums of consecutive primes

For each positive integer n, let p_n denote the nth prime. In [S13b] the author conjectured that for any positive integer m there are consecutive primes $p_k, \ldots, p_n$ $(k < n)$ not exceeding $2m + 2.2\sqrt{m}$ such that $m = p_n - p_{n-1} + \cdots + (-1)^{n-k} p_k$. Here we give a variant of this conjecture involving practical numbers.

Conjecture 5.1 (2012-02-25). (i) *Any integer m can be written in the form $p_n - p_{n-1} + \cdots + (-1)^{n-k} p_k$ with $k < n$ and $p_n \leqslant 3m$, and $p_n + 1$ and $p_k - 1$ both practical.*

(ii) *For each $m \in \mathbb{Z}^+$, let $f(m)$ be the least prime p_n with $p_n + 1$ practical such that $m = p_n - p_{n-1} + \cdots + (-1)^{n-k} p_k$ for some $k < n$ with $p_k - 1$ practical. Then*
$$\lim_{n\to\infty} \frac{f(2n-1)}{2n-1} = 1 \quad \textit{and} \quad \lim_{n\to\infty} \frac{f(2n)}{2n} = 2.$$

Remark 5.1. The reader may consult [S, A222579 and A222580] for related data and sequences. Here we give a concrete example:
$$806 = p_{358} - p_{357} + \cdots + p_{150} - p_{149}$$
with $f(806) = p_{358} = 2411 < 3 \times 806$, and $p_{358} + 1 = 2412$ and $p_{149} - 1 = 858$ both practical.

In this section we set $s_n = p_n - p_{n-1} + \cdots + (-1)^{n-k} p_k$ for $n = 1, 2, 3, \ldots$.

Conjecture 5.2 (2013-02-27). *For any $m \in \mathbb{Z}^+$ and $r \in \mathbb{Z}$, there are infinitely many positive integers n such that $s_n \equiv r \pmod{m}$.*

Remark 5.2. This is an analogy of Dirichlet's theorem on primes in arithmetic progressions.

Conjecture 5.3 (2013-02-27). *Every $n = 3, 4, \ldots$ can be written as $p + s_k$ $(k > 0)$, where p is a Sophie Germain prime.*

Remark 5.3. Let $r(n)$ be the number of ways to write n in the form $p + s_k$ $(k > 0)$ with p a Sophie Germain prime. The sequence $r(n)$ $(n = 1, 2, 3, \ldots)$ is available from [S, A213202]. We have verified Conj. 5.3 for n up to 3.35×10^7.

Conjecture 5.4. (i) (2013-03-01) *Any integer $n > 8$ can be written as $q + s_k$ $(k > 0)$, where q is a practical number with $q - 4$ and $q + 4$ also practical.*

(ii) (2013-02-27) *Any integer $n > 3$ different from 65 and 365 can be written as $p + s_k$ $(k > 0)$, where p is a prime with $p - 1$ and $p + 1$ both practical.*

Remark 5.4. We have verified Conj. 5.4(i) for n up to 5×10^6.

Conjecture 5.5. (2013-02-27) *Any integer $n > 1$ can be written as $j(j+1)/2+s_k$, where j and k are positive integers.*

Remark 5.5. We have verified this for n up to 6×10^6.

Conjecture 5.6. (2013-03-05) *For each $\lambda = 1, 2, 3$, any integer $n > \lambda$ can be written as $s_k + \lambda s_l$ with $k, l \in \mathbb{Z}^+$.*

Remark 5.6. We also have some similar conjectures, for example, any integer $n > 12$ can be written as $s_k + 6s_l$ with $k, l \in \mathbb{Z}^+$.

Conjecture 5.7. (i) (2013-02-27) *Each $n = 6, 7, \ldots$ can be written as $p + s_k$ $(k > 0)$, where p is a prime with $p + 6$ also prime.*

(ii) (2013-03-01) *Any integer $n > 2$ can be written as $q + s_k$ $(k > 0)$ with $3q - 1$ and $3q + 1$ both prime. Also, each integer $n > 3$ can be written as $q + s_k$ $(k > 0)$ with $3q - 2$ and $3q + 2$ both prime.*

6. Some other conjectures involving quadratic forms

Conjecture 6.1. *Let $d \in \mathbb{Z}^+$ and $d \not\equiv 2 \pmod 6$.*

(i) (2011-11-05) *If d is odd, then there is a prime $p(d)$ such that for any prime $p > p(d)$ there is a prime $q < p$ with $p^2 + dpq + q^2$ prime.*

(ii) (2011-11-07) *If d is even, then there is a prime $p(d)$ such that for any prime $p > p(d)$ there is a prime $q < p$ with $p^2 + dq^2$ prime.*

(iii) *We may take*

$$\begin{gathered}
p(1)=5,\ p(3)=2,\ p(4)=3,\ p(5)=61,\ p(6)=p(7)=3,\ p(9)=13,\\
p(10)=5,\ p(11)=7,\ p(12)=p(13)=3,\ p(15)=163,\ p(16)=2,\\
p(17)=13,\ p(18)=3,\ p(19)=5,\ p(21)=2,\ p(22)=11,\ p(23)=2,\\
p(24)=17,\ p(25)=89,\ p(27)=3,\ p(28)=7,\ p(29)=53,\ p(30)=7.
\end{gathered}$$

Remark 6.1. Actually the least prime $q < p$ having the described property in Conj. 6.1 is rather small compared with p.

It is well known that any prime $p \equiv 1 \pmod 4$ can be written uniquely in the form $a_p^2 + b_p^2$ with $a_p, b_p \in \mathbb{Z}^+$ and $a_p > b_p$. (This was found by Fermat and proved by Euler.) During Oct. 3-4, 2012, Tomasz Ordowski [O] conjectured that

$$\lim_{N\to\infty} \frac{\sum_{p\leqslant N,\ p\equiv 1 \pmod 4} a_p}{\sum_{p\leqslant N,\ p\equiv 1 \pmod 4} b_p} = 1 + \sqrt{2}$$

and

$$\lim_{N\to\infty}\frac{\sum_{p\leqslant N,\ p\equiv1\pmod 4}a_p^2}{\sum_{p\leqslant N,\ p\equiv1\pmod 4}b_p^2}=\frac{9}{2}.$$

The following two conjectures have the same nature.

Conjecture 6.2 (2012-11-03). *For any prime $p\equiv 1\pmod 3$ write $p=x_p^2+x_py_p+y_p^2$ with $x_p,y_p\in\mathbb{Z}^+$ and $x_p>y_p$. Then*

$$\lim_{N\to\infty}\frac{\sum_{p\leqslant N,\ p\equiv1\pmod 3}x_p}{\sum_{p\leqslant N,\ p\equiv1\pmod 3}y_p}=1+\sqrt{3}$$

and

$$\lim_{N\to\infty}\frac{\sum_{p\leqslant N,\ p\equiv1\pmod 3}x_p^2}{\sum_{p\leqslant N,\ p\equiv1\pmod 3}y_p^2}=\frac{52}{9}.$$

Remark 6.2. It seems that

$$\lim_{N\to\infty}\frac{\sum_{p\leqslant N,\ p\equiv1\pmod 3}x_p^3}{\sum_{p\leqslant N,\ p\equiv1\pmod 3}y_p^3}\approx 11.15\quad\text{and}\quad\lim_{N\to\infty}\frac{\sum_{p\leqslant N,\ p\equiv1\pmod 3}x_p^3}{\sum_{p\leqslant N,\ p\equiv1\pmod 3}y_p^3}\approx 20.6.$$

Conjecture 6.3 (2012-11-04). *For any prime $p\equiv\pm1\pmod 5$ write $p=u_p^2+3u_pv_p+v_p^2$ with $u_p,v_p\in\mathbb{Z}^+$ and $u_p>v_p$. Then*

$$\lim_{N\to\infty}\frac{\sum_{p\leqslant N,\ p\equiv\pm1\pmod 5}u_p}{\sum_{p\leqslant N,\ p\equiv\pm1\pmod 5}v_p}=1+\sqrt{5}.$$

Remark 6.3. It seems that

$$\lim_{N\to\infty}\frac{\sum_{p\leqslant N,\ p\equiv\pm1\pmod 5}u_p^2}{\sum_{p\leqslant N,\ p\equiv\pm1\pmod 5}v_p^2}\approx 8.185.$$

Recall that the Fibonacci numbers $F_0,F_1,F_2,\ldots$ and the Lucas numbers $L_0,L_1,L_2,\ldots$ are given by

$$F_0=0,\ F_1=1,\ F_{n+1}=F_n+F_{n-1}\ (n=1,2,3,\ldots)$$

and

$$L_0=2,\ L_1=1,\ L_{n+1}=L_n+L_{n-1}\ (n=1,2,3,\ldots)$$

respectively.

Conjecture 6.4 (2012-11-03)**.** *Let $p\ne 2,5$ be a prime. If $(\frac{-1}{p})=(\frac{5}{p})=1$ (i.e., $p\equiv 1,9 \pmod{20}$) and $p=x^2+5y^2$ with $x,y\in\mathbb{Z}$, then*

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}F_{6k}\equiv&0 \pmod{p^3},\\ \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}L_{6k}\equiv&(-1)^y(8x^2-4p) \pmod{p^2},\\ \sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^3}{64^k}F_{6k}\equiv&\frac{(-1)^y}{10}(3p-4x^2) \pmod{p^2}.\end{aligned}$$

If $(\frac{-5}{p})=-1$ (i.e., $p\equiv 11,13,17,19 \pmod{20}$), then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}F_{6k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}L_{6k}\equiv 0 \pmod{p^2},\ \text{and}\ \sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^3}{64^k}F_{6k}\equiv 0 \pmod{p}.$$

Conjecture 6.5 (2012-11-03)**.** *Let $p\ne 2,5$ be a prime. If $(\frac{-2}{p})=(\frac{5}{p})=1$ (i.e., $p\equiv 1,9,11,19 \pmod{40}$) and $p=x^2+10y^2$ with $x,y\in\mathbb{Z}$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}L_{12k}\equiv\left(\frac{-1}{p}\right)(8x^2-4p) \pmod{p^2};$$

if $p\equiv 1,9 \pmod{40}$ then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}F_{12k}\equiv 0 \pmod{p^3}.$$

If $(\frac{-2}{p})=(\frac{5}{p})=-1$ (i.e., $p\equiv 7,13,23,37 \pmod{40}$) and $p=2x^2+5y^2$ with $x,y\in\mathbb{Z}$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}F_{12k}\equiv 16\left(\frac{-1}{p}\right)(4x^2-p) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}L_{12k}\equiv 36\left(\frac{-1}{p}\right)(p-4x^2) \pmod{p^2}.$$

If $(\frac{-10}{p})=-1$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}F_{12k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}L_{12k}\equiv 0 \pmod{p^2}.$$

Conjecture 6.6 (2012-11-03)**.** *Let $p \neq 2,5$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}F_{24k}\equiv\begin{cases}0 \pmod{p^3} & \text{if } p\equiv 1,9 \pmod{20},\\ 0 \pmod{p^2} & \text{if } p\equiv 3,7,11,19 \pmod{20},\\ 288(p-2x^2) \pmod{p^2} & \text{if } p=x^2+4y^2\equiv 13,17 \pmod{20},\end{cases}$$

and

$$\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^3}{64^k}F_{24k}\equiv\begin{cases}(-1)^y(3p-4x^2)/6 \pmod{p^2} & \text{if } p=x^2+25y^2\equiv 1,9 \pmod{20},\\ 110x^2/3 \pmod{p} & \text{if } p=x^2+4y^2\ \&\ (\frac{p}{5})=-1,\\ 0 \pmod{p} & \text{if } p\equiv 3 \pmod{4}.\end{cases}$$

Also,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}L_{24k}\equiv\begin{cases}(81-80(\frac{p}{5}))(8x^2-4p) \pmod{p^2} & \text{if } p=x^2+4y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod{4},\end{cases}$$

and

$$\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^3}{64^k}L_{24k}\equiv\begin{cases}(-1)^y(3p-4x^2)/2 \pmod{p^2} & \text{if } p=x^2+25y^2\equiv 1,9 \pmod{20},\\ -82x^2 \pmod{p} & \text{if } p=x^2+4y^2\ \&\ (\frac{p}{5})=-1,\\ 0 \pmod{p} & \text{if } p>3\ \&\ p\equiv 3 \pmod{4}.\end{cases}$$

The Pell sequence $(P_n)_{n\geqslant 0}$ and its companion $(Q_n)_{n\geqslant 0}$ are given by

$$P_0=0,\ P_1=1,\ \text{and } P_{n+1}=2P_n+P_{n-1}\ (n=1,2,3,\ldots)$$

and

$$Q_0=2,\ Q_1=2,\ \text{and } Q_{n+1}=2Q_n+Q_{n-1}\ (n=1,2,3,\ldots).$$

Conjecture 6.7 (2012-11-02)**.** *Let p be an odd prime. When $p\equiv 1,3 \pmod 8$ and $p=x^2+2y^2$ with $x,y\in\mathbb{Z}$, we have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-8)^k}Q_{3k}\equiv\left(2-\left(\frac{-1}{p}\right)\right)(8x^2-4p) \pmod{p^2},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-8)^k}P_{3k}\equiv\begin{cases}0 \pmod{p^3} & \text{if } p\equiv 1 \pmod 8,\\ 4p-8x^2 \pmod{p^2} & \text{if } p\equiv 3 \pmod 8,\end{cases}$$

$$14\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^3}{(-8)^k}P_{3k}\equiv\begin{cases}3p-4x^2 \pmod{p^2} & \text{if } p\equiv 1 \pmod 8,\\ 20x^2+21p \pmod{p^2} & \text{if } p\equiv 3 \pmod 8.\end{cases}$$

If $p\equiv 1\pmod 8$, *then*

$$\sum_{k=0}^{p-1}(7k+2)\frac{\binom{2k}{k}^3}{(-8)^k}Q_{3k}\equiv 4p\pmod{p^3};$$

if $p\equiv 3\pmod 8$, *then*

$$\sum_{k=0}^{p-1}(21k+4)\frac{\binom{2k}{k}^3}{(-8)^k}Q_{3k}\equiv -132p\pmod{p^3}$$

and

$$\sum_{k=0}^{p-1}(28k+5)\frac{\binom{2k}{k}^3}{(-8)^k}P_{3k}\equiv 62p\pmod{p^3}.$$

If $p\equiv 5,7\pmod 8$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-8)^k}P_{3k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-8)^k}Q_{3k}\equiv 0\pmod{p^2},$$

$$14\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^3}{(-8)^k}P_{3k}\equiv -p\left(16+15\left(\frac{-1}{p}\right)\right)\pmod{p^2},$$

and

$$\sum_{k=0}^{p-1}(21k+4)\frac{\binom{2k}{k}^3}{(-8)^k}Q_{3k}\equiv 12p\left(5+6\left(\frac{-1}{p}\right)\right)\pmod{p^2}.$$

Conjecture 6.8 (2013-03-12). *Let* p *be an odd prime. If* $(\frac{-6}{p})=-1$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}P_{4k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}Q_{4k}\equiv 0\pmod{p^2}.$$

If $p\equiv 1,7\pmod{24}$ *and* $p=x^2+6y^2$ $(x,y\in\mathbb{Z})$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}P_{4k}\equiv 0\pmod{p^3}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}Q_{4k}\equiv(-1)^y(8x^2-4p)\pmod{p^2}.$$

When $p\equiv 5,11\pmod{24}$ *and* $p=2x^2+3y^2$ $(x,y\in\mathbb{Z})$, *we have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}P_{4k}\equiv 4\left(\frac{-1}{p}\right)(p-4x^2)\pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}Q_{4k}\equiv 12\left(\frac{-1}{p}\right)(4x^2-p)\pmod{p^2}.$$

Conjecture 6.9 (2013-03-11). *Let p be an odd prime. If $(\frac{-22}{p})=-1$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}P_{12k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}Q_{12k}\equiv 0 \pmod{p^2}.$$

If $(\frac{2}{p})=(\frac{p}{11})=1$ and $p=x^2+22y^2$ $(x,y\in\mathbb{Z})$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}P_{12k}\equiv 0 \pmod{p^3}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}Q_{12k}\equiv(-1)^y(8x^2-4p) \pmod{p^2}.$$

When $(\frac{2}{p})=(\frac{p}{11})=-1$ and $p=2x^2+11y^2$ $(x,y\in\mathbb{Z})$, we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}P_{12k}\equiv 140\left(\frac{-1}{p}\right)(p-4x^2) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}Q_{12k}\equiv 396\left(\frac{-1}{p}\right)(4x^2-p) \pmod{p^2}.$$

Let A and B be integers. The Lucas sequence $u_n=u_n(A,B)$ $(n\in\mathbb{N})$ and its companion $v_n=v_n(A,B)$ $(n\in\mathbb{N})$ are defined by

$$u_0=0,\ u_1=1,\ \text{and}\ u_{n+1}=Au_n-Bu_{n-1}\ (n=1,2,3,\ldots);$$
$$v_0=2,\ v_1=A,\ \text{and}\ \ v_{n+1}=Av_n-Bv_{n-1}\ (n=1,2,3,\ldots).$$

Conjecture 6.10 (2011-11-03). *Let p be an odd prime. If $(\frac{-13}{p})=-1$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}u_{6k}(3,-1)\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}v_{6k}(3,-1)\equiv 0 \pmod{p^2}.$$

If $(\frac{-1}{p})=(\frac{p}{13})=1$ and $p=x^2+13y^2$ $(x,y\in\mathbb{Z})$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}u_{6k}(3,-1)\equiv 0 \pmod{p^3}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}v_{6k}(3,-1)\equiv(-1)^y(8x^2-4p) \pmod{p^2}.$$

Conjecture 6.11 (2012-11-03). *Let p be an odd prime. If $(\frac{-58}{p})=-1$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}u_{12k}(5,-1)\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}v_{12k}(5,-1)\equiv 0 \pmod{p^2}.$$

If $(\frac{-2}{p})=(\frac{29}{p})=1$ and $p=x^2+58y^2$ $(x,y\in\mathbb{Z})$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}u_{12k}(5,-1)\equiv 0 \pmod{p^3}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}v_{12k}(5,-1)\equiv\left(\frac{-1}{p}\right)(8x^2-4p) \pmod{p^2}.$$

If $(\frac{-2}{p})=(\frac{29}{p})=-1$ and $p=2x^2+29y^2$ $(x,y\in\mathbb{Z})$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}u_{12k}(5,-1)\equiv 7280\left(\frac{-1}{p}\right)(4x^2-p) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}v_{12k}(5,-1)\equiv 39204\left(\frac{-1}{p}\right)(p-4x^2) \pmod{p^2}.$$

Conjecture 6.12 (2012-11-03). *Let p be an odd prime. If $(\frac{-37}{p})=-1$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}u_{6k}(12,-1)\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}v_{6k}(12,-1)\equiv 0 \pmod{p^2}.$$

If $(\frac{-1}{p})=(\frac{37}{p})=1$ and $p=x^2+37y^2$ $(x,y\in\mathbb{Z})$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}u_{6k}(12,-1)\equiv 0 \pmod{p^3}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}v_{6k}(12,-1)\equiv(-1)^y(8x^2-4p) \pmod{p^2}.$$

Conjecture 6.13 (2013-03-12). *Let p be an odd prime. If $p \equiv 5, 7 \pmod 8$, then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(-64)^k} u_{4k}(10,1) \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(-64)^k} v_{4k}(10,1) \equiv 0 \pmod{p^2}.$$

If $p \equiv 1,\ 19 \pmod{24}$ and $p = x^2 + 2y^2$ $(x, y \in \mathbb{Z})$, then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(-64)^k} u_{4k}(10,1) \equiv 0 \pmod{p^3}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(-64)^k} v_{4k}(10,1) \equiv (-1)^y (8x^2 - 4p) \pmod{p^2}.$$

If $p \equiv 11,\ 17 \pmod{24}$ and $p = x^2 + 2y^2$ $(x, y \in \mathbb{Z})$, then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(-64)^k} u_{4k}(10,1) \equiv 20 \left(\frac{-1}{p}\right) (p - 2x^2) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(-64)^k} v_{4k}(10,1) \equiv 196 \left(\frac{-1}{p}\right) (2x^2 - p) \pmod{p^2}.$$

Remark 6.4. We also note that for any prime $p \equiv \pm 1 \pmod{12}$ we have the congruence $\sum_{k=0}^{p-1} \binom{2k}{k}^3 u_{4k}(4,1)/(-64)^k \equiv 0 \pmod p$.

Conjecture 6.14 (2013-03-12). *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(-2^{12})^k} u_{4k}(5,8) \equiv 0 \pmod{p^2}.$$

When $(\frac{p}{7}) = 1$ (i.e., $p \equiv 1, 2, 4 \pmod 7$), we even have

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(-2^{12})^k} u_{4k}(5,8) \equiv 0 \pmod{p^3}.$$

Also,

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(-4096)^k} v_{4k}(5,8) \equiv \begin{cases} 8x^2 - 4p \pmod{p^2} & \text{if } (\frac{p}{7}) = 1 \ \&\ p = x^2 + 7y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{7}) = -1. \end{cases}$$

If $(\frac{p}{7})=-1$ *and* $p>3$*, then*

$$\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^3}{(-4096)^k}u_{4k}(5,8)\equiv\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^3}{(-4096)^k}v_{4k}(5,8)\equiv 0 \pmod p.$$

If $(\frac{p}{7})=1$ *and* $p=x^2+7y^2$ $(x,y\in\mathbb{Z})$*, then*

$$\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^3}{(-4096)^k}u_{4k}(5,8)\equiv\frac{3p-4x^2}{42} \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^3}{(-4096)^k}v_{4k}(5,8)\equiv\frac{3}{2}p-2x^2 \pmod{p^2}.$$

Conjecture 6.15 (2013-03-13)**.** *Let* p *be an odd prime. If* $p>3$ *and* $(\frac{p}{7})=-1$*, then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}^3(-1)^ku_{3k}(16,1)\equiv\sum_{k=0}^{p-1}\binom{2k}{k}^3(-1)^kv_{3k}(16,1)\equiv 0 \pmod{p^2}.$$

When $(\frac{p}{7})=1$ *and* $p=x^2+7y^2$ $(x,y\in\mathbb{Z})$*, we have*

$$\sum_{k=0}^{p-1}\binom{2k}{k}^3(-1)^ku_{3k}(16,1)\equiv\begin{cases}0 \pmod{p^3} & \text{if } p\equiv 1 \pmod 4,\\ (-1)^y32(p-2x^2) \pmod{p^2} & \text{if } p\equiv 3 \pmod 4,\end{cases}$$

and

$$\sum_{k=0}^{p-1}\binom{2k}{k}^3(-1)^kv_{3k}(16,1)\equiv\left(64\left(\frac{-1}{p}\right)-63\right)(8x^2-4p) \pmod{p^2}.$$

If $(\frac{p}{7})=-1$ *and* $p\neq 3,19$*, then*

$$\sum_{k=0}^{p-1}k\binom{2k}{k}^3(-1)^ku_{3k}(16,1)\equiv\sum_{k=0}^{p-1}k\binom{2k}{k}^3(-1)^kv_{3k}(16,1)\equiv 0 \pmod p.$$

If $(\frac{p}{7})=(\frac{-1}{p})=1$*, then*

$$\sum_{k=0}^{p-1}k\binom{2k}{k}^3(-1)^ku_{3k}(16,1)\equiv\frac{8(3p-4x^2)}{399} \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} k\binom{2k}{k}^3(-1)^k v_{3k}(16,1) \equiv \frac{32(3p-4x^2)}{57} \pmod{p^2}.$$

If $(\frac{p}{7})=1$ *and* $(\frac{-1}{p})=-1$, *then*

$$\sum_{k=0}^{p-1} k\binom{2k}{k}^3(-1)^k u_{3k}(16,1) \equiv -\frac{8}{3591}(3492x^2+4535p) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} k\binom{2k}{k}^3(-1)^k v_{3k}(16,1) \equiv \frac{32}{171}(660x^2+857p) \pmod{p^2}.$$

Conjecture 6.16 (2013-03-14). *Let p be an odd prime. If $p>7$ and $p\equiv 3 \pmod 4$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-72)^k}u_k(24,-3) \equiv \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-72)^k}v_k(24,-3) \equiv 0 \pmod{p^2}.$$

If $p\equiv 1 \pmod{12}$ and $p=x^2+9y^2$ with $x,y\in\mathbb{Z}$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-72)^k}u_k(24,-3) \equiv 0 \pmod{p^3}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-72)^k}v_k(24,-3) \equiv 8x^2-4p \pmod{p^2}.$$

If $p\equiv 5 \pmod{12}$ and $p=x^2+y^2$ with $x,y\in\mathbb{Z}$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-72)^k}u_k(24,-3) \equiv \frac{8}{7}\left(\frac{xy}{3}\right)xy \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-72)^k}v_k(24,-3) \equiv -32\left(\frac{xy}{3}\right)xy \pmod{p^2}.$$

Remark 6.5. Conjectures 6.4-6.15 are similar to the author's previous conjectures on $\sum_{k=0}^{p-1}\binom{2k}{k}^3/m^k$ modulo p^2 mentioned in [S12], and we can prove most of them modulo p. We also have some other conjectures (involving $\binom{2k}{k}^3$ or $\binom{2k}{k}^2\binom{3k}{k}$ or $\binom{2k}{k}^2\binom{4k}{2k}$ or $\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}$) similar to Conj. 6.4-6.16.

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